CHAPTER 1

Tempered distributions and Fourier transform

Microlocal analysis is a geometric theory of distributions, or a theory of geometric distributions. Rather than study general distributions – which are like general continuous functions but worse – we consider more specific types of distributions which actually arise in the study of differential and integral equations. Distributions are usually defined by duality, starting from very "good" test functions; correspondingly a general distribution is everywhere "bad". The *conormal distributions* we shall study implicitly for a long time, and eventually explicitly, are usually good, but like (other) people have a few interesting faults, i.e. singularities. These singularities are our principal target of study. Nevertheless we need the general framework of distribution theory to work in, so I will start with a brief introduction. This is designed either to remind you of what you already know or else to send you off to work it out. Proofs of some of the main theorems are outlined in the problems at the end of each lecture.

1.1. Schwartz test functions

To fix matters at the beginning we shall work in the space of tempered distributions. These are defined by duality from the space of Schwartz test functions of rapid decrease (the goody two-shoes of distributions). We can think of analysis as starting off from algebra, which gives us the polynomials. Thus in $\mathbb{R}^n$ we have the coordinate functions, $x_1, \dots, x_n$ and the constant functions and then the polynomials are obtained by taking (finite) sums and products:

$$(1.1)\quad \phi(x) = \sum_{|\alpha|\le k} p_\alpha x^\alpha,\ p_\alpha \in \mathbb{C},\ \alpha \in \mathbb{N}_0^n,\ \alpha = (\alpha_1, \dots, \alpha_n),$$

$$\text{where } x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n} = \prod_{j=1}^n x_j^{\alpha_j} \text{ and } \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

A general function $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is differentiable at $\bar{x}$ if there is a linear function $\ell_{\bar{x}}(x) = c + \sum_{j=1}^{n} (x_j - \bar{x}_j) d_j$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$(1.2)\qquad |\phi(x) - \ell_{\bar{x}}(x)| \le \epsilon |x - \bar{x}| \quad \forall\ |x - \bar{x}| < \bar{\delta}.$$

The coefficients d_j are the partial derivative of ϕ at the point $\bar{x}$. Then, ϕ is said to be differentiable on $\mathbb{R}^n$ if it is differentiable at each point $\bar{x} \in \mathbb{R}^n$; the partial derivatives are then also functions on $\mathbb{R}^n$ and ϕ is *twice* differentiable if the partial derivatives are differentiable. In general it is k times differentiable if its partial derivatives are $k - 1$ times differentiable.

If ϕ is k times differentiable then, for each $\bar{x} \in \mathbb{R}^n$, there is a polynomial of degree k, $p_k(x; \bar{x}) = \sum_{|\alpha|\le k} a_\alpha i^{|\alpha|} (x - \bar{x})^\alpha / \alpha!$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\phi(x) - p_k(x, \bar{x})| \leq \epsilon |x - \bar{x}|^k \quad \text{if } |x - \bar{x}| < \delta. \tag{1.3}$$

Then we set

$$D^\alpha \phi(\bar{x}) = a_\alpha. \tag{1.4}$$

If ϕ is infinitely differentiable all the $D^\alpha \phi$ are infinitely differentiable (hence continuous!) functions.

Definition 1.1. *The space of Schwartz test functions of rapid decrease consists of those $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that for every $\alpha, \beta \in \mathbb{N}_0^n$*

$$\sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha \phi(x)| < \infty; \tag{1.5}$$

it is denoted $\mathcal{S}(\mathbb{R}^n)$.

From (1.5) we construct norms on $\mathcal{S}(\mathbb{R}^n)$:

$$\|\phi\|_k = \max_{|\alpha|+|\beta| \leq k} \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)|. \tag{1.6}$$

It is straightforward to check the conditions for a norm:

1. $\|\phi\|_k \geq 0,\ \|\phi\|_k = 0 \Longleftrightarrow \phi \equiv 0$
2. $\|t\phi\|_k = |t| \|\phi\|_k$
3. $\|\phi + \psi\|_k \leq \|\phi\|_k + \|\psi\|_k \ \forall\ \phi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

The topology on $\mathcal{S}(\mathbb{R}^n)$ is given by the metric

$$d(\phi, \psi) = \sum_k 2^{-k} \frac{\|\phi - \psi\|_k}{1 + \|\phi - \psi\|_k}. \tag{1.7}$$

Proposition 1.1. *With the distance function* (1.7), *$\mathcal{S}(\mathbb{R}^n)$ becomes a complete metric space (hence a Fréchet space).*

Of course one needs to check that $\mathcal{S}(\mathbb{R}^n)$ is non-trivial; however one can easily see that

$$\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n).$$

The two elementary operators of differentiation and coordinate multiplication give continuous linear operators:

$$x_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \tag{1.8}$$

$$D_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{1.9}$$

Other important operations we shall encounter include the exterior product,

$$\begin{gathered} \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m) \ni (\phi, \psi) \mapsto \phi \boxtimes \psi \in \mathcal{S}(\mathbb{R}^{n+m}) \\ \phi \boxtimes \psi(x, y) = \phi(x)\psi(y). \end{gathered} \tag{1.10}$$

1.2. Tempered distributions

As well as the exterior multiplication (1.10) there is the even more obvious multiplication operation

$$\begin{gathered} \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \\ (\phi, \psi) \mapsto \phi(x)\psi(x) \end{gathered} \tag{1.11}$$

and integration:

$$\int : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}. \tag{1.12}$$

Combining these gives a *pairing*:

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi, \psi) \longmapsto \int\limits_{\mathbb{R}^n} \phi(x)\psi(x)dx.$$

If we fix $\phi \in \mathcal{S}(\mathbb{R}^n)$ this defines a continuous linear map:

$$T_\phi : \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \int \phi(x)\psi(x)dx. \tag{1.13}$$

Continuity becomes the condition:

$$\exists\ k, C_k \text{ s.t. } |T_\phi(\psi)| \leq C_k \|\psi\|_k\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$$

We generalize this by denoting by $\mathcal{S}'(\mathbb{R}^n)$ the dual space, i.e. the space of all continuous linear functionals

$$u \in \mathcal{S}'(\mathbb{R}^n) \Longleftrightarrow u : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$$
$$\exists\ k, C_k \text{ such that } |u(\psi)| \leq C_k \|\psi\|_k\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$$

LEMMA 1.1. *The map*

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto T_\phi \in \mathcal{S}'(\mathbb{R}^n) \tag{1.14}$$

is an injection.

PROOF. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $T_\phi(\phi) = \int |\phi(x)|^2 dx$, so $T_\phi = 0$ implies $\phi \equiv 0$. □

If we wish to consider a topology on $\mathcal{S}'(\mathbb{R}^n)$ it will normally be the *weak* topology, that is the weakest topology with respect to which all the linear maps

$$\mathcal{S}'(\mathbb{R}^n) \ni u \longmapsto u(\phi) \in \mathbb{C}, \quad \phi \in \mathcal{S}(\mathbb{R}^n)$$

are continuous. This just means that it is given by the seminorms

$$\mathcal{S}(\mathbb{R}^n) \ni u \longmapsto |u(\phi)| \in \mathbb{R} \tag{1.15}$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$ is fixed but arbitrary. The sets

$$\{u \in \mathcal{S}'(\mathbb{R}^n); |u(\phi_j)| < \epsilon_j,\ \phi_j \in \Phi\} \tag{1.16}$$

form a basis of the neighbourhoods of 0 as $\Phi \subset \mathcal{S}(\mathbb{R}^n)$ runs over finite open sets and the ϵ_j are positive numbers.

PROPOSITION 1.2. *The injection* $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, *given by* (1.14)*, has dense range in the weak topology.*

See Problem 1.4 for the outline of a proof.

The maps x_i, D_j extend by continuity (and hence uniquely) to operators

$$x_j,\ D_j : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{1.17}$$

This is easily seen by defining then by duality. Thus if $\phi \in \mathcal{S}(\mathbb{R}^n)$ set $D_j T_\phi = T_{D_j\phi}$, then

$$T_{D_j\phi}(\psi) = \int D_j\phi\psi = -\int \phi D_j\psi,$$

the integration by parts formula. The definition

$$D_j u(\psi) = u(-D_j\psi),\ x_j u(\psi) = u(x_j\psi), \quad u \in \mathcal{S}'(\mathbb{R}^n), \psi \in \mathcal{S}(\mathbb{R}^n)$$

satisfies all requirements of the proposition.

1.3. Two big theorems

The association, by (1.14), of a distribution to a function can be extended considerably. For example if $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a bounded and continuous function then

$$T_u(\psi) = \int u(x)\psi(x)dx$$

still defines a distribution which vanishes if and only if u vanishes identically. Using the operations (1.17) we conclude that for any $\alpha, \beta \in \mathbb{N}_0^n$

$$x^\beta D_x^\alpha u \in \mathcal{S}'(\mathbb{R}^n) \text{ if } u : \mathbb{R}^n \longrightarrow \mathbb{C} \text{ is bounded and continuous.}$$

Conversely we have the *Schwartz representation Theorem:*

THEOREM 1.1. *For any $u \in \mathcal{S}'(\mathbb{R}^n)$ there is a finite collection $u_{\alpha\beta} : \mathbb{R}^n \longrightarrow \mathbb{C}$ of bounded continuous functions, $|\alpha| + |\beta| \leq k$, such that*

$$u = \sum_{|\alpha|+|\beta|\leq k} x^\beta D_x^\alpha u_{\alpha\beta}.$$

Thus tempered distributions are just products of polynomials and derivatives of bounded continuous functions. This is important because it says that distributions are "not too bad".

The second important result (long considered very difficult to prove, but there is a relatively straightforward proof using the Fourier transform) is the *Schwartz kernel theorem.* To show this we need to use the exterior product (1.10). If $K \in \mathcal{S}'(\mathbb{R}^{n+m})$ this allows us to define a linear map

$$O_K : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

by

$$O_K(\psi)(\phi) = \int K \cdot \phi \boxtimes \psi \ dxdy.$$

THEOREM 1.2. *There is a 1-1 correspondence between continuous linear operators*

$$A : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

and $\mathcal{S}'(\mathbb{R}^{n+m})$ given by $A = O_K$.

Brief outlines of the proofs of these two results can be found in Problems 1.10 and 1.11.

1.4. Examples

Amongst tempered distributions we think of $\mathcal{S}(\mathbb{R}^n)$ as being the 'trivial' examples, since they are the test functions. One can say that the study of the singularities of tempered distributions amounts to the study of the quotient

$$\mathcal{S}'(\mathbb{R}^n)/\mathcal{S}(\mathbb{R}^n)$$

which could, reasonably, be called the space of tempered microfunctions.

The sort of distributions we are interested in are those like the Dirac delta "function"

$$\delta(x) \in \mathcal{S}'(\mathbb{R}^n), \quad \delta(\phi) = \phi(0).$$

The definition here shows that δ is just the Schwartz kernel of the operator

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto \phi(0) \in \mathbb{C} = \mathcal{S}(\mathbb{R}^0).$$

This is precisely one reason it is interesting. More generally we can consider the map

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto D^\alpha\phi(0), \quad \alpha \in \mathbb{N}_0^n.$$

These have Schwartz kernels $(-D)^\alpha\delta$ since

$$(-D)^\alpha\delta(\phi) = \delta(D^\alpha\phi) = D^\alpha\phi(0). \tag{1.18}$$

If we write the relationship $A = O_K \longleftrightarrow K$ as

$$(A\psi)(\phi) = \int K(x,y)\phi(x)\psi(y)dxdy$$

then (1.18) becomes

$$D^\alpha\phi(0) = \int(-D)^\alpha\delta(x)\phi(x)dx.$$

More generally, if $K(x,y)$ is the kernel of an operator A then the kernel of $A\cdot D^\alpha$ is $(-D)_y^\alpha K(x,y)$ whereas the kernel of $D^\alpha \circ A$ is $D_x^\alpha K(x,y)$.

1.5. Two little lemmas

Above, I outlined some of the basic properties of tempered distributions. The main "raison d'être" for $\mathcal{S}'(\mathbb{R}^n)$ is the *Fourier transform* which I proceed to discuss. We shall use the Fourier transform as an indispensable tool in the treatment of pseudodifferential operators. The description of differential operators, via their Schwartz kernels, using the Fourier transform is an essential motivation for the extension to pseudodifferential operators.

Partly as simple exercises in the theory of distributions, and more significantly as preparation for the proof of the inversion formula for the Fourier transform we consider two lemmas.

First recall that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then we have defined $D_j u \in \mathcal{S}'(\mathbb{R}^n)$ by

$$D_j u(\phi) = u(-D_j\phi) \ \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

In this sense it is a "weak derivative". Let us consider the simple question of the form of the solutions to

$$D_j u = 0, \qquad u \in \mathcal{S}'(\mathbb{R}^n). \tag{1.19}$$

Let I_j be the integration operator:

$$\begin{gathered} I_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{n-1}) \\ I_j(\phi)(y_1,\dots,y_{n-1}) = \int \phi(y_1,\dots y_{j-1},x,y_j,\dots y_{n-1})dx. \end{gathered} \tag{1.20}$$

Then if $\pi_j : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ is the map $\pi_j(x) = (x_1,\dots,x_{j-1},x_{j+1}\dots,x_n)$, we *define*, for $v \in \mathcal{S}'(\mathbb{R}^{n-1})$,

$$\pi_j^* v(\phi) = v(I_j\phi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

It is clear from (1.20) that $I_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{n-1})$ is continuous and hence $\pi_j^* v \in \mathcal{S}'(\mathbb{R}^n)$ is well-defined for each $v \in \mathcal{S}'(\mathbb{R}^{n-1})$.

Lemma 1.2. *The identity* (1.19) *holds if and only if* $u = \pi_j^* v$ *for some* $v \in \mathcal{S}'(\mathbb{R}^{n+1})$.

PROOF. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi = D_j\psi$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ then $I_j\phi = I_j(D_j\psi) = 0$. Thus if $u = \pi_j^* v$ then

$$u(-D_j\phi) = \pi_j^* v(-D_j\phi) = v(I_j(-D_j\phi)) = 0.$$

Thus $u = \pi_j^* v$ does always satisfy (1.19).

Conversely suppose (1.19) holds. Choose $\rho \in \mathcal{S}(\mathbb{R})$ with the property

$$\int \rho(x)dx = 1.$$

Then each $\phi \in \mathcal{S}(\mathbb{R}^n)$ can be decomposed as

$$\phi(x) = \rho(x_j)I_j\phi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n) + D_j\psi, \quad \psi \in \mathcal{S}(\mathbb{R}^n). \tag{1.21}$$

Indeed this is just the statement

$$\zeta \in \mathcal{S}(\mathbb{R}^n),\ I_j\zeta = 0 \Longrightarrow \psi(x) \in \mathcal{S}(\mathbb{R}^n) \text{ where}$$

$$\psi(x) = \begin{cases} \int\limits_{-\infty}^{x_j} \zeta(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)dt \\ \int\limits_{\infty}^{x_j} \zeta(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)dt \end{cases}$$

Using (1.21) and (1.19) we have

$$u(\phi) = u\left(\rho(x_j)I_j\phi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n)\right).$$

Thus if

$$v(\psi) = u\left(\rho(x_j)\psi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n)\right)\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^{n-1})$$

then $v \in \mathcal{S}'(\mathbb{R}^{n-1})$ and $u = \pi_j^* v$. This proves the lemma. □

Of course the notation $u = \pi_j^* v$ is much too heavy-handed. We just write $u(x) = v(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$.

The second, related, lemma is just a special case of a general result of Schwartz concerning the support of a distribution. The particular results is:

LEMMA 1.3. *Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ and $x_j u = 0, j = 1, \dots n$ then $u = c\delta(x)$ for some constant c.*

PROOF. Again we use the definition of multiplication and a dual result for test functions. Namely, choose $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\rho(x) = 1$ in $|x| < \frac{1}{2}$, $\rho(x) = 0$ in $|x| \geq 3/4$. Then any $\phi \in \mathcal{S}(\mathbb{R}^n)$ can be written

$$\phi = \phi(0) \cdot \rho(x) + \sum_{j=1}^{n} x_j\psi_j(x), \qquad \psi_j \in \mathcal{S}(\mathbb{R}^n). \tag{1.22}$$

This in turn can be proved using Taylor's formula as I shall proceed to show. Thus

$$\phi(x) = \phi(0) + \sum_{j=1}^{n} x_j\zeta_j(x) \text{ in } |x| \leq 1, \text{ with } \zeta_j \in \mathcal{C}^\infty.$$

Then,

$$\rho(x)\phi(x) = \phi(0)\rho(x) + \sum_{j=1}^{n} x_j\rho_j\zeta_j(x)$$

and $\rho\zeta_j \in \mathcal{S}(\mathbb{R}^n)$. Thus it suffices to check (1.22) for $(1-\rho)\phi$, which vanishes identically near 0. Then $\zeta = |x|^{-2}(1-\rho)\phi \in \mathcal{S}(\mathbb{R}^n)$ and so

$$(1-\rho)\phi = |x|^2\zeta = \sum_{j=1}^n x_j(x_j\zeta)$$

finally gives (1.22) with $\psi_j(x) = \rho(x)\zeta_j(x) + x_j\zeta(x)$. Having proved the existence of such a decomposition we see that if $x_j u = 0$ for all j the

$$u(\phi) = u(\phi(0)\rho(x)) + \sum_{j=1}^n u(x_j\psi_j) = c\phi(0), \quad c = u(\rho(x)),$$

i.e. $u = c\delta(x)$. □

1.6. Fourier transform

Our normalization of the Fourier transform will be

$$\mathcal{F}\phi(\xi) = \int e^{-i\xi\cdot x}\phi(x)dx. \tag{1.23}$$

As you all know the inverse Fourier transform is given by

$$\mathcal{G}\psi(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\psi(\xi)\,d\xi. \tag{1.24}$$

Since it is so important here I will give a proof of this invertibility. First however, let us note some of the basic properties.

Both $\mathcal{F}$ and $\mathcal{G}$ give continuous linear maps

$$\mathcal{F}, \mathcal{G} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{1.25}$$

To see this observe first that the integrals in (1.23) and (1.24) are absolutely convergent:

$$|\mathcal{F}\phi(\xi)| \le \int |\phi(x)|dx \le \int (1+|x|^2)^{-n}\,dx \times \sup_{x\in\mathbb{R}^n}(1+|x|^2)^n|\phi(x)|,$$

where we use the definition of $\mathcal{S}(\mathbb{R}^n)$. In fact this shows that $\sup|\mathcal{F}\phi| < \infty$ if $\phi \in \mathcal{S}(\mathbb{R}^n)$. Formal differentiation under the integral sign gives an absolutely convergent integral:

$$D_j\mathcal{F}\phi(\xi) = \int D_{\xi_j}e^{-ix\xi}\phi(x)dx \tag{1.26}$$

$$= \int e^{-ix\cdot\xi}(-x_j\phi)dx \tag{1.27}$$

since $\sup_x(1+|x|^2)^n|x_j\phi| < \infty$. Then it follows that $D_j\mathcal{F}\phi$ is also bounded, i.e. $\mathcal{F}\phi$ is differentiable, and (1.27) holds. This argument can be extended to show that $\mathcal{F}\phi$ is $\mathcal{C}^\infty$,

$$D^\alpha\mathcal{F}\phi(\xi) = \mathcal{F}\big((-x)^\alpha\phi\big). \tag{1.28}$$

Similarly, starting from (1.23), we can use integration by parts to show that

$$\xi_j\mathcal{F}\phi(\xi) = \int e^{-ix\xi}\xi_j\phi(x)dx \tag{1.29}$$

$$= \int e^{-ix\cdot\xi}(D_j\phi)(x)dx \tag{1.30}$$

i.e. $\xi_j \mathcal{F}\phi = \mathcal{F}(D_j\phi)$. Combining this with (1.28) gives

$$\xi^\alpha D_\xi^\beta \mathcal{F}\phi = \mathcal{F}\big(D^\alpha \cdot [(-x)^\beta \phi]\big). \tag{1.31}$$

Since $D_x^\alpha((-x)^\beta\phi) \in \mathcal{S}(\mathbb{R}^n)$ we conclude

$$\sup |\xi^\alpha D_\zeta^\beta \mathcal{F}\phi| < \infty \Longrightarrow \mathcal{F}\phi \in \mathcal{S}(\mathbb{R}^n).$$

This map is continuous since

$$\sup |\xi^\alpha D_\xi^\beta \mathcal{F}\phi| \le C \cdot \sup_x |(1+|x|^2)^n D_x^\alpha [(-x)^\beta \phi]$$
$$\Longrightarrow \|\mathcal{F}\phi\|_k \le C_k \|\phi\|_{k+2n}, \quad \forall\ k.$$

The identity (1.31), written in the form

$$\mathcal{F}(D_j\phi) = \xi_j \mathcal{F}\phi \tag{1.32}$$
$$\mathcal{F}(x_j\phi) = -D_{\xi_j}\mathcal{F}\phi \tag{1.33}$$

is already the key to the proof of invertibility:

THEOREM 1.3. *The Fourier transform gives an isomorphism* $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longleftrightarrow \mathcal{S}(\mathbb{R}^n)$ *with inverse* $\mathcal{G}$.

PROOF. We shall use the *idea* of the Schwartz kernel theorem. It is important *not* to use this theorem itself, since the Fourier transform is a key tool in the (simplest) proof of the kernel theorem. Thus we consider the composite map

$$\mathcal{G} \circ \mathcal{F} :\ \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

and write down its kernel. Namely

$$K(\phi) = (2\pi)^{-n} \iiint e^{iy\cdot\xi - ix\cdot\xi} \phi(y,x)\, dx\, d\xi\, dy \tag{1.34}$$

$$\forall\ \phi \in \mathcal{S}(\mathbb{R}_y^n \times \mathbb{R}_x^n) \Longrightarrow K \in \mathcal{S}'(\mathbb{R}^{2n}). \tag{1.35}$$

The integrals in (1.35) are iterated, i.e. should be performed in the order indicated. Notice that if $\psi, \zeta \in \mathcal{S}(\mathbb{R}^n)$ then indeed

$$(\mathcal{G}\cdot\mathcal{F}(\psi))(\zeta) = \int \zeta(y)(2\pi)^{-n} \int e^{iy\cdot\xi} \int e^{-ix\cdot\xi}\psi(x)dx \tag{1.36}$$
$$= K(\zeta \boxtimes \psi). \tag{1.37}$$

The two identities (1.33) translate (with essentially the same proofs) to the conditions on K :

$$\begin{cases} (D_{x_j} + D_{y_j})K(x,y) = 0 \\ (x_j - y_j)K(x,y) = 0 \end{cases} \quad j = 1, \dots, n. \tag{1.38}$$

Next we use the freedom to make linear changes of variables, setting

$$\begin{gathered} K_L(x,z) = K(x, x-z),\ K_L \in \mathcal{S}'(\mathbb{R}^{2n}) \\ \text{i.e. } K_L(\phi) = K(\psi),\ \psi(x,y) = \phi(x, x-y) \end{gathered} \tag{1.39}$$

where the notation will be explained later. Then (1.37) becomes

$$D_{x_j} K_L(x,z) = 0 \text{ and } z_j K_L(x,z) = 0 \text{ for } j = 1, \dots n \tag{1.40}$$

This puts us in a position to apply the two little lemmas. The first says $K_L(x,z) = f(z)$ for some $f \in \mathcal{S}'(\mathbb{R}^n)$ and then the second says $f(z) = c\delta(z)$. Thus

$$K(x,y) = c\delta(x-y) \Longrightarrow \mathcal{G}\cdot\mathcal{F} = c\,\mathrm{Id}\,.$$

It remains only to show that $c = 1$. That $c \neq 0$ is obvious (since $\mathcal{F}(\delta) = 1$). The easiest way to compute the constant is to use the integral identity

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \pi^{\frac{1}{2}}$$

to show that

$$\begin{aligned} \mathcal{F}(e^{-|x|^2}) &= \pi^{\frac{n}{2}} e^{-|\xi|^2/4} \\ \Longrightarrow \mathcal{G}(e^{-|\xi|^2/4}) &= \pi^{-\frac{n}{2}} e^{-|x|^2} \\ \Longrightarrow \mathcal{G} \cdot \mathcal{F} &= \mathrm{Id}\,. \end{aligned}$$

□

Now $(2\pi)^n \mathcal{G}$ is actually the *adjoint* of $\mathcal{F}$:

$$\int \phi(\zeta)\overline{\mathcal{F}\psi}(\zeta) d\zeta = (2\pi)^n \int (\mathcal{G}\phi) \cdot \overline{\psi} dx \ \forall\ \phi, \psi \in \mathcal{S}(\mathbb{R}^n). \tag{1.41}$$

It follows that we can extend $\mathcal{F}$ to a map on tempered distributions

$$\begin{aligned} \mathcal{F} : \mathcal{S}'(\mathbb{R}^n) &\longrightarrow \mathcal{S}'(\mathbb{R}^n) \\ \mathcal{F}u(\overline{\phi}) &= u((2\pi)^n \overline{\mathcal{G}\phi}) \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \end{aligned}$$

Then we conclude

Corollary 1.1. *The Fourier transform extends by continuity to an isomorphism*

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longleftrightarrow \mathcal{S}'(\mathbb{R}^n)$$

with inverse $\mathcal{G}$, *satisfying the identities* (1.33).

Proposition 1.3. *From the dense subspace* $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, *the Fourier transform extends by continuity to an isomorphism*

$$\mathcal{F} : L^2(\mathbb{R}^n) \longleftrightarrow L^2(\mathbb{R}^n)$$

satisfying $\|\mathcal{F}u\|_{L^2} = (2\pi)^{\frac{1}{2}n}\|u\|_{L^2}$.

Proof. Given the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, this also a consequence of (1.41), since setting $\phi = \mathcal{F}u$, for $u \in \mathcal{S}(\mathbb{R}^n)$, gives Plancherel's formula

$$\int \mathcal{F}u(\zeta)\overline{\mathcal{F}v(\zeta)} = (2\pi)^n \int u(x)\overline{v(x)} dx.$$

Setting $v = u$ gives norm equality (which is Parseval's formula).

An outline of the proof of the density statement is given in the problems below.

□

1.7. Differential operators

The simplest examples of the Fourier transform of distributions are immediate consequences of the definition and (1.33). Thus

$$\mathcal{F}(\delta) = 1$$

as already noted and hence, from (1.33),

$$\mathcal{F}(D^\alpha \delta(x)) = \xi^\alpha \quad \forall\ \alpha \in \mathbb{N}_0^n.$$

Now, recall that the space of distributions with support the point 0 is just:

$$\big\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{sup}(u) \subset \{0\}\big\} = \big\{u = \sum_{\text{finite}} c_\alpha D^\alpha \delta\big\}. \tag{1.42}$$

Thus we conclude that the Fourier transform gives an isomorphism

$$\mathcal{F} : \big\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \subset \{0\}\big\} \longleftrightarrow \mathbb{C}[\xi] = \{\text{ polynomials in } \xi\}. \tag{1.43}$$

Another way of looking at this same isomorphism is to consider partial differential operators with constant coefficients:

$$\begin{gathered} P(D) : \ \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \\ P(D) = \sum c_\alpha D^\alpha. \end{gathered} \tag{1.44}$$

The identity becomes

$$\mathcal{F}(P(D)\phi)(\zeta) = P(\xi)\mathcal{F}(\phi)(\zeta) \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.45}$$

and indeed the same formula holds for all $\phi \in \mathcal{S}'(\mathbb{R}^n)$. Using the simpler notation $\hat{u}(\xi) = \mathcal{F}u(\xi)$ this can be written

$$\widehat{P(D)u}(\xi) = P(\xi)\hat{u}(\xi), \ P(\xi) = \sum c_\alpha \xi^\alpha. \tag{1.46}$$

The polynomial P is called the (full) characteristic polynomial of $P(D)$; of course it determines $P(D)$ uniquely.

It is important for us to extend this formula to differential operators with variable coefficients. Using (1.23) and the inverse Fourier transform we get

$$P(D)u(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} P(\xi)u(y)dyd\xi \tag{1.47}$$

where again this is an *iterated* integral. In particular the inversion formula is just the case $P(\xi) = 1$. Consider the space

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) = \big\{u : \mathbb{R}^n \longrightarrow \mathbb{C};\ \sup_x |D^\alpha u(x)| < \infty \ \forall\ \alpha\big\}$$

the space of $\mathcal{C}^\infty$ function with all derivatives bounded on $\mathbb{R}^n$. Of course

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n)$$

but $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ is much bigger, in particular $1 \in \mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Now by Leibniz' formula

$$D^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u \cdot D^{\alpha-\beta} v$$

it follows that $\mathcal{S}(\mathbb{R}^n)$ is a *module* over $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. That is,

$$u \in \mathcal{C}^\infty_\infty(\mathbb{R}^n), \phi \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow u\phi \in \mathcal{S}(\mathbb{R}^n).$$

From this it follows that if

$$P(x, D) = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha, \ p_\alpha \in \mathcal{C}^\infty_\infty(\mathbb{R}^n)$$

then $P(x, D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$. The formula (1.47) extends to

$$P(x, D)\phi = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} P(x, \xi)\phi(y)dyd\xi \tag{1.48}$$

where again this is an iterated. Here

$$P(x, \xi) = \sum_{|\alpha| \le m} p_\alpha(x)\xi^\alpha$$

is the (full) characteristic polynomial of P.

1.8. Radial compactification

For later purposes, and general propaganda, consider the raidal compactification of $\mathbb{R}^n$. The smooth map

$$\mathrm{RC} : \mathbb{R}^n \ni x \longmapsto \frac{x}{(1+|x|^2)^{\frac{1}{2}}} \in \mathbb{R}^n \tag{1.49}$$

is 1-1 and maps onto the interior of the until ball, $\mathbb{B}^n = \{|x| \leq 1\}$. Consider the subspace

$$\dot{\mathcal{C}}^\infty(\mathbb{B}^n) = \{u \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(u) \subset \mathbb{B}^n\}. \tag{1.50}$$

This is just the set of smooth functions on $\mathbb{R}^n$ which vanish outside the unit ball. Then the composite ('pull-back') map

$$\mathrm{RC}^* : \dot{\mathcal{C}}^\infty(\mathbb{B}^n) \nu u \longmapsto u \circ \mathrm{RC} \in \mathcal{S}(\mathbb{R}^n) \tag{1.51}$$

is a topological isomorphism. A proof is indicated in the problems below.

The dual space of $\dot{\mathcal{C}}^\infty(\mathbb{B}^n)$ is generally called the space of 'extendible distributions' on $\mathbb{B}^n$ – because they are all given by restricting elements of $\mathcal{S}'(\mathbb{R}^n)$ to $\dot{\mathcal{C}}^\infty(\mathbb{B}^n)$. Thus RC also identifies the tempered distributions on $\mathbb{R}^n$ with the extendible distributions on $\mathbb{B}^n$. We shall see below that various spaces of functions on $\mathbb{R}^n$ take interesting forms when pulled back to $\mathbb{B}^n$. I often find it useful to 'bring infinity in' in this way.

1.9. Problems

PROBLEM 1.1. Suppose $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a function such that for *each* point $\bar{x} \in \mathbb{R}^n$ and each $k \in \mathbb{N}_0$ there exists a constant $\epsilon_k > 0$ and a polynomial $p_k(x;\bar{x})$ (in x) for which

$$|\phi(x) - p_k(x;\bar{x})| \leq \frac{1}{\epsilon_k}|x-\bar{x}|^{k+1} \quad \forall\ |x-\bar{x}| \leq \epsilon_k.$$

Does it follow that ϕ is infinitely differentiable – either prove this or give a counterexample.

PROBLEM 1.2. Show that the function $u(x) = \exp(x)\cos[e^x]$ 'is' a tempered distribution. Part of the question is making a precise statement as to what this means!

PROBLEM 1.3. Write out a careful (but not necessarily long) proof of the 'easy' direction of the Schwartz kernel theorem, that any $K \in \mathcal{S}'(\mathbb{R}^{n+m})$ defines a continuous linear operator

$$O_K : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

[with respect to the weak topology on $\mathcal{S}'(\mathbb{R}^n)$ and the metric topology on $\mathcal{S}(\mathbb{R}^m)$] by

$$O_K\phi(\psi) = K(\psi \boxtimes \phi).$$

[Hint: Work out what the continuity estimate on the kernel, K, means when it is paired with an exterior product $\psi \boxtimes \phi$.]

PROBLEM 1.4. Prove Proposition 1.2. [Hint: It is only necessary to show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ is fixed then for any of the open sets in (1.1), B, (with all the $\epsilon_j > 0$) there is an element $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $u - T_\phi \in B$. First show that if $\phi'_1, \dots \phi'_p$ is a basis for Φ then the set

$$B' = \{v \in \mathcal{S}'(\mathbb{R}^n); |\langle v, \phi'_j\rangle| < \delta_j \tag{1.52}$$

is contained in B if the $\delta_j > 0$ are chosen small enough. Taking the basis to be orthonormal, show that $u - \psi \in B'$ can be arranged for some $\psi \in \Phi$.]

PROBLEM 1.5. Show that (1.13) makes sense for $\phi \in L^2(\mathbb{R}^n)$ (the space of (equivalence classes of) Lebesgue square-integrable functions and that the resulting map $L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is an injection.

PROBLEM 1.6. Suppose $u \in L^2(\mathbb{R}^n)$ and that

$$D_1 D_2 \cdots D_n u \in (1 + |x|)^{-n-1} L^2(\mathbb{R}^n),$$

where the derivatives are defined using Problem 1.5. Using repeated integration, show that u is necessarily a bounded continuous function. Conclude further that for $u \in \mathcal{S}'(\mathbb{R}^n)$

$$\begin{aligned} &D^\alpha u \in (1 + |x|)^{-n-1} L^2(\mathbb{R}^n) \ \forall \ |\alpha| \le k + n \\ &\Longrightarrow D^\alpha u \text{ is bounded and continuous for } |\alpha| \le k. \end{aligned} \tag{1.53}$$

[This is a weak form of the Sobolev embedding theorem.]

PROBLEM 1.7. The *support* of a (tempered) distribution can be defined in terms of the support of a test function. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ the support, $\operatorname{supp}(\phi)$, is the closure of the set of points at which it takes a non-zero value. For $u \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\operatorname{supp}(u) = O^{\complement}, \ O = \bigcup \{O' \subset \mathbb{R}^n \text{ open}; \operatorname{supp}(\phi) \subset O' \Longrightarrow u(\phi) = 0\}.$$

Show that the definitions for $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are consistent with the inclusion $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Prove that $\operatorname{supp}(\delta) = \{0\}$.

PROBLEM 1.8. For simplicity in $\mathbb{R}$, i.e. with $n = 1$, prove Schwartz theorem concerning distributions with support the origin. Show that with respect to the norm $\|\cdot\|_k$ the space

$$\{\phi \in \mathcal{S}(\mathbb{R}); \phi(x) = 0 \text{ in } |x| < \epsilon, \ \epsilon = \epsilon(\phi) > 0\}$$

is dense in

$$\left\{\phi \in \mathcal{S}(\mathbb{R}); \phi(x) = x^{k+1}\psi(x), \ \psi \in \mathcal{S}(\mathbb{R})\right\}.$$

Use this to show that

$$u \in \mathcal{S}'(\mathbb{R}), \ \operatorname{supp}(u) \subset \{0\} \Longrightarrow u = \sum_{\ell, \text{ finite}} c_\ell D_x^\ell \delta(x).$$

PROBLEM 1.9. Show that if P is a differential operator with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ then P is local in the sense that

$$\operatorname{supp}(Pu) \subset \operatorname{supp}(u) \quad \forall \ u \in \mathcal{S}'(\mathbb{R}^n).$$

The converse of this, for an operator $P : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ where (for simplicity) we assume

$$\operatorname{supp}(Pu) \subset K \subset \mathbb{R}^n$$

for a fixed compact set K, is *Peetre's theorem.* How would you try to prove this? (No full proof required.)

PROBLEM 1.10. (Schwartz representation theorem) Show that, for any $p \in \mathbb{R}$ the map

$$R_p : \mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto (1|x|^2)^{-p/2}\mathcal{F}^{-1}[(1+|\xi|^2)^{-p/2}\mathcal{F}\phi] \in \mathcal{S}(\mathbb{R}^n) \tag{1.54}$$

is an isomorphism and, using Problem 1.6 or otherwise,

$$p \geq n+1+k \Longrightarrow \|R_p\phi\|_k \leq C_k\|\phi\|_{L^2},\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.55}$$

Let $R_p^t \colon \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ be the dual map (defined by $T_p^t u(\phi) = u(R_p\phi)$). Show that R_p^t is an isomorphism and that if $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$|u(\phi)| \leq C\|\phi\|_k,\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.56}$$

then $R_p^t u \in L^2(\mathbb{R}^n)$, if $p \geq n+1+k$, in the sense that it is in the image of the map in Problem 1.5. Using Problem 1.6 show that $R_{n+1}(R_{n+1+k}^t u)$ is bounded and continuous and hence that

$$u = \sum_{|\alpha|+|\beta|\leq 2n+2+k} x^\beta D^\alpha u_{\alpha,\beta} \tag{1.57}$$

for some bounded continuous functions $u_{\alpha,\beta}$.

PROBLEM 1.11. (Schwartz kernel theorem.) Show that any continuous linear operator

$$T : \mathcal{S}(\mathbb{R}_y^m) \longrightarrow \mathcal{S}'(\mathbb{R}_x^n)$$

extends to a continuous linear operator

$$T : (1+|y|^2)^{-k/2}H^k(\mathbb{R}_y^m) \longrightarrow (1+|x|^2)^{-q/2}H^q(\mathbb{R}_x^n)$$

for some k and q. Deduce that the operator

$$\tilde{T} = (1+|D_x|^2)^{(-n-1-q)/2}(1+|x|^2)^{q/2} \circ T \circ (1+|y|^2)^{k/2}(1+|D|^2)^{-k/2} : \\ L^2(\mathbb{R}^m) \longrightarrow \mathcal{C}_\infty(\mathbb{R}^n)$$

is continuous with values in the bounded continuous functions on $\mathbb{R}^n$. Deduce that $\tilde{T}$ has Schwartz kernel in $\mathcal{C}_\infty(\mathbb{R}^n; L^2(\mathbb{R}^m)) \subset \mathcal{S}'(\mathbb{R}^{n+m})$ and hence that T itself has a tempered Schwartz kernel.