CHAPTER 2

Pseudodifferential operators

Formula (1.48) for the action of a differential operator (with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$) on $\mathcal{S}(\mathbb{R}^n)$ can be written

$$P(x,D)u = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}P(x,\xi)u(y)dyd\xi \tag{2.1}$$

$$= (2\pi)^{-n}\int e^{ix\cdot\xi}P(x,\xi)\hat{u}(\xi)d\xi \tag{2.2}$$

where $\hat{u}(\xi)=\mathcal{F}u(\xi)$ is the Fourier transform of u. We shall generalize this formula by generalizing $P(x,\xi)$ from a polynomial in ξ to a *symbol.* In fact we shall also allow the symbol, or rather the *amplitude*, in the integral (2.2) to depend on the 'incoming' variables, y :

$$A(x,D)u = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi, u\in\mathcal{S}(\mathbb{R}^n). \tag{2.3}$$

Of course it is not clear that this integral is well-defined. To interpret (2.3) we shall first look into the properties of symbols.

2.1. Symbols

A polynomial, p, in ξ, of degree at most m, satisfies a bound

$$|p(\xi)|\le C(1+|\xi|)^m \quad \forall\ \xi\in\mathbb{R}^n. \tag{2.4}$$

Since successive derivatives, $D^\alpha_\xi p(\xi)$, are polynomials of degree $m-|\alpha|$, for any multiindex α, we get the family of estimates

$$|D^\alpha_\xi p(\xi)|\le C_\alpha(1+|\xi|)^{m-|\alpha|} \quad \forall\ \xi\in\mathbb{R}^n, \alpha\in\mathbb{N}_0^n. \tag{2.5}$$

Of course if $|\alpha|>m$ then $D^\alpha_\xi p\equiv 0$, so we can even take the constant C_α to be independent of α. If we consider the characteristic polynomial $P(x,\xi)$ of a differential operator of order m with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ (i.e. all derivatives of the coefficients are bounded) (2.5) is replaced by

$$\left| D^\alpha_x D^\beta_\xi P(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \quad \forall\ (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,\ \alpha,\beta\in\mathbb{N}_0^n. \tag{2.6}$$

There is no particular reason to have the same number of x variables as of ξ variables, so in general we define:

Definition 2.1. *The space $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ of symbols of order m consists of those functions $a\in\mathcal{C}^\infty(\mathbb{R}^p\times\mathbb{R}^n)$ satisfying all the estimates*

$$\left| D^\alpha_z D^\beta_\xi a(z,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \text{ on } \mathbb{R}^p\times\mathbb{R}^n \quad \forall\ \alpha\in\mathbb{N}_0^p, \beta\in\mathbb{N}_0^n. \tag{2.7}$$

For later reference we even define $S^m_\infty(\Omega;\mathbb{R}^n)$ when $\Omega\subset\mathbb{R}^p$ and $\Omega\subset\operatorname{clos}(\operatorname{int}(\Omega))$ as consisting of those $a\in\mathcal{C}^\infty(\operatorname{int}(\Omega)\times\mathbb{R}^n)$ satisfying (2.7) for $(z,\xi)\in\operatorname{int}(\Omega)\times\mathbb{R}^n$.

The estimates (2.7) can be rewritten

$$\|a\|_{N,m} = \sup_{\substack{z\in\mathrm{int}(\Omega)\\ \xi\in\mathbb{R}^n}} \max_{|\alpha|+|\beta|\le N} (1+|\xi|)^{-m+|\beta|} \times \left| D_z^\alpha D_\xi^\beta a(z,\xi) \right| < \infty. \tag{2.8}$$

With these *norms* $S^m_\infty(\Omega;\mathbb{R}^n)$ is a Fréchet space, rather similar in structure to $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Thus the topology is given by the metric

$$d(a,b) = \sum_{N\ge 0} 2^{-N} \frac{\|a-b\|_{N,m}}{1+\|a-b\|_{N,m}},\ a,b\in S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.9}$$

The subscript '∞' here is *not* standard notation. It refers to the assumption of uniform boundedness of the derivatives of the 'coefficients'. More standard notation would be just $S^m(\Omega\times\mathbb{R}^n)$, especially for $\Omega=\mathbb{R}^p$, but I think this is too confusing.

Now to some basic properties. First notice that

$$(1+|\xi|)^m \le (1+|\xi|)^{m'} \quad \forall\, \xi\in\mathbb{R}^n \Longleftrightarrow m\le m'.$$

Thus we have an inclusion

$$S^m_\infty(\Omega;\mathbb{R}^n) \hookrightarrow S^{m'}_\infty(\Omega;\mathbb{R}^n) \quad \forall\ m'\ge m. \tag{2.10}$$

Moreover this inclusion is continuous, since from (2.8), $\|a\|_{N,m'} \le \|a\|_{N,m}$ if $a\in S^m(\Omega;\mathbb{R}^n)$ and $m'\ge m$. Since these spaces increase with m we think of them as a *filtration* of the big space

$$S^\infty_\infty(\Omega;\mathbb{R}^n) = \bigcup_m S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.11}$$

The *residual* space of this filtration is

$$S^{-\infty}_\infty(\Omega;\mathbb{R}^n) = \bigcap_m S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.12}$$

A rather useful technical result is:

LEMMA 2.1. *For any $m'>m$ the space $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ is dense in the space $S^m_\infty(\Omega;\mathbb{R}^n)$ in the topology of $S^{m'}_\infty(\Omega;\mathbb{R}^n)$.*

PROOF. One *cannot* take $m'=m$ here, for the same reason that $\mathcal{S}(\mathbb{R}^n)$ is *not* dense in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$.

Choose $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(\xi)=1$ if $|\xi|<1, \phi(\xi)=0$ if $|\xi|>2$ and consider the sequence

$$a_k(z,\xi) = \phi(\xi/n)a(z,\xi), \quad a\in S^m_\infty(\Omega;\mathbb{R}^n)$$

I claim that $a_k\in S^{-\infty}_\infty(\Omega,\mathbb{R}^n)$ and $a_k\longrightarrow a$ in $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for any $m'>m$. [This gives a slightly stronger result than stated, i.e. the same sequence works for all $m'>m$.] Certainly for each N

$$|a_k(z,\xi)| \le C_{N,n}(1+|\xi|)^{-N}$$

and by Leibniz formula we easily conclude that

$$\left| D_z^\alpha D_\xi^\beta a_k(z,\xi) \right| \le C_{N,\alpha,\beta,n}(1+|\xi|)^{-N} \quad \forall\ \alpha,\beta,N,n.$$

Thus $a_k\in S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$.

So consider the difference

$$(a-a_k)(z,\xi) = (1-\phi)(\xi/n)\ a(z,\xi). \tag{2.13}$$

If $\frac{1}{2} > \epsilon > 0$ is given then

$$\mid (1-\phi)(\frac{\xi}{n})(1+|\xi|)^{-\epsilon} \mid\leq C_\epsilon n^{-\epsilon/2},\ n\geq 1. \tag{2.14}$$

Indeed, in $|\xi|\geq n^{\frac{1}{2}}$, $(1+|\xi|)^{-\epsilon}\leq n^{-\frac{\epsilon}{2}}$ and $(1-\phi)$ is bounded, and in $|\xi|\leq n^{\frac{1}{2}}$, by Taylor's formula

$$\begin{gathered}(1-\phi)(\frac{\xi}{n})\leq C|\frac{\xi}{n}|\leq Cn^{-\frac{1}{2}},\\ (1+|\xi|)^{-\epsilon}\leq C.\end{gathered} \tag{2.15}$$

This gives (2.14) and hence

$$(1+|\xi|)^{-m'} \mid (a-a_k)(z,\xi) \mid\leq C(1+|\xi|)^{-\epsilon}|(1-\phi)(\frac{\xi}{n})|\leq Cn^{-\frac{\epsilon}{2}}\longrightarrow 0.$$

This is the first symbol estimate. Next consider the ξ derivatives of (2.13). Using Leibniz' formula

$$D_\xi^\beta(a-a_k)=\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}D_\xi^{\beta-\gamma}(1-\phi)(\frac{\xi}{n})\cdot D_\xi^\gamma a(z,\xi) \tag{2.16}$$

$$=(1-\phi)(\frac{\xi}{n})\cdot D_\xi^\beta a(z,\xi)-\sum_{\gamma<\beta}\binom{\beta}{\gamma}(D^{\beta-\gamma}\phi)(\frac{\xi}{n})\cdot n^{-|\beta-\gamma|}D_\xi^\alpha a(z,\xi). \tag{2.17}$$

In the first term, $D_\xi^\beta a(z,\xi)$ is a symbol of order $m-|\beta|$, so

$$\sup_\xi(1+|\xi|)^{-m'+|\beta|}(1-\phi)(\frac{\xi}{n})D_\xi^\beta a(x,\xi)\longrightarrow 0$$

as $n\longrightarrow\infty$ if $m'>m$. In all the other terms, $(D^{\beta-\gamma}\phi)(\zeta)$ has compact support, i.e. $1\leq|\zeta|\leq 2$. Thus for each term we get a bound

$$\sup_{n\leq|\xi|\leq 2n}(1+|\xi|)^{-m'+|\beta|}\cdot n^{-|\beta-\gamma|}C\cdot(1+|\xi|)^{m-|\gamma|}\leq Cn^{-m'+m}.$$

The variables z play the rôle of parameters so we have in fact shown that

$$\sup_{\substack{z\in\Omega\\ \xi\in\mathbb{R}^n}}(1+|\xi|)^{-m'} \mid D_z^\alpha D_\xi^\beta(a-a_k) \mid\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

This means $a_k\longrightarrow a$ in $S_\infty^{m'}(\mathbb{R};\mathbb{R}^n)$ as required. □

In fact this proof suggests a couple of other 'obvious' results. Namely

$$S_\infty^m(\Omega;\mathbb{R}^n)\cdot S_\infty^{m'}(\Omega;\mathbb{R}^n)\subset S_\infty^{m+m'}(\Omega;\mathbb{R}^n). \tag{2.18}$$

This can be proved directly using Leibniz' formula:

$$\begin{aligned}&\sup_\xi(1+|\xi|)^{-m-m'+|\beta|} \mid D_z^\alpha D_\xi^\beta\left(a(z,\xi)\cdot b(z,\xi)\right) \mid\\ \leq&\sum_{\substack{\mu\leq\alpha\\ \gamma\leq\beta}}\binom{\alpha}{\mu}\binom{\beta}{\gamma}\sup_\xi(1+|\xi|)^{-m+|\gamma|} \mid D_z^\mu D_\xi^\gamma a(z,\xi) \mid\\ &\times\sup_\xi(1+|\xi|)^{-m'+|\beta-\gamma|} \mid D_z^{\alpha-\mu}D_\xi^{\beta-\gamma}b(z,\xi) \mid<\infty.\end{aligned}$$

We note the obvious action of differentiation:

$$
\begin{aligned}
D_z^\alpha &: S^m_\infty(\Omega;\mathbb{R}^n) \longrightarrow S^m_\infty(\Omega;\mathbb{R}^n)\\
D_\xi^\beta &: S^m_\infty(\Omega;\mathbb{R}^n) \longrightarrow S^{m-|\beta|}_\infty(\Omega;\mathbb{R}^n).
\end{aligned}
\tag{2.19}
$$

In fact, while we are thinking about these things we might as well show the important consequence of *ellipticity*. A symbol $a \in S^m_\infty(\Omega;\mathbb{R}^n)$ is said to be (globally) elliptic if

$$
|a(z,\xi)| \geq \epsilon(1+|\xi|)^m - C(1+|\xi|)^{m-1},\ \epsilon > 0 \tag{2.20}
$$

or equivalently

$$
|a(z,\xi)| \geq \epsilon(1+|\xi|)^m \text{ in } |\xi| \geq C_\epsilon. \tag{2.21}
$$

Lemma 2.2. *If $a \in S^m_\infty(\Omega;\mathbb{R}^n)$ is elliptic there exists $b \in S^{-m}_\infty(\Omega;\mathbb{R}^n)$ such that*

$$
a \cdot b - 1 \in S^{-\infty}_\infty(\Omega;\mathbb{R}^n). \tag{2.22}
$$

Proof. Using (2.21) choose ϕ as in the proof of Lemma 2.1 and set

$$
b(z,\xi) = \begin{cases} \frac{1-\phi(2\xi/C)}{a(z,\xi)} & |\xi| \geq C \\ 0 & |\xi| \leq C. \end{cases} \tag{2.23}
$$

Then b is $\mathcal{C}^\infty$ since $b = 0$ in $C \leq |\xi| \leq C + \epsilon$ for $\epsilon > 0$. The symbol estimates follow by noting that, in $|\xi| \geq C$,

$$
D_z^\alpha D_\xi^\beta b = a^{-1-2|\alpha|-|\beta|} \cdot G_{\alpha\beta}
$$

where $G_{\alpha\beta}$ is a symbol of order $(|\alpha|+|\beta|)m - |\beta|$. This can be proved by induction. □

2.2. Pseudodifferential operators

Now we proceed to discuss the formula (2.3) where we shall assume that, for some $p, m \in \mathbb{R}$

$$
\begin{aligned}
a(x,y,\xi) &= (1+|x-y|^2)^{p/2}\tilde{a}(x,y,\xi)\\
\tilde{a} &\in S^m_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi).
\end{aligned}
\tag{2.24}
$$

The extra factor (which allows polynomial growth in the direction of $x - y$) turns out, somewhat enigmatically, to both make no difference and be very useful! Notice that if $a \in \mathcal{C}^\infty(\mathbb{R}^{2n} \times \mathbb{R}^n)$ then $a \in (1+|x-y|^2)^{p/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ if and only if

$$
|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \leq C_{\alpha,\beta,\gamma}(1+|x-y|)^p(1+|\xi|)^m \ \forall\ \alpha,\beta,\gamma \in \mathbb{N}_0^n. \tag{2.25}
$$

If $m < -n$ then, for each $u \in \mathcal{S}(\mathbb{R}^n)$ the integral in (2.3) is absolutely convergent, uniformly in x, since

$$
\begin{aligned}
|a(x,y,\xi)u(y)| &\leq C(1+|x-y|)^p(1+|\xi|)^m(1+|y|)^{-N}\\
&\leq C(1+|x|)^p(1+|\xi|)^m(1+|y|)^m,\ m < -n.
\end{aligned}
$$

Thus we conclude that, provided $m < -n$,

$$
A : \mathcal{S}(\mathbb{R}^n) \longrightarrow (1+|x|^2)^p\mathcal{C}^0_\infty(\mathbb{R}^n).
$$

To show that, for general m, A exists as an operator, we prove that its Schwartz kernel exists.

PROPOSITION 2.1. *The map, defined for $m < -n$ as a convergent integral,*

$$(2.26)\quad \begin{aligned}(1+|x-y|^2)^{p/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \longmapsto I(a) = \\ (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi \in \mathcal{S}'(\mathbb{R}^{2n})\end{aligned}$$

extends by continuity to all m.

PROOF. Since we already have the density of $S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ in S^m_∞ we only need to show the continuity on this subspace. What we shall show is that, for each $m \in \mathbb{R}$ there are integers $N, k \in \mathbb{N}$ such that

$$(2.27)\quad \mid I(a)(\phi) \mid \le C\|a\|_{N,m}\|\phi\|_k \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^{2n}).$$

To see this we just use integration by parts!

Observe that

$$(2.28)\quad (1+\xi\cdot D_x)e^{i(x-y)\cdot\xi} = (1+|\xi|^2)e^{i(x-y)\cdot\xi}$$

$$(2.29)\quad (1-\xi\cdot D_y)e^{i(x-y)\cdot\xi} = (1+|\xi|^2)e^{i(x-y)\cdot\xi}.$$

Thus we can write (for $a \in S^{-\infty}_\infty$),

$$(2.30)\quad \begin{aligned}I(a)(\phi) = \iint (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-2q} \\ (1-\xi\cdot D_x)^q(1+\xi\cdot D_y)^q\big[a(x,y,\xi)\phi(x,y)\big]d\xi dx dy\end{aligned}$$

$$(2.31)\quad = \sum_{|\gamma|\le 2q}\iint \Big(\int e^{i(x-y)\cdot\xi}a_\gamma(x,y,\xi)d\xi\Big)D^\gamma_{(x,y)}\phi(x,y)dxdy.$$

Here $a_\gamma = (1+|\xi|^2)^{-2q}\sum_{\substack{\mu\le\gamma\\|\gamma|\le 2q}} C_\mu\xi^\gamma D^\mu_{(x,y)}a$. So if we take $-2q+m < -n$, e.g. $q \ge \max(\frac{n+m}{2},0)$ then

$$\mid I(a)(\phi) \mid \le C\|(1+|x-y|^2)^{-p/2}a\|_{2q,m}\|\phi\|_{2q+p}.$$

This proves the desired continuity. □

In fact we do *not* really need to integrate by parts in both x *and* y, either will do the trick.

LEMMA 2.3. *If $a \in (1+|x-y|^2)^{p/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ then the operator A, with Schwartz kernel $I(a)$, is a continuous linear map*

$$(2.32)\quad A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n).$$

We shall denote by $\Psi^m_\infty(\mathbb{R}^n)$ the linear spaces of operator (2.32), corresponding to $(1+|x-y|^2)^{p/2}a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for some p. They are called *pseudodifferential operators.*

PROOF. Proceeding as in (2.31) but only integrating by parts in y we deduce that, for q large depending on m,

$$Au(\psi) = \sum_{\gamma\le 2q}(2\pi)^{-n}\iint\int e^{i(x-y)\cdot\xi}a_\gamma(x,y,\xi)D^\gamma_y u(y)d\xi\phi(x)dydx.$$

This shows that Au is given by the convergent integral

$$Au(x) = \sum_{\gamma\le 2q}(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}a_\gamma(x,y,\xi)D^\gamma_y u(y)d\xi dy.$$

Notice that

$$D_{x_j}Au(x) = (2\pi)^{-n}\sum_{|\gamma|\le 2q}\iint e^{i(x-y)\cdot\xi}\xi_j a_\gamma\cdot D_y^\gamma u(y)dyd\xi$$

and

$$x_jAu(x) = (2\pi)^n\sum_{|\gamma|\le 2q}\iint e^{i(x-y)\cdot\xi}(-D_{\xi_j}+y_j)a_\gamma\cdot D_y^\gamma u(y)dyd\xi.$$

Proceeding inductively we find (2.32). □

2.3. Composition

There are two extreme cases of $I(a)$, namely where a is independent of x and of y. Below we shall prove:

THEOREM 2.1 (Reduction). *Each $A\in\Psi_\infty^m(\mathbb{R}^n)$ can be written uniquely as $I(a')$ where $a'\in S_\infty^m(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$.*

This is the main step in proving the fundamental result of this Chapter, which is that two pseudodifferential operators can be composed to give a pseudodifferential operator and that the orders are additive.

THEOREM 2.2. *The space $\Psi_\infty^\infty(\mathbb{R}^n)$ is a filtered $*$-algebra on $\mathcal{S}(\mathbb{R}^n)$.*

More explicitly we have to show that each $A\in\Psi_\infty^\infty(\mathbb{R}^n)$ defines a continuous linear map

$$A:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n) \tag{2.33}$$

and that

$$A\in\Psi_\infty^m(\mathbb{R}^n)\Longrightarrow A^*\in\Psi_\infty^m(\mathbb{R}^n) \tag{2.34}$$

$$A\in\Psi_\infty^m(\mathbb{R}^n),\ B\in\Psi_\infty^{m'}(\mathbb{R}^n)\Longrightarrow A\circ B\in\Psi_\infty^{m+m'}(\mathbb{R}^n). \tag{2.35}$$

In fact we will pick up some more information along the way.

2.4. Reduction

We proceed to prove Theorem 2.1, which we can restate as:

PROPOSITION 2.2. *The range of* (2.26) *(for any p) is the same as the range of I restricted to the image of the inclusion map*

$$S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)\ni a\longmapsto a(x,\xi)\in S_\infty^m(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n).$$

PROOF. Suppose $a\in\left(1+|x-y|^2\right)^{p/2}S_\infty^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ for some p, then

$$I\big((x_j-y_j)a\big) = I\big(-D_{\xi_j}a\big)\ j=1,\dots,n. \tag{2.36}$$

Indeed this is just the result of inserting the identity

$$D_{\xi_j}e^{i(x-y)\cdot\xi} = (x_j-y_j)e^{i(x-y)\cdot\xi}$$

into (2.26), with $N=0$, and integrating by parts. Since both sides of (2.36) are continuous on $\left(1+|x-y|^2\right)^{p/2}S_\infty^\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ the identity holds in general. Notice that if a is of order m then $D_{\xi_j}a$ is of order $m-1$, so (2.36) shows that even though the operator with amplitude $(x_j-y_j)a(x,y,\xi)$ *appears* to have order m, it actually has order $m-1$.

To exploit (2.36) consider the Taylor series (with Legendre's remainder) for $a(x, y, \xi)$ around $x = y$:

$$\begin{aligned} a(x,y,\xi) = &\sum_{|\alpha|\le N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \left(D_y^\alpha a\right)(x,x,\xi) \\ &+ \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \cdot R_{N,\alpha}(x,y,\xi). \end{aligned} \tag{2.37}$$

Here,

$$R_{N,\alpha}(x,y,\xi) = \int_0^1 (1-t)^{N-1}\left(D_y^\alpha a\right)(x,(1-t)x+ty,\xi)dt. \tag{2.38}$$

Now,

$$(x-y)^\alpha \left(D_y^\alpha a\right)(x,y,\xi) \in \left(1+|x-y|^2\right)^{\frac{(p+|\alpha|)}{2}} S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n).$$

Applying (2.36) repeatedly we see that if A is the operator with kernel $I(a)$ then

$$A = \sum_{j=0}^{N-1} A_j + R_N, \quad A_j \in \Psi_\infty^{m-j}(\mathbb{R}^n),\ R_N \in \Psi_\infty^{m-N}(\mathbb{R}^n) \tag{2.39}$$

where the A_j have kernels

$$I\Big(\sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!}\left(D_y^\alpha D_\xi^\alpha a\right)(x,x,\xi)\Big). \tag{2.40}$$

To proceed further we need somehow to *sum* this series. Of course we cannot really do this! □

2.5. Asymptotic summation

Suppose $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. The fact that the *orders* are decreasing means that those symbols are getting very small, for $|\xi|$ large. The infinite series

$$\sum_j a_j(z,\xi)$$

need not converge. However we shall say that it converges asymptotically, or since it is a series we say it is 'asymptotically summable,' if there exists $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ such that

$$\text{for every } N, \quad a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n). \tag{2.41}$$

We write this relation as

$$a \sim \sum_{j=0}^{\infty} a_j. \tag{2.42}$$

Proposition 2.3. *Any series $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ is asymptotically summable, in the sense of (2.41), and the asymptotic sum is well defined up to an additive term in $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.*

PROOF. The uniqueness part is easy. Suppose a and a' both satisfy (2.41). Taking the difference

$$a - a' = \big(a - \sum_{j=0}^{N-1} a_j\big) - \big(a' - \sum_{j=0}^{N-1} a_j\big) \in S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n). \tag{2.43}$$

Since $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ is just the intersection of the $S^{-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ over N it follows that $a - a' \in S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, proving the uniqueness.

So to the existence of an asymptotic sum. To construct this (by Borel's method[1]) we cut off each term "near infinity in ξ". Thus fix $\phi \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ with $\phi(\xi) = 0$ in $|\xi| \le 1$, $\phi(\xi) = 1$ in $|\xi| \ge 2$, $0 \le \phi(\xi) \le 1$. Consider a decreasing sequence

$$\epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \downarrow 0. \tag{2.44}$$

We want to set

$$a(z,\xi) = \sum_{j=0}^{\infty} \phi(\epsilon_j \xi) a_j(z,\xi). \tag{2.45}$$

Of course we need to consider the seminorms, in $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ of each term. The first of these is

$$\sup_z \sup_\xi (1+|\xi|)^{-m} \;\big|\, \phi(\epsilon_j\xi) \,\big|\;\big|\, a_j(z,\xi) \,\big|\,. \tag{2.46}$$

Now $\phi(\epsilon_j\xi) = 0$ unless $|\epsilon_j\xi| \le 1$, i.e. $|\xi| \le \frac{1}{\epsilon_j}$. Since a_j is of order $m-j$ this allows us to estimate (2.46) by

$$\sup_z \sup_{|\xi|\le\frac{1}{\epsilon_j}} (1+|\xi|)^{-j} \cdot \big[\big(1+|\xi|\big)^{-m+j} \;\big|\, a_j(z,\xi) \,\big|\big]$$

$$\le \big(1+\frac{1}{\epsilon_j}\big)^{-j} \cdot C_j \le \epsilon_j^j \cdot C_j$$

where the C_j's are *fixed* constants, independent of ϵ_j, Let's look at the higher symbol estimates. As usual we can apply Leibniz' formula:

$$\sup_z \sup_\xi (1+|\xi|)^{-m+|\beta|} \;\big|\, D^\alpha_z D^\beta_\xi \phi(\epsilon_j\xi) a_j(z,\xi) \,\big|$$

$$\le \sum_{\mu\le\beta} \sup_z \sup_\xi (1+|\xi|)^{|\beta|-|\mu|-j} \epsilon_j^{|\beta|-|\mu|} \;\big|\, (D^{\beta-\mu}\phi)(\epsilon_j\xi) \,\big|$$

$$\times (1+|\xi|)^{-m+j+|\mu|} \;\big|\, D^\alpha_z D^\mu_\xi a_j(z,\xi) \,\big|\,.$$

The term with $\mu = \beta$ we estimate as before and the others, with $\mu \ne \beta$ are supported in $\frac{1}{\epsilon_j} \le |\xi| \le \frac{2}{\epsilon_j}$. Then we find that for all j

$$\|\phi(\epsilon_j\xi) a_j(z,\xi)\|_{N,m} \le C_{N,j}\epsilon_j^j \tag{2.47}$$

where C_N is independent of ϵ_j.

So we see that for a given N we can arrange that, for instance,

$$\|\phi(\epsilon_j\xi) a_j(z,\xi)\|_{N,m} \le C_N \frac{1}{j^2}$$

[1] Émile Borel

by choosing the ϵ_j to satify

$$C_{N,j}\epsilon_j^j \le \frac{1}{j^2} \quad \forall\ j \ge j(N).$$

Notice the crucial point here, we can arrange that for *each* N the sequence of norms in (2.47) is dominated by $C_N j^{-2}$ by fixing $\epsilon_j < \epsilon_{j,N}$ for large j. Thus we can arrange convergence of *all* the sums

$$\sum_j \|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,n}$$

by diagonalization of the ϵ, N sequences. Thus by choosing $\epsilon_j \downarrow 0$ rapidly enough we ensure that the series (2.45) converges. In fact the same argument allows us to ensure that for every N

$$\sum_{j\ge N} \phi(\epsilon_j\xi)a_j(z,\xi) \text{ converges in } S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n). \tag{2.48}$$

This certainly gives (2.41) with a defined by (2.45). □

2.6. Residual terms

Now we can apply Proposition 2.3 to the series (2.40), i.e. we can find $b \in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$

$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D^\alpha_y a)(x,x,\xi). \tag{2.49}$$

Let $B = I(b)$ be the operator defined by the symbol (which is independent of y). Now (2.39) says that

$$A - B = \sum_{j=0}^{N-1} A_j + R_N - B$$

and from (2.41) applied to (2.49)

$$B = \sum_{j=0}^{N-1} A_j + R'_N,\ R'_N \in \Psi^{m-N}_\infty(\mathbb{R}^n)$$

Thus

$$A - B \in \Psi^{-\infty}_\infty(\mathbb{R}^n) = \bigcap_N \Psi^N_\infty(\mathbb{R}^n). \tag{2.50}$$

Notice that, at this stage, we do *not* know that $A - B$ has kernel $I(c)$ with $c \in S^{-\infty}_\infty(\mathbb{R}^{2n},\mathbb{R}^n)$, just that it has kernel $I(c_N)$ with $c_N \in S^N_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for each N.

However:

Proposition 2.4. *An operator $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is an element of the space $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies the estimates*

$$\left| D^\alpha_x D^\beta_y K(x,y) \right| \le C_{N,\alpha,\beta}(1+|x-y|)^{-N}\ \forall\ N. \tag{2.51}$$

PROOF. Suppose first that $A \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$, which means that $A \in \Psi_\infty^N(\mathbb{R}^n)$ for every N. The Schwartz kernel, K_A, of A is therefore given by (2.26) with the amplitude $C_N \in S_\infty^N(\mathbb{R}^{2n};\mathbb{R}^n)$. For $N << -n-1-p$ the integral certainly converges absolutely and we can integrate by parts to show that

$$\begin{aligned}(x-y)^\alpha D_x^\beta D_y^\gamma K_A(x,y)\\ = (2\pi)^{-N}\int e^{i(x-y)\cdot\xi}(-D_\xi)^\alpha(D_x+i\xi)^\beta(D_y-i\xi)^\gamma a(x,y,\xi)d\zeta\end{aligned}$$

which converges absolutely provided $|\beta|+|\gamma|+N-|\alpha|<-n$. Thus

$$\sup \left| (x-y)^\alpha D_x^\beta D_y^\gamma K \right| < \infty \qquad \forall\ \alpha,\beta,\gamma$$

which is another way of writing (2.51) i.e.

$$\sup\bigl(1+|x-y|^2\bigr)^N \left| D_x^\beta D_y^\gamma K \right| < \infty \qquad \forall\ \beta,\gamma,N.$$

Conversely suppose that (2.51) holds. Define

$$g(x,z) = K(x,x-z). \tag{2.52}$$

The estimates (2.51) become

$$\sup \left| D_x^\alpha z^\gamma D_z^\beta g(x,z) \right| < \infty \qquad \forall\ \alpha,\beta,\gamma. \tag{2.53}$$

That is, g is rapidly decreasing with all its derivatives in z. Taking the Fourier transform,

$$b(x,\xi) = \int e^{-iz\cdot\xi} g(x,z)dz \tag{2.54}$$

the estimate (2.53) translates to

$$\begin{aligned}\left| D_x^\alpha \xi^\beta D_\xi^\gamma b(x,\xi) \right| < \infty \qquad \forall\ \alpha,\beta,\gamma\\ \iff b \in S_\infty^{-\infty}(\mathbb{R}_x^n;\mathbb{R}_\xi^n).\end{aligned} \tag{2.55}$$

Now the inverse Fourier transform in (2.54), combined with (2.52) gives

$$K(x,y) = g(x,x-y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}b(x,\xi)d\xi \tag{2.56}$$

i.e. $K = I(b)$. This certainly proves the proposition and actually gives the stronger result.

$$A \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \iff A = I(c),\ c \in S_\infty^{-\infty}(\mathbb{R}_x^n;\mathbb{R}_\xi^n). \tag{2.57}$$

□

This also finishes the proof of Proposition 2.2 since in (2.49), (2.50) we have shown that

$$A = B + R, \quad B = I(b), \quad R \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \tag{2.58}$$

so in fact

$$A = I(e),\ e \in S_\infty^m(\mathbb{R}_x^n;\mathbb{R}_\xi^n),\ e \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\bigl(D_y^\alpha D_\xi^\alpha a\bigr)(x,x,\xi). \tag{2.59}$$

2.7. Proof of Theorem 2.2

First consider the adjoint formula. If

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

the adjoint is the operator

$$A^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

defined by duality:

$$A^*u(\bar{\phi}) = u(\overline{A\phi}) \qquad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{2.60}$$

Certainly $A^*u \in \mathcal{S}'(\mathbb{R}^n)$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ since

$$\begin{gathered} A^*u(\psi) = u(\overline{A\overline{\psi}}) \text{ and} \\ \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \overline{A\bar{\psi}} \in \mathcal{S}(\mathbb{R}^n) \end{gathered} \tag{2.61}$$

is clearly continuous. In terms of Schwartz kernels,

$$\begin{aligned} A\phi(x) &= \int K_A(x,y)\phi(y)dy \\ A^*u(x) &= \int K_{A^*}(x,y)u(y)dy \end{aligned} \tag{2.62}$$

we see that

$$\begin{gathered} \int K_{A^*}(x,y)u(y)\overline{\phi(x)}dydx \\ = \int \overline{K_A(x,y)\phi(y)}dyu(x)dx \\ \Longrightarrow K_{A^*}(x,y) = \overline{K_A(y,x)} \end{gathered} \tag{2.63}$$

where we are using the uniqueness of Schwartz kernels. This proves the first part of (2.35) since

$$\begin{aligned} \overline{K_A(y,x)} &= \overline{[\frac{1}{(2\pi)^n}\int e^{i(y-x)\cdot\xi}a(y,x,\xi)d\xi]} \\ &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}\bar{a}(y,x,\xi)d\xi \end{aligned} \tag{2.64}$$

i.e. $A^* = I(\bar{a}(y,x,\xi))$. Thus one advantage of allowing general operators (2.26) is that closure under adjoint is immediate.

For the composition formula we need to apply Proposition 2.2 twice. First to $A \in \Psi^m_\infty(\mathbb{R}^n)$, to write it with symbol $a(x,\xi)$

$$A\phi(x) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)\phi(y)dyd\xi \tag{2.65}$$

$$= (2\pi)^{-n}\int e^{ix\cdot\xi}a(x,\xi)\hat{\phi}(\xi)d\xi. \tag{2.66}$$

Then we also apply Proposition 2.2 to B^*,

$$B^*u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\bar{b}(x,\xi)\hat{u}(\xi)d\xi$$

Integrating this against a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ gives

$$\langle \phi, B^* u\rangle = (2\pi)^{-n} \int \int e^{-ix\cdot\xi}\phi(x)b(x,\xi)\overline{\hat{u}(\xi)}d\xi dx$$

$$\Longrightarrow \widehat{Bu}(\xi) = \int e^{-iy\cdot\xi}b(y,\xi)u(y)dy.$$

Inserting this into the formula for $A\phi$ shows that

$$\Longrightarrow AB(u) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi}a(x,\xi)b(y,\xi)u(y)dyd\xi.$$

Since $a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi)$ this shows that $AB \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ as claimed.

2.8. Problems

PROBLEM 2.1. Show, in detail, that for each $m \in \mathbb{R}$

$$(1+|\xi|^2)^{-\frac{1}{2}m} \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$$

for any p. Use this to show that

$$S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\cdot S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) = S^{m+m'}_\infty(\mathbb{R}^p;\mathbb{R}^n).$$

PROBLEM 2.2. Consider $p = 0$ and $n = 2$ in the definition of symbols and show that if $a \in S^1_\infty(\mathbb{R}^2)$ is elliptic then for $r > 0$ sufficiently large the integral

$$\int_0^{2\pi} \frac{1}{a(re^{i\theta})}\frac{d}{d\theta}a(re^{i\theta})d\theta$$

exists and is independent of r, where $z = \xi_1 + i\xi_2$ is the complex variable in $\mathbb{R}^2 = \mathbb{C}$. Conclude that there is an elliptic symbol, a on $\mathbb{R}^2$, such that there does not exist b, a symbol with

$$b \neq 0 \text{ on } \mathbb{R}^2 \text{ and } a(\xi) = b(\xi) \text{ for } |\xi| > r$$

for any r.

PROBLEM 2.3. Show that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies all the estimates

$$|D^\alpha_x D^\beta_y a(x,y)| < C_{\alpha,\beta,N}(1+|x-y|)^{-N}$$

for multiindices $\alpha, \beta \in \mathbb{N}^n_0$ and $N \in \mathbb{N}_0$.