CHAPTER 3

Ellipticity and boundedness

So, what did we in the previous chapter? We showed that there is an 'oscillatory integral' interpretation of

$$K(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi) d\xi = I(a) \tag{3.1}$$

which defines, for any p, a continuous linear map

$$I\,(1+|x-y|^2)^{\frac{1}{2}} S^{\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

the range of which is the space of pseudodifferential operators on $\mathbb{R}^n$;

$$\begin{gathered} A \in \Psi^m_\infty(\mathbb{R}^n) \Longleftrightarrow A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ and} \\ \exists\, p \text{ s.t. } K_A(x,y) = I(a),\ a \in \left(1+|x-y|^2\right)^{p/2} S^m_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right). \end{gathered} \tag{3.2}$$

We then showed in Proposition 2.2 that the special case, $p=0$ and $\partial_y a \equiv 0$, gives an isomorphism

$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_L}{\overset{\sigma_L}{\rightleftarrows}} S^m_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{3.3}$$

The maps here, $q_L = I$, is the *left quantization map* and its inverse σ_L is the *left symbol map.* We also showed that $\Psi^\infty_\infty(\mathbb{R}^n)$ forms a filtered $*$-algebra. Next shall consider invertibility, modulo $\Psi^{-\infty}_\infty(\mathbb{R}^n)$. First we consider some more consequences of the reduction theorem.

3.1. Quantization and symbols

We have already noted that as well as the left quantization map leading to the isomorphism (3.3) there is a right quantization map, similarly derived from (3.1):

$$q_R(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(y,\xi) d\xi, \quad a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right). \tag{3.4}$$

In fact using the adjoint operator, $*$, on operators and writing as well $*$ for complex conjugation of symbols shows that

$$q_R = * \cdot q_L \cdot * \tag{3.5}$$

is also an isomorphism, with inverse σ_R

$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_R}{\overset{\sigma_R}{\rightleftarrows}} S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right). \tag{3.6}$$

These are the two "extreme" quantization procedures, see Problem 3.1 for another (more centrist). Using the proof of the reduction theorem we find:

LEMMA 3.1. *For any* $a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$,

$$\sigma_L\left(q_R(a)\right)(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi a(x,\xi). \tag{3.7}$$

PROOF. This follows from the general formula (2.59). □

One important thing to note from (3.7) is that

$$D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-|\alpha|}_\infty(\mathbb{R}^n;\mathbb{R}^n)$$

so that for *any* pseudodifferential operator

$$A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow \sigma_L(A) - \sigma_R(A) \in S^{m-1}_\infty(\mathbb{R}^n;\mathbb{R}^n).$$

For this reason we consider the general quotient spaces

$$S^{m-[1]}_\infty(\mathbb{R}^p;\mathbb{R}^n) = S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\,/S^{m-1}_\infty(\mathbb{R}^p;\mathbb{R}^n)$$

and, for $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, write $[a]$ for its image, i.e. equivalence class, in the quotient space $S^{m-[1]}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. The "principal symbol map"

$$\begin{gathered}\sigma_m : \Psi^m_\infty(\mathbb{R}^n) \longrightarrow S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)\\ \text{is defined by } \sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)].\end{gathered} \tag{3.8}$$

As distinct from σ_L or σ_R, σ_m depends on m, i.e. one needs to know that the order is at most m before it is defined.

The isomorphism (3.3) is replaced by the weaker exact sequence.

LEMMA 3.2. *For every* $m \in \mathbb{R}$

$$0 \hookrightarrow \Psi^{m-1}_\infty(\mathbb{R}^n) \hookrightarrow \Psi^m_\infty(\mathbb{R}^n) \xrightarrow{\sigma_m} S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow 0$$

is a short exact sequence (the 'principal symbol sequence' or simply the 'symbol sequence').

PROOF. This is just the statement that the range of each map is the null space of the next i.e. that σ_m is surjective, which follows from (3.3), and that the null space of σ_m is just $\Psi^{m-1}_\infty(\mathbb{R}^n)$ and this is again (3.3) and the definition of σ_m. □

The fundamental result proved last time is that

$$\Psi^m_\infty(\mathbb{R}^n)\cdot\Psi^{m'}_\infty(\mathbb{R}^n) \subset \Psi^{m+m'}_\infty(\mathbb{R}^n). \tag{3.9}$$

In fact we showed that if $A = q_L(a)$, $a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $B = q_R(b), b \in S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ then the composite operator has Schwartz kernel

$$K_{A\cdot B}(x,y) = I\left(a(x,\xi)b(y,\xi)\right)$$

Using the formula (2.59) again we see that

$$\sigma_L(A\cdot B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi\left[a(x,\xi)D^\alpha_x b(x,\xi)\right]. \tag{3.10}$$

Of course $b = \sigma_R(B)$ so we really want to rewrite (3.10) in terms of $\sigma_L(B)$.

LEMMA 3.3. *If* $A \in \Psi^m_\infty(\mathbb{R}^n)$ *and* $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ *then* $A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ *and*

$$\sigma_{m+m'}(A\circ B) = \sigma_m(A)\cdot\sigma_{m'}(B), \tag{3.11}$$

$$\sigma_L(A\circ B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi\sigma_L(A)\cdot D^\alpha_x\sigma_L(B). \tag{3.12}$$

PROOF. The simple formula (3.11) is already immediate from (3.10) since all terms with $|\alpha| \geq 1$ are of order $m+m'-|\alpha| \leq m+m'-1$. To get the "full" formula (3.12) we can insert into (3.10) the inverse of (3.7), namely

$$\sigma_R(x,\xi) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \sigma_L(x,\xi).$$

This gives the double sum (still asymptotically convergent)

$$\sigma_L(A \circ B) \sim \sum_\beta \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha \left[\sigma_L(A) D_x^\alpha \frac{i^{|\beta|}}{\beta!} D_x^\beta D_\xi^\beta \sigma_L(B)\right]$$

Setting $\gamma = \alpha + \beta$ this becomes

$$\sigma_L(A \circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} \sum_{0 \leq \alpha \leq \gamma} \frac{\gamma!(-1)^{|\gamma-\alpha|}}{\alpha!(\gamma-\alpha)!} D_\xi^\alpha \left[\sigma_L(A) \times D_\xi^{\gamma-\alpha} D_x^\gamma \sigma_L(B)\right].$$

Then Leibniz formula shows that this sum over α can be rewritten as

$$\sigma_L(A \circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} D_\xi^\gamma \sigma_L(A) \cdot D_x^\gamma \sigma_L(B),$$

this is just (3.12). □

The simplicity of (3.11) over (3.12) is achieved at the expense of enormous loss of information. Still many problems can be solved using (3.11) which we can think of as saying that the principal symbol maps give a homomorphism for instance from the filtered algebra $\Psi_\infty^0(\mathbb{R}^n)$ to the *commutative* algebra $S_\infty^{0-[1]}(\mathbb{R}^n;\mathbb{R}^n)$.

3.2. Ellipticity

We say that an element of $\Psi_\infty^m(\mathbb{R}^n)$ is *elliptic* if it is invertible modulo an error in $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ with the approximate inverse of order $-m$ i.e.

$$\begin{aligned} &A \in \Psi_\infty^m(\mathbb{R}^n) \text{ is elliptic} \\ &\iff \exists \quad B \in \Psi_\infty^{-m}(\mathbb{R}^n) \text{ s.t. } \; A \circ B - \mathrm{Id} \in \Psi_\infty^{-\infty}(\mathbb{R}^n). \end{aligned} \tag{3.13}$$

Thus ellipticity, here by definition, is invertibility in $\Psi_\infty^m(\mathbb{R}^n)/\Psi_\infty^{-\infty}(\mathbb{R}^n)$, with the inverse in $\Psi_\infty^{-m}(\mathbb{R}^n)/\Psi_\infty^{-\infty}(\mathbb{R}^n)$. The point about ellipticity is that it is a phenomenon of the *principal symbol.*

THEOREM 3.1. *The following conditions on $A \in \Psi_\infty^m(\mathbb{R}^n)$ are equivalent*

$$A \text{ is elliptic} \tag{3.14}$$

$$\exists\ [b] \in S_\infty^{-m-[1]}(\mathbb{R}^n;\mathbb{R}^n) \text{ s.t. } \sigma_m(A)\cdot[b] \equiv 1 \text{ in } S_\infty^{0-[1]}(\mathbb{R}^n;\mathbb{R}^n) \tag{3.15}$$

$$\exists\ b \in S_\infty^{-m}(\mathbb{R}^n;\mathbb{R}^n) \text{ s.t. } \sigma_L(A)\cdot b - 1 \in S_\infty^{-\infty}(\mathbb{R}^n;\mathbb{R}^n) \tag{3.16}$$

$$\exists\ \epsilon > 0 \text{ s.t. } \left|\, \sigma_L(A)(x,\xi)\, \right| \geq \epsilon(1+|\xi|)^m \text{ in } |\xi| > \frac{1}{\epsilon}. \tag{3.17}$$

PROOF. We shall show

$$(3.14) \Longrightarrow (3.15) \Longrightarrow (3.16) \Longleftrightarrow (3.17) \Longrightarrow (3.14). \tag{3.18}$$

Since we know that $\sigma_0(\mathrm{Id}) = 1$ applying the identity (3.11) to the definition of ellipticity in (3.13) gives

$$\sigma_m(A)\cdot\sigma_{-m}(B) \equiv 1 \text{ in } S_\infty^{0-[1]}(\mathbb{R}^n,\mathbb{R}^n), \tag{3.19}$$

i.e. that (3.14) $\Longrightarrow$ (3.15).

Now assuming (3.19) (i.e. (3.15)), and recalling that $\sigma_m(A) = [\sigma_L(A)]$ we find that a representation b_1 of the class $[b]$ must satisfy

$$\sigma_L(A) \cdot b_1 = 1 + e_1, \quad e_1 \in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n), \tag{3.20}$$

this being the meaning of the equality of residue classes. Now, since the remainder $e_1 \in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ the Neumann series

$$f \sim \sum_{j\geq 1} (-1)^j e_1^j$$

is asymptotically convergent, so $f \in S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ exists, and

$$(1+f)\cdot(1+e_1) = 1 + e_\infty, \quad e_\infty \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{3.21}$$

Then multiplying (3.21) by $(1+f)$ gives

$$\sigma_L(A)\cdot\{b_1(1+f)\} = 1 + e_\infty \tag{3.22}$$

which proves (3.16), since $b = b_1(1+f) \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Of course

$$\sup(1+|\xi|)^N |e_\infty| < \infty \quad \forall\ N$$

so

$$\exists\ C \text{ s.t. } \ |e_\infty(x,\xi)| < \frac{1}{2} \text{ in } |\xi| > C.$$

From (3.22) this means

$$\big|\ \sigma_L(A)(x,\xi)\ \big| \cdot \big|\ b(x,\xi)\ \big| \geq \frac{1}{2}, \ |\xi| > C. \tag{3.23}$$

Since $|b(x,\xi)| \leq C(1+|\xi|)^{-m}$ (being a symbol of order $-m$), (3.23) implies

$$\inf_{|\xi|\geq C} \big|\ \sigma_L(A)(x,\xi)\ \big|\ (1+|\xi|)^{-m} \leq C. \tag{3.24}$$

which shows that (3.16) implies (3.17).

Conversely, suppose (3.24) holds, i.e. (3.17) holds with $C = 1/\epsilon$. Then if $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, has $\phi(\xi) = 1$ when $|\xi| \leq 1$, and $\phi(\xi) = 0$ when $|\xi| \geq 2$,

$$\frac{(1-\phi(\xi/C))}{\sigma_L(A)(x,\xi)} = b(x,\xi) \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n) \tag{3.25}$$

as follows from (2.35). This gives (3.16), so (3.17) implies (3.16).

Now suppose (3.16) holds. Set $B_1 = q_L(b)$ then from (3.11) again

$$\sigma_0(A \circ B_1) = [q_m(A)] \cdot [b] \equiv 1.$$

That is,

$$A \circ B_1 - \mathrm{Id} = E_1 \in \Psi^{-1}_\infty(\mathbb{R}^n). \tag{3.26}$$

Consider the Neumann series of operators

$$\sum_{j\geq 0} (-1)^j E_1^j.$$

The corresponding series of symbols is asymptotically summable so we can choose $F \in \Psi^{-1}_\infty(\mathbb{R}^n)$ with

$$\sigma_L(F) \sim \sum_{j\geq 0} (-1)^j \sigma_L(E_1^j). \tag{3.27}$$

Then

$$(\mathrm{Id} + E_1)(\mathrm{Id} + F) = \mathrm{Id} + E_\infty,\, E_\infty \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{3.28}$$

Then $B = B_1(\mathrm{Id} + F) \in \Psi^{-m}_\infty(\mathbb{R}^n)$ satisfies (3.13), i.e. A is elliptic. $\square$

In the definition of ellipticity in (3.13) we have taken B to be a 'right parametrix', i.e. a right inverse modulo $\Psi^{-\infty}_\infty(\mathbb{R}^n)$. We can just as well take it to be a *left* parametrix.

Lemma 3.4. *$A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic if and only if there exists $B' \in \Psi^{-m}_\infty(\mathbb{R}^n)$ such that*

$$B' \circ A = \mathrm{Id} + E',\ \ E' \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \tag{3.29}$$

and then if B satisfies (3.13), $B - B' \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$.

Proof. Certainly (3.29) implies $\sigma_{-m}(B') \cdot \sigma_m(A) \equiv 1$, and the multiplication here is commutative so (3.15) holds and A is elliptic. Conversely if A is elliptic we get in place of (3.26)

$$B_1 \circ A - \mathrm{Id} = E'_1 \in \Psi^{-1}_\infty(\mathbb{R}^n).$$

Then defining F' as in (3.27) with E'_1 in place of E_1 we get $(\mathrm{Id} + F')(\mathrm{Id} + E'_1) = \mathrm{Id} + E'_\infty$ and then $B' = (\mathrm{Id} + F') \circ B_1$ satisfies (3.29). Thus 'left' ellipticity as in (3.29) is equivalent to right ellipticity. Applying B to (3.29) gives

$$B' \circ (\mathrm{Id} + E) = B' \circ (A \circ B) = (\mathrm{Id} + E') \circ B \tag{3.30}$$

which shows that $B - B' \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. $\square$

Thus a left parametrix of an elliptic element of $\Psi^m_\infty(\mathbb{R}^n)$ is always a right, hence two-sided, parametrix.

3.3. The Laplacian

Suppose that $g_{ij}(x)$ are the components of an "∞-metric" on $\mathbb{R}^n$, i.e.

$$\begin{gathered} g_{ij}(x) \in \mathcal{C}^\infty_\infty(\mathbb{R}^n), i, j = 1, \dots, n \\ \Big| \sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j \Big| \geq \epsilon|\xi| \quad \forall\ x \in \mathbb{R}^n, \xi \in \mathbb{R}^n. \end{gathered} \tag{3.31}$$

The *Laplacian* of the metric is the second order differential operator

$$\Delta_g = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} D_{x_i} g^{ij} \sqrt{g} D_{x_j} \tag{3.32}$$

where

$$g(x) = \det g^{ij}(x),\ g^{ij}(x) = (g_{ij}(x))^{-1}.$$

The Laplacian is determined by the integration by parts formula

$$\int_{\mathbb{R}^n} \sum g^{ij}(x) D_{x_i}\phi \cdot \overline{D_{x_i}\psi} dg = \int \Delta_g \phi \cdot \overline{\psi} dg\ \forall\ \phi, \psi \in \mathcal{S}(\mathbb{R}^n) \tag{3.33}$$

where

$$dg = \sqrt{g} dx. \tag{3.34}$$

Our assumption in (3.31) shows that $\Delta = \varDelta_g \in \mathrm{Diff}^2_\infty(\mathbb{R}^n) \subset \Psi^2_\infty(\mathbb{R}^n)$ is in fact elliptic, since

$$\sigma_2(\Delta) = \sum_{i,j=1} g^{ij}\xi_i\xi_j. \tag{3.35}$$

Thus Δ has a two-sided parametrix $B \in \Psi^{-2}_\infty(\mathbb{R}^n)$

$$\Delta \circ B \equiv B \circ \Delta \equiv \mathrm{Id} \quad \mod \quad \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{3.36}$$

Corollary 3.1. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\Delta u = 0$ then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.*

Proof. $B \circ \Delta = \mathrm{Id} + E$, $E \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ means, given $\Delta u = 0$,

$$u = -Eu. \tag{3.37}$$

So it suffices to prove:

Lemma 3.5. *If $E \in \Psi^{-\infty}_\infty$ then*

$$E : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n). \tag{3.38}$$

Proof. This follows from Proposition 2.4 since we can regard the kernel as a $\mathcal{C}^\infty$ function of x taking values in $\mathcal{S}(\mathbb{R}^n_y)$. □

□

3.4. L^2 boundedness

So far we have thought of pseudodifferential operators, the elements of $\Psi^m_\infty(\mathbb{R}^n)$ for some m, as defining continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$, and by duality, on $\mathcal{S}'(\mathbb{R}^n)$. Now that we have proved the composition formula we can use it to prove other 'finite order' regularity results. The basic one of these is L^2 boundedness:

Proposition 3.1. *If $A \in \Psi^0_\infty(\mathbb{R}^n)$ then, by continuity from $\mathcal{S}(\mathbb{R}^n)$, A defines a bounded linear operator*

$$A : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n). \tag{3.39}$$

As usual our proof is by *reductio ad absurdum*; we actually start there.

3.5. Residual terms

To prove boundedness (3.39) we start by assuming that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. In that case we know exactly what the Schwartz kernel looks like:

$$A \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \Longleftrightarrow \Big| \sup_{x,y} |(x-y)^\alpha D^\beta_x D^\gamma_y K(x,y)\Big| < \infty \quad \forall\ \alpha, \beta, \gamma. \tag{3.40}$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (see Problem) we only need to show the existence of a constant, C, such that

$$\int \big|\ A\phi(x)\ \big|^2\, dx \le C \int |\phi|^2 \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{3.41}$$

Writing out the integral on the left

$$\begin{aligned} &\int \Big|\ \int K_A(x,y)\phi(y)dy\ \Big|^2\, dx \\ &= \iiint K_A(x,y)\overline{K_A(x,z)}\phi(y)\overline{\phi(z)}\ dydzdx \end{aligned} \tag{3.42}$$

is certainly absolutely convergent and

$$\begin{aligned}&\int \mid A\phi(x) \mid^2 dx \\ &\le \left(\iiint \mid K_A(x,y)K_A(x,z) \mid\mid \phi(y) \mid^2 dydxdz\right)^{\frac{1}{2}} \\ &\times \left(\iiint \mid K_A(x,y)K_A(x,z) \mid\mid \phi(z) \mid^2 \ dzdxdy\right)^{\frac{1}{2}}.\end{aligned}$$

These two factors are the same. Changing variables to $y, t = x - y$ and $x - z = s$ gives

$$\int \mid A\phi \mid^2 dx \le \sup_y \iint \mid K_A(t+y,y) \mid\mid K_A(t+y,t+y-s) \mid dtds \cdot \int |\phi|^2 dy.$$

which gives (3.41), since from (3.40)

$$\begin{aligned}\mid K_A(t+y,y) \mid &\le C\,(1+|t|)^{-n-1}, \\ \mid K_A\,(t+y,t+y-s) \mid &\le C\,(1+|s|)^{-n-1}.\end{aligned}$$

3.6. Square root

To prove the general result, (3.39), we shall use the clever idea, due to Hörmander, of using the (approximate) square root of an operator. We shall say that an element $[a] \in S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ is positive if there is some $0 < a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ in the equivalence class.

Proposition 3.2. *Suppose $A \in \Psi^m_\infty(\mathbb{R}^n)$ is self-adjoint, $A = A^*$, and elliptic with a positive principal symbol, then there exists $B \in \Psi^{m/2}_\infty(\mathbb{R}^n), B = B^*$, such that*

$$A = B^2 + G, \qquad G \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{3.43}$$

Proof. This is a good exercise in the use of the symbol calculus. Let $a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n), a > 0$, be a positive representative of the principal symbol of A. Now

$$b_0 = a^{\frac{1}{2}} \in S^{m/2}_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{3.44}$$

Let $B_0 \in \Psi^{m/2}_\infty(\mathbb{R}^n)$ have principal symbol b_0. We can assume that $B_0 = B_0^*$, since if not we just replace it by $\frac{1}{2}(B_0 + B_0^*)$ which has the same principal symbol.

The symbol calculus shows that $B_0^2 \in \Psi^m_\infty(\mathbb{R}^n)$ and

$$\sigma_m(B_0^2) = \left(\sigma_{m/2}(B_0)\right)^2 = b_0^2 = a_0 \mod S^{m-1}_\infty.$$

Thus

$$A - B_0^2 = E_1 \in \Psi^{m-1}_\infty(\mathbb{R}^n). \tag{3.45}$$

Then we proceed inductively. Suppose we have chosen $B_j \in \Psi^{m/2-j}_\infty(\mathbb{R}^n)$, with $B_j^* = B_j$, for $j \le N$ such that

$$A - \left(\sum_{j=0}^N B_j\right)^2 = E_{N+1} \in \Psi^{m-N-1}_\infty(\mathbb{R}^n). \tag{3.46}$$

Of course we *have* done this for $N = 0$. Then see the effect of adding $B_{N+1} \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$:

$$A - \left(\sum_{j=0}^{N+1} B_j\right)^2 = E_{N+1} - \left(\sum_{j=0}^{N} B_j\right) B_{N+1} \tag{3.47}$$
$$-B_{N+1}\left(\sum_{j=0}^{N} B_j\right) - B_{N+1}^2.$$

On the right side all terms are of order $m - N - 2$, except for

$$E_{N+1} - B_0 B_{N+1} - B_{N+1} B_0 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n). \tag{3.48}$$

The principal symbol, of order $m - N - 1$, of this is just

$$\sigma_{m-N-1}(E_{N+1}) - 2\ b_0 \cdot \sigma_{\frac{m}{2}-N-1}(B_{N+1}). \tag{3.49}$$

Thus if we choose

$$\sigma_{m/2-N-1}(B_{N+1}) = \frac{1}{2}\frac{1}{b_0} \cdot \sigma_{m-N-1}(E_{N+1})$$

and replace B_{N+1} by $\frac{1}{2}(B_{N+1} + B_{N+1}^*)$, we get the inductive hypothesis for $N + 1$. Thus we have arranged (3.46) for every N. Now define $B = \frac{1}{2}(B' + (B')^*)$ where

$$\sigma_L(B') \sim \sum_{j=0}^{\infty} \sigma_L(B_j). \tag{3.50}$$

Since all the B_j are self-adjoint B also satisfies (3.50) and from (3.46)

$$A - B^2 = A - \left(\sum_{j=0}^{N} B_j + B_{(N+1)}\right)^2 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n)$$

for every N, since $B_{(N+1)} = B - \sum_{j=0}^{N} B_j \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$. Thus $A - B^2 \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ and we have proved (3.43), and so Proposition 3.2. $\square$

3.7. Proof of Proposition 3.1

Here is Hörmander's clever argument. We want to show that

$$\|A\phi\| \le C\|\phi\| \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{3.51}$$

where $A \in \Psi_\infty^0(\mathbb{R}^n)$. The square of the left side can be written

$$\int A\phi \cdot \overline{A\phi} dx = \int \phi \cdot (A^* A\phi)\, dx.$$

So it suffices to show that

$$\langle \phi, A^* A\phi\rangle \le C\|\phi\|^2. \tag{3.52}$$

Now $A^* A \in \Psi_\infty^0(\mathbb{R}^n)$. If $C > 0$ is a large constant,

$$C > \sup_{x,\xi} \left|\ \sigma_L(A^* A)(x, \xi)\ \right|$$

then $C - A^*A$ has a positive representative of its principal symbol. We can therefore apply Proposition 3.2 to it:

$$C - A^*A = B^*B + G, \quad G \in \Psi_\infty^{-\infty}(\mathbb{R}^n). \tag{3.53}$$

This gives

$$\begin{aligned} \langle \phi, A^*A\phi\rangle =& C\langle \phi, \phi\rangle - \langle \phi, B^*B\phi\rangle - \langle \phi, G\phi\rangle \\ =& C\|\phi\|^2 - \|B\phi\|^2 - \langle \phi, G\phi\rangle. \end{aligned} \tag{3.54}$$

The second term on the right is negative and, since $G \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$, we can use the residual case discussed above to conclude that

$$|\langle \phi, G\phi\rangle| \le C'\|\phi\|^2 \Longrightarrow \|A\phi\|^2 \le C\|\phi\|^2 + C'\|\phi\|^2,$$

so (3.39) holds and Proposition 3.1 is proved.

3.8. Sobolev boundedness

Using the basic boundedness result, Proposition 3.1, and the calculus of pseudodifferential operators we can prove more general results on the action of pseudodifferential operators on Sobolev spaces.

Recall that for any positive integer, k,

$$H^k(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n);\ D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \le k\right\}. \tag{3.55}$$

Using the Fourier transform we find

$$u \in H^k(\mathbb{R}^n) \Longrightarrow \xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n) \quad \forall\ |\alpha| \le k. \tag{3.56}$$

Now these finitely many conditions can be written just

$$\left(1 + |\xi|^2\right)^{k/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n). \tag{3.57}$$

Notice that $a(\xi) = (1 + |\xi|^2)^{k/2} = \langle\xi\rangle^k \in S_\infty^k(\mathbb{R}^n)$. Here we use the notation

$$\langle\xi\rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}} \tag{3.58}$$

for a smooth (symbol) of the size of $1 + |\xi|$, thus (3.57) just says

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n). \tag{3.59}$$

For *negative* integers

$$H^k(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{|\beta| \le -k} D^\beta u_\beta, u_\beta \in L^2(\mathbb{R}^n)\right\},\ -k \in \mathbb{N}. \tag{3.60}$$

The same sort of discussion applies, showing that

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n). \tag{3.61}$$

In view of this we define the Sobolev space $H^m(\mathbb{R}^n)$, for any real order, by

$$u \in H^k, (\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^m u \in L^2(\mathbb{R}^n). \tag{3.62}$$

It is a Hilbert space with

$$\|u\|_m^2 = \left\|\langle D\rangle^m u\right\|_{L^2}^2 = \int (1 + |\xi|^2)^m |\hat{u}(\xi)| d\xi. \tag{3.63}$$

Clearly we have

$$H^m(\mathbb{R}^n) \supseteq H^{m'}(\mathbb{R}^n) \text{ if } m' \ge m. \tag{3.64}$$

Notice that it is rather unfortunate that these spaces get *smaller* as m gets bigger, as opposed to the spaces $\Psi^m_\infty(\mathbb{R}^n)$ which get *bigger* with m. Anyway that's life and we have to think of

$$\begin{cases} H^\infty(\mathbb{R}^n) = \bigcap\limits_m H^m(\mathbb{R}^n) & \text{as the residual space} \\ H^{-\infty}(\mathbb{R}^n) = \bigcup\limits_m H^m(\mathbb{R}^n) & \text{as the big space.} \end{cases} \tag{3.65}$$

It is important to note that

$$\mathcal{S}(\mathbb{R}^n) \subsetneq H^\infty(\mathbb{R}^n) \subsetneq H^{-\infty}(\mathbb{R}^n) \subsetneq \mathcal{S}'(\mathbb{R}^n). \tag{3.66}$$

In particular we do *not* capture all the tempered distributions in $H^{-\infty}(\mathbb{R}^n)$. We therefore consider *weighted* versions of these Sobolev spaces:

$$\langle x\rangle^q H^m(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle x\rangle^{-q} u \in H^m(\mathbb{R}^n)\right\}. \tag{3.67}$$

THEOREM 3.2. *For each $q, m, M \in \mathbb{R}$ each $A \in \Psi^M_\infty(\mathbb{R}^n)$ defines a continuous linear map*

$$A : \langle x\rangle^q H^m(\mathbb{R}^n) \longrightarrow \langle x\rangle^q H^{m-M}(\mathbb{R}^n). \tag{3.68}$$

PROOF. Let us start off with $q = 0$, so we want to show that

$$A : H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n),\ A \in \Psi^M_\infty(\mathbb{R}^n) \tag{3.69}$$

Now from (3.62) we see that

$$\begin{aligned} u \in H^m(\mathbb{R}^n) &\Longleftrightarrow \langle D\rangle^m u \in L^2(\mathbb{R}^n) \\ &\Longleftrightarrow \langle D\rangle^{m-M}\langle D\rangle^M u \in L^2(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^M u \in H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M. \end{aligned} \tag{3.70}$$

That is,

$$\langle D\rangle^M : H^m(\mathbb{R}^n) \longleftrightarrow H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M. \tag{3.71}$$

To prove (3.69) it suffices to show that

$$B = \langle D\rangle^{-M+m} \cdot A \cdot \langle D\rangle^{-m} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \tag{3.72}$$

since then $A = \langle D\rangle^{-m+M} \cdot B \cdot \langle D\rangle^m$ maps $H^m(\mathbb{R}^n)$ to $H^{m-M}(\mathbb{R}^n)$:

$$\begin{array}{ccc} H^m(\mathbb{R}^n) & \xrightarrow{A} & H^{m-M}(\mathbb{R}^n) \\ {\scriptstyle\langle D\rangle^m}\big\downarrow & & \big\downarrow{\scriptstyle\langle D\rangle^{m-M}} \\ L^2(\mathbb{R}^n) & \xrightarrow[B]{} & L^2(\mathbb{R}^n). \end{array} \tag{3.73}$$

Since $B \in \Psi^0_\infty(\mathbb{R}^n)$, by the composition theorem, we already know (3.72).

Thus we have proved (3.69). To prove the general case, (3.68), we proceed in the same spirit. Thus $\langle x\rangle^q$ is an isomorphism from $H^m(\mathbb{R}^n)$ to $\langle x\rangle^q H^m(\mathbb{R}^n)$, by definition. So to get (3.68) we need to show that

$$Q = \langle x\rangle^q \cdot A \cdot \langle x\rangle^{-q} :\ H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n), \tag{3.74}$$

i.e. satisfies (3.69). Consider the Schwartz kernel of Q. Writing A in left-reduced form, with symbol a,

$$K_Q(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \langle x\rangle^q a(x, \xi) d\xi \cdot \langle y\rangle^{-q}. \tag{3.75}$$

Now if we check that

$$\langle x\rangle^q\langle y\rangle^{-q}a(x,\xi)\in\left(1+|x-y|^2\right)^{\frac{|q|}{2}}S^M_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right) \tag{3.76}$$

then we know that $Q\in\Psi^M_\infty(\mathbb{R}^n)$ and we get (3.74) from (3.69). Thus we want to show that

$$\langle x-y\rangle^{-|q|}\frac{\langle x\rangle^q}{\langle y\rangle^q}a(x,\xi)\in S^M_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \tag{3.77}$$

assuming of course that $a(x,\xi)\in S^M_\infty(\mathbb{R}^n;\mathbb{R}^n)$. By interchanging the variables x and y if necessary we can assume that $q>0$. Consider separately the two regions

$$\begin{aligned}&\left\{(x,y);|x-y|<\frac{1}{4}(|x|+|y|)\right\}=\Omega_1\\&\left\{(x,y);|x-y|>\frac{1}{8}(|x|+|y|)\right\}=\Omega_2\end{aligned} \tag{3.78}$$

In Ω_1, x is "close" to y, in the sense that

$$|x|\le|x-y|+|y|\le\frac{1}{4}(|x|+|y|)+|y|\Longrightarrow|x|\le\frac{4}{3}\cdot\frac{5}{4}|y|\le2|y|.$$

Thus

$$\langle x-y\rangle^{-q}\cdot\frac{\langle x\rangle^q}{\langle y\rangle^q}\le C\ \text{ in }\Omega_1. \tag{3.79}$$

On the other hand in Ω_2,

$$|x|+|y|<8|x-y|\Longrightarrow|x|<8|x-y|$$

so again

$$\langle x-y\rangle^{-q}\frac{\langle x\rangle^q}{\langle y\rangle^q}\le C. \tag{3.80}$$

In fact we easily conclude that

$$\langle x-y\rangle^{-q}\frac{\langle x\rangle^q}{\langle y\rangle^q}\in\mathcal{C}^\infty_\infty(\mathbb{R}^n)\quad\forall\ q, \tag{3.81}$$

since differentiation by x or y makes all terms "smaller". This proves (3.77), hence (3.76) and (3.74) and therefore (3.68), i.e. the theorem is proved. □

3.9. Consequences

We can capture any tempered distribution in a weighted Sobolev space; this is really Schwartz' representation theorem which says that any $u\in\mathcal{S}'(\mathbb{R}^n)$ is of the form

$$u=\sum_{\text{finite}}x^\alpha D^\beta_x u_{\alpha\beta},\ u_{\alpha\beta}\text{ bounded and continuous.}$$

Clearly $\mathcal{C}^0_\infty(\mathbb{R}^n)\subset\langle x\rangle^{-n}L^2(\mathbb{R}^n)$. Thus

Lemma 3.6.

$$\mathcal{S}'(\mathbb{R}^n)=\bigcup_M\langle x\rangle^M H^{-M}(\mathbb{R}^n).$$

The elliptic regularity result we found before can now be refined:

PROPOSITION 3.3. *If* $A \in \Psi_\infty^m(\mathbb{R}^n)$ *is elliptic then*

$$
\tag{3.82}
\begin{aligned}
&Au \in \langle x\rangle^p H^q(\mathbb{R}^n),\ u \in \langle x\rangle^{p'} H^{q'}(\mathbb{R}^n) \\
&\Longrightarrow u \in \langle x\rangle^{p''} H^{q+m}(\mathbb{R}^n),\ p'' = \max(p, p').
\end{aligned}
$$

PROOF. The existence of a left parametrix for A, $B \in \Psi_\infty^{-m}(\mathbb{R}^n)$,

$$
B \cdot A = \mathrm{Id} + G,\ G \in \Psi_\infty^{-\infty}(\mathbb{R}^n)
$$

means that

$$
\begin{aligned}
u &= B(Au) + Gu \\
&\in \langle x\rangle^p H^{q+m}(\mathbb{R}^n) + \langle x\rangle^{p'} H^{\infty}(\mathbb{R}^n) \\
&\subset \langle x\rangle^{p''} H^{q+m}(\mathbb{R}^n).
\end{aligned}
$$

□

3.10. Calculus of supports

Recall that we have already defined the support of a tempered distribution in the slightly round-about way:

$$
\tag{3.83}
\text{if } u \in \mathcal{S}'(\mathbb{R}^n),\ \operatorname{supp}(u) = \{x \in \mathbb{R}^n;\ \exists\ \phi \in \mathcal{S}(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^{\complement}.
$$

Now if $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is any continuous linear operator we can consider the support of the kernel:

$$
\tag{3.84}
\operatorname{supp}(A) = \operatorname{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.
$$

We write out the space as a product here to point to the fact that *any* subset of the product defines (is) a *relation* i.e. a map on subsets:

$$
\tag{3.85}
\begin{gathered}
G \subset \mathbb{R}^n \times \mathbb{R}^n, \quad S \subset \mathbb{R}^n \Longrightarrow \\
G \circ S = \big\{x \in \mathbb{R}^n;\ \exists\ y \in S \text{ s.t. } (x, y) \in G\big\}.
\end{gathered}
$$

One can write this much more geometrically in terms of the two projection maps

$$
\tag{3.86}
\begin{tikzcd}
& \mathbb{R}^{2n} \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\
\mathbb{R}^n & & \mathbb{R}^n.
\end{tikzcd}
$$

Thus $\pi_R(x, y) = y$, $\pi_L(x, y) = x$. Then (3.85) can be written in terms of the action of maps on sets as

$$
\tag{3.87}
G \circ S = \pi_L\left(\pi_R^{-1}(S) \cap G\right).
$$

From this it follows that if S is compact and G is closed, then $G \circ S$ is closed, since its intersection with any compact set is the image of a compact set under a continuous map, hence compact. Now, by the *calculus of supports* we mean the 'trivial' result.

PROPOSITION 3.4. *If* $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ *is a continuous linear map then*

$$
\tag{3.88}
\operatorname{supp}(A\phi) \subset \operatorname{supp}(A) \circ \operatorname{supp}(\phi)\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n).
$$

PROOF. Since we want to *bound* $\operatorname{supp}(A\phi)$ we can use (3.83) directly, i.e. show that

$$x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi) \Longrightarrow x \notin \operatorname{supp}(A\phi). \tag{3.89}$$

Since we know $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$ to be closed, the assumption that x is outside this set means that there exists $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with

$$\psi(x) \neq 0 \text{ and } \operatorname{supp}(\psi) \cap \operatorname{supp}(A) \circ \operatorname{supp}(\phi) = \emptyset.$$

From (3.85) or (3.87) this means

$$\operatorname{supp}(A) \cap (\operatorname{supp}(\psi) \times \operatorname{supp}(\phi)) = \emptyset \text{ in } \mathbb{R}^{2n}. \tag{3.90}$$

But this certainly implies that

$$\begin{gathered} K_A(x,y)\psi(x)\phi(x) = 0 \\ \Longrightarrow \psi A(\phi) = \int K_A(x,y)\psi(x)\phi(y)dy = 0. \end{gathered} \tag{3.91}$$

Thus we have proved (3.88) and the lemma. □

3.11. Singular supports

As well as the support of a tempered distribution we can consider the singular support:

$$\operatorname{sing\,supp}(u) = \left\{x \in \mathbb{R}^n; \exists\ \phi \in \mathcal{S}(\mathbb{R}^n), \phi(x) \neq 0, \phi u \in \mathcal{S}(\mathbb{R}^n)\right\}^{\complement}. \tag{3.92}$$

Again this is a closed set since $x \notin \operatorname{sing\,supp}(u) \Longrightarrow \exists\ \phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi u \in \mathcal{S}(\mathbb{R}^n)$ and $\phi(x) \neq 0$ so $\phi(x') \neq 0$ for $|x - x'| < \epsilon$, some $\epsilon > 0$ and hence $x' \notin \operatorname{sing\,supp}(u)$ i.e. the complement of $\operatorname{sing\,supp}(u)$ is open.

Directly from the definition we have

$$\operatorname{sing\,supp}(u) \subset \operatorname{supp}(u)\ \forall\ u \in \mathcal{S}(\mathbb{R}^n) \text{ and} \tag{3.93}$$

$$\operatorname{sing\,supp}(u) = \emptyset \Longleftrightarrow u \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{3.94}$$

3.12. Pseudolocality

We would like to have a result like (3.88) for singular support, and indeed we can get one for pseudodifferential operators. First let us work out the singular support of the kernels of pseudodifferential operators.

PROPOSITION 3.5. *If* $A \in \Psi_\infty^m(\mathbb{R}^n)$ *then*

$$\operatorname{sing\,supp}(A) = \operatorname{sing\,supp}(K_A) \subset \left\{(x,y) \in \mathbb{R}^{2n}; x = y\right\}. \tag{3.95}$$

PROOF. The kernel is defined by an oscillatory integral

$$I(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi)d\xi.$$

If the order m is $< -n$ we can show by integration by parts that

$$(x-y)^\alpha I(a) = I\left((-D_\xi)^\alpha a\right), \tag{3.96}$$

and then this must hold by continuity for all orders. If a is of order m and $|\alpha| > m + n$ then $(-D_\xi)^\alpha a$ is of order less than $-n$, so

$$(x-y)^\alpha I(a) \in \mathcal{C}_\infty^0(\mathbb{R}^n), |\alpha| > m + n. \tag{3.97}$$

In fact we can also differentiate under the integral sign:

$$D_x^\beta D_y^\gamma (x-y)^\alpha I(a) = I\left(D_x^\beta D_y^\gamma (-D_\xi)^\alpha a\right) \tag{3.98}$$

so generalizing (3.97) to

$$(x-y)^\alpha I(a) \in \mathcal{C}^k_\infty(\mathbb{R}^n) \text{ if } |\alpha| > m+n+k. \tag{3.99}$$

This implies that $I(A)$ is $\mathcal{C}^\infty$ on the complement of the diagonal, $\{x=y\}$. This proves (3.95). ☐

An operator is said to be *pseudolocal* if it satisfies the condition

$$\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u)\ \forall\ u \in \mathcal{C}^{-\infty}(\mathbb{R}^n). \tag{3.100}$$

PROPOSITION 3.6. *Pseudodifferential operators are pseudolocal.*

PROOF. Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ has compact support and $\overline{x} \notin \operatorname{sing\,supp}(u)$. Then we can choose $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi \equiv 1$ near $\overline{x}$ and $\phi u \in \mathcal{S}(\mathbb{R}^n)$ (by definition). Thus

$$u = u_1 + u_2,\ u_1 = (1-\phi)u, \quad u_2 \in \mathcal{S}(\mathbb{R}^n). \tag{3.101}$$

Since $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, $Au_2 \in \mathcal{S}(\mathbb{R}^n)$ so

$$\operatorname{sing\,supp}(Au) = \operatorname{sing\,supp}(Au_1) \text{ and } \overline{x} \notin \operatorname{supp}(u_1). \tag{3.102}$$

Choose $\psi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $\psi(\overline{x}) = 1$ and

$$\operatorname{supp}(\psi) \cap \operatorname{supp}(1-\phi) = \emptyset. \tag{3.103}$$

Thus

$$\psi A u_1 = \psi A(1-\phi)u = \tilde{A}u \tag{3.104}$$

where

$$K_{\tilde{A}}(x,y) = \psi(x) K_A(x,y)(1-\phi(y)).$$

Combining (3.103) and (3.95) shows that $K_{\tilde{A}} \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ so, by Lemma 3.5, $\tilde{A}u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\overline{x} \notin \operatorname{sing\,supp}(Au)$ by (3.95). This proves the proposition. ☐

3.13. Coordinate invariance

If $\Omega \subset \mathbb{R}^n$ is an open set, put

$$\mathcal{C}^\infty_c(\Omega) = \left\{u \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\right\}$$
$$\mathcal{C}^{-\infty}_c(\Omega) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\right\}$$

respectively the space of $\mathcal{C}^\infty$ functions of compact support in Ω and of distributions of compact support in Ω. Here $K \Subset \Omega$ indicates that K is a compact subset of Ω. Notice that if $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ then u defines a continuous linear functional

$$\mathcal{C}^\infty(\Omega) \ni \phi \longmapsto u(\phi) = u(\psi\phi) \in \mathbb{C} \tag{3.105}$$

where if $\psi \in \mathcal{C}^\infty_c(\Omega)$ is chosen to be identically one near $\operatorname{supp}(u)$ then (3.105) is independent of ψ. [Think about what continuity means here!]

Now suppose

$$F : \Omega \longrightarrow \Omega' \tag{3.106}$$

is a diffeomorphism between open sets of $\mathbb{R}^n$. The pull-back operation is

$$F^* :\ \mathcal{C}^\infty_c(\Omega') \longleftrightarrow \mathcal{C}^\infty_c(\Omega)$$
$$F^*\phi = \phi \circ F.$$

LEMMA 3.7. *If F is a diffeomorphism,* (3.106)*, between open sets of* $\mathbb{R}^n$ *then*

$$F^* : \mathcal{C}_c^{-\infty}(\Omega') \longleftrightarrow \mathcal{C}_c^{-\infty}(\Omega) \tag{3.107}$$

by continuity.

PROOF. The density of $\mathcal{C}_c^{\infty}(\Omega)$ in $\mathcal{C}_c^{-\infty}(\Omega)$, in the weak topology given by the seminorms from (3.105), can be proved in the same way as the density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ (see Problem 3.14). Thus, we only need to show continuity of (3.106) in this sense. Suppose $u \in \mathcal{C}_c^{\infty}(\Omega)$ and $\phi \in \mathcal{C}_c^{\infty}(\Omega')$ then

$$\begin{aligned} (F^*u)(\phi) &= \int u(F(x))\phi(x)dx \\ &= \int u(y)\phi(G(y))|J_G(y)|dy \end{aligned} \tag{3.108}$$

where $J_G(y) = \left(\frac{\partial G(y)}{\partial y}\right)$ is the Jacobian of G, the inverse of F. Thus (3.107) can be written

$$F^*u(\phi) = (|J_G|u)\,(G^*\phi) \tag{3.109}$$

and since $G^* : \mathcal{C}^{\infty}(\Omega) \longrightarrow \mathcal{C}^{\infty}(\Omega')$ is continuous (!) we conclude that F^* is continuous as desired. □

Now suppose that

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

has

$$\operatorname{supp}(A) \Subset \Omega \times \Omega \subset \mathbb{R}^{2n}. \tag{3.110}$$

Then

$$A : \mathcal{C}_c^{\infty}(\Omega) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega) \tag{3.111}$$

by Proposition 3.4. Applying a diffeomorphism, F, as in (3.7) set

$$A_F : \mathcal{C}_c^{\infty}(\Omega') \longrightarrow \mathcal{C}_c^{-\infty}(\Omega'),\ A_F = G^* \circ A \circ F^*. \tag{3.112}$$

LEMMA 3.8. *If A satisfies* (3.110) *and F is a diffeomorphism* (3.7) *then*

$$K_{A_F}(x,y) = (G \times G)^*K \cdot |J_G(y)| \textit{ on } \Omega' \times \Omega' \tag{3.113}$$

has compact support in $\Omega' \times \Omega'$.

PROOF. Same as (3.110). □

PROPOSITION 3.7. *Suppose $A \in \Psi_\infty^m(\mathbb{R}^n)$ has kernel satisfying* (3.110) *and F is a diffeomorphism as in* (3.7) *then A_F, defined by* (3.112)*, is an element of* $\Psi_\infty^m(\mathbb{R}^n)$.

PROOF. Since $A \in \Psi_\infty^m(\mathbb{R}^n)$,

$$K_A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,\xi)d\xi \tag{3.114}$$

for some $a \in S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)$. Now choose $\psi \in \mathcal{C}_c^{\infty}(\Omega)$ such that $\psi(x)\psi(y) = 1$ on $\operatorname{supp}(K_A)$, possible by (3.110). Then

$$K_A = I\left(\psi(x)\psi(y)a(x,\xi)\right). \tag{3.115}$$

In fact suppose $\mu_\epsilon(x,y) \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ and $\mu \equiv 1$ in $|x-y| < \epsilon$ for $\epsilon > 0$, $\mu(x,y) = 0$ in $|x-y| > 2\epsilon$. Then if

$$K_{A_\epsilon} = I\left(\mu_\epsilon(x,y)\psi(x)\psi(y)a(x,\xi)\right) \tag{3.116}$$

we know that if

$$A'_\epsilon = A - A_\epsilon \text{ then } K_{A'_\epsilon} = \left(1 - \mu_\epsilon(x,y)\right) K_A \in \Psi_\infty^{-\infty}(\mathbb{R}^n), \tag{3.117}$$

by (3.95). Certainly A'_ϵ also satisfies (3.110) and from (3.113)

$$(A'_\epsilon)_F \in \Psi_\infty^{-\infty}(\mathbb{R}^n). \tag{3.118}$$

So we only need to consider A_ϵ defined by (3.116). Again using (3.113) (assuming $m < -n$)

$$K_{(A_\epsilon)_F}(x,y) = (2\pi)^{-n} \int e^{i(G(x)-G(y))\cdot\xi} b(G(x), G(y), \xi) \left| \frac{\partial G}{\partial y} \right| d\xi \tag{3.119}$$

where $b(x,y,\xi) = \mu_\epsilon(x-y)\psi(x)\psi(y)a(x,\xi)$. Applying Taylor's formula,

$$G(x) - G(y) = (x-y)\cdot T(x,y) \tag{3.120}$$

where $T(x,y)$ is an invertible $\mathcal{C}^\infty$ matrix on $K \times K \cap \{|x-y| < \epsilon\}$ for $\epsilon < \epsilon(K)$, where $\epsilon(K) > 0$ depends on the compact set $K \Subset \Omega'$. Thus we can set

$$\eta = T^t(x,y)\cdot\xi$$

and rewrite (3.120) as

$$\begin{gathered} K_{(A_\epsilon)_F}(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\eta} c(x,y,\eta) d\eta \\ c(x,y,\eta) = b\left(G(x), G(y), (T^t)^{-1}(x,y)\eta\right) \left| \frac{\partial G}{\partial y} \right| \cdot \left| \det T(x,y) \right|^{-1}. \end{gathered} \tag{3.121}$$

So it only remains to show that $c \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ and the proof is complete. We can drop all the $\mathcal{C}^\infty$ factors, given by $\left| \partial G/\partial y \right|$ etc. and proceed to show that

$$\left| D_x^\alpha D_y^\beta D_\xi^\gamma a\left(G(x), G(y), S(x,y)\xi\right) \right| \le C(1+|\xi|)^{m-|\gamma|} \quad \text{on } K \times K \times \mathbb{R}^n$$

where $K \subset\subset \Omega'$ and S is $\mathcal{C}^\infty$ with $|\det S| \ge \epsilon$. The estimates with $\alpha = \beta = 0$ follow easily and the general case by induction:

$$D_x^\alpha D_y^\beta D_\xi^\gamma a\left(G(x), G(y), S(x,y)\xi\right) \tag{3.122}$$

$$= \sum_{\substack{|\mu| \le |\alpha|+|\beta|+|\gamma| \\ |\alpha'| \le |\alpha|, |\beta| \le |\beta| \\ |\nu|+|\gamma| \le |\mu|}} M^{\alpha',\rho',\mu'}_{\alpha,\beta,\gamma,\nu}(x,y)\xi^\nu \left(D^{\alpha'} D^{\beta'} D^\mu a\right)(G(x), G(y), S\xi) \tag{3.123}$$

where the coefficients are $\mathcal{C}^\infty$. □

3.14. Problems

Problem 3.1. Show that *Weyl quantization*

$$S^\infty_\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto q_W(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) d\xi$$

is well-defined by continuity from $S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ and induces an isomorphism

$$S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \Psi^m_\infty(\mathbb{R}^n) \ \forall\ m \in \mathbb{R}.$$

Find an asymptotic formula relating $q_W(A)$ to $q_L(A)$ for any $A \in \Psi^m_\infty(\mathbb{R}^n)$.

PROBLEM 3.2. Show that if $A \in \Psi^m_\infty(\mathbb{R}^n)$ then $A^* = A$ if and only if $\sigma_W(A)$ is real-valued.

PROBLEM 3.3. Is it true that every $E \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ defines a map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$?

PROBLEM 3.4. Show that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ by proving that if $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ has compact support and is identically equal to 1 near the origin then

$$u_n(x) = (2\pi)^{-n}\phi(\frac{x}{n})\int e^{ix\cdot\xi}\phi(\xi/n)\hat{u}(\xi)d\xi \in \mathcal{S}(\mathbb{R}^n) \text{ if } u \in L^2(\mathbb{R}^n)$$

and $u_n \to u$ in $L^2(\mathbb{R}^n)$. Can you see any relation to pseudodifferential operators here?

PROBLEM 3.5. Check carefully that with the definition

$$H^k(\mathbb{R}^n) = \left\{ u \in \mathcal{S}(\mathbb{R}^n); u = \sum_{|\alpha|\le -k} D^\alpha u_\alpha,\ u_\alpha \in L^2(\mathbb{R}^n) \right\}$$

for $-k \in \mathbb{N}$ one does have

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n)$$

as claimed in the lecture.

PROBLEM 3.6. Suppose that $a(x) \in \mathcal{C}^\infty_\infty(\mathbb{R}^n)$ and that $a(x) \ge 0$. Show that the operator

$$A = \sum_{j=1}^n D^2_{x_j} + a(x)$$

can have no solution which is in $L^2(\mathbb{R}^n)$.

PROBLEM 3.7. Show that for any open set $\Omega \subset \mathbb{R}^n$, $\mathcal{C}^\infty_c(\Omega)$ is dense in $\mathcal{C}^{-\infty}_c(\Omega)$ in the weak topology.