Introduction to Microlocal Analysis

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0.5B; Revised: 27-1-2003; Run: February 20, 2003

1991 *Mathematics Subject Classification.* All over the shop

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Preface

This is a somewhat revised version of the lecture notes from various courses taught at MIT, now for Spring 2003.

There are many people to thank, including:

Benoit Charbonneau.
Sine Rikke Jensen.
Edith Mooers.
Mark Joshi.
Raul Tataru.
Jonathan Kaplan.

Introduction

I shall assume some familiarity with distribution theory, with basic analysis knowledge of the theory of manifolds would also be useful. Any one or two of these prerequisites can be easily picked up along the way, but the prospective student with none of them should perhaps do some preliminary reading:

Distributions: I good introduction is Friedlander's book [**4**]. For a more exhaustive treatment see Volume I of Hörmander's treatise [**8**].

Analysis on manifolds: Most of what we need can be picked up from Munkres' book [**9**] or Spivak's little book [**12**].

CHAPTER 1

Tempered Tempered distributions and the Fourier transform

Microlocal analysis is a geometric theory of distributions, or a theory of geometric distributions. Rather than study general distributions – which are like general continuous functions but worse – we consider more specific types of distributions which actually arise in the study of differential and integral equations. Distributions are usually defined by duality, starting from very "good" test functions; correspondingly a general distribution is everywhere "bad". The *conormal distributions* we shall study implicitly for a long time, and eventually explicitly, are usually good, but like (other) people have a few interesting faults, i.e. singularities. These singularities are our principal target of study. Nevertheless we need the general framework of distribution theory to work in, so I will start with a brief introduction. This is designed either to remind you of what you already know or else to send you off to work it out.[1] Proofs of some of the main theorems are outlined in the problems at the end of the chapter.

S.Schwartz.Test 1.1. Schwartz test functions

To fix matters at the beginning we shall work in the space of tempered distributions. These are defined by duality from the space of Schwartz functions, also called the space of test functions of rapid decrease. We can think of analysis as starting off from algebra, which gives us the polynomials. Thus in $\mathbb{R}^n$ we have the coordinate functions, $x_1, \dots, x_n$ and the constant functions and then the polynomials are obtained by taking (finite) sums and products:

1.1 (1.1) $$\phi(x) = \sum_{|\alpha| \le k} p_\alpha x^\alpha,\ p_\alpha \in \mathbb{C},\ \alpha \in \mathbb{N}_0^n,\ \alpha = (\alpha_1, \dots, \alpha_n),$$

$$\text{where } x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n} = \prod_{j=1}^n x_j^{\alpha_j} \text{ and } \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

A general function $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is differentiable at $\bar{x}$ if there is a linear function $\ell_{\bar{x}}(x) = c + \sum_{j=1}^n (x_j - \bar{x}_j) d_j$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that

1.2 (1.2) $$|\phi(x) - \ell_{\bar{x}}(x)| \le \epsilon |x - \bar{x}| \quad \forall\ |x - \bar{x}| < \bar{\delta}.$$

The coefficients d_j are the partial derivative of ϕ at the point $\bar{x}$. Then, ϕ is said to be differentiable on $\mathbb{R}^n$ if it is differentiable at each point $\bar{x} \in \mathbb{R}^n$; the partial derivatives are then also functions on $\mathbb{R}^n$ and ϕ is *twice* differentiable if the partial

[1] I suggest Friedlander's little book MR86h:46002 [3] (there is also a newer edition) as a good introduction to distributions. Volume 1 of Hörmander's treatise Hörmander1 [7] has all that you would need and a good deal more; it is a good general reference.

derivatives are differentiable. In general it is k times differentiable if its partial derivatives are $k-1$ times differentiable.

If ϕ is k times differentiable then, for each $\bar{x} \in \mathbb{R}^n$, there is a polynomial of degree k,

$$p_k(x;\bar{x}) = \sum_{|\alpha|\le k} a_\alpha i^{|\alpha|}(x-\bar{x})^\alpha/\alpha!,\ \ |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

1.3 (1.3) $$|\phi(x) - p_k(x,\bar{x})| \le \epsilon |x-\bar{x}|^k \quad \text{if } |x-\bar{x}| < \delta.$$

Then we set

1.4 (1.4) $$D^\alpha \phi(\bar{x}) = a_\alpha.$$

If ϕ is infinitely differentiable all the $D^\alpha \phi$ are infinitely differentiable (hence continuous!) functions.

1.5 Definition 1.1. *The space of Schwartz test functions of rapid decrease consists of those* $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ *such that for every* $\alpha, \beta \in \mathbb{N}_0^n$

1.6 (1.5) $$\sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha \phi(x)| < \infty;$$

it is denoted $\mathcal{S}(\mathbb{R}^n)$.

From (1.5) we construct norms on $\mathcal{S}(\mathbb{R}^n)$:

1.7 (1.6) $$\|\phi\|_k = \max_{|\alpha|+|\beta|\le k} \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \phi(x)|.$$

It is straightforward to check the conditions for a norm:

(1) $\|\phi\|_k \ge 0,\ \|\phi\|_k = 0 \Longleftrightarrow \phi \equiv 0$
(2) $\|t\phi\|_k = |t|\|\phi\|_k,\ t \in \mathbb{C}$
(3) $\|\phi + \psi\|_k \le \|\phi\|_k + \|\psi\|_k\ \forall\ \phi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

The topology on $\mathcal{S}(\mathbb{R}^n)$ is given by the metric

1.8 (1.7) $$d(\phi,\psi) = \sum_k 2^{-k} \frac{\|\phi-\psi\|_k}{1 + \|\phi-\psi\|_k}.$$

See Problem 1.4.

1.9 Proposition 1.1. *With the distance function (1.7),* $\mathcal{S}(\mathbb{R}^n)$ *becomes a complete metric space (in fact it is a Fréchet space).*

Of course one needs to check that $\mathcal{S}(\mathbb{R}^n)$ is non-trivial; however one can easily see that

(1.8) $$\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n).$$

In fact there are lots of smooth functions of compact support and

1.2.2000.266 (1.9) $$\mathcal{C}_c^\infty(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n); u = 0 \text{ in } |x| > R = R(u)\} \subset \mathcal{S}(\mathbb{R}^n) \text{ is dense.}$$

The two elementary operations of differentiation and coordinate multiplication give continuous linear operators:

31.1.2000.263 (1.10) $$\begin{aligned} x_j &: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \\ D_j &: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \end{aligned}$$

Other important operations we shall encounter include the exterior product,

1.10 (1.11)
$$\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^m)\ni(\phi,\psi)\mapsto\phi\boxtimes\psi\in\mathcal{S}(\mathbb{R}^{n+m})$$
$$\phi\boxtimes\psi(x,y)=\phi(x)\psi(y).$$

and pull-back or restriction. If $\mathbb{R}^k\subset\mathbb{R}^n$ is identified as the subspace $x_j=0$, $j>k$, then the restriction map

31.1.2000.264 (1.12)
$$\pi_k^*:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^k),\ \pi_k^*f(y)=f(y_1,\dots,y_k,0,\dots,0)$$

is continuous (and surjective).

S.Linear.transformations

1.2. Linear transformations

A linear transformation acts on $\mathbb{R}^n$ as a matrix[2]

1.2.2000.267 (1.13)
$$L:\mathbb{R}^n\longrightarrow\mathbb{R}^n,\ (Lx)_j=\sum_{k=1}^n L_{jk}x_k.$$

The Lie group of invertible linear transformations, $\mathrm{GL}(n,\mathbb{R})$ is fixed by several equivalent conditions

1.2.2000.268 (1.14)
$$\begin{aligned}L\in\mathrm{GL}(n,\mathbb{R})&\Longleftrightarrow\det(L)\neq0\\&\Longleftrightarrow\ \exists\ L^{-1}\text{ s.t. }(L^{-1})Lx=x\ \forall\ x\in\mathbb{R}^n\\&\Longleftrightarrow\ \exists\ c>0\text{ s.t. }c|x|\le|Lx|\le c^{-1}|x|\ \forall\ x\in\mathbb{R}^n.\end{aligned}$$

Pull-back of functions is defined by

$$L^*\phi(x)=\phi(Lx)=(\phi\circ L)(x).$$

The chain rule for differentiation shows that if ϕ is diffferentiable then[3]

1.2.2000.269 (1.15)
$$D_jL^*\phi(x)=D_j\phi(Lx)=\sum_{k=1}^n L_{kj}(D_k\phi)(Lx)=L^*((L_*D_j)\phi)(x),$$
$$L_*D_j=\sum_{k=1}^n L_{kj}D_k.$$

From this it follows that

1.2.2000.270 (1.16)
$$L^*:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n)\text{ is an isomorphism for }L\in\mathrm{GL}(n,\mathbb{R}).$$

To characterize the action of $L\in\mathrm{GL}(n,\mathbb{R})$ on $\mathcal{S}'(\mathbb{R}^n)$ consider, as usual, the distribution associated to $L^*\phi$:

1.2.2000.271 (1.17)
$$\begin{aligned}T_{L^*\phi}(\psi)&=\int_{\mathbb{R}^n}\phi(Lx)\psi(x)dx\\&=\int_{\mathbb{R}^n}\phi(y)\psi(L^{-1}y)|\det L|^{-1}dy=T_\phi(|\det L|^{-1}(L^{-1})^*\psi).\end{aligned}$$

[2] This is the standard action, but it is potentially confusing since it means that for the basis elements $e_j\in\mathbb{R}^n$, $Le_j=\sum\limits_{k=1}^n L_{kj}e_k$.

[3] So D_j transforms as a basis of $\mathbb{R}^n$ as it should, despite the factors of i.

Since the operator $|\det L|^{-1}(L^{-1})^*$ is an ismorphism of $\mathcal{S}(\mathbb{R}^n)$ it follows that if we take the definition by duality

1.2.2000.272 (1.18) $$L^*u(\psi) = u(|\det L|^{-1}(L^{-1})^*\psi),\ u \in \mathcal{S}'(\mathbb{R}^n),\ \psi \in \mathcal{S}(\mathbb{R}^n),\ L \in \mathrm{GL}(n,\mathbb{R}) \Longrightarrow L^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

is an isomorphism which extends (1.16) and satisfies

1.2.2000.273 (1.19) $$D_jL^*u = L^*((L_*D_j)u), L^*(x_ju) = (L^*x_j)(L^*u),\ u \in \mathcal{S}'(\mathbb{R}^n),\ L \in \mathrm{GL}(n,\mathbb{R}),$$

as in (1.15).

S.Tempered.distributions

1.3. Tempered distributions

As well as exterior multiplication (1.11) there is the even more obvious multiplication operation

1.11 (1.20) $$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$ $$(\phi,\psi) \mapsto \phi(x)\psi(x)$$

which turns $\mathcal{S}(\mathbb{R}^n)$ into a commutative algebra without identity. There is also integration

1.12 (1.21) $$\int : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}.$$

Combining these gives a *pairing*, a bilinear map

(1.22) $$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi,\psi) \longmapsto \int_{\mathbb{R}^n} \phi(x)\psi(x)dx.$$

If we fix $\phi \in \mathcal{S}(\mathbb{R}^n)$ this defines a continuous linear map:

1.13 (1.23) $$T_\phi : \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \int \phi(x)\psi(x)dx.$$

Continuity becomes the condition:

(1.24) $$\exists\ k, C_k \text{ s.t. } |T_\phi(\psi)| \le C_k\|\psi\|_k\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$$

We generalize this by denoting by $\mathcal{S}'(\mathbb{R}^n)$ the dual space, i.e. the space of all continuous linear functionals

$$u \in \mathcal{S}'(\mathbb{R}^n) \Longleftrightarrow u : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$$
$$\exists\ k, C_k \text{ such that } |u(\psi)| \le C_k\|\psi\|_k\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$$

1.14 Lemma 1.1. *The map*

1.15 (1.25) $$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto T_\phi \in \mathcal{S}'(\mathbb{R}^n)$$

is an injection.

Proof. For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $T_{\bar\phi}(\phi) = \int |\phi(x)|^2 dx$, so $T_\phi = 0$ implies $\phi \equiv 0$. □

If we wish to consider a topology on $\mathcal{S}'(\mathbb{R}^n)$ it will normally be the *weak* topology, that is the weakest topology with respect to which all the linear maps

(1.26) $$\mathcal{S}'(\mathbb{R}^n) \ni u \longmapsto u(\phi) \in \mathbb{C}, \quad \phi \in \mathcal{S}(\mathbb{R}^n)$$

are continuous. This just means that it is given by the seminorms

1.16 (1.27) $$\mathcal{S}(\mathbb{R}^n) \ni u \longmapsto |u(\phi)| \in \mathbb{R}$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$ is fixed but arbitrary. The sets

1.101 $$\{u \in \mathcal{S}'(\mathbb{R}^n); |u(\phi_j)| < \epsilon_j,\ \phi_j \in \Phi\} \tag{1.28}$$

form a basis of the neighbourhoods of 0 as $\Phi \subset \mathcal{S}(\mathbb{R}^n)$ runs over finite sets and the ϵ_j are positive numbers.

1.17 Proposition 1.2. *The continuous injection $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, given by (1.25), has dense range in the weak topology.*

See Problem 1.8 for the outline of a proof.

The maps x_i, D_j extend by continuity (and hence uniquely) to operators

1.18 $$x_j,\ D_j : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{1.29}$$

This is easily seen by defining them by duality. Thus if $\phi \in \mathcal{S}(\mathbb{R}^n)$ set $D_j T_\phi = T_{D_j\phi}$, then

$$T_{D_j\phi}(\psi) = \int D_j\phi\psi = -\int \phi D_j\psi, \tag{1.30}$$

the integration by parts formula. The definitions

$$D_j u(\psi) = u(-D_j\psi),\ x_j u(\psi) = u(x_j\psi), \quad u \in \mathcal{S}'(\mathbb{R}^n), \psi \in \mathcal{S}(\mathbb{R}^n) \tag{1.31}$$

satisfy all requirements, in that they give continuous maps (1.29) which extend the standard definitions on $\mathcal{S}(\mathbb{R}^n)$.

S.Two.big.theorems

1.4. Two big theorems

The association, by (1.25), of a distribution to a function can be extended considerably. For example if $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a bounded and continuous function then

$$T_u(\psi) = \int u(x)\psi(x)dx \tag{1.32}$$

still defines a distribution which vanishes if and only if u vanishes identically. Using the operations (1.29) we conclude that for any $\alpha, \beta \in \mathbb{N}_0^n$

$$x^\beta D_x^\alpha u \in \mathcal{S}'(\mathbb{R}^n) \text{ if } u : \mathbb{R}^n \longrightarrow \mathbb{C} \text{ is bounded and continuous.} \tag{1.33}$$

Conversely we have the *Schwartz representation Theorem:*

1.19 Theorem 1.1. *For any $u \in \mathcal{S}'(\mathbb{R}^n)$ there is a finite collection $u_{\alpha\beta} : \mathbb{R}^n \longrightarrow \mathbb{C}$ of bounded continuous functions, $|\alpha| + |\beta| \le k$, such that*

$$u = \sum_{|\alpha|+|\beta|\le k} x^\beta D_x^\alpha u_{\alpha\beta}. \tag{1.34}$$

Thus tempered distributions are just products of polynomials and derivatives of bounded continuous functions. This is important because it says that distributions are "not too bad".

The second important result (long considered very difficult to prove, but there is a relatively straightforward proof using the Fourier transform) is the *Schwartz kernel theorem.* To show this we need to use the exterior product (1.11). If $K \in \mathcal{S}'(\mathbb{R}^{n+m})$ this allows us to define a linear map

$$O_K : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{1.35}$$

by

$$O_K(\psi)(\phi) = \int K \cdot \phi \boxtimes \psi \; dxdy. \tag{1.36}$$

1.20 Theorem 1.2. *There is a* 1-1 *correspondence between continuous linear operators*

$$A : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{1.37}$$

and $\mathcal{S}'(\mathbb{R}^{n+m})$ *given by* $A = O_K$.

Brief outlines of the proofs of these two results can be found in Problems P2.4 1.15 and P2.5 1.16.

S.Examples 1.5. Examples

Amongst tempered distributions we think of $\mathcal{S}(\mathbb{R}^n)$ as being the 'trivial' examples, since they are the test functions. One can say that the study of the singularities of tempered distributions amounts to the study of the quotient

$$\mathcal{S}'(\mathbb{R}^n)/\mathcal{S}(\mathbb{R}^n) \tag{1.38}$$

which could, reasonably, be called the space of tempered microfunctions.

The sort of distributions we are interested in are those like the Dirac delta "function"

$$\delta(x) \in \mathcal{S}'(\mathbb{R}^n), \quad \delta(\phi) = \phi(0). \tag{1.39}$$

The definition here shows that δ is just the Schwartz kernel of the operator

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto \phi(0) \in \mathbb{C} = \mathcal{S}(\mathbb{R}^0). \tag{1.40}$$

This is precisely one reason it is interesting. More generally we can consider the maps

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto D^\alpha \phi(0), \quad \alpha \in \mathbb{N}_0^n. \tag{1.41}$$

These have Schwartz kernels $(-D)^\alpha \delta$ since

1.21 $$(-D)^\alpha \delta(\phi) = \delta(D^\alpha \phi) = D^\alpha \phi(0). \tag{1.42}$$

If we write the relationship $A = O_K \longleftrightarrow K$ as

$$(A\psi)(\phi) = \int K(x,y)\phi(x)\psi(y) dxdy \tag{1.43}$$

then (1.21 1.42) becomes

$$D^\alpha \phi(0) = \int (-D)^\alpha \delta(x) \phi(x) dx. \tag{1.44}$$

More generally, if $K(x,y)$ is the kernel of an operator A then the kernel of $A \cdot D^\alpha$ is $(-D)_y^\alpha K(x,y)$ whereas the kernel of $D^\alpha \circ A$ is $D_x^\alpha K(x,y)$.

S.Two.little.lemmas

1.6. Two little lemmas

Above, some of the basic properties of tempered distributions have been outlined. The main "raison d'être" for $\mathcal{S}'(\mathbb{R}^n)$ is the *Fourier transform* which we proceed to discuss. We shall use the Fourier transform as an almost indispensable tool in the treatment of pseudodifferential operators. The description of differential operators, via their Schwartz kernels, using the Fourier transform is an essential motivation for the extension to pseudodifferential operators.

Partly as simple exercises in the theory of distributions, and more significantly as preparation for the proof of the inversion formula for the Fourier transform we consider two lemmas.

First recall that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then we have defined $D_j u \in \mathcal{S}'(\mathbb{R}^n)$ by

(1.45) $$D_j u(\phi) = u(-D_j\phi) \ \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n).$$

In this sense it is a "weak derivative". Let us consider the simple question of the form of the solutions to

2.1 (1.46) $$D_j u = 0,\ u \in \mathcal{S}'(\mathbb{R}^n).$$

Let I_j be the integration operator:

2.2 (1.47) $$\begin{gathered} I_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{n-1}) \\ I_j(\phi)(y_1, \dots, y_{n-1}) = \int \phi(y_1, \dots y_{j-1}, x, y_j, \dots y_{n-1}) dx. \end{gathered}$$

Then if $\pi_j : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ is the map $\pi_j(x) = (x_1, \dots, x_{j-1}, x_{j+1} \dots, x_n)$, we *define*, for $v \in \mathcal{S}'(\mathbb{R}^{n-1})$,

(1.48) $$\pi_j^* v(\phi) = v(I_j \phi) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n).$$

It is clear from (1.47) that $I_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{n-1})$ is continuous and hence $\pi_j^* v \in \mathcal{S}'(\mathbb{R}^n)$ is well-defined for each $v \in \mathcal{S}'(\mathbb{R}^{n-1})$.

2.3 Lemma 1.2. *The equation (1.46) holds if and only if $u = \pi_j^* v$ for some $v \in \mathcal{S}'(\mathbb{R}^{n-1})$.*

Proof. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi = D_j\psi$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ then $I_j\phi = I_j(D_j\psi) = 0$. Thus if $u = \pi_j^* v$ then

(1.49) $$u(-D_j\phi) = \pi_j^* v(-D_j\phi) = v(I_j(-D_j\phi)) = 0.$$

Thus $u = \pi_j^* v$ does always satisfy (1.46).

Conversely suppose (1.46) holds. Choose $\rho \in \mathcal{S}(\mathbb{R})$ with the property

(1.50) $$\int \rho(x) dx = 1.$$

Then each $\phi \in \mathcal{S}(\mathbb{R}^n)$ can be decomposed as

2.4 (1.51) $$\phi(x) = \rho(x_j) I_j\phi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n) + D_j\psi, \quad \psi \in \mathcal{S}(\mathbb{R}^n).$$

Indeed this is just the statement

$$\zeta \in \mathcal{S}(\mathbb{R}^n),\ I_j\zeta = 0 \Longrightarrow \psi(x) \in \mathcal{S}(\mathbb{R}^n) \text{ where}$$
$$\psi(x) = \int_{-\infty}^{x_j} \zeta(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)dt$$
$$= \int_{\infty}^{x_j} \zeta(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)dt.$$

Using (1.51) and (1.46) we have

$$u(\phi) = u\left(\rho(x_j) I_j\phi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n)\right). \tag{1.52}$$

Thus if

$$v(\psi) = u\left(\rho(x_j)\psi(x_1, \dots, x_{j-1}, x_{j+1}, \dots x_n)\right)\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^{n-1}) \tag{1.53}$$

then $v \in \mathcal{S}'(\mathbb{R}^{n-1})$ and $u = \pi_j^* v$. This proves the lemma. □

Of course the notation $u = \pi_j^* v$ is much too heavy-handed. We just write $u(x) = v(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ and regard 'v as a distribution in one additional variable'.

The second, related, lemma is just a special case of a general result of Schwartz concerning the support of a distribution. The particular result is:

2.5 LEMMA 1.3. *Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ and $x_j u = 0, j = 1, \dots n$ then $u = c\delta(x)$ for some constant c.*

PROOF. Again we use the definition of multiplication and a dual result for test functions. Namely, choose $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\rho(x) = 1$ in $|x| < \frac{1}{2}$, $\rho(x) = 0$ in $|x| \geq 3/4$. Then any $\phi \in \mathcal{S}(\mathbb{R}^n)$ can be written

2.6 $$\phi = \phi(0) \cdot \rho(x) + \sum_{j=1}^{n} x_j \psi_j(x), \qquad \psi_j \in \mathcal{S}(\mathbb{R}^n). \tag{1.54}$$

This in turn can be proved using Taylor's formula as I proceed to show. Thus

$$\phi(x) = \phi(0) + \sum_{j=1}^{n} x_j \zeta_j(x) \text{ in } |x| \leq 1, \text{ with } \zeta_j \in \mathcal{C}^\infty. \tag{1.55}$$

Then,

$$\rho(x)\phi(x) = \phi(0)\rho(x) + \sum_{j=1}^{n} x_j \rho\zeta_j(x) \tag{1.56}$$

and $\rho\zeta_j \in \mathcal{S}(\mathbb{R}^n)$. Thus it suffices to check (1.54) for $(1-\rho)\phi$, which vanishes identically near 0. Then $\zeta = |x|^{-2}(1-\rho)\phi \in \mathcal{S}(\mathbb{R}^n)$ and so

$$(1-\rho)\phi = |x|^2\zeta = \sum_{j=1}^{n} x_j (x_j\zeta) \tag{1.57}$$

finally gives (1.54) with $\psi_j(x) = \rho(x)\zeta_j(x) + x_j\zeta(x)$. Having proved the existence of such a decomposition we see that if $x_j u = 0$ for all j then

$$u(\phi) = u(\phi(0)\rho(x)) + \sum_{j=1}^{n} u(x_j\psi_j) = c\phi(0), \quad c = u(\rho(x)), \tag{1.58}$$

i.e. $u = c\delta(x)$. □

S.Fourier.transform

1.7. Fourier transform

Our normalization of the Fourier transform will be

2.7
$$\mathcal{F}\phi(\xi) = \int e^{-i\xi\cdot x}\phi(x)dx. \tag{1.59}$$

As you all know the inverse Fourier transform is given by

2.8
$$\mathcal{G}\psi(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\psi(\xi)\,d\xi. \tag{1.60}$$

Since it is so important here I will give a proof of this invertibility. First however, let us note some of the basic properties.

Both $\mathcal{F}$ and $\mathcal{G}$ give continuous linear maps

2.9
$$\mathcal{F}, \mathcal{G} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{1.61}$$

To see this observe first that the integrals in (1.59) and (1.60) are absolutely convergent:

$$|\mathcal{F}\phi(\xi)| \le \int |\phi(x)|dx \le \int (1+|x|^2)^{-n}\,dx \times \sup_{x\in\mathbb{R}^n}(1+|x|^2)^n|\phi(x)|, \tag{1.62}$$

where we use the definition of $\mathcal{S}(\mathbb{R}^n)$. In fact this shows that $\sup|\mathcal{F}\phi| < \infty$ if $\phi \in \mathcal{S}(\mathbb{R}^n)$. Formal differentiation under the integral sign gives an absolutely convergent integral:

$$D_j\mathcal{F}\phi(\xi) = \int D_{\xi_j}e^{-ix\xi}\phi(x)dx = \int e^{-ix\cdot\xi}(-x_j\phi)dx$$

since $\sup\limits_x(1+|x|^2)^n|x_j\phi| < \infty$. Then it follows that $D_j\mathcal{F}\phi$ is also bounded, i.e. $\mathcal{F}\phi$ is differentiable, and (1.7) holds. This argument can be extended to show that $\mathcal{F}\phi$ is $\mathcal{C}^\infty$,

2.11
$$D^\alpha\mathcal{F}\phi(\xi) = \mathcal{F}\big((-x)^\alpha\phi\big). \tag{1.63}$$

Similarly, starting from (1.59), we can use integration by parts to show that

$$\xi_j\mathcal{F}\phi(\xi) = \int e^{-ix\xi}\xi_j\phi(x)dx = \int e^{-ix\cdot\xi}(D_j\phi)(x)dx$$

i.e. $\xi_j\mathcal{F}\phi = \mathcal{F}(D_j\phi)$. Combining this with (1.63) gives

2.12
$$\xi^\alpha D_\xi^\beta\mathcal{F}\phi = \mathcal{F}\big(D^\alpha\cdot[(-x)^\beta\phi]\big). \tag{1.64}$$

Since $D_x^\alpha((-x)^\beta\phi) \in \mathcal{S}(\mathbb{R}^n)$ we conclude

$$\sup|\xi^\alpha D_\zeta^\beta\mathcal{F}\phi| < \infty \Longrightarrow \mathcal{F}\phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.65}$$

This map is continuous since

$$\begin{aligned}\sup|\xi^\alpha D_\xi^\beta\mathcal{F}\phi| &\le C\cdot\sup_x|(1+|x|^2)^n D_x^\alpha[(-x)^\beta\phi]\\ &\Longrightarrow \|\mathcal{F}\phi\|_k \le C_k\|\phi\|_{k+2n},\ \forall\ k.\end{aligned}$$

The identity (1.64), written in the form

2.13 (1.66)
$$\mathcal{F}(D_j\phi) = \xi_j\mathcal{F}\phi$$
$$\mathcal{F}(x_j\phi) = -D_{\xi_j}\mathcal{F}\phi$$

is already the key to the proof of invertibility:

2.14 THEOREM 1.3. *The Fourier transform gives an isomorphism $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longleftrightarrow \mathcal{S}(\mathbb{R}^n)$ with inverse $\mathcal{G}$.*

PROOF. We shall use the *idea* of the Schwartz kernel theorem. It is important *not* to use this theorem itself, since the Fourier transform is a key tool in the (simplest) proof of the kernel theorem. Thus we consider the composite map

(1.67)
$$\mathcal{G}\circ\mathcal{F} :\ \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

and write down its kernel. Namely

2.15 (1.68)
$$K(\phi) = (2\pi)^{-n}\iiint e^{iy\cdot\xi - ix\cdot\xi}\phi(y,x)dxd\xi dy$$
$$\forall\ \phi\in\mathcal{S}(\mathbb{R}^n_y\times\mathbb{R}^n_x) \Longrightarrow K\in\mathcal{S}'(\mathbb{R}^{2n}).$$

The integrals in (1.68) are iterated, i.e. should be performed in the order indicated. Notice that if $\psi,\zeta\in\mathcal{S}(\mathbb{R}^n)$ then indeed

2.16 (1.69)
$$(\mathcal{G}\cdot\mathcal{F}(\psi))(\zeta) = \int\zeta(y)(2\pi)^{-n}\left(\int e^{iy\cdot\xi}\int e^{-ix\cdot\xi}\psi(x)dx\,d\xi\right)dy\,d\xi\,dy$$
$$= K(\zeta\boxtimes\psi)$$

so K is the Schwartz kernel of $\mathcal{G}\cdot\mathcal{F}$.

The two identities (1.66) translate (with essentially the same proofs) to the conditions on K :

2.17 (1.70)
$$\begin{cases}(D_{x_j} + D_{y_j})K(x,y) = 0\\ (x_j - y_j)K(x,y) = 0\end{cases}\quad j = 1,\dots,n.$$

Next we use the freedom to make linear changes of variables, setting

2.18 (1.71)
$$K_L(x,z) = K(x,x-z),\ \ K_L\in\mathcal{S}'(\mathbb{R}^{2n})$$
$$\text{i.e. } K_L(\phi) = K(\psi),\ \ \psi(x,y) = \phi(x,x-y)$$

where the notation will be explained later. Then (1.70) becomes

2.19 (1.72)
$$D_{x_j}K_L(x,z) = 0 \text{ and } z_jK_L(x,z) = 0 \text{ for } j = 1,\dots n$$

This puts us in a position to apply the two little lemmas. The first says $K_L(x,z) = f(z)$ for some $f\in\mathcal{S}'(\mathbb{R}^n)$ and then the second says $f(z) = c\delta(z)$. Thus

(1.73)
$$K(x,y) = c\delta(x-y) \Longrightarrow \mathcal{G}\cdot\mathcal{F} = c\,\mathrm{Id}\,.$$

It remains only to show that $c = 1$. That $c \neq 0$ is obvious (since $\mathcal{F}(\delta) = 1$). The easiest way to compute the constant is to use the integral identity

(1.74)
$$\int_{-\infty}^{\infty} e^{-x^2}dx = \pi^{\frac{1}{2}}$$

to show that[4]

[4]See Problem 1.9.

$$
\begin{aligned}
&\mathcal{F}(e^{-|x|^2}) = \pi^{\frac{n}{2}} e^{-|\xi|^2/4} \\
(1.75) \qquad &\Longrightarrow \mathcal{G}(e^{-|\xi|^2/4}) = \pi^{-\frac{n}{2}} e^{-|x|^2} \\
&\Longrightarrow \mathcal{G} \cdot \mathcal{F} = \mathrm{Id}\,.
\end{aligned}
$$

□

Now $(2\pi)^n\mathcal{G}$ is actually the *adjoint* of $\mathcal{F}$:

1.103 (1.76) $$\int \phi(\zeta)\overline{\mathcal{F}\psi}(\zeta)d\zeta = (2\pi)^n \int (\mathcal{G}\phi) \cdot \overline{\psi} dx \ \forall \ \phi, \psi \in \mathcal{S}(\mathbb{R}^n).$$

It follows that we can extend $\mathcal{F}$ to a map on tempered distributions

(1.77) $$\begin{aligned} &\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \\ &\mathcal{F}u(\overline{\phi}) = u((2\pi)^n \overline{\mathcal{G}\phi}) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n) \end{aligned}$$

Then we conclude

2.20 Corollary 1.1. *The Fourier transform extends by continuity to an isomorphism*

(1.78) $$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longleftrightarrow \mathcal{S}'(\mathbb{R}^n)$$

with inverse $\mathcal{G}$, *satisfying the identities* (2.13 1.66).

Although I have not discussed Lebesgue integrability I assume familiarity with the basic Hilbert space

$$
\begin{aligned}
&L^2(\mathbb{R}^n) = \\
&\quad \left\{ u : \mathbb{R}^n \longrightarrow \mathbb{C}; f \text{ is measurable and } \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty \right\} / \sim, \\
&\qquad\qquad f \sim g \Longleftrightarrow f = g \text{ almost everywhere.}
\end{aligned}
$$

This also injects by the same integration map (1.2.2000.274 1.102) with $\mathcal{S}(\mathbb{R}^n)$ as a dense subset

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^p) \hookrightarrow \mathcal{S}(\mathbb{R}^n).$$

1.102 Proposition 1.3. *The Fourier transform extends by continuity from the dense subspace* $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, *to an isomorphism*

$$\mathcal{F} : L^2(\mathbb{R}^n) \longleftrightarrow L^2(\mathbb{R}^n)$$

satisfying $\|\mathcal{F}u\|_{L^2} = (2\pi)^{\frac{1}{2}n}\|u\|_{L^2}$.

Proof. Given the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, this is also a consequence of (1.103 1.76), since setting $\phi = \mathcal{F}u$, for $u \in \mathcal{S}(\mathbb{R}^n)$, gives Parseval's formula

$$\int \mathcal{F}u(\zeta)\overline{\mathcal{F}v(\zeta)} = (2\pi)^n \int u(x)\overline{v(x)} dx.$$

Setting $v = u$ gives norm equality (which is Plancherel's formula).

An outline of the proof of the density statement is given in the problems below.

□

S.Differential.operators

1.8. Differential operators

The simplest examples of the Fourier transform of distributions are immediate consequences of the definition and (1.66). Thus

$$\mathcal{F}(\delta) = 1 \tag{1.79}$$

as already noted and hence, from (1.66),

$$\mathcal{F}(D^\alpha \delta(x)) = \xi^\alpha \quad \forall\ \alpha \in \mathbb{N}_0^n. \tag{1.80}$$

Now, recall that the space of distributions with support the point 0 is just:

2.21
$$\big\{u \in \mathcal{S}'(\mathbb{R}^n); \sup(u) \subset \{0\}\big\} = \big\{u = \sum_{\text{finite}} c_\alpha D^\alpha \delta\big\}. \tag{1.81}$$

Thus we conclude that the Fourier transform gives an isomorphism

2.22
$$\mathcal{F} : \big\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \subset \{0\}\big\} \longleftrightarrow \mathbb{C}[\xi] = \{\text{polynomials in } \xi\}. \tag{1.82}$$

Another way of looking at this same isomorphism is to consider partial differential operators with constant coefficients:

2.23
$$\begin{gathered} P(D) :\ \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \\ P(D) = \sum c_\alpha D^\alpha. \end{gathered} \tag{1.83}$$

The identity becomes

2.24
$$\mathcal{F}(P(D)\phi)(\xi) = P(\xi)\mathcal{F}(\phi)(\xi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.84}$$

and indeed the same formula holds for all $\phi \in \mathcal{S}'(\mathbb{R}^n)$. Using the simpler notation $\hat{u}(\xi) = \mathcal{F}u(\xi)$ this can be written

2.25
$$\widehat{P(D)u}(\xi) = P(\xi)\hat{u}(\xi),\ P(\xi) = \sum c_\alpha \xi^\alpha. \tag{1.85}$$

The polynomial P is called the (full) characteristic polynomial of $P(D)$; of course it determines $P(D)$ uniquely.

It is important for us to extend this formula to differential operators with variable coefficients. Using (1.59) and the inverse Fourier transform we get

2.26
$$P(D)u(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} P(\xi) u(y) dy d\xi \tag{1.86}$$

where again this is an *iterated* integral. In particular the inversion formula is just the case $P(\xi) = 1$. Consider the space

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) = \big\{u : \mathbb{R}^n \longrightarrow \mathbb{C};\ \sup_x |D^\alpha u(x)| < \infty\ \forall\ \alpha\big\} \tag{1.87}$$

the space of $\mathcal{C}^\infty$ function with all derivatives bounded on $\mathbb{R}^n$. Of course

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n) \tag{1.88}$$

but $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ is much bigger, in particular $1 \in \mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Now by Leibniz' formula

$$D^\alpha(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u \cdot D^{\alpha-\beta} v \tag{1.89}$$

it follows that $\mathcal{S}(\mathbb{R}^n)$ is a *module* over $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. That is,

$$u \in \mathcal{C}^\infty_\infty(\mathbb{R}^n), \phi \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow u\phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.90}$$

From this it follows that if

$$P(x,D)=\sum_{|\alpha|\le m}p_\alpha(x)D^\alpha,\ p_\alpha\in\mathcal{C}^\infty_\infty(\mathbb{R}^n) \tag{1.91}$$

then $P(x,D):\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n)$. The formula (2.26 1.86) extends to

2.27
$$P(x,D)\phi=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}P(x,\xi)\phi(y)dyd\xi \tag{1.92}$$

where again this is an iterated integral. Here

$$P(x,\xi)=\sum_{|\alpha|\le m}p_\alpha(x)\xi^\alpha \tag{1.93}$$

is the (full) characteristic polynomial of P.

.radial.compactification

1.9. Radial compactification

For later purposes, and general propaganda, consider the quadratic radial compactification of $\mathbb{R}^n$. The smooth map

1.104
$$\mathrm{QRC}:\mathbb{R}^n\ni x\longmapsto\frac{x}{(1+|x|^2)^{\frac12}}\in\mathbb{R}^n \tag{1.94}$$

is 1-1 and maps onto the interior of the unit ball, $\mathbb{B}^n=\{|x|\le 1\}$. Consider the subspace

1.105
$$\dot{\mathcal{C}}^\infty(\mathbb{B}^n)=\{u\in\mathcal{S}(\mathbb{R}^n);\operatorname{supp}(u)\subset\mathbb{B}^n\}. \tag{1.95}$$

This is just the set of smooth functions on $\mathbb{R}^n$ which vanish outside the unit ball. Then the composite ('pull-back') map

1.106
$$\mathrm{QRC}^*:\dot{\mathcal{C}}^\infty(\mathbb{B}^n)\ni u\longmapsto u\circ\mathrm{QRC}\in\mathcal{S}(\mathbb{R}^n) \tag{1.96}$$

is a topological isomorphism. A proof is indicated in the problems below.

The dual space of $\dot{\mathcal{C}}^\infty(\mathbb{B}^n)$ is generally called the space of 'extendible distributions' on $\mathbb{B}^n$ – because they are all given by restricting elements of $\mathcal{S}'(\mathbb{R}^n)$ to $\dot{\mathcal{C}}^\infty(\mathbb{B}^n)$. Thus QRC also identifies the tempered distributions on $\mathbb{R}^n$ with the extendible distributions on $\mathbb{B}^n$. We shall see below that various spaces of functions on $\mathbb{R}^n$ take interesting forms when pulled back to $\mathbb{B}^n$. I often find it useful to 'bring infinity in' in this way.

Why is this the 'quadratic' radial compactification, and not just the radial compactification? There is a good reason which is discussed in the problems below. The actual radial compactification is a closely related map which identifies Euclidean space, $\mathbb{R}^n$, with the interior of the upper half of the n-sphere in $\mathbb{R}^{n+1}$:

1.2.2000.275
$$\begin{aligned}\mathrm{RC}:\mathbb{R}^n\ni x\longmapsto&\left(\frac{1}{(1+|x|^2)^{\frac12}},\frac{x}{(1+|x|^2)^{\frac12}}\right)\\&\in\mathbb{S}^{n,1}=\{X=(X_0,X')\in\mathbb{R}^{n+1};X_0\ge0,X_0^2+|X'|^2=1\}\end{aligned} \tag{1.97}$$

Since the half-sphere is diffeomorphic to the ball (as compact manifolds with boundary) these two maps can be compared – they are not the same. However it is true that RC also identifies $\mathcal{S}(\mathbb{R}^n)$ with $\dot{\mathcal{C}}^\infty(\mathbb{S}^{n,1})$.

S.Problems.1

1.10. Problems

P1.1 PROBLEM 1.1. Suppose $\phi : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a function such that for *each* point $\bar{x} \in \mathbb{R}^n$ and each $k \in \mathbb{N}_0$ there exists a constant $\epsilon_k > 0$ and a polynomial $p_k(x; \bar{x})$ (in x) for which

$$|\phi(x) - p_k(x; \bar{x})| \le \frac{1}{\epsilon_k} |x - \bar{x}|^{k+1} \quad \forall\ |x - \bar{x}| \le \epsilon_k. \tag{1.98}$$

Does it follow that ϕ is infinitely differentiable – either prove this or give a counter-example.

P1.2 PROBLEM 1.2. Show that the function $u(x) = \exp(x)\cos[e^x]$ 'is' a tempered distribution. Part of the question is making a precise statement as to what this means!

P1.3 PROBLEM 1.3. Write out a careful (but not necessarily long) proof of the 'easy' direction of the Schwartz kernel theorem, that any $K \in \mathcal{S}'(\mathbb{R}^{n+m})$ defines a continuous linear operator

$$O_K : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{1.99}$$

[with respect to the weak topology on $\mathcal{S}'(\mathbb{R}^n)$ and the metric topology on $\mathcal{S}(\mathbb{R}^m)$] by

$$O_K\phi(\psi) = K(\psi \boxtimes \phi). \tag{1.100}$$

[Hint: Work out what the continuity estimate on the kernel, K, means when it is paired with an exterior product $\psi \boxtimes \phi$.]

31.1.2000.260 PROBLEM 1.4. Show that d in (1.6) is a metric on $\mathcal{S}(\mathbb{R}^n)$. [Hint: If $\|\cdot\|$ is a norm on a vector space show that

$$\frac{\|u+v\|}{1+\|u+v\|} \le \frac{\|u\|}{1+\|u\|} + \frac{\|v\|}{1+\|v\|}.]$$

31.1.2000.261 PROBLEM 1.5. Show that a sequence ϕ_n in $\mathcal{S}(\mathbb{R}^n)$ is Cauchy, resp. converges to ϕ, with respect to the metric d in Problem 1.4 if and only if ϕ_n is Cauchy, resp. converges to ϕ, with respect to each of the norms $\|\cdot\|_k$.

31.1.2000.262 PROBLEM 1.6. Show that a linear map $F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^p)$ is continuous with respect to the metric topology given in Problem 1.4 if and only if for each k there exists $N(k) \in \mathbb{N}$ a constant C_k such that

$$\|F\phi\|_k \le C_k \|\phi\|_{N(k)}\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

Give similar equivalent conditions for continuity of a linear map $f : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ and for a bilinear map $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^p) \longrightarrow \mathbb{C}$.

31.1.2000.265 PROBLEM 1.7. Check the continuity of (1.12).

P1.4 PROBLEM 1.8. Prove Proposition 1.2. [Hint: It is only necessary to show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ is fixed then for any of the open sets in (1.1), B, (with all the $\epsilon_j > 0$) there is an element $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $u - T_\phi \in B$. First show that if $\phi'_1, \dots \phi'_p$ is a basis for Φ then the set

$$B' = \{v \in \mathcal{S}'(\mathbb{R}^n); |\langle v, \phi'_j \rangle| < \delta_j \tag{1.101}$$

is contained in B if the $\delta_j > 0$ are chosen small enough. Taking the basis to be orthonormal, show that $u - \psi \in B'$ can be arranged for some $\psi \in \Phi$.]

1.2.2000.278 Problem 1.9. Compute the Fourier transform of $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$. [Hint: The Fourier integral is a product of 1-dimensional integrals so it suffices to assume $x \in \mathbb{R}$. Then

$$\int e^{-i\xi x} e^{-x^2} dx = e^{-\xi^2/4} \int e^{-(x+\frac{i}{2}\xi)^2} dx.$$

Interpret the integral as a contour integral and shift to the new contour where $x + \frac{i}{2}\xi$ is real.]

P1.5 Problem 1.10. Show that (1.23) makes sense for $\phi \in L^2(\mathbb{R}^n)$ (the space of (equivalence classes of) Lebesgue square-integrable functions and that the resulting map $L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is an injection.

P1.6 Problem 1.11. Suppose $u \in L^2(\mathbb{R}^n)$ and that

$$D_1 D_2 \cdots D_n u \in (1 + |x|)^{-n-1} L^2(\mathbb{R}^n),$$

where the derivatives are defined using Problem 1.10. Using repeated integration, show that u is necessarily a bounded continuous function. Conclude further that for $u \in \mathcal{S}'(\mathbb{R}^n)$

1.2.2000.274
$$\begin{gathered} D^\alpha u \in (1 + |x|)^{-n-1} L^2(\mathbb{R}^n) \ \forall \ |\alpha| \le k + n \\ \Longrightarrow D^\alpha u \text{ is bounded and continuous for } |\alpha| \le k. \end{gathered} \tag{1.102}$$

[This is a weak form of the Sobolev embedding theorem.]

P2.1 Problem 1.12. The *support* of a (tempered) distribution can be defined in terms of the support of a test function. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ the support, $\operatorname{supp}(\phi)$, is the closure of the set of points at which it takes a non-zero value. For $u \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\operatorname{supp}(u) = O^{\complement},\ O = \bigcup \{O' \subset \mathbb{R}^n \text{ open}; \operatorname{supp}(\phi) \subset O' \Longrightarrow u(\phi) = 0\}. \tag{1.103}$$

Show that the definitions for $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are consistent with the inclusion $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Prove that $\operatorname{supp}(\delta) = \{0\}$.

P2.2 Problem 1.13. For simplicity in $\mathbb{R}$, i.e. with $n = 1$, prove Schwartz theorem concerning distributions with support the origin. Show that with respect to the norm $\|\cdot\|_k$ the space

$$\{\phi \in \mathcal{S}(\mathbb{R}); \phi(x) = 0 \text{ in } |x| < \epsilon,\ \epsilon = \epsilon(\phi) > 0\} \tag{1.104}$$

is dense in

$$\left\{\phi \in \mathcal{S}(\mathbb{R}); \phi(x) = x^{k+1}\psi(x),\ \psi \in \mathcal{S}(\mathbb{R})\right\}. \tag{1.105}$$

Use this to show that

$$u \in \mathcal{S}'(\mathbb{R}),\ \operatorname{supp}(u) \subset \{0\} \Longrightarrow u = \sum_{\ell,\text{ finite}} c_\ell D_x^\ell \delta(x). \tag{1.106}$$

P2.3 Problem 1.14. Show that if P is a differential operator with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ then P is local in the sense that

$$\operatorname{supp}(Pu) \subset \operatorname{supp}(u) \quad \forall\ u \in \mathcal{S}'(\mathbb{R}^n). \tag{1.107}$$

The converse of this, for an operator $P : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ where (for simplicity) we assume

$$\operatorname{supp}(Pu) \subset K \subset \mathbb{R}^n \tag{1.108}$$

for a fixed compact set K, is *Peetre's theorem.* How would you try to prove this? (No full proof required.)

P2.4 Problem 1.15. (Schwartz representation theorem) Show that, for any $p \in \mathbb{R}$ the map

2.28 $$R_p \colon \mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto (1+|x|^2)^{-p/2}\mathcal{F}^{-1}[(1+|\xi|^2)^{-p/2}\mathcal{F}\phi] \in \mathcal{S}(\mathbb{R}^n) \tag{1.109}$$

is an isomorphism and, using Problem 1.11 or otherwise,

$$p \ge n+1+k \Longrightarrow \|R_p\phi\|_k \le C_k\|\phi\|_{L^2},\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.110}$$

Let $R_p^t \colon \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ be the dual map (defined by $T_p^t u(\phi) = u(R_p\phi)$). Show that R_p^t is an isomorphism and that if $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$|u(\phi)| \le C\|\phi\|_k,\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.111}$$

then $R_p^t u \in L^2(\mathbb{R}^n)$, if $p \ge n+1+k$, in the sense that it is in the image of the map in Problem 1.10. Using Problem 1.11 show that $R_{n+1}(R^t_{n+1+k}u)$ is bounded and continuous and hence that

2.29 $$u = \sum_{|\alpha|+|\beta| \le 2n+2+k} x^\beta D^\alpha u_{\alpha,\beta} \tag{1.112}$$

for some bounded continuous functions $u_{\alpha,\beta}$.

P2.5 Problem 1.16. (Schwartz kernel theorem.) Show that any continuous linear operator

$$T : \mathcal{S}(\mathbb{R}^m_y) \longrightarrow \mathcal{S}'(\mathbb{R}^n_x)$$

extends to a continuous linear operator

$$T : (1+|y|^2)^{-k/2}H^k(\mathbb{R}^m_y) \longrightarrow (1+|x|^2)^{-q/2}H^q(\mathbb{R}^n_x)$$

for some k and q. Deduce that the operator

$$\tilde{T} = (1+|D_x|^2)^{(-n-1-q)/2}(1+|x|^2)^{q/2} \circ T \circ (1+|y|^2)^{k/2}(1+|D|^2)^{-k/2} : L^2(\mathbb{R}^m) \longrightarrow \mathcal{C}_\infty(\mathbb{R}^n)$$

is continuous with values in the bounded continuous functions on $\mathbb{R}^n$. Deduce that $\tilde{T}$ has Schwartz kernel in $\mathcal{C}_\infty(\mathbb{R}^n; L^2(\mathbb{R}^m)) \subset \mathcal{S}'(\mathbb{R}^{n+m})$ and hence that T itself has a tempered Schwartz kernel.

1.2.2000.301 Problem 1.17. Radial compactification and symbols.

PolyDouble Problem 1.18. Series of problems discussing double polyhomogeneous symbols.

CHAPTER 2

C.Euclidean Pseudodifferential operators on Euclidean space

Formula (1.92) for the action of a differential operator (with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$) on $\mathcal{S}(\mathbb{R}^n)$ can be written

3.1 (2.1)
$$\begin{aligned} P(x,D)u &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}P(x,\xi)u(y)dyd\xi \\ &= (2\pi)^{-n}\int e^{ix\cdot\xi}P(x,\xi)\hat{u}(\xi)d\xi \end{aligned} \tag{2.1}$$

where $\hat{u}(\xi) = \mathcal{F}u(\xi)$ is the Fourier transform of u. We shall generalize this formula by generalizing $P(x,\xi)$ from a polynomial in ξ to a *symbol*, which is to say a smooth function satisfying certain uniformity conditions at infinity. In fact we shall also allow the symbol, or rather the *amplitude*, in the integral (2.1) to depend in addition on the 'incoming' variables, y :

3.2 (2.2)
$$A(x,D)u = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi, u \in \mathcal{S}(\mathbb{R}^n). \tag{2.2}$$

Of course it is not clear that this integral is well-defined.

To interpret (2.2) we shall first look into the definition and properties of symbols. Then we show how this integral can be interpreted as an oscillatory integral and that it thereby defines an operator on $\mathcal{S}(\mathbb{R}^n)$. We then investigate the properties of these *pseudodifferential operators* at some length.

S.Symbols 2.1. Symbols

A polynomial, p, in ξ, of degree at most m, satisfies a bound

3.3 (2.3)
$$|p(\xi)| \le C(1+|\xi|)^m \quad \forall\ \xi \in \mathbb{R}^n. \tag{2.3}$$

Since successive derivatives, $D^\alpha_\xi p(\xi)$, are polynomials of degree $m - |\alpha|$, for any multiindex α, we get the family of estimates

3.4 (2.4)
$$|D^\alpha_\xi p(\xi)| \le C_\alpha(1+|\xi|)^{m-|\alpha|} \quad \forall\ \xi \in \mathbb{R}^n, \alpha \in \mathbb{N}^n_0. \tag{2.4}$$

Of course if $|\alpha| > m$ then $D^\alpha_\xi p \equiv 0$, so we can even take the constant C_α to be independent of α. If we consider the characteristic polynomial $P(x,\xi)$ of a differential operator of order m with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ (i.e. all derivatives of the coefficients are bounded) (2.4) is replaced by

3.5 (2.5)
$$\left|D^\alpha_x D^\beta_\xi P(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \quad \forall\ (x,\xi) \in \mathbb{R}^n\times\mathbb{R}^n,\ \alpha,\beta \in \mathbb{N}^n_0. \tag{2.5}$$

There is no particular reason to have the same number of x variables as of ξ variables, so in general we define:

3.6 Definition 2.1. *The space $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ of symbols of order m consists of those functions $a \in \mathcal{C}^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ satisfying all the estimates*

3.7
$$\left|D^\alpha_z D^\beta_\xi a(z,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \text{ on } \mathbb{R}^p \times \mathbb{R}^n \quad \forall\ \alpha \in \mathbb{N}^p_0, \beta \in \mathbb{N}^n_0. \tag{2.6}$$

For later reference we even define $S^m_\infty(\Omega;\mathbb{R}^n)$ when $\Omega \subset \mathbb{R}^p$ and $\Omega \subset \operatorname{clos}(\operatorname{int}(\Omega))$ as consisting of those $a \in \mathcal{C}^\infty(\operatorname{int}(\Omega) \times \mathbb{R}^n)$ satisfying (2.6) for $(z,\xi) \in \operatorname{int}(\Omega) \times \mathbb{R}^n$.

The estimates (2.6) can be rewritten

3.8
$$\|a\|_{N,m} = \sup_{\substack{z \in \operatorname{int}(\Omega)\\ \xi \in \mathbb{R}^n}} \max_{|\alpha|+|\beta| \le N} (1+|\xi|)^{-m+|\beta|}\left|D^\alpha_z D^\beta_\xi a(z,\xi)\right| < \infty. \tag{2.7}$$

With these *norms* $S^m_\infty(\Omega;\mathbb{R}^n)$ is a Fréchet space, rather similar in structure to $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Thus the topology is given by the metric

3.9
$$d(a,b) = \sum_{N \ge 0} 2^{-N} \frac{\|a-b\|_{N,m}}{1+\|a-b\|_{N,m}},\ a,b \in S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.8}$$

The subscript '∞' here is *not* standard notation. It refers to the assumption of uniform boundedness of the derivatives of the 'coefficients'. More standard notation would be just $S^m(\Omega \times \mathbb{R}^n)$, especially for $\Omega = \mathbb{R}^p$, but I think this is too confusing.

A more significant issue is: Why this class precisely? As we shall see below, there are other choices which are not only possible but even profitable to make. However, the present one has several virtues. It is large enough to cover most of the straightforward things we want to do (at least initially) and small enough to 'work' easily. It leads to what I shall refer to as the 'traditional' algebra of pseudodifferential operators.

Now to some basic properties. First notice that

$$(1+|\xi|)^m \le C(1+|\xi|)^{m'}\ \forall\ \xi \in \mathbb{R}^n \Longleftrightarrow m \le m'. \tag{2.9}$$

Thus we have an inclusion

3.10
$$S^m_\infty(\Omega;\mathbb{R}^n) \hookrightarrow S^{m'}_\infty(\Omega;\mathbb{R}^n)\ \forall\ m' \ge m. \tag{2.10}$$

Moreover this inclusion is continuous, since from (2.7), $\|a\|_{N,m'} \le \|a\|_{N,m}$ if $a \in S^m(\Omega;\mathbb{R}^n)$ and $m' \ge m$. Since these spaces increase with m we think of them as a *filtration* of the big space

3.11
$$S^\infty_\infty(\Omega;\mathbb{R}^n) = \bigcup_m S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.11}$$

Notice that the two '∞s' here are quite different. The subscript refers to the fact that the 'coefficients' are bounded and stands for L^∞ whereas the superscript ∞ stands really for $\mathbb{R}$. The *residual* space of this filtration is

3.12
$$S^{-\infty}_\infty(\Omega;\mathbb{R}^n) = \bigcap_m S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.12}$$

In fact the inclusion (2.10) is *never* dense if $m' > m$. Instead we have the following rather technical, but nevertheless very useful, result.

3.13 Lemma 2.1. *For any $m \in \mathbb{R}$ and any $a \in S^m_\infty(\Omega;\mathbb{R}^n)$ there is a sequence in $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ which is bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$ and converges to a in the topology of $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for any $m' > m$; in particular $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ is dense in the space $S^m_\infty(\Omega;\mathbb{R}^n)$ in the topology of $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for $m' > m$.*

The reason one cannot take $m' = m$ here is essentially the same reason that underlies the fact that $\mathcal{S}(\mathbb{R}^n)$ is not dense in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Namely any uniform limit obtained from a converging Schwartz sequence must vanish at infinity. In particular the constant function $1 \in S^0_\infty(\mathbb{R}^p;\mathbb{R}^n)$ cannot be in the closure in this space of $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ if $n > 0$.

Proof. Choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $0 \le \phi(\xi) \le 1$, $\phi(\xi) = 1$ if $|\xi| < 1, \phi(\xi) = 0$ if $|\xi| > 2$ and consider the sequence

$$a_k(z,\xi) = \phi(\xi/k)a(z,\xi), \quad a \in S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.13}$$

We shall show that $a_k \in S^{-\infty}_\infty(\Omega,\mathbb{R}^n)$ is a bounded sequence in $S^m_\infty(\Omega;\mathbb{R}^n)$ and that $a_k \longrightarrow a$ in $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for any $m' > m$. Certainly for each N

$$|a_k(z,\xi)| \le C_{N,k}(1+|\xi|)^{-N} \tag{2.14}$$

since ϕ has compact support. Leibniz' formula gives

1.2.2000.353

$$D^\alpha_z D^\beta_\xi a_k(z,\xi) = \sum_{\beta' \le \beta} \binom{\beta'}{\beta} k^{-|\beta'|}(D^{\beta'}\phi)(\xi/k) D^\alpha_z D^{\beta-\beta'}_\xi a(z,\xi). \tag{2.15}$$

On the support of $\phi(\xi/k)$, $|\xi| \le k$ so, using the symbol estimates on a, it follows that a_k is bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$. We easily conclude that

$$\left|D^\alpha_z D^\beta_\xi a_k(z,\xi)\right| \le C_{N,\alpha,\beta,k}(1+|\xi|)^{-N} \quad \forall\ \alpha,\beta,N,k. \tag{2.16}$$

Thus $a_k \in S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$.

So consider the difference

3.14

$$(a - a_k)(z,\xi) = (1-\phi)(\xi/k)\ a(z,\xi). \tag{2.17}$$

Now, $|(1-\phi)(\xi/k)| = 0$ in $|\xi| \le k$ so we only need estimate the difference in $|\xi| \ge k$ where this factor is bounded by 1. In this region $1 + |\xi| \ge 1 + k$ so, since $-m' + m < 0$,

$$\begin{gathered}(1+|\xi|)^{-m'}\left|(a-a_k)(z,\xi)\right| \le \\ (1+k)^{-m'+m}\sup_{z,\xi}|(1+|\xi|)^{-m}|a(z,\xi)| \le (1+k)^{-m'+m}\|a\|_{0,m} \longrightarrow 0.\end{gathered} \tag{2.18}$$

This is convergence with respect to the first symbol norm.

Next consider the ξ derivatives of (2.17). Using Leibniz' formula

$$\begin{aligned} D^\beta_\xi(a-a_k) &= \sum_{\gamma\le\beta}\binom{\beta}{\gamma} D^{\beta-\gamma}_\xi(1-\phi)(\frac{\xi}{k})\cdot D^\gamma_\xi a(z,\xi) \\ &= (1-\phi)(\frac{\xi}{k})\cdot D^\beta_\xi a(z,\xi) - \sum_{\gamma<\beta}\binom{\beta}{\gamma}\big(D^{\beta-\gamma}\phi\big)(\frac{\xi}{k})\cdot k^{-|\beta-\gamma|}D^\gamma_\xi a(z,\xi). \end{aligned}$$

In the first term, $D^\beta_\xi a(z,\xi)$ is a symbol of order $m - |\beta|$, so by the same argument as above

$$\sup_\xi (1+|\xi|)^{-m'+|\beta|}(1-\phi)(\frac{\xi}{k})D^\beta_\xi a(x,\xi) \longrightarrow 0 \tag{2.19}$$

as $k \longrightarrow \infty$ if $m' > m$. In all the other terms, $(D^{\beta-\gamma}\phi)(\zeta)$ has compact support, in fact $1 \le |\zeta| \le 2$ on the support. Thus for each term we get a bound

$$\sup_{k\le|\xi|\le 2k}(1+|\xi|)^{-m'+|\beta|}\cdot k^{-|\beta-\gamma|}C\cdot(1+|\xi|)^{m-|\gamma|} \le Ck^{-m'+m}. \tag{2.20}$$

The variables z play the rôle of parameters so we have in fact shown that

$$\sup_{\substack{z\in\Omega\\ \xi\in\mathbb{R}^n}} (1+|\xi|)^{-m'+|\beta|}\left|D_z^\alpha D_\xi^\beta(a-a_k)\right| \longrightarrow 0 \text{ as } k\longrightarrow\infty. \tag{2.21}$$

This means $a_k \longrightarrow a$ in each of the symbol norms, and hence in the topology of $S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ as desired. □

In fact this proof suggests a couple of other 'obvious' results. Namely

3.17 $$S^m_\infty(\Omega;\mathbb{R}^n)\cdot S^{m'}_\infty(\Omega;\mathbb{R}^n)\subset S^{m+m'}_\infty(\Omega;\mathbb{R}^n). \tag{2.22}$$

This can be proved directly using Leibniz' formula:

$$\begin{aligned}&\sup_\xi(1+|\xi|)^{-m-m'+|\beta|}\left|D_z^\alpha D_\xi^\beta\left(a(z,\xi)\cdot b(z,\xi)\right)\right|\\ &\le\sum_{\substack{\mu\le\alpha\\ \gamma\le\beta}}\binom{\alpha}{\mu}\binom{\beta}{\gamma}\sup_\xi(1+|\xi|)^{-m+|\gamma|}\left|D_z^\mu D_\xi^\gamma a(z,\xi)\right|\\ &\qquad\times\sup_\xi(1+|\xi|)^{-m'+|\beta-\gamma|}\left|D_z^{\alpha-\mu}D_\xi^{\beta-\gamma}b(z,\xi)\right|<\infty.\end{aligned}$$

We also note the action of differentiation:

3.18 $$\begin{gathered}D_z^\alpha : S^m_\infty(\Omega;\mathbb{R}^n)\longrightarrow S^m_\infty(\Omega;\mathbb{R}^n) \text{ and}\\ D_\xi^\beta : S^m_\infty(\Omega;\mathbb{R}^n)\longrightarrow S^{m-|\beta|}_\infty(\Omega;\mathbb{R}^n).\end{gathered} \tag{2.23}$$

In fact, while we are thinking about these things we might as well show the important consequence of *ellipticity*. A symbol $a\in S^m_\infty(\Omega;\mathbb{R}^n)$ is said to be (globally) elliptic if

3.19 $$|a(z,\xi)|\ge\epsilon(1+|\xi|)^m - C(1+|\xi|)^{m-1},\ \epsilon>0 \tag{2.24}$$

or equivalently[1]

3.20 $$|a(z,\xi)|\ge\epsilon(1+|\xi|)^m \text{ in } |\xi|\ge C_\epsilon,\ \epsilon>0. \tag{2.25}$$

3.21 Lemma 2.2. *If $a\in S^m_\infty(\Omega;\mathbb{R}^n)$ is elliptic there exists $b\in S^{-m}_\infty(\Omega;\mathbb{R}^n)$ such that*

3.22 $$a\cdot b-1\in S^{-\infty}_\infty(\Omega;\mathbb{R}^n). \tag{2.26}$$

Proof. Using (2.25) choose ϕ as in the proof of Lemma 2.1 and set

3.23 $$b(z,\xi)=\begin{cases}\frac{1-\phi(\xi/2C)}{a(z,\xi)} & |\xi|\ge C\\ 0 & |\xi|\le C.\end{cases} \tag{2.27}$$

Then b is $\mathcal{C}^\infty$ since $b=0$ in $C\le|\xi|\le C+\delta$ for some $\delta>0$. The symbol estimates follow by noting that, in $|\xi|\ge C$,

1.2.2000.277 $$D_z^\alpha D_\xi^\beta b = a^{-1-|\alpha|-|\beta|}\cdot G_{\alpha\beta} \tag{2.28}$$

where $G_{\alpha\beta}$ is a symbol of order $(|\alpha|+|\beta|)m-|\beta|$. This may be proved by induction. Indeed, it is true when $\alpha=\beta=0$. Assuming (2.28) for some α and β, differentiation of (2.28) gives

$$\begin{aligned}D_{z_j}D_z^\alpha D_\xi^\beta b = D_{z_j}a^{-1-|\alpha|-|\beta|}\cdot G_{\alpha\beta} = a^{-2-|\alpha|-|\beta|}G',&\\ G'=(-1-|\alpha|-|\beta|)(D_{z_j}a)G_{\alpha\beta}+aD_{z_j}G_{\alpha\beta}.&\end{aligned}$$

[1]Note it is required that ϵ be chosen to be independent of z here, so this is a notion of uniform ellipticity.

By the inductive hypothesis, G' is a symbol of order $(|\alpha|+1+|\beta|)m-|\beta|$. A similar argument applies to derivatives with respect to the ξ variables. □

dodifferential.operators

2.2. Pseudodifferential operators

Now we proceed to discuss the formula (3.2 2.2) where we shall assume that, for some $w, m \in \mathbb{R}$,

3.24 (2.29)
$$a(x,y,\xi) = (1+|x-y|^2)^{w/2}\tilde{a}(x,y,\xi)$$
$$\tilde{a} \in S^m_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi).$$

The extra 'weight' factor (which allows polynomial growth in the direction of $x-y$) turns out, somewhat enigmatically, to both make no difference and be very useful! Notice[2] that if $a \in \mathcal{C}^\infty(\mathbb{R}^{2n}\times\mathbb{R}^n)$ then $a \in (1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ if and only if

3.32 (2.30)
$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1+|x-y|)^w(1+|\xi|)^{m-|\gamma|}\ \forall\ \alpha,\beta,\gamma \in \mathbb{N}^n_0.$$

If $m < -n$ then, for each $u \in \mathcal{S}(\mathbb{R}^n)$ the integral in (3.2 2.2) is absolutely convergent, locally uniformly in x, since

(2.31)
$$\begin{aligned}|a(x,y,\xi)u(y)| &\le C(1+|x-y|)^w(1+|\xi|)^m(1+|y|)^{-N}\\ &\le C(1+|x|)^w(1+|\xi|)^m(1+|y|)^m,\ m<-n.\end{aligned}$$

Here we have used the following simple consequence of the triangle inequality

$$(1+|x-y|) \le (1+|x|)(1+|y|)$$

from which it follows that

19.2.1998.102 (2.32)
$$(1+|x-y|)^w \le \begin{cases}(1+|x|)^w(1+|y|)^w & \text{if } w>0\\ (1+|x|)^w(1+|y|)^{-w} & \text{if } w \le 0.\end{cases}$$

Thus we conclude that, provided $m < -n$,

3.33 (2.33)
$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow (1+|x|^2)^{w/2}\mathcal{C}^0_\infty(\mathbb{R}^n).$$

To show that, for general m, A exists as an operator, we prove that its Schwartz kernel exists.

3.25 Proposition 2.1. *The map, defined for $m<-n$ as a convergent integral,*

3.26 (2.34)
$$(1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \longmapsto I(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi \in (1+|x|^2+|y|^2)^{w/2}\mathcal{C}^0_\infty(\mathbb{R}^{2n})$$

extends by continuity to

1.2.2000.302 (2.35)
$$I : (1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

for each w, $m \in \mathbb{R}$ in the topology of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any $m' > m$.

[2]See Problem 1.2.2000.276 2.5.

Proof. Since we already have the density of $S^{-\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ in the toplogy of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any $m' > m$, we only need to show the continuity of the map ([3.26]2.34) on this residual subspace with respect to the topology of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any m', which we may as well write as m. What we shall show is that, for each $w, m \in \mathbb{R}$, there are integers $N, k \in \mathbb{N}$ such that, in terms of the norms in ([3.8]2.7) and ([1.7]1.6)

3.27 (2.36)
$$\big|I(a)(\phi)\big| \le C\|\tilde a\|_{N,m}\|\phi\|_k \quad \forall\ \phi\in\mathcal{S}(\mathbb{R}^{2n}),\qquad a = (1+|x-y|^2)^{w/2}\tilde a,\ \ \tilde a\in S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n).$$

To see this we just use integration by parts.

Set $\tilde\phi(x,y) = (1+|x-y|^2)^{w/2}\phi(x,y)$. Observe that

$$\begin{aligned}(1+\xi\cdot D_x)e^{i(x-y)\cdot\xi} &= (1+|\xi|^2)e^{i(x-y)\cdot\xi}\\ (1-\xi\cdot D_y)e^{i(x-y)\cdot\xi} &= (1+|\xi|^2)e^{i(x-y)\cdot\xi}.\end{aligned}$$

Thus we can write, for $\tilde a\in S^{-\infty}_\infty$, with $a=(1+|x-y|^2)^{w/2}\tilde a$ and for any $q\in\mathbb{N}$

3.28 (2.37)
$$\begin{aligned}I(a)(\phi) = \iint (2\pi)^{-n}\int & e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-2q}\\ &(1-\xi\cdot D_x)^q(1+\xi\cdot D_y)^q\big[\tilde a(x,y,\xi)\tilde\phi(x,y)\big]d\xi dxdy\\ = \sum_{|\gamma|\le 2q}\iint &\big(\int e^{i(x-y)\cdot\xi}a^{(q)}_\gamma(x,y,\xi)d\xi\big)D^\gamma_{(x,y)}\tilde\phi(x,y)dxdy.\end{aligned}$$

Here the $a^{(q)}_\gamma$ arise by expanding the powers of the operator

$$(1-\xi\cdot D_x)^q(1+\xi\cdot D_x)^q = \sum_{|\mu|,|\nu|\le q} C_{\mu,\nu}\xi^{\mu+\nu}D^\mu_x D^\nu_y$$

and applying Leibniz' formula. Thus $a^{(q)}_\gamma$ arises from terms in which $2q-|\gamma|$ derivatives act on $\tilde a$ so it is of the form

$$\begin{aligned}a_\gamma = (1+|\xi|^2)^{-2q}&\sum_{|\mu|\le|\gamma|,|\gamma|\le 2q} C_{\mu,\gamma}\xi^\gamma D^\mu_{(x,y)}\tilde a\\ \Longrightarrow \|a_\gamma\|_{N,m} &\le C_{m,q,N}\|\tilde a\|_{N+2q,m-2q}\ \forall\ m,N,q.\end{aligned}$$

So (for given m) if we take $-2q+m<-n$, e.g. $q>\max(\frac{n+m}{2},0)$ and use the integrability of $(1+|x|+|y|)^{-2n-1}$ on $\mathbb{R}^{2n}$, then

$$\big|I(a)(\phi)\big| \le C\|\tilde a\|_{2q,m}\|\tilde\phi\|_{2q+2n+1} \le C\|\tilde a\|_{2q,m}\|\phi\|_{2q+w+2n+1}. \tag{2.38}$$

This is the estimate ([3.27]2.36), which proves the desired continuity. □

In showing the existence of the Schwartz' kernel in this proof we do *not* really need to integrate by parts in both x *and* y; either separately will do the trick. We can use this observation to show that these pseudodifferential operator act on $\mathcal{S}(\mathbb{R}^n)$.

3.29 Lemma 2.3. *If* $a\in(1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ *then the operator* A, *with Schwartz kernel* $I(a)$, *is a continuous linear map*

3.30 (2.39)
$$A:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n).$$

We shall denote by $\Psi^m_\infty(\mathbb{R}^n)$ the linear space of operators ([3.30]2.39), corresponding to $(1+|x-y|^2)^{-w/2}a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w. I call them *pseudodifferential operators 'of traditional type' – or type '1,0'*, the meaning of which is explained in Problem [1.2.2000.279]2.16 below.

Proof. Proceeding as in ([3.28]2.37) but only integrating by parts in y we deduce that, for q large depending on m,

$$Au(\psi) = \sum_{\gamma\le 2q}(2\pi)^{-n}\iint\int e^{i(x-y)\cdot\xi}a_\gamma(x,y,\xi)D^\gamma_y u(y)d\xi\psi(x)dydx,$$
$$a_\gamma \in (1+|x-y|^2)^{w/2}S^{m-q}(\mathbb{R}^{2n};\mathbb{R}^n) \text{ if } a\in(1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n).$$

The integration by parts is justified by continuity from $S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. Taking $-q+m<-n-|w|$, this shows that Au is given by the convergent integral

1.2.2000.282

$$Au(x) = \sum_{\gamma\le 2q}(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}a_\gamma(x,y,\xi)D^\gamma_y u(y)d\xi dy, \tag{2.40}$$
$$A:\mathcal{S}(\mathbb{R}^n)\longrightarrow(1+|x|^2)^{\frac{|w|}{2}}\mathcal{C}^0_\infty(\mathbb{R}^n)$$

which is really just ([3.33]2.33) again. Here $\mathcal{C}^0_\infty(\mathbb{R}^n)$ is the Banach space of bounded continuous functions on $\mathbb{R}^n$, with the supremum norm. The important point is that the weight depends on w but not on m. Notice that

$$D_{x_j}Au(x) = (2\pi)^{-n}\sum_{|\gamma|\le 2q}\iint e^{i(x-y)\cdot\xi}\big(\xi_j + D_{x_j}\big)a_\gamma\cdot D^\gamma_y u(y)dyd\xi$$

and

$$x_jAu(x) = (2\pi)^n\sum_{|\gamma|\le 2q}\iint e^{i(x-y)\cdot\xi}\big(-D_{\xi_j}+y_j\big)a_\gamma\cdot D^\gamma_y u(y)dyd\xi.$$

Proceeding inductively ([3.30]2.39) follows from ([3.33]2.33) or ([1.2.2000.282]2.40) since we conclude that

$$x^\alpha D^\beta_x Au \in (1+|x|^2)^{\frac{|w|}{2}}\mathcal{C}^0_\infty(\mathbb{R}^n),\ \forall\ \alpha,\beta\in\mathbb{N}^n_0$$

and this implies that $Au\in\mathcal{S}(\mathbb{R}^n)$. □

S.Composition

2.3. Composition

There are two extreme cases of $I(a)$, namely where a is independent of either x or of y. Below we shall prove:

3.31 Theorem 2.1 (Reduction). *Each $A\in\Psi^m_\infty(\mathbb{R}^n)$ can be written uniquely as $I(a')$ where $a'\in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$.*

This is the main step in proving the fundamental result of this Chapter, which is that two pseudodifferential operators can be composed to give a pseudodifferential operator and that the orders are additive. Thus our aim is to demonstrate the fundamental

4.1 Theorem 2.2. *[Composition] The space $\Psi^\infty_\infty(\mathbb{R}^n)$ is an order-filtered $*$-algebra on $\mathcal{S}(\mathbb{R}^n)$.*

We have already shown that each $A \in \Psi^{\infty}_{\infty}(\mathbb{R}^n)$ defines a continuous linear map (2.39). We now want to show that

4.35 $$A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A^* \in \Psi^m_\infty(\mathbb{R}^n) \tag{2.41}$$

4.3 $$A \in \Psi^m_\infty(\mathbb{R}^n),\ B \in \Psi^{m'}_\infty(\mathbb{R}^n) \Longrightarrow A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n), \tag{2.42}$$

since this is what is meant by an order-filtered (the orders add on composition) $*$-algebra (meaning (2.41) holds). In fact we will pick up some more information along the way.

S.Reduction

2.4. Reduction

We proceed to prove Theorem 2.1, which we can restate as:

4.4 Proposition 2.2. *The range of (2.34) (for any w) is the same as the range of I restricted to the image of the inclusion map*

$$S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto a(x,\xi) \in S^m_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n).$$

Proof. Suppose $a \in \big(1+|x-y|^2\big)^{w/2} S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w, then

4.5 $$I\big((x_j-y_j)a\big) = I\big(-D_{\xi_j}a\big)\ j=1,\dots,n. \tag{2.43}$$

Indeed this is just the result of inserting the identity

$$D_{\xi_j} e^{i(x-y)\cdot\xi} = (x_j-y_j)e^{i(x-y)\cdot\xi}$$

into (2.34) and integrating by parts. Since both sides of (2.43) are continuous on $\big(1+|x-y|^2\big)^{w/2} S^\infty_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ the identity holds in general. Notice that if a is of order m then $D_{\xi_j}a$ is of order $m-1$, so (2.43) shows that even though the operator with amplitude $(x_j-y_j)a(x,y,\xi)$ *appears* to have order m, it actually has order $m-1$.

To exploit (2.43) consider the Taylor series (with Legendre's remainder) for $a(x,y,\xi)$ around $x=y$:

4.6 $$a(x,y,\xi) = \sum_{|\alpha|\le N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \big(D^\alpha_y a\big)(x,x,\xi) + \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \cdot R_{N,\alpha}(x,y,\xi). \tag{2.44}$$

Here,

4.7 $$R_{N,\alpha}(x,y,\xi) = \int_0^1 (1-t)^{N-1}\big(D^\alpha_y a\big)(x,(1-t)x+ty,\xi)dt. \tag{2.45}$$

Now,

$$(x-y)^\alpha \big(D^\alpha_y a\big)(x,y,\xi) \in \big(1+|x-y|^2\big)^{\frac{(w+|\alpha|)}{2}} S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n). \tag{2.46}$$

Applying (2.43) repeatedly we see that if A is the operator with kernel $I(a)$ then

4.8 $$A = \sum_{j=0}^{N-1} A_j + R_N, \quad A_j \in \Psi^{m-j}_\infty(\mathbb{R}^n),\ R_N \in \Psi^{m-N}_\infty(\mathbb{R}^n) \tag{2.47}$$

where the A_j have kernels

4.9 (2.48)
$$I\big(\sum_{|\alpha|=j}\frac{i^{|\alpha|}}{\alpha!}\big(D_y^\alpha D_\xi^\alpha a\big)(x,x,\xi)\big).$$

To proceed further we need somehow to *sum* this series. Of course we cannot really do this, but we can come close!

S.Asymptotic.summation

2.5. Asymptotic summation

Suppose $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. The fact that the *orders* are decreasing means that those symbols are getting very small, for $|\xi|$ large. The infinite series

(2.49)
$$\sum_j a_j(z,\xi)$$

need not converge. However we shall say that it converges asymptotically, or since it is a series we say it is 'asymptotically summable,' if there exists $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ such that,

4.10 (2.50)
$$\text{for every } N,\ a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n).$$

We write this relation as

4.11 (2.51)
$$a \sim \sum_{j=0}^{\infty} a_j.$$

4.12 Proposition 2.3. *Any series $a_j \in S^{m-j}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ is asymptotically summable, in the sense of* (4.10 2.50)*, and the asymptotic sum is well defined up to an additive term in $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.*

Proof. The uniqueness part is easy. Suppose a and a' both satisfy (4.10 2.50). Taking the difference

4.13 (2.52)
$$a - a' = \big(a - \sum_{j=0}^{N-1} a_j\big) - \big(a' - \sum_{j=0}^{N-1} a_j\big) \in S^{m-N}_\infty(\mathbb{R}^p;\mathbb{R}^n).$$

Since $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ is just the intersection of the $S^{-N}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ over N it follows that $a - a' \in S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$, proving the uniqueness.

So to the existence of an asymptotic sum. To construct this (by Borel's method[3]) we cut off each term 'near infinity in ξ'. Thus fix $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $\phi(\xi) = 0$ in $|\xi| \le 1$, $\phi(\xi) = 1$ in $|\xi| \ge 2$, $0 \le \phi(\xi) \le 1$. Consider a decreasing sequence

4.14 (2.53)
$$\epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \downarrow 0.$$

We shall set

4.15 (2.54)
$$a(z,\xi) = \sum_{j=0}^{\infty} \phi(\epsilon_j\xi) a_j(z,\xi).$$

Since $\phi(\epsilon_j\xi) = 0$ in $|\xi| < 1/\epsilon_j \to \infty$ as $j \to \infty$, only finitely many of these terms are non-zero in any ball $|\xi| \le R$. Thus $a(z,\xi)$ is a well-defined $\mathcal{C}^\infty$ function. Of course we need to consider the seminorms, in $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, of each term.

[3]Émile Borel

The first of these is

4.16 (2.55)
$$\sup_z \sup_\xi (1+|\xi|)^{-m} \, \big|\phi(\epsilon_j\xi)\big| \, \big|a_j(z,\xi)\big|.$$

Now $|\xi| \le \frac{1}{\epsilon_j}$ on the support of $\phi(\epsilon_j\xi)a_j(z,\xi)$ and since a_j is a symbol of order $m-j$ this allows us to estimate (2.55) by

$$\begin{aligned}\sup_z \sup_{|\xi|\le\frac{1}{\epsilon_j}} &(1+|\xi|)^{-j}\cdot\big[\big(1+|\xi|\big)^{-m+j}\big|a_j(z,\xi)\big|\big]\\ &\le \big(1+\frac{1}{\epsilon_j}\big)^{-j}\cdot C_j \le \epsilon_j^j\cdot C_j\end{aligned}$$

where the C_j's are *fixed* constants, independent of ϵ_j.

Let us look at the higher symbol estimates. As usual we can apply Leibniz' formula:

$$\begin{aligned}&\sup_z\sup_\xi(1+|\xi|)^{-m+|\beta|}\big|D_z^\alpha D_\xi^\beta\phi(\epsilon_j\xi)a_j(z,\xi)\big|\\ \le\sum_{\mu\le\beta}&\sup_z\sup_\xi(1+|\xi|)^{|\beta|-|\mu|-j}\epsilon_j^{|\beta|-|\mu|}\big|\big(D^{\beta-\mu}\phi\big)(\epsilon_j\xi)\big|\\ &\times(1+|\xi|)^{-m+j+|\mu|}\big|D_z^\alpha D_\xi^\mu a_j(z,\xi)\big|.\end{aligned}$$

The term with $\mu=\beta$ we estimate as before and the others, with $\mu\ne\beta$ are supported in $\frac{1}{\epsilon_j}\le|\xi|\le\frac{2}{\epsilon_j}$. Then we find that for all j

4.17 (2.56)
$$\|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m}\le C_{N,j}\epsilon_j^j$$

where $C_{N,j}$ is independent of ϵ_j.

So we see that for each given N we can arrange that, for instance,

$$\|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m}\le C_N\frac{1}{j^2}$$

by choosing the ϵ_j to satify

$$C_{N,j}\epsilon_j^j\le\frac{1}{j^2}\ \forall\ j\ge j(N).$$

Notice the crucial point here, we can arrange that for *each* N the sequence of norms in (2.56) is dominated by C_Nj^{-2} by fixing $\epsilon_j<\epsilon_{j,N}$ for large j. Thus we can arrange convergence of *all* the sums

$$\sum_j\|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m}$$

by diagonalization, for example setting $\epsilon_j=\frac{1}{2}\epsilon_{j,j}$. Thus by choosing $\epsilon_j\downarrow 0$ rapidly enough we ensure that the series (2.54) converges. In fact the same argument allows us to ensure that for every N

4.18 (2.57)
$$\sum_{j\ge N}\phi(\epsilon_j\xi)a_j(z,\xi)\text{ converges in }S_\infty^{m-N}(\mathbb{R}^p;\mathbb{R}^n).$$

This certainly gives (2.50) with a defined by (2.54). □

S.Residual.terms

2.6. Residual terms

Now we can apply Proposition 2.3 to the series in (2.48), that is we can find $b \in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$ satisfying

4.19 (2.58)
$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\big(D^\alpha_y a\big)(x,x,\xi).$$

Let $B = I(b)$ be the operator defined by this amplitude (which is independent of y). Now (2.47) says that

$$A - B = \sum_{j=0}^{N-1} A_j + R_N - B$$

and from (2.50) applied to (2.58)

$$B = \sum_{j=0}^{N-1} A_j + R'_N,\ \ R'_N \in \Psi^{m-N}_\infty(\mathbb{R}^n)$$

Thus

4.20 (2.59)
$$A - B \in \Psi^{-\infty}_\infty(\mathbb{R}^n) = \bigcap_N \Psi^N_\infty(\mathbb{R}^n).$$

Notice that, at this stage, we do *not* know that $A-B$ has kernel $I(c)$ with $c \in S^{-\infty}_\infty(\mathbb{R}^{2n},\mathbb{R}^n)$, just that it has kernel $I(c_N)$ with $c_N \in S^N_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for each N.

However:

4.21 Proposition 2.4. *An operator $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is an element of the space $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies the estimates*

4.22 (2.60)
$$\big|D^\alpha_x D^\beta_y K(x,y)\big| \le C_{N,\alpha,\beta}(1+|x-y|)^{-N}\ \forall\ \alpha,\beta,N.$$

Proof. Suppose first that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$, which means that $A \in \Psi^N_\infty(\mathbb{R}^n)$ for every N. The Schwartz kernel, K_A, of A is therefore given by (2.34) with the amplitude $a_N \in S^N_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. For $N << -n-1-p$ the integral converges absolutely and we can integrate by parts to show that

$$\begin{aligned}&(x-y)^\alpha D^\beta_x D^\gamma_y K_A(x,y)\\ &= (2\pi)^{-N}\int e^{i(x-y)\cdot\xi}(-D_\xi)^\alpha(D_x+i\xi)^\beta(D_y-i\xi)^\gamma a_N(x,y,\xi)d\xi\end{aligned}$$

which converges absolutely, and uniformly in x,y, provided $|\beta|+|\gamma|+N-|\alpha| < -n$. Thus

$$\sup\big|(x-y)^\alpha D^\beta_x D^\gamma_y K\big| < \infty\ \forall\ \alpha,\beta,\gamma$$

which is another way of writing (2.60) i.e.

$$\sup\big(1+|x-y|^2\big)^N\big|D^\beta_x D^\gamma_y K\big| < \infty\ \forall\ \beta,\gamma,N.$$

Conversely suppose that (2.60) holds. Define

4.23 (2.61)
$$g(x,z) = K(x,x-z).$$

The estimates (2.60) become

4.24 (2.62)
$$\sup\big|D^\alpha_x z^\gamma D^\beta_z g(x,z)\big| < \infty\ \forall\ \alpha,\beta,\gamma.$$

That is, g is rapidly decreasing with all its derivatives in z. Taking the Fourier transform,

4.25 (2.63)
$$b(x,\xi) = \int e^{-iz\cdot\xi} g(x,z)dz$$

the estimate (2.62) translates to

4.26 (2.64)
$$\sup_{x,\xi} \left| D_x^\alpha \xi^\beta D_\xi^\gamma b(x,\xi) \right| < \infty \ \forall \ \alpha, \beta, \gamma$$
$$\Longleftrightarrow b \in S_\infty^{-\infty}(\mathbb{R}_x^n; \mathbb{R}_\xi^n).$$

Now the inverse Fourier transform in (2.63), combined with (2.61) gives

4.27 (2.65)
$$K(x,y) = g(x, x-y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} b(x,\xi) d\xi$$

i.e. $K = I(b)$. This certainly proves the proposition and actually gives the stronger result.

4.127 (2.66)
$$A \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \Longleftrightarrow A = I(c), \ c \in S_\infty^{-\infty}(\mathbb{R}_x^n; \mathbb{R}_\xi^n).$$

□

This also finishes the proof of Proposition 2.2 since in (2.58), (2.59) we have shown that

4.28 (2.67)
$$A = B + R, \ B = I(b), \quad R \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$$

so in fact

4.29 (2.68)
$$A = I(e), \ e \in S_\infty^m(\mathbb{R}_x^n; \mathbb{R}_\xi^n), \ e \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \left(D_y^\alpha D_\xi^\alpha a \right)(x,x,\xi).$$

□

f.of.Composition.Theorem

2.7. Proof of Composition Theorem

First consider the adjoint formula. If

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

the adjoint is the operator

$$A^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

defined by duality:

4.30 (2.69)
$$A^* u(\bar{\phi}) = u(\overline{A\phi}) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n).$$

Certainly $A^*u \in \mathcal{S}'(\mathbb{R}^n)$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ since

4.31 (2.70)
$$A^* u(\psi) = u(\overline{A\overline{\psi}}) \text{ and } \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \overline{A\overline{\psi}} \in \mathcal{S}(\mathbb{R}^n)$$

is clearly continuous. In terms of Schwartz kernels,

4.32 (2.71)
$$A\phi(x) = \int K_A(x,y)\phi(y)\,dy, \ \phi \in \mathcal{S}(\mathbb{R}^n)$$
$$A^* u(x) = \int K_{A^*}(x,y) u(y)\,dy, \ u \in \mathcal{S}(\mathbb{R}^n).$$

We then see that

4.33 (2.72)
$$\int K_{A^*}(x,y)u(y)\overline{\phi(x)}dydx = \int \overline{K_A(x,y)\phi(y)}dyu(x)dx$$
$$\Longrightarrow K_{A^*}(x,y) = \overline{K_A(y,x)}$$

where we are using the uniqueness of Schwartz' kernels.

This proves (2.41) since

4.34 (2.73)
$$\overline{K_A(y,x)} = \overline{\left[\frac{1}{(2\pi)^n}\int e^{i(y-x)\cdot\xi}a(y,x,\xi)d\xi\right]}$$
$$= \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}\bar{a}(y,x,\xi)d\xi$$

i.e. $A^* = I(\bar{a}(y,x,\xi))$. Thus one advantage of allowing general operators (2.34) is that closure under the passage to adjoint is immediate.

For the composition formula we need to apply Proposition 2.2 twice. First to $A \in \Psi^m_\infty(\mathbb{R}^n)$, to write it with symbol $a(x,\xi)$

$$A\phi(x) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)\phi(y)dyd\xi$$
$$= (2\pi)^{-n}\int e^{ix\cdot\xi}a(x,\xi)\hat{\phi}(\xi)d\xi.$$

Then we also apply Proposition 2.2 to B^*,

$$B^*u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\bar{b}(x,\xi)\hat{u}(\xi)d\xi.$$

Integrating this against a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ gives

21.2.1998.112 (2.74)
$$\langle B\phi, u\rangle = \langle \phi, B^*u\rangle = (2\pi)^{-n}\int\int e^{-ix\cdot\xi}\phi(x)b(x,\xi)\overline{\hat{u}(\xi)}d\xi dx$$
$$\Longrightarrow \widehat{B\phi}(\xi) = \int e^{-iy\cdot\xi}b(y,\xi)\phi(y)dy.$$

Inserting this into the formula for $A\phi$ shows that

$$\Longrightarrow AB(u) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)b(y,\xi)u(y)dyd\xi.$$

Since $a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi)$ this shows that $AB \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ as claimed.

Quantization.and.symbols 2.8. Quantization and symbols

So, we have now shown that there is an 'oscillatory integral' interpretation of

5.1 (2.75)
$$K(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi = I(a)$$

which defines, for any $w \in \mathbb{R}$, a continuous linear map

$$I : (1+|x-y|^2)^{\frac{w}{2}}S^\infty_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

the range of which is the space of pseudodifferential operators on $\mathbb{R}^n$;

5.2 (2.76)
$$A \in \Psi^m_\infty(\mathbb{R}^n) \Longleftrightarrow A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ and}$$
$$\exists\ w \text{ s.t. } K_A(x,y) = I(a),\ a \in \left(1+|x-y|^2\right)^{\frac{w}{2}}S^m_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right).$$

Furthermore, we have shown in Proposition 2.2 that the special case, $w = 0$ and $\partial_y a \equiv 0$, gives an isomorphism

5.3 (2.77)
$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_L}{\overset{\sigma_L}{\rightleftarrows}} S^m_\infty(\mathbb{R}^n;\mathbb{R}^n).$$

The map here, $q_L = I$ on symbols independent of y, is the *left quantization map* and its inverse σ_L is the *left full symbol map.* Next we consider some more consequences of this reduction theorem.

As well as the left quantization map leading to the isomorphism (2.77) there is a right quantization map, similarly derived from (2.75):

5.4 (2.78)
$$q_R(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(y,\xi) d\xi, \quad a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right).$$

In fact using the adjoint operator, $*$, on operators and writing as well $*$ for complex conjugation of symbols shows that

5.5 (2.79)
$$q_R = * \cdot q_L \cdot *$$

is also an isomorphism, with inverse σ_R

5.6 (2.80)
$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_R}{\overset{\sigma_R}{\rightleftarrows}} S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right).$$

These are the two 'extreme' quantization procedures, see Problem 4.1 for another (more centrist) approach. Using the proof of the reduction theorem we find:

5.7 Lemma 2.4. *For any $a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right)$,*

5.8 (2.81)
$$\sigma_L\left(q_R(a)\right)(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi a(x,\xi) \sim e^{i<D_x,D_\xi>}a.$$

For the moment the last asymptotic equality is just to help in remembering the formula, which is the same as given by the formal Taylor series expansion at the origin of the exponential.

Proof. This follows from the general formula (2.68). □

S.Principal.symbol

2.9. Principal symbol

One important thing to note from (2.81) is that

(2.82)
$$D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-|\alpha|}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right)$$

so that for *any* pseudodifferential operator

(2.83)
$$A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow \sigma_L(A) - \sigma_R(A) \in S^{m-1}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right).$$

For this reason we consider the general quotient spaces

(2.84)
$$S^{m-[1]}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right) = S^m_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right) \big/ S^{m-1}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right)$$

and, for $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, write $[a]$ for its image, i.e. equivalence class, in the quotient space $S^{m-[1]}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right)$. The 'principal symbol map'

5.9 (2.85)
$$\sigma_m : \Psi^m_\infty(\mathbb{R}^n) \longrightarrow S^{m-[1]}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right)$$
is defined by $\sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)]$.

As distinct from σ_L or σ_R, σ_m depends on m, i.e. one needs to know that the order is at most m before it is defined.

The isomorphism (2.77) is replaced by a weaker (but very useful) exact sequence.

5.10 LEMMA 2.5. *For every* $m \in \mathbb{R}$

$$0 \hookrightarrow \Psi_\infty^{m-1}(\mathbb{R}^n) \hookrightarrow \Psi_\infty^m(\mathbb{R}^n) \xrightarrow{\sigma_m} S_\infty^{m-[1]}(\mathbb{R}^n; \mathbb{R}^n) \longrightarrow 0$$

is a short exact sequence (the 'principal symbol sequence' or simply the 'symbol sequence').

PROOF. This is just the statement that the range of each map is the null space of the next i.e. that σ_m is surjective, which follows from (2.77), and that the null space of σ_m is just $\Psi_\infty^{m-1}(\mathbb{R}^n)$ and this is again (2.77) and the definition of σ_m. □

The fundamental result proved above is that

5.11 (2.86) $$\Psi_\infty^m(\mathbb{R}^n) \cdot \Psi_\infty^{m'}(\mathbb{R}^n) \subset \Psi_\infty^{m+m'}(\mathbb{R}^n).$$

In fact we showed that if $A = q_L(a)$, $a \in S_\infty^m(\mathbb{R}^n; \mathbb{R}^n)$ and $B = q_R(b)$, $b \in S_\infty^{m'}(\mathbb{R}^n; \mathbb{R}^n)$ then the composite operator has Schwartz kernel

$$K_{A \cdot B}(x, y) = I\left(a(x, \xi) b(y, \xi)\right)$$

Using the formula (2.68) again we see that

5.12 (2.87) $$\sigma_L(A \cdot B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha \big[a(x, \xi) D_x^\alpha b(x, \xi)\big].$$

Of course $b = \sigma_R(B)$ so we really want to rewrite (2.87) in terms of $\sigma_L(B)$.

5.13 LEMMA 2.6. *If* $A \in \Psi_\infty^m(\mathbb{R}^n)$ *and* $B \in \Psi_\infty^{m'}(\mathbb{R}^n)$ *then* $A \circ B \in \Psi_\infty^{m+m'}(\mathbb{R}^n)$ *and*

5.14 (2.88) $$\sigma_{m+m'}(A \circ B) = \sigma_m(A) \cdot \sigma_{m'}(B),$$

5.15 (2.89) $$\sigma_L(A \circ B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha \sigma_L(A) \cdot D_x^\alpha \sigma_L(B).$$

PROOF. The simple formula (2.88) is already immediate from (2.87) since all terms with $|\alpha| \geq 1$ are of order $m + m' - |\alpha| \leq m + m' - 1$. To get the 'full' formula (2.89) we can insert into (2.87) the inverse of (2.81), namely

$$\sigma_R(x, \xi) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \sigma_L(x, \xi) \sim e^{-i<D_x, D_\xi>} \sigma_L(x, \xi).$$

This gives the double sum (still asymptotically convergent)

$$\sigma_L(A \circ B) \sim \sum_\beta \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha \big[\sigma_L(A) D_x^\alpha \frac{i^{|\beta|}}{\beta!} D_x^\beta D_\xi^\beta \sigma_L(B)\big].$$

Setting $\gamma = \alpha + \beta$ this becomes

$$\sigma_L(A \circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} \sum_{0 \leq \alpha \leq \gamma} \frac{\gamma!(-1)^{|\gamma - \alpha|}}{\alpha!(\gamma - \alpha)!} D_\xi^\alpha \big[\sigma_L(A) \times D_\xi^{\gamma - \alpha} D_x^\gamma \sigma_L(B)\big].$$

Then Leibniz' formula shows that this sum over α can be rewritten as

$$\sigma_L(A \circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} D_\xi^\gamma \sigma_L(A) \cdot D_x^\gamma \sigma_L(B)$$

$$\sim e^{i<D_y, D_\xi>} \sigma_L(A)(x, \xi) \sigma_L(B)(y, \eta)\big|_{y=x, \eta=\xi}.$$

This is just (2.89). □

The simplicity of (2.88) over (2.89) is achieved at the expense of enormous loss of information. Still, many problems can be solved using (2.88) which we can think of as saying that the principal symbol maps give a homomorphism, for instance from the filtered algebra $\Psi^0_\infty(\mathbb{R}^n)$ to the *commutative* algebra $S^{0-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)$.

S.Ellipticity

2.10. Ellipticity

We say that an element of $\Psi^m_\infty(\mathbb{R}^n)$ is *elliptic* if it is invertible modulo an error in $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ with the approximate inverse of order $-m$ i.e.

5.16 (2.90)
$$\begin{gathered} A \in \Psi^m_\infty(\mathbb{R}^n) \text{ is elliptic} \\ \Longleftrightarrow \exists \quad B \in \Psi^{-m}_\infty(\mathbb{R}^n) \text{ s.t. } \quad A \circ B - \operatorname{Id} \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \end{gathered} \tag{2.90}$$

Thus ellipticity, here by definition, is invertibility in $\Psi^m_\infty(\mathbb{R}^n)\big/\Psi^{-\infty}_\infty(\mathbb{R}^n)$, so the inverse lies in $\Psi^{-m}_\infty(\mathbb{R}^n)\big/\Psi^{-\infty}_\infty(\mathbb{R}^n)$. The point about ellipticity is that it is a phenomenon of the *principal symbol.*

5.17 THEOREM 2.3. *The following conditions on $A \in \Psi^m_\infty(\mathbb{R}^n)$ are equivalent*

5.18 (2.91)
$$A \textit{ is elliptic} \tag{2.91}$$

5.19 (2.92)
$$\exists\ [b] \in S^{-m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n) \ \textit{ s.t. } \sigma_m(A)\cdot[b] \equiv 1 \textit{ in } S^{0-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n) \tag{2.92}$$

5.20 (2.93)
$$\exists\ b \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n) \ \textit{ s.t. } \sigma_L(A)\cdot b - 1 \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n) \tag{2.93}$$

5.21 (2.94)
$$\exists\ \epsilon > 0 \textit{ s.t. } \big|\sigma_L(A)(x,\xi)\big| \ge \epsilon(1+|\xi|)^m \textit{ in } |\xi| > \frac{1}{\epsilon}. \tag{2.94}$$

PROOF. We shall show

5.22 (2.95)
$$(2.91) \Longrightarrow (2.92) \Longrightarrow (2.93) \Longleftrightarrow (2.94) \Longrightarrow (2.91). \tag{2.95}$$

In fact Lemma 2.2 shows the equivalence of (2.93) and (2.94). Since we know that $\sigma_0(\operatorname{Id}) = 1$ applying the identity (2.88) to the definition of ellipticity in (2.90) gives

5.44 (2.96)
$$\sigma_m(A)\cdot\sigma_{-m}(B) \equiv 1 \text{ in } S^{0-[1]}_\infty(\mathbb{R}^n,\mathbb{R}^n), \tag{2.96}$$

i.e. that (2.91) $\Longrightarrow$ (2.92).

Now assuming (2.96) (i.e. (2.92)), and recalling that $\sigma_m(A) = [\sigma_L(A)]$ we find that a representative b_1 of the class $[b]$ must satisfy

5.23 (2.97)
$$\sigma_L(A)\cdot b_1 = 1 + e_1, \quad e_1 \in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n), \tag{2.97}$$

this being the meaning of the equality of residue classes. Now for the remainder, $e_1 \in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$, the Neumann series

$$f \sim \sum_{j\ge 1}(-1)^j e_1^j \tag{2.98}$$

is asymptotically convergent, so $f \in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ exists, and

5.24 (2.99)
$$(1+f)\cdot(1+e_1) = 1 + e_\infty, \quad e_\infty \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.99}$$

Then multiplying (2.97) by $(1+f)$ gives

5.25 (2.100)
$$\sigma_L(A)\cdot\{b_1(1+f)\} = 1 + e_\infty \tag{2.100}$$

which proves (2.93), since $b = b_1(1+f) \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Of course

$$\sup(1+|\xi|)^N|e_\infty| < \infty \quad \forall\ N \tag{2.101}$$

so

$$\exists\ C \text{ s.t. } \quad |e_\infty(x,\xi)| < \frac{1}{2} \text{ in } |\xi| > C. \tag{2.102}$$

From (2.100) this means

5.26 $$\big|\sigma_L(A)(x,\xi)\big| \cdot \big|b(x,\xi)\big| \geq \frac{1}{2},\ |\xi| > C. \tag{2.103}$$

Since $|b(x,\xi)| \leq C(1+|\xi|)^{-m}$ (being a symbol of order $-m$), (2.103) implies

5.27 $$\inf_{|\xi|\geq C} \big|\sigma_L(A)(x,\xi)\big|(1+|\xi|)^{-m} \geq C > 0. \tag{2.104}$$

which shows that (2.93) implies (2.94).

Conversely, as already remarked, (2.94) implies (2.93).

Now suppose (2.93) holds. Set $B_1 = q_L(b)$ then from (2.88) again

$$\sigma_0(A \circ B_1) = [q_m(A)] \cdot [b] \equiv 1. \tag{2.105}$$

That is,

5.29 $$A \circ B_1 - \mathrm{Id} = E_1 \in \Psi^{-1}_\infty(\mathbb{R}^n). \tag{2.106}$$

Consider the Neumann series of operators

$$\sum_{j\geq 1} (-1)^j E_1^j. \tag{2.107}$$

The corresponding series of (left-reduced) symbols is asymptotically summable so we can choose $F \in \Psi^{-1}_\infty(\mathbb{R}^n)$ with

5.45 $$\sigma_L(F) \sim \sum_{j\geq 1} (-1)^j \sigma_L(E_1^j). \tag{2.108}$$

Then

5.30 $$(\mathrm{Id} + E_1)(\mathrm{Id} + F) = \mathrm{Id} + E_\infty,\ E_\infty \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.109}$$

Thus $B = B_1(\mathrm{Id} + F) \in \Psi^{-m}_\infty(\mathbb{R}^n)$ satisfies (2.90) and it follows that A is elliptic. □

In the definition of ellipticity in (2.90) we have taken B to be a 'right parametrix', i.e. a right inverse modulo $\Psi^{-\infty}_\infty(\mathbb{R}^n)$. We can just as well take it to be a *left* parametrix.

5.31 Lemma 2.7. *$A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic if and only if there exists $B' \in \Psi^{-m}_\infty(\mathbb{R}^n)$ such that*

5.32 $$B' \circ A = \mathrm{Id} + E',\ E' \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \tag{2.110}$$

and then if B satisfies (2.90), $B - B' \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$.

Proof. Certainly (2.110) implies $\sigma_{-m}(B') \cdot \sigma_m(A) \equiv 1$, and the multiplication here is commutative so (2.92) holds and A is elliptic. Conversely if A is elliptic we get in place of (2.106)

$$B_1 \circ A - \mathrm{Id} = E_1' \in \Psi^{-1}_\infty(\mathbb{R}^n).$$

Then defining F' as in (2.108) with E_1' in place of E_1 we get $(\mathrm{Id} + F')(\mathrm{Id} + E_1') = \mathrm{Id} + E_\infty'$ and then $B' = (\mathrm{Id} + F') \circ B_1$ satisfies (2.110). Thus 'left' ellipticity as in (2.110) is equivalent to right ellipticity. Applying B to (2.110) gives

5.33 $$B' \circ (\mathrm{Id} + E) = B' \circ (A \circ B) = (\mathrm{Id} + E') \circ B \tag{2.111}$$

which shows that $B - B' \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n)$. □

Thus a left parametrix of an elliptic element of $\Psi^m_\infty(\mathbb{R}^n)$ is always a right, hence two-sided, parametrix and such a parametrix is unique up to an additive term in $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$.

S.Elliptic.regularity 2.11. Elliptic regularity

One of the main reasons that the 'residual' terms *are* residual is that they are smoothing operators.

5.42 Lemma 2.8. *If $E \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n)$ then*

5.43 (2.112)
$$E : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n).$$

Proof. This follows from Proposition 2.4 since we can regard the kernel as a $\mathcal{C}^\infty$ function of x taking values in $\mathcal{S}(\mathbb{R}^n_y)$. □

Directly from the existence of parametrices for elliptic operators we can deduce the regularity of solutions to elliptic (pseudodifferential) equations.

20.2.1998.103 Proposition 2.5. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic and $u \in \mathcal{S}'(\mathbb{R}^n)$ satifies $Au = 0$ then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.*

Proof. Let $B \in \Psi^{-m}_\infty(\mathbb{R}^n)$ be a parametrix for A. Then $B \circ A = \operatorname{Id} + E$, $E \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Thus,

5.41 (2.113)
$$u = (BA - E)u = -Eu$$

and the conclusion follows from Lemma 2.8. □

S.The.Laplacian 2.12. The Laplacian

Suppose that $g_{ij}(x)$ are the components of an '∞-metric' on $\mathbb{R}^n$, i.e.

$$g_{ij}(x) \in \mathcal{C}^\infty_\infty(\mathbb{R}^n), i, j = 1, \dots, n$$

5.34 (2.114)
$$\Big|\sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j\Big| \ge \epsilon|\xi|^2 \quad \forall\ x \in \mathbb{R}^n, \xi \in \mathbb{R}^n,\ \epsilon > 0.$$

The *Laplacian* of the metric is the second order differential operator

5.35 (2.115)
$$\Delta_g = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} D_{x_i} g^{ij} \sqrt{g} D_{x_j}$$

where

$$g(x) = \det g^{ij}(x),\ g^{ij}(x) = \left(g_{ij}(x)\right)^{-1}.$$

The Laplacian is determined by the integration by parts formula

5.37 (2.116)
$$\int_{\mathbb{R}^n} \sum_{i,j} g^{ij}(x) D_{x_i}\phi \cdot \overline{D_{x_j}\psi} dg = \int \Delta_g \phi \cdot \overline{\psi} dg \ \forall\ \phi, \psi \in \mathcal{S}(\mathbb{R}^n)$$

where

5.38 (2.117)
$$dg = \sqrt{g} dx.$$

Our assumption in (2.114) shows that $\Delta = \varDelta_g \in \mathrm{Diff}^2_\infty(\mathbb{R}^n) \subset \Psi^2_\infty(\mathbb{R}^n)$ is in fact elliptic, since

5.39 (2.118)
$$\sigma_2(\Delta) = \sum_{i,j=1} g^{ij}\xi_i\xi_j.$$

Thus Δ has a two-sided parametrix $B \in \Psi^{-2}_\infty(\mathbb{R}^n)$

5.40 (2.119)
$$\Delta \circ B \equiv B \circ \Delta \equiv \mathrm{Id} \quad \mod \quad \Psi^{-\infty}_\infty(\mathbb{R}^n).$$

In particular we see from Proposition 2.5 that $\Delta u = 0$, $u \in \mathcal{S}'(\mathbb{R}^n)$ implies $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

S.L2.boundedness

2.13. L^2 boundedness

So far we have thought of pseudodifferential operators, the elements of $\Psi^m_\infty(\mathbb{R}^n)$ for some m, as defining continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$ and, by duality, on $\mathcal{S}'(\mathbb{R}^n)$. Now that we have proved the composition formula we can use it to prove other 'finite order' regularity results. The basic one of these is L^2 boundedness:

6.1 PROPOSITION 2.6. *[Boundedness] If $A \in \Psi^0_\infty(\mathbb{R}^n)$ then, by continuity from $\mathcal{S}(\mathbb{R}^n)$, A defines a bounded linear operator*

6.2 (2.120)
$$A : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

Our proof will be in two stages, the first part is by direct estimation. Namely, *Schur's lemma*[4] gives a useful criterion for an integral operator to be bounded on L^2.

5.50 LEMMA 2.9 (Schur). *If $K(x,y)$ is locally integrable on $\mathbb{R}^{2n}$ and is such that*

5.51 (2.121)
$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|dy,\ \sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|dx < \infty$$

then the operator $K : \phi \longmapsto \int_{\mathbb{R}^n} K(x,y)\phi(y)dy$ is bounded on $L^2(\mathbb{R}^n)$.

PROOF. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (see Problem 2.18) we only need to show the existence of a constant, C, such that

6.4 (2.122)
$$\int \left|K\phi(x)\right|^2 dx \le C\int |\phi|^2 \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

Writing out the integral on the left

6.5 (2.123)
$$\begin{aligned}&\int\Big|\int K(x,y)\phi(y)dy\Big|^2 dx\\ &= \iiint K(x,y)\overline{K(x,z)}\phi(y)\overline{\phi(z)}\ dydzdx\end{aligned}$$

is certainly absolutely convergent and

$$\begin{aligned}&\int \left|K\phi(x)\right|^2 dx\\ &\le \left(\iiint \left|K(x,y)K(x,z)\right|\left|\phi(y)\right|^2 dydxdz\right)^{\frac12}\\ &\times\left(\iiint \left|K(x,y)K(x,z)\right|\left|\phi(z)\right|^2 dzdxdy\right)^{\frac12}.\end{aligned}$$

[4]Schur

These two factors are the same. Since

$$\int \big|K(x,y)\big|\big|K(x,z)\big|\,dx\,dz \le \sup_{x\in\mathbb{R}^n}\int \big|K(x,z)\big|dz \cdot \sup_{y\in\mathbb{R}^n}\int \big|K(x,y\big|dx$$

(2.122) follows. Thus (2.121) gives (2.122). □

This standard lemma immediately implies the L^2 boundedness of the 'residual terms.' Thus, if $K \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ then its kernel satisfies (2.60). This in particular implies

$$\big|K(x,y)\big| \le C\,(1+|x-y|)^{-n-1}$$

and hence that K satisfies (2.121). Thus

5.52 (2.124) $$\text{each } K \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \text{ is bounded on } L^2(\mathbb{R}^n).$$

S.Square.root

2.14. Square root

To prove the general result, (2.120), we shall use the clever idea, due to Hörmander, of using the (approximate) square root of an operator. We shall say that an element $[a] \in S_\infty^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)$ is positive if there is some $0 < a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ in the equivalence class.

6.6 Proposition 2.7. *Suppose $A \in \Psi_\infty^m(\mathbb{R}^n)$, $m > 0$, is self-adjoint, $A = A^*$, and elliptic with a positive principal symbol, then there exists $B \in \Psi_\infty^{m/2}(\mathbb{R}^n)$, $B = B^*$, such that*

6.7 (2.125) $$A = B^2 + G, \quad G \in \Psi_\infty^{-\infty}(\mathbb{R}^n).$$

Proof. This is a good exercise in the use of the symbol calculus. Let $a \in S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)$, $a > 0$, be a positive representative of the principal symbol of A. Now (See Problem 2.19 for an outline of the proof)

6.8 (2.126) $$b_0 = a^{\frac{1}{2}} \in S_\infty^{m/2}(\mathbb{R}^n;\mathbb{R}^n).$$

Let $B_0 \in \Psi_\infty^{m/2}(\mathbb{R}^n)$ have principal symbol b_0. We can assume that $B_0 = B_0^*$, since if not we just replace B_0 by $\frac{1}{2}(B_0 + B_0^*)$ which has the same principal symbol.

The symbol calculus shows that $B_0^2 \in \Psi_\infty^m(\mathbb{R}^n)$ and

$$\sigma_m(B_0^2) = \big(\sigma_{m/2}(B_0)\big)^2 = b_0^2 = a_0 \mod S_\infty^{m-1}.$$

Thus

6.9 (2.127) $$A - B_0^2 = E_1 \in \Psi_\infty^{m-1}(\mathbb{R}^n).$$

Then we proceed inductively. Suppose we have chosen $B_j \in \Psi_\infty^{m/2-j}(\mathbb{R}^n)$, with $B_j^* = B_j$, for $j \le N$ such that

6.10 (2.128) $$A - \left(\sum_{j=0}^{N} B_j\right)^2 = E_{N+1} \in \Psi_\infty^{m-N-1}(\mathbb{R}^n).$$

Of course we *have* done this for $N = 0$. Then see the effect of adding $B_{N+1} \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$:

6.11
$$A - \left(\sum_{j=0}^{N+1} B_j\right)^2 = E_{N+1} - \left(\sum_{j=0}^{N} B_j\right) B_{N+1} - B_{N+1}\left(\sum_{j=0}^{N} B_j\right) - B_{N+1}^2. \tag{2.129}$$

On the right side all terms are of order $m - N - 2$, except for

6.12
$$E_{N+1} - B_0 B_{N+1} - B_{N+1} B_0 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n). \tag{2.130}$$

The principal symbol, of order $m - N - 1$, of this is just

6.13
$$\sigma_{m-N-1}(E_{N+1}) - 2\ b_0 \cdot \sigma_{\frac{m}{2}-N-1}(B_{N+1}). \tag{2.131}$$

Thus if we choose

$$\sigma_{m/2-N-1}(B_{N+1}) = \frac{1}{2}\frac{1}{b_0} \cdot \sigma_{m-N-1}(E_{N+1})$$

and replace B_{N+1} by $\frac{1}{2}(B_{N+1} + B_{N+1}^*)$, we get the inductive hypothesis for $N+1$. Thus we have arranged (2.128) for every N. Now define $B = \frac{1}{2}(B' + (B')^*)$ where

6.14
$$\sigma_L(B') \sim \sum_{j=0}^{\infty} \sigma_L(B_j). \tag{2.132}$$

Since all the B_j are self-adjoint B also satisfies (2.132) and from (2.128)

$$A - B^2 = A - \left(\sum_{j=0}^{N} B_j + B_{(N+1)}\right)^2 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n) \tag{2.133}$$

for every N, since $B_{(N+1)} = B - \sum_{j=0}^{N} B_j \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$. Thus $A - B^2 \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ and we have proved (2.125), and so Proposition 2.7. □

S.Proof.of.Boundedness

2.15. Proof of Boundedness

Here is Hörmander's argument. We want to show that

6.15
$$\|A\phi\| \le C\|\phi\| \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{2.134}$$

where $A \in \Psi_\infty^0(\mathbb{R}^n)$. The square of the left side can be written

$$\int A\phi \cdot \overline{A\phi} dx = \int \phi \cdot \overline{(A^*A\phi)} dx.$$

So it suffices to show that

6.16
$$\langle \phi, A^*A\phi \rangle \le C\|\phi\|^2. \tag{2.135}$$

Now $A^*A \in \Psi_\infty^0(\mathbb{R}^n)$ with $\sigma_0(A^*A) = \overline{\sigma_0(A)}\sigma_0(A) \in \mathbb{R}$. If $C > 0$ is a large constant,

$$C > \sup_{x,\xi} \left|\sigma_L(A^*A)(x,\xi)\right|$$

then $C - A^*A$ has a positive representative of its principal symbol. We can therefore apply Proposition 2.7 to it:

6.17
$$C - A^*A = B^*B + G, \quad G \in \Psi_\infty^{-\infty}(\mathbb{R}^n). \tag{2.136}$$

This gives

6.18
$$\begin{aligned}\langle \phi, A^*A\phi\rangle =& C\langle \phi, \phi\rangle - \langle \phi, B^*B\phi\rangle - \langle \phi, G\phi\rangle\\ =& C\|\phi\|^2 - \|B\phi\|^2 - \langle \phi, G\phi\rangle.\end{aligned} \tag{2.137}$$

The second term on the right is negative and, since $G \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$, we can use the residual case discussed above to conclude that

$$\big|\langle \phi, G\phi\rangle\big| \le C'\|\phi\|^2 \Longrightarrow \|A\phi\|^2 \le C\|\phi\|^2 + C'\|\phi\|^2,$$

so (2.120) holds and Proposition 2.6 is proved.

S.Sobolev.boundedness

2.16. Sobolev boundedness

Using the basic boundedness result, Proposition 2.6, and the calculus of pseudodifferential operators we can prove more general results on the action of pseudodifferential operators on Sobolev spaces.

Recall that for any positive integer, k,

6.19
$$H^k(\mathbb{R}^n) = \big\{u \in L^2(\mathbb{R}^n);\ D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \le k\big\}. \tag{2.138}$$

Using the Fourier transform we find

6.20
$$u \in H^k(\mathbb{R}^n) \Longrightarrow \xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n) \quad \forall\ |\alpha| \le k. \tag{2.139}$$

Now these finitely many conditions can be written as just the one condition

6.21
$$\big(1 + |\xi|^2\big)^{k/2}\, \hat{u}(\xi) \in L^2(\mathbb{R}^n). \tag{2.140}$$

Notice that $a(\xi) = (1 + |\xi|^2)^{k/2} = \langle\xi\rangle^k \in S^k_\infty(\mathbb{R}^n)$. Here we use the notation

6.22
$$\langle\xi\rangle = \big(1 + |\xi|^2\big)^{\frac{1}{2}} \tag{2.141}$$

for a smooth (symbol) of the size of $1 + |\xi|$, thus (2.140) just says

6.23
$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n). \tag{2.142}$$

For *negative* integers

6.24
$$H^k(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{|\beta| \le -k} D^\beta u_\beta,\ u_\beta \in L^2(\mathbb{R}^n)\big\},\ -k \in \mathbb{N}. \tag{2.143}$$

The same sort of discussion applies, showing that

6.25
$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n). \tag{2.144}$$

In view of this we define the Sobolev space $H^m(\mathbb{R}^n)$, for any real order, by

6.26
$$u \in H^m, (\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^m u \in L^2(\mathbb{R}^n). \tag{2.145}$$

It is a Hilbert space with

6.27
$$\|u\|_m^2 = \|\langle D\rangle^m u\|^2_{L^2} = \int (1 + |\xi|^2)^m |\hat{u}(\xi)|^2 d\xi. \tag{2.146}$$

Clearly we have

6.28
$$H^m(\mathbb{R}^n) \supseteq H^{m'}(\mathbb{R}^n) \text{ if } m' \ge m. \tag{2.147}$$

Notice that it is rather unfortunate that these spaces get *smaller* as m gets bigger, as opposed to the spaces $\Psi^m_\infty(\mathbb{R}^n)$ which get *bigger* with m. Anyway that's life and we have to think of

6.29 (2.148) $$\begin{cases} H^\infty(\mathbb{R}^n) = \bigcap_m H^m(\mathbb{R}^n) & \text{as the residual space} \\ H^{-\infty}(\mathbb{R}^n) = \bigcup_m H^m(\mathbb{R}^n) & \text{as the big space.} \end{cases}$$

It is important to note that

6.30 (2.149) $$\mathcal{S}(\mathbb{R}^n) \subsetneq H^\infty(\mathbb{R}^n) \subsetneq H^{-\infty}(\mathbb{R}^n) \subsetneq \mathcal{S}'(\mathbb{R}^n).$$

In particular we do *not* capture all the tempered distributions in $H^{-\infty}(\mathbb{R}^n)$. We therefore consider *weighted* versions of these Sobolev spaces:

6.31 (2.150) $$\langle x\rangle^q H^m(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle x\rangle^{-q} u \in H^m(\mathbb{R}^n)\big\}.$$

6.32 THEOREM 2.4. *For each $q, m, M \in \mathbb{R}$ each $A \in \Psi^M_\infty(\mathbb{R}^n)$ defines a continuous linear map*

6.33 (2.151) $$A : \langle x\rangle^q H^m(\mathbb{R}^n) \longrightarrow \langle x\rangle^q H^{m-M}(\mathbb{R}^n).$$

PROOF. Let us start off with $q = 0$, so we want to show that

6.34 (2.152) $$A : H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n),\ A \in \Psi^M_\infty(\mathbb{R}^n)$$

Now from (2.145) we see that

6.35 (2.153) $$u \in H^m(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^m u \in L^2(\mathbb{R}^n) \\ \Longleftrightarrow \langle D\rangle^{m-M}\langle D\rangle^M u \in L^2(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^M u \in H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M.$$

That is,

6.36 (2.154) $$\langle D\rangle^M : H^m(\mathbb{R}^n) \longleftrightarrow H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M.$$

To prove (2.152) it suffices to show that

6.37 (2.155) $$B = \langle D\rangle^{-M+m} \cdot A \cdot \langle D\rangle^{-m} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

since then $A = \langle D\rangle^{-m+M} \cdot B \cdot \langle D\rangle^m$ maps $H^m(\mathbb{R}^n)$ to $H^{m-M}(\mathbb{R}^n)$:

6.38 (2.156) $$\begin{array}{ccc} H^m(\mathbb{R}^n) & \xrightarrow{A} & H^{m-M}(\mathbb{R}^n) \\ {\scriptstyle \langle D\rangle^m}\downarrow & & \downarrow{\scriptstyle \langle D\rangle^{m-M}} \\ L^2(\mathbb{R}^n) & \xrightarrow[B]{} & L^2(\mathbb{R}^n). \end{array}$$

Since $B \in \Psi^0_\infty(\mathbb{R}^n)$, by the composition theorem, we already know (2.155).

Thus we have proved (2.152). To prove the general case, (2.151), we proceed in the same spirit. Thus $\langle x\rangle^q$ is an isomorphism from $H^m(\mathbb{R}^n)$ to $\langle x\rangle^q H^m(\mathbb{R}^n)$, by definition. So to get (2.151) we need to show that

6.39 (2.157) $$Q = \langle x\rangle^{-q} \cdot A \cdot \langle x\rangle^q :\ H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n),$$

i.e. satisfies (2.152). Consider the Schwartz kernel of Q. Writing A in left-reduced form, with symbol a,

6.40 (2.158) $$K_Q(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \langle x\rangle^{-q} a(x,\xi) d\xi \cdot \langle y\rangle^q.$$

Now if we check that

6.41
$$\langle x\rangle^{-q}\langle y\rangle^{q}a(x,\xi)\in\big(1+|x-y|^2\big)^{\frac{|q|}{2}}\,S^{M}_{\infty}\left(\mathbb{R}^{2n};\mathbb{R}^n\right) \tag{2.159}$$

then we know that $Q\in\Psi^M_\infty(\mathbb{R}^n)$ and we get (2.157) from (2.152). Thus we want to show that

6.42
$$\langle x-y\rangle^{-|q|}\frac{\langle y\rangle^q}{\langle x\rangle^q}a(x,\xi)\in S^M_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \tag{2.160}$$

assuming of course that $a(x,\xi)\in S^M_\infty(\mathbb{R}^n;\mathbb{R}^n)$. By interchanging the variables x and y if necessary we can assume that $q<0$. Consider separately the two regions

6.43
$$\begin{gathered}\big\{(x,y);|x-y|<\frac{1}{4}(|x|+|y|)\big\}=\Omega_1\\ \big\{(x,y);|x-y|>\frac{1}{8}(|x|+|y|)\big\}=\Omega_2.\end{gathered} \tag{2.161}$$

In Ω_1, x is "close" to y, in the sense that

$$|x|\le|x-y|+|y|\le\frac{1}{4}(|x|+|y|)+|y|\Longrightarrow|x|\le\frac{4}{3}\cdot\frac{5}{4}|y|\le 2|y|. \tag{2.162}$$

Thus

6.44
$$\langle x-y\rangle^{-q}\cdot\frac{\langle x\rangle^{-q}}{\langle y\rangle^{-q}}\le C\ \text{ in }\Omega_1. \tag{2.163}$$

On the other hand in Ω_2,

$$|x|+|y|<8|x-y|\Longrightarrow|x|<8|x-y| \tag{2.164}$$

so again

6.45
$$\langle x-y\rangle^{-q}\frac{\langle x\rangle^{-q}}{\langle y\rangle^{-q}}\le C. \tag{2.165}$$

In fact we easily conclude that

6.46
$$\langle x-y\rangle^{-q}\frac{\langle y\rangle^q}{\langle x\rangle^q}\in\mathcal{C}^\infty_\infty(\mathbb{R}^n)\qquad\forall\ q, \tag{2.166}$$

since differentiation by x or y makes all terms "smaller". This proves (2.160), hence (2.159) and (2.157) and therefore (2.151), i.e. the theorem is proved. □

S.Consequences

2.17. Consequences

We can capture any tempered distribution in a weighted Sobolev space; this is really Schwartz' representation theorem which says that any $u\in\mathcal{S}'(\mathbb{R}^n)$ is of the form

$$u=\sum_{\text{finite}}x^\alpha D^\beta_x u_{\alpha\beta},\ u_{\alpha\beta}\text{ bounded and continuous.} \tag{2.167}$$

Clearly $\mathcal{C}^0_\infty(\mathbb{R}^n)\subset\langle x\rangle^{1+n}L^2(\mathbb{R}^n)$. Thus as a special case of Theorem 2.4,

$$D^\beta_x:\langle x\rangle^{1+n}L^2(\mathbb{R}^n)\longrightarrow\langle x\rangle^{1+n}H^{-|\beta|}(\mathbb{R}^n)$$

so

6.47 Lemma 2.10.

$$\mathcal{S}'(\mathbb{R}^n)=\bigcup_M\langle x\rangle^M H^{-M}(\mathbb{R}^n). \tag{2.168}$$

The elliptic regularity result we found before can now be refined:

6.48 PROPOSITION 2.8. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic then*

6.49
$$
\begin{aligned}
&Au \in \langle x\rangle^p H^q(\mathbb{R}^n),\ u \in \langle x\rangle^{p'} H^{q'}(\mathbb{R}^n)\\
&\Longrightarrow u \in \langle x\rangle^{p''} H^{q''}(\mathbb{R}^n),\ p'' = \max(p,p'),\ q'' = \max(q+m,q').
\end{aligned}
\tag{2.169}
$$

PROOF. The existence of a left parametrix for A, $B \in \Psi^{-m}_\infty(\mathbb{R}^n)$,

$$B \cdot A = \mathrm{Id} + G,\ G \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$$

means that

6.50
$$u = B(Au) + Gu \in \langle x\rangle^p H^{q+m}(\mathbb{R}^n) + \langle x\rangle^{p'} H^\infty(\mathbb{R}^n) \subset \langle x\rangle^{p''} H^{q+m}(\mathbb{R}^n). \tag{2.170}$$

□

S.Polyhomogeneity

2.18. Polyhomogeneity

So far we have been considering operators $A \in \Psi^m_\infty(\mathbb{R}^n)$ which correspond, via (2.2), to amplitudes satisfying the symbol estimates (2.6), i.e., in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. As already remarked, there are many variants of these estimates and corresponding spaces of pseudodifferential operators. Some *weakening* of the estimates is discussed in the problems below, starting with Problem 2.16. Here we consider a restriction of the spaces, in that we define

eq:P.1
$$S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}) \subset S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\,. \tag{2.171}$$

The definition of the subspace (2.171) is straightforward. First we note that if $a \in \mathcal{C}^\infty(\mathbb{R}^p;\mathbb{R}^n)$ is homogeneous of degree $m \in \mathbb{R}$ in $|\xi| \ge 1$, then

eq:P.2
$$a(z,t\xi) = t^m a(z,\xi),\ |t|,|\xi| \ge 1. \tag{2.172}$$

If it also satisfies the uniform regularity estimates

eq:P.3
$$\sup_{z\in\mathbb{R}^n,\ |\xi|\le 2} |D^\alpha_z D^\beta_\xi a(z,\xi)| < \infty\ \forall\ \alpha,\beta\,, \tag{2.173}$$

then in fact

eq:P.4
$$a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\,. \tag{2.174}$$

Indeed, (2.173) is exactly the restriction of the symbol estimates to $z \in \mathbb{R}^p$, $|\xi| \le 2$. On the other hand, in $|\xi| \ge 1$, $a(z,\xi)$ is homogeneous so

$$|D^\alpha_z D^\beta_\xi a(z,\xi)| = |\xi|^{m-|\beta|} |D^\alpha_z D^\beta_\xi a(z,\hat\xi)|\,, \quad \hat\xi = \frac{\xi}{|\xi|}$$

from which the symbol estimates follow.

DEFINITION 2.2. *For any $m \in \mathbb{R}$, the subspace of (one-step)*[5] *polyhomogeneous symbols is defined as a subset (2.171) by the requirement that $a \in S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ if and only if there exist elements $a_{m-j}(z,\xi) \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ which are homogeneous of degree $m-j$ in $|\xi| \ge 1$, for $j \in \mathbb{N}_0$, such that*

eq:P.5
$$a \sim \sum_j a_{m-j}\,. \tag{2.175}$$

[5]For a somewhat more general class of polyhomogeneous symbols, see problem 2.8.

Clearly

eq:P.6
$$S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)\cdot S^{m'}_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)\subset S^{m+m'}_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n),\tag{2.176}$$

since the asymptotic expansion of the product is given by the formal product of the asymtotic expansion. In fact there is equality here, because

eq:P.7
$$(1+|\xi|^2)^{m/2}\in S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)\tag{2.177}$$

and multiplication by $(1+|\xi|^2)^{m/2}$ is an isomorphism of the space $S^0_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ onto $S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$. Furthermore differentiation with respect to z_j or ξ_l preserves asymptotic homogeneity so

$$\begin{matrix} D_{x_j}:S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)\longrightarrow S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)\\ D_{\xi_l}:S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)\longrightarrow S^{m-1}_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)\end{matrix}\quad \forall j=1,\dots,n.$$

It is therefore no surprise that the polyhomogeneous operators form a subalgebra.

Proposition 2.9. *The spaces $\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)\subset\Psi^m_\infty(\mathbb{R}^n)$ defined by the condition that the kernel of $A\in\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ should be of the form $I(a)$ for some*

eq:P.9
$$a\in(1+|x-y|^2)^{w/2}S^m_{\mathrm{ph}}(\mathbb{R}^{2n};\mathbb{R}^n)\,,\tag{2.178}$$

form an order-filtered $*$-algebra.

Proof. Since the definition shows that

$$\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)\subset\Psi^m_\infty(\mathbb{R}^n)$$

we know already that

$$\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)\cdot\Psi^{m'}_{\mathrm{ph}}(\mathbb{R}^n)\subset\Psi^{m+m'}_\infty(\mathbb{R}^n)\,.$$

To see that products are polyhomogeneous it suffices to use (2.176) and (2.178) which together show that the asymptotic formulæ describing the left symbols of $A\in\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ and $B\in\Psi^{m'}_{\mathrm{ph}}(\mathbb{R}^m)$, e.g.

$$\sigma_L(A)\sim\sum_\alpha\frac{i^{|\alpha|}}{\alpha!}D^\alpha_\xi D^\alpha_y a(x,y,\xi)|_{y=x}$$

imply that $\sigma_L(A)\in S^m_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$, $\sigma_L(B)\in S^m_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$. Then the asymptotic formula for the product shows that $\sigma_L(A\cdot B)\in S^{m+m'}_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$.

The proof of $*$-invariance is similarly elementary, since if $A=I(a)$ then $A^*=I(b)$ with $b(x,y,z)=\overline{a(y,x,\xi)}\in S^m_{\mathrm{ph}}(\mathbb{R}^{2n};\mathbb{R}^n)$. □

This subalgebra is usually denoted simply $\Psi^m(\mathbb{R}^n)$ and its elements are often said to be 'classical' pseudodifferential operators. As a small exercise in the use of the principal symbol map we shall show that

eq:P.10
$$\begin{gathered} A\in\Psi^m_{\mathrm{ph}}(\mathbb{R}^n),\ A\text{ elliptic }\Longrightarrow\exists\text{ a parametrix}\\ B\in\Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n),\ A\cdot B-\mathrm{Id},\ B\cdot A-\mathrm{Id}\in\Psi^{-\infty}_\infty(\mathbb{R}^n)\,.\end{gathered}\tag{2.179}$$

In fact we already know that $B\in\Psi^{-m}_\infty(\mathbb{R}^n)$ exists with these properties, and even that it is unique modulo $\Psi^{-\infty}_\infty(\mathbb{R}^n)$. To show that $B\in\Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n)$ we can use the principal symbol map.

For elements $A \in \Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ the principal symbol $\sigma_m(A) \in S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ has a preferred class of representatives, namely the leading term in the expansion of $\sigma_L(A)$

$$\sigma_m(A) = \sigma(\xi)a_m(x,\xi) \qquad \mod S^{m-1}_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$$

where $\sigma|\xi| = 1$ in $|\xi| \geq 1$, $\sigma|\xi| = 0$ in $|\xi| \leq 1/2$. It is even natural to identify the principal symbol with $a_m(x,\xi)$ as a *homogeneous* function. Then we can see that

eq:P.11
$$A \in \Psi^m_\infty(\mathbb{R}^n),\ \sigma_m(A) \text{ homogeneous of degree m} \iff \Psi^m_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{m-1}_\infty(\mathbb{R}^n)\,. \tag{2.180}$$

Indeed, we just subtract from A an element $A_1 \in \Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ with $\sigma_m(A_1) = \sigma_m(A)$, then $\sigma_m(A - A_1) = 0$ so $A - A_1 \in \Psi^{m-1}_\infty(\mathbb{R}^n)$.

So, returning to the proof of (2.179) note straight away that

$$\sigma_{-m}(B) = \sigma_m(A)^{-1}$$

has a homogeneous representative, namely $a_m(x,\xi)^{-1}$. Thus we have shown that for $j = 1$

eq:P.12
$$B \in \Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{-m-j}_\infty(\mathbb{R}^n)\,. \tag{2.181}$$

We take (2.181) as an inductive hypthesis for general j. Writing this decomposition $B = B' + B_j$ it follows from the identity (2.179) that

$$A \cdot B = A \cdot B' + AB_j = \mathrm{Id} \quad \mod \Psi^{-\infty}_\infty(\mathbb{R}^n)$$

so

$$A \cdot B_j = \mathrm{Id} - AB' \in \Psi^0_{\mathrm{ph}}(\mathbb{R}^n) \cap \Psi^{-j}_\infty(\mathbb{R}^n) = \Psi^{-j}_{\mathrm{ph}}(\mathbb{R}^n).$$

Now applying B on the left, or using the principal symbol map, it follows that $B_j \in \Psi^{-m-j}_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{-m-j-1}_\infty(\mathbb{R}^n)$ which gives the inductive hypothesis (2.181) for $j+1$.

It is usually the case that a construction in $\Psi^*_\infty(\mathbb{R}^n)$, applied to an element of $\Psi^*_{\mathrm{ph}}(\mathbb{R}^n)$ will yield an element of $\Psi^*_{\mathrm{ph}}(\mathbb{R}^n)$ and when this is the case it can generally be confirmed by an inductive argument like that used above to check (2.179).

As a subspace[6]

$$S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \subset S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$$

is not closed. Indeed, since it contains $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$, its closure contains all of $S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ for $m' < m$. In fact it is a dense subspace.[7] To capture its properties we can strengthen the topology $S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ inherits from $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

Then, as well as the symbol norms $\|\cdot\|_{N,m}$ in (2.7) we can add norms on the terms in the expansions in (2.175)

eq:P.14
$$\|D^\alpha_x D^\beta_\xi a_{m-j}(x,\xi)\|_{L^\infty(G)},\ G = \mathbb{R}^p \times \{1 \leq |\xi| \leq 2\}\,. \tag{2.182}$$

Then we can further add the symbol norms ensuring (2.175), i.e.,

eq:P.15
$$\|a - \sum_{j=0}^k a_{m-j}\|_{m-k-1,N} \quad \forall\ k, N\,. \tag{2.183}$$

[6]Polyhomogeneous symbols may seem to be quite sophisticated objects but they are really smooth functions on manifolds with boundary; see Problems 2.8–2.7.

[7]See Problem 2.9.

Together these give a countable number of norms on $S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$. With respect to the metric topology defined as in (3.9 2.8) the spaces $S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ are then *complete*.[8]. Later when we wish to topologize $\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$, or rather related algebras, it is this type of topology we will use. Namely we identify

eq:P.16

$$\sigma_L : \Psi^m_{\mathrm{ph}}(\mathbb{R}^n) \longleftrightarrow S^m_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)\,. \tag{2.184}$$

S.Linear.invariance

2.19. Linear invariance

It is rather straightforward to see that the algebra $\Psi^\infty_\infty(\mathbb{R}^n)$ is invariant under affine transformations of $\mathbb{R}^n$. In particular if $T_a x = x + a$, for $a \in \mathbb{R}^n$, is translation by a and

$$T_a^* f(x) = f(x+a),\ T_a^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

is the isomorphism on functions then a new operator is defined by

$$T_a^* A_a f = A T_a^* f \text{ and } A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A_a \in \Psi^m_\infty(\mathbb{R}^n).$$

In fact the left-reduced symbols satisfy

$$\sigma_L(A_a)(x,\xi) = \sigma_L(A)(x+a,\xi),\ A_a = T^*_{-a} A T^*_a.$$

Similarly if $T \in \mathrm{GL}(n)$ is an invertible linear transformation of $\mathbb{R}^n$ then

21.2.1998.104

$$\begin{aligned} A_T f = T^* A (T^*)^{-1} f,\ A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A_T \in \Psi^m_\infty(\mathbb{R}^n) \\ \text{and } \sigma_L(A_T)(x,\xi) = \sigma_L(A)(Tx,(T^t)^{-1}\xi)|\det(T)| \end{aligned} \tag{2.185}$$

where T^t is the transpose of T (so $Tx\cdot\xi = x\cdot T^t\xi$) and $\det(T)$ the determinant.

This invariance means that we can define the spaces $\Psi^m_\infty(V)$ and $\Psi^m_{\mathrm{ph}}(V)$ for any vector space V (or even affine space) as operators on $\mathcal{S}(V)$. We are much more interested in full coordinate invariance which is discussed in Chapter C.Microlocalization 4.

S.Chapter.2.Problems

2.20. Problems

P3.1

Problem 2.1. Show, in detail, that for each $m \in \mathbb{R}$

$$(1+|\xi|^2)^{\frac12 m} \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \tag{2.186}$$

for any p. Use this to show that

$$S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\cdot S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) = S^{m+m'}_\infty(\mathbb{R}^p;\mathbb{R}^n).$$

P3.2

Problem 2.2. Consider $w = 0$ and $n = 2$ in the definition of symbols and show that if $a \in S^1_\infty(\mathbb{R}^2)$ is elliptic then for $r > 0$ sufficiently large the integral

$$\int_0^{2\pi} \frac{1}{2\pi}\frac{1}{a(re^{i\theta})}\frac{d}{d\theta}a(re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{d}{d\theta}\log a(re^{i\theta})d\theta,$$

exists and is an integer independent of r, where $z = \xi_1 + i\xi_2$ is the complex variable in $\mathbb{R}^2 = \mathbb{C}$. Conclude that there is an elliptic symbol, a on $\mathbb{R}^2$, such that there does not exist b, a symbol with

$$b \neq 0 \text{ on } \mathbb{R}^2 \text{ and } a(\xi) = b(\xi) \text{ for } |\xi| > r \tag{2.187}$$

for any r.

---

[8] See Problem prob:CC 2.10.

22.2.1998.139 Problem 2.3. Show that a symbol $a \in S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ which satisfies an estimate

22.2.1998.140
$$|a(z,\xi)| \le C(1+|\xi|)^{m'},\ m'<m \tag{2.188}$$

is necessarily in the space $S^{m'+\epsilon}_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ for all $\epsilon>0$.

22.2.1998.142 Problem 2.4. Show that if $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^p_z \times \mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\psi(\xi)=1$ in $|\xi|<1$ then

22.2.1998.143
$$c_\phi(z,\xi) = \phi(z,\frac{\xi}{|\xi|})(1-\psi)(\xi) \in S^0(\mathbb{R}^p_z;\mathbb{R}^n_\xi). \tag{2.189}$$

If $a \in S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ define the cone support of a in terms of its complement

22.2.1998.144
$$\begin{aligned}\operatorname{cone\,supp}(a)^{\complement} = \{(\bar z,\bar\xi) \in \mathbb{R}^p_z \times (\mathbb{R}^n_\xi \setminus \{0\}); \exists \\ \phi \in \mathcal{C}^\infty_c(\mathbb{R}^p_z;\mathbb{R}^n),\ \phi(\bar z,\bar\xi) \neq 0, \text{ such that } c_\phi a \in S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)\}.\end{aligned} \tag{2.190}$$

Show that if $a \in S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ and $b \in S^{m'}_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ then

27.1.2003.30
$$\operatorname{cone\,supp}(ab) \subset \operatorname{cone\,supp}(a) \cap \operatorname{cone\,supp}(b). \tag{2.191}$$

If $a \in S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$ and cone supp$(a)\emptyset$ does it follow that $a \in S^{-\infty}_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi)$?

1.2.2000.276 Problem 2.5. Prove that ([3.32]2.30) is a characterization of functions $a \in (1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$. [Hint: Use Liebniz' formula to show instead that the equivalent estimates

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1+|x-y|^2)^{w/2}(1+|\xi|)^{m-|\gamma|}\ \forall\ \alpha,\beta,\gamma \in \mathbb{N}^n_0$$

characterize this space.]

P3.3 Problem 2.6. Show that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies all the estimates

$$|D^\alpha_x D^\beta_y a(x,y)| < C_{\alpha,\beta,N}(1+|x-y|)^{-N} \tag{2.192}$$

for multiindices $\alpha,\beta \in \mathbb{N}^n_0$ and $N \in \mathbb{N}_0$.

prob:NN Problem 2.7. Polyhomogeneous symbols as smooth functions.

prob:MM Problem 2.8. General polyhomogeneous symbols and operators.

prob:DD Problem 2.9. Density of polyhomogeneous symbols in L^∞ symbols of the same order.

prob:CC Problem 2.10. Completeness of the spaces of polyhomogeneous symbols.

prob:FF Problem 2.11. Fourier transform??

Problem 2.12. Show that the kernel of any element of $\Psi^\infty_\infty(\mathbb{R}^n)$ is $\mathcal{C}^\infty$ away from the diagonal. Hint: Prove that $(x-y)^\alpha K(x,y)$ becomes increasingly smooth as $|\alpha|$ increases.

21.2.1998.117 Problem 2.13. Show that for any $m \ge 0$ the unit ball in $H^m(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is *not* precompact, i.e. there is a sequence $f_j \in H^m(\mathbb{R}^n)$ which has $\|f_j\|_m \le 1$ and has no subsequence convergent in $L^2(\mathbb{R}^n)$.

21.2.1998.118 Problem 2.14. Show that for any $R > 0$ there exists $N > 0$ such that the Hilbert subspace of $H^N(\mathbb{R}^n)$

21.2.1998.119 (2.193) $$\{u \in H^N(\mathbb{R}^n); u(x) = 0 \text{ in } |x| > R\}$$

is compactly included in $L^2(\mathbb{R}^n)$, i.e. the intersection of the unit ball in $H^N(\mathbb{R}^n)$ with the subspace (2.193) is precompact in $L^2(\mathbb{R}^n)$. Hint: This is true for *any* $N > 0$, taking $N >> 0$ will allow you to use the Sobolev embedding theorem and Arzela-Ascoli.

21.2.1998.120 Problem 2.15. Using Problem 2.14 (or otherwise) show that for any $\epsilon > 0$

$$(1 + |x|)^\epsilon H^\epsilon(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$$

is a compact inclusion, i.e. any infinite sequence f_n such that $(1+|x|^2)^{-\epsilon}$ is bounded in $H^\epsilon(\mathbb{R}^n)$ has a subsequence convergent in $L^2(\mathbb{R}^n)$. Hint: Choose $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi(x) = 1$ in $|x| < 1$ and, for each k, consider the sequence $\phi(x/k)f_j$. Show that the Fourier transform converts this into a sequence which is bounded in $(1 + |\xi|^2)^{-\frac{1}{2}\epsilon} H^N(\mathbb{R}^n_\xi)$ for any N. Deduce that it has a convergent subsequence in $L^2(\mathbb{R}^n)$. By diagonalization (and using the rest of the assumption) show that f_j itself has a convergent subsequence.

1.2.2000.279 Problem 2.16. About ρ and δ.

Problem 2.17. Prove the formula (2.185) for the left-reduced symbol of the operator A_T obtained from the pseudodifferential operator A by linear change of variables. How does the right-reduced symbol transform?

1.2.2000.280 Problem 2.18. Density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$.

1.2.2000.281 Problem 2.19. Square-root of a positive elliptic symbol is a symbol.

21.2.1998.108 Problem 2.20. Write out a proof to Proposition 3.2. Hint (just to do it elegantly, it is straightforward enough): Write A in right-reduced form as in (2.74) and apply it to $\hat{u}$; this gives a formula for $\hat{A}u$.

21.2.1998.110 Problem 2.21. Show that any continuous linear operator

$$\mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

has Schwartz kernel in $\mathcal{S}(\mathbb{R}^{2n})$.

CHAPTER 3

C.IandS.calculi

Isotropic and scattering calculi

In this chapter many of the general constructions with pseudodifferential operators are carried out in the context of two global calculi on $\mathbb{R}^n$. Partly this is done for the obvious reason, that these calculi and results have interesting applications, and partly it is preparatory to the discussion of the geometric algebras of pseudodifferential operators on a compact manifold without boundary and for the scattering algebra on a compact manifold with boundary. Thus, while this chapter is somewhat interstitial, it is designed to clarify the later discussions by separating the generalities of the construction from the particulars of the calculus involved. It should be noted that in this chapter it is generally the *polyhomogeneous* calculus which is under discussion unless it is explicitly stated to the contrary.

.Sect.isotropic.calculus

3.1. Isotropic operators

As noted in the discussion in Chapter 2 [C.Euclidean], there are other sensible choices of the class of amplitudes which can be admitted in the definition of a space of pseudodifferential operators than the basic case of $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ discussed there. One of the smallest such choices is the class which is completely symmetric in the variables x and ξ and consists of the symbols on $\mathbb{R}^{2n}$. Thus, $a \in S^m_\infty(\mathbb{R}^{2n}_{x,\xi})$ satisfies the estimates

1.2.2000.304

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^{m-|\alpha|-|\beta|} \tag{3.1}$$

for all multiindices α and β. If $m \le 0$ this is in the space $S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$; if $m > 0$ it is not, however,

1.2.2000.303

Lemma 3.1. *For any p and n*

8.2.1998.98

$$S^m_\infty(\mathbb{R}^{p+n}) \subset \begin{cases} \bigcap\limits_{0\ge r\ge m} (1+|x|^2)^r S^{m-r}(\mathbb{R}^p_x;\mathbb{R}^n_\xi) & m \le 0 \\ (1+|x|^2)^{m/2} S^m(\mathbb{R}^p_x;\mathbb{R}^n_\xi), & m > 0. \end{cases} \tag{3.2}$$

Proof. This follows from (3.1) [1.2.2000.304] and the inequalities

$$1+|x|+|\xi| \le (1+|x|)(1+|\xi|),$$
$$1+|x|+|\xi| \ge (1+|x|)^t(1+|\xi|)^{1-t},\ 0 \le t \le 1.$$

□

In view of these estimates the following definition makes sense.

1.2.2000.313

Definition 3.1. *For any $m \in \mathbb{R}$ we define*

1.2.2000.314

$$\Psi^m_{\text{iso}}(\mathbb{R}^n) \subset \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n) \subset \langle x\rangle^{m_+}\Psi^m_\infty(\mathbb{R}^n) \tag{3.3}$$

as the subspaces determined by

1.2.2000.315 (3.4)
$$A \in \Psi^m_{\text{iso}}(\mathbb{R}^n) \iff \sigma_L(A) \in S^m_{\text{ph}}(\mathbb{R}^{2n})$$
$$A \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n) \iff \sigma_L(A) \in S^m_{\infty}(\mathbb{R}^{2n}).$$

As in the discussion of the traditional algebra in Chapter 2 we show the $*$-invariance and composition properties of these spaces of operators by proving an appropriate 'reduction' theorem. Note however that there is a small difficulty here. Namely it might be supposed that it is enough to analyse $I(a)$ for $a \in S^m_\infty(\mathbb{R}^{3n})$. This however is not the case. Indeed the definition above is in terms of left-reduced symbols. If $a \in S^m_\infty(\mathbb{R}^{2n})$ is regarded as a function on $\mathbb{R}^{3n}$ which is independent of one of the variables then it is in general *not* an element of $S^m_\infty(\mathbb{R}^{3n})$. For this reason we consider some more 'hybrid' estimates.

Consider a subdivision of $\mathbb{R}^{3n}$ into two closed regions:

1.2.2000.316 (3.5)
$$R_1(\epsilon) = \{(x,y,\xi) \in \mathbb{R}^{3n}; |x-y| \le \epsilon(1+|x|^2+|y|^2+|\xi|^2)^{\frac12}\}$$
$$R_2(\epsilon) = \{(x,y,\xi) \in \mathbb{R}^{3n}; |x-y| \ge \epsilon(1+|x|^2+|y|^2+|\xi|^2)^{\frac12}\}.$$

If $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ consider the estimates

1.2.2000.317 (3.6)
$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma} \begin{cases} \langle (x,y,\xi)\rangle^{m-|\alpha|-|\beta|-|\gamma|} & \text{in } R_1(\frac18) \\ \langle (x,y)\rangle^{m_+}\langle\xi\rangle^{m-|\gamma|} & \text{in } R_2(\frac18). \end{cases}$$

The choice $\epsilon = \frac18$ here is rather arbitrary. However if ϵ is decreased, but kept positive the same estimates continue to hold for the new subdivision, since the estimates in R_1 are stronger than those in R_2 (which is increasing at the expense of R_1 as ϵ decreases). Notice too that these estimates do in fact imply that $a \in \langle (x,y)\rangle^{m_+} S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)$.

1.2.2000.318 Proposition 3.1. *If $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ satisfies the estimates* (3.6) *then* $A = I(a) \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$ *and* (2.58) *holds for* $\sigma_L(A)$.

Proof. We separate a into two pieces. Choose $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $0 \le \chi \le 1$, with support in $[-\frac18, \frac18]$ and with $\chi \equiv 1$ on $[-\frac19, \frac19]$. Then consider the cutoff function on $\mathbb{R}^{3n}$

1.2.2000.319 (3.7)
$$\psi(x,y,\xi) = \chi\left(\frac{|x-y|^2}{1+|x|^2+|y|^2+|\xi|^2}\right).$$

Clearly, ψ has support in $R_1(\frac18)$ and $\psi \in S^0_\infty(\mathbb{R}^{3n})$. It follows then that $a' = \psi a \in S^m_{\text{iso}}(\mathbb{R}^{3n})$. On the other hand, $a'' = (1-\psi)a$ has support in $R_2(\frac19)$. In this region $|x-y|$, $\langle (x,y)\rangle$ and $\langle (x,y,\xi)\rangle$ are bounded by constant multiples of each other. Thus a'' satisfies the estimates

1.2.2000.320 (3.8)
$$|D^\alpha_x D^\beta_y D^\gamma_\xi a''(x,y,\xi)| \le C_{\alpha,\beta,\gamma}|x-y|^{m_+}\langle\xi\rangle^{m-|\gamma|}$$
$$\le C'_{\alpha,\beta,\gamma}\langle (x,y,\xi)\rangle^{m_+}\langle\xi\rangle^{m-|\gamma|},\ \operatorname{supp}(a'') \subset R_2(\frac19).$$

First we check that $I(a'') \in \mathcal{S}(\mathbb{R}^{2n})$. On $R_2(\frac19)$ it is certainly the case that $|x-y| \ge \frac19\langle (x,y)\rangle$ and by integration by parts

$$|x-y|^{2p} D^\alpha_x D^\beta_y I(a'') = I(|D_\xi|^{2p} D^\alpha_x D^\beta_y a'').$$

For all sufficiently large p it follows from (3.8) that this is the product of $\langle(x,y)\rangle^{m_+}$ and a bounded continuous function. Thus, $I(a'') \in \mathcal{S}(\mathbb{R}^{2n})$ is the kernel of an operator in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

So it remains only to show that $A' = I(a') \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Certainly this is an element of $\langle x\rangle^{m_+}\Psi^m_\infty(\mathbb{R}^n)$. The left-reduced symbol of A' has an asymptotic expansion, as $\xi \to \infty$, given by the usual formula, namely (2.58). Each of the terms in this expansion

$$a_L(A') \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_y D^\alpha_\xi a(x,x,\xi)$$

is in the space $S^{m-2|\alpha|}_\infty(\mathbb{R}^{2n})$. Thus we can actually choose an asymptotic sum in the stronger sense that

$$b' \in S^m_\infty(\mathbb{R}^{2n}),\ b_N = b' - \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-2N}_\infty(\mathbb{R}^{2n})\ \forall\ N.$$

Consider the remainder term in (2.47), given by (2.44) and (2.45). Integrating by parts in ξ to remove the factors of $(x-y)^\alpha$ the remainder, R_N, can be written as a pseudodifferential operator with amplitude

$$r_N(x,y,\xi) = \sum_{|\alpha|=N} \frac{i^{|\alpha|}}{\alpha!} \int_0^1 dt(1-t)^N (D^\alpha_\xi D^\alpha_y a)((1-t)x+ty,\xi).$$

This satisfies the estimates (3.6) with m replaced by $m-2N$. Indeed from the symbol estimates on a' the integrand satisfies the bounds

$$|D^\beta_x D^\gamma_y D^\delta_\xi D^\alpha_\xi D^\alpha_y a'((1-t)x+ty,\xi)| \\ \le C(1+|(x+t(x-y)|+|\xi|)^{m-2N-|\beta|-|\gamma|-|\delta|}.$$

In $R_1(\frac18)$, $|x-y| \le \frac18\langle(x,y,\xi)\rangle$ so $|x+t(x-y)|+|\xi| \ge \frac12\langle(x,y,\xi)\rangle$ and these estimates imply the full symbol estimates there. On R_2 we immediately get the weaker estimates in (3.7).

Thus, for large N, the remainder term gives an operator in $\langle x\rangle^{\frac{m}{2}-N}\Psi^{\frac{m}{2}-N}_{\mathrm{iso}}(\mathbb{R}^n)$. The difference between A' and the operator $B' \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$, which is R_N plus an operator in $\Psi^{m-2N}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for any N is therefore in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. □

This is a perfectly adequate replacement in this context for our previous reduction theorem, so now we can show the basic result.

8.2.1998.99 THEOREM 3.1. *The spaces $\Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ (resp. $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$) of isotropic (resp. polyhomogeneous isotropic) pseudodifferential operators on $\mathbb{R}^n$, defined by (3.4) form an order-filtered $*$-algebra with residual space $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n})$ (resp. the same) as spaces of kernels.*

PROOF. The condition that a continuous linear operator A on $\mathcal{S}(\mathbb{R}^n)$ be an element of $\Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is that it be an element of $(1+|x|^2)^{m/2}\Psi^m_\infty(\mathbb{R}^n)$ if $m \ge 0$ or $\Psi^m_\infty(\mathbb{R}^n)$ if $m < 0$ with left-reduced symbol an element of $S^m_\infty(\mathbb{R}^{2n}_{x,\xi})$:

8.2.1998.101 (3.9) $$q_l : S^m_\infty(\mathbb{R}^{2n}) \longleftrightarrow \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n).$$

Thus A^* has right-reduced symbol in $S^m_\infty(\mathbb{R}^{2n})$. This satisfies the estimates (3.6) as a function of x, y and ξ. Thus Propostion 3.1 shows that $A^* \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$, since its left-reduced symbol is in $S^m_\infty(\mathbb{R}^{2n})$. This shows the $*$-invariance. Moreover it also follows that any $B \in \Psi^{m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ has right-reduced symbol in $S^{m'}_\infty(\mathbb{R}^{2n})$. Thus if $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{m'}_{\mathrm{iso}}(\mathbb{R}^n)$ then using this result to right-reduce B we see that the composite opertor has kernel $I(a_L(x,\xi)b_R(y,\xi))$ where $a_L \in S^m_\infty(\mathbb{R}^{2n})$ and $b_R \in S^{m'}_\infty(\mathbb{R}^{2n})$. Now it again follows that this product satisifes the estimates (3.6) of order $m+m'$. Hence, again applying Proposition 3.1, we conclude that $A \circ B \in \Psi^{m+m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. This proves the theorem for $\Psi^*_{\infty-\mathrm{iso}}(\mathbb{R}^n)$.

The proof for the polyhomogeneous space $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ follows immediately, since the symbol expansions all preserve polyhomogeneity. □

One further property of the isotropic calculus that distinguishes it strongly from the traditional calculus is that it is invariant under Fourier transformation.

21.2.1998.107 Proposition 3.2. *If* $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$*) then* $\hat{A} \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$*) where* $\widehat{\hat{A}u} = A\hat{u}$ *with* $\hat{u}$ *being the Fourier transform of* $u \in \mathcal{S}(\mathbb{R}^n)$.

The proof of this is outlined in Problem 2.20.

3.2. Scattering operators

S.Scattering.operators

There is another calculus of pseudodifferential operators which is 'smaller' than the traditional calculus. It arises by taking amplitudes in (2.2) which treat the base and fibre variables symmetrically, but not 'simultaneously.' Thus consider the spaces

21.2.1998.113
$$S^{l,m}_\infty(\mathbb{R}^p_z,\mathbb{R}^n_\xi) = \big\{a \in \mathcal{C}^\infty(\mathbb{R}^{p+n}); \sup_{\mathbb{R}^{p+n}} (1+|z|)^{-l+|\alpha|}(1+|\xi|)^{-m+|\beta|}|D^\alpha_z D^\beta_\xi a(z,\xi)| < \infty,\ \forall\ \alpha,\beta\big\}. \tag{3.10}$$

Observe that

21.2.1998.115
$$S^{l,m}_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi) \subset (1+|z|^2)^{l/2} S^m_\infty(\mathbb{R}^p_z;\mathbb{R}^n_\xi). \tag{3.11}$$

We can then define

21.2.1998.114
$$A \in \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \Longleftrightarrow A = (1+|x|^2)^{l/2}B, \quad B \in \Psi^m_\infty(\mathbb{R}^n) \text{ and } \sigma_L(B) \in S^{0,m}_\infty(\mathbb{R}^n_x,\mathbb{R}^n_\xi). \tag{3.12}$$

It follows directly from this definition and the properties of the 'traditional' operators that the left symbol map is an isomorphism

1.2.2000.300
$$\sigma_L : \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n) \longrightarrow S^{l,m}_\infty(\mathbb{R}^n_x,\mathbb{R}^n_\xi). \tag{3.13}$$

To prove that this is an algebra, we need first the analogue of the asymptotic completeness, Proposition 2.3, for symbols in $S^{*,*}_\infty(\mathbb{R}^p;\mathbb{R}^n)$.

1.2.2000.292 Lemma 3.2. *If* $a_j \in S^{l-j,m-j}_\infty(\mathbb{R}^p,\mathbb{R}^n)$ *for* $j \in \mathbb{N}_0$ *then there exists*

1.2.2000.293
$$a \in S^{l,m}(\mathbb{R}^p,\mathbb{R}^n) \text{ s.t. } a - \sum_{j=0}^N a_j \in S^{l-N,m-N}_\infty(\mathbb{R}^p,\mathbb{R}^n)\ \forall\ N \in \mathbb{N}_0. \tag{3.14}$$

Even though there is some potential for confusion we write $a \sim \sum_j a_j$ for a symbol a satisfying (3.14).

Proof. We use the same strategy as in the proof of Proposition 2.3 with the major difference that there are essentially two different symbolic variables. Thus with the same notation as in (2.54) we set

1.2.2000.294 (3.15)
$$a=\sum_j \phi(\epsilon_j z)\phi(\epsilon_j\xi)a_j(z,\xi)$$

and we proceed to check that if the $\epsilon_j \downarrow 0$ fast enough as $j\to\infty$ then the series converges in $S^{l,m}_\infty(\mathbb{R}^p,\mathbb{R}^n)$ and the limit satisfies (3.14).

The first of the seminorms, for convergence, is

$$A_j=\sup_z\sup_\xi(1+|z|)^{-l}(1+|\xi|)^{-m}\phi(\epsilon_j z)\phi(\epsilon_j\xi)|a_j(z,\xi)|.$$

On the support of this function either $|z|\ge 1/\epsilon_j$ or $|\xi|\ge 1/\epsilon$. Thus

$$\begin{aligned}A_j\le \sup_z\sup_\xi(1+|z|)^{-l+j}(1+|\xi|)^{-m+j}|a_j(z,\xi)|\\ \times\sup_z\sup_\xi(1+|z|)^{-j}(1+|\xi|)^{-j}\phi(\epsilon_j z)\phi(\epsilon_j\xi)\\ \le\epsilon_j^j\sup_z\sup_\xi(1+|z|)^{-l+j}(1+|\xi|)^{-m+j}|a_j(z,\xi)|\end{aligned}$$

The last term on the right is a seminorm on $S^{l-j,m-j}_\infty(\mathbb{R}^p,\mathbb{R}^n)$ so convergence follows by choosing the ϵ_j *eventually* smaller than a certain sequence of positive numbers. The same argument follows, as in the discussion leading to (2.56), for convergence of the series for the derivatives and also for the stronger convergence leading to (3.14). Since overall this is a countable collection of conditions, all can be arranged by diagonalization and the result follows. □

With this result on asymptotic completeness the proof of Theorem 3.1 can be followed closely to yield the analogous result on products. In fact we can also define polyhomogeneous operators. This requires a little work if we try to do it directly. However see (1.97) and Problem 1.17 which encourages us to identify

1.2.2000.297 (3.16)
$$\begin{gathered}\mathrm{RC}^*_p\times\mathrm{RC}^*_n:S^{0,0}_{\mathrm{ph}}(\mathbb{R}^p,\mathbb{R}^n)\longleftrightarrow\mathcal{C}^\infty(\mathbb{S}^{p,1}\times\mathbb{S}^{n,1}),\\ S^{l,m}_{\mathrm{ph}}(\mathbb{R}^p,\mathbb{R}^n)=(1+|z|^2)^{l/2}(1+|\xi|^2)^{m/2}S^{0,0}_{\mathrm{ph}}(\mathbb{R}^p,\mathbb{R}^n),\ l,m\in\mathbb{R}.\end{gathered}$$

These definitions are discussed as problems starting at Problem 1.18. Thus we simply define

1.2.2000.299 (3.17)
$$\Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)=\left\{A\in\Psi^{l,m}_{\infty-\mathrm{sc}};\sigma_L(A)\in S^{l,m}_{\mathrm{ph}}(\mathbb{R}^n,\mathbb{R}^n)\right\}.$$

1.2.2000.295 Theorem 3.2. *The spaces $\Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)$ (resp. $\Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)$) of scattering (resp. polyhomogeneous scattering) pseudodifferential operators on $\mathbb{R}^n$, form an order-bifiltered $*$-algebra*

1.2.2000.296 (3.18)
$$\Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)\circ\Psi^{l',m'}_{\infty-\mathrm{sc}}(\mathbb{R}^n)\subset\Psi^{l+l',m+m'}_{\infty-\mathrm{sc}}(\mathbb{R}^n)$$

with residual spaces

1.2.2000.355 (3.19)
$$\bigcap_{l,m}\Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)=\bigcap_{l,m}\Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)=\mathcal{S}(\mathbb{R}^{2n}).$$

S.Residual.isotropic

3.3. The residual algebra isotropic algebra

The residual isotropic (and scattering) algebra has two important properties not shared by the residual algebra $\Psi_{\infty}^{-\infty}(\mathbb{R}^n)$, of which it is a subalgebra (and in fact in which it is an ideal). The first is that as operators on $L^2(\mathbb{R}^n)$ the residual isotropic operators are compact.

1.2.2000.309 Proposition 3.3. *Elements of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ are characterized amongst continous operators on $\mathcal{S}(\mathbb{R}^n)$ by the fact that they extend by continuity to define continuous linear maps*

1.2.2000.356
$$A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{3.20}$$

In particular the image of a bounded subset of $L^2(\mathbb{R}^n)$ under an element of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ is contained in a compact subset.

Proof. The kernels of elements of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ are in $\mathcal{S}(\mathbb{R}^{2n})$ so the mapping property (1.2.2000.356 3.20) follows.

The norm $\sup_{|\alpha|\le 1} |\langle x\rangle^{n+1} D^\alpha u(x)|$ is continuous on $\mathcal{S}(\mathbb{R}^n)$. Thus if $S \subset L^2(\mathbb{R}^n)$ is bounded and $A \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ the continuity of $A : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ implies that $A(S)$ is bounded with respect to this norm. The theorem of Arzela-Ascoli shows that any sequence in $A(S)$ has a strongly convergent subsequence in $\langle x\rangle^n \mathcal{C}_\infty^0(\mathbb{R}^n)$ and such a sequence converges in $L^2(\mathbb{R}^n)$. Thus $A(S)$ has compact closure in $L^2(\mathbb{R}^n)$. □

The second important property of the residual algebra is that it is 'bi-ideal' or a 'corner' in the bounded operators on $L^2(\mathbb{R}^n)$. Note that it is not an ideal.

1.2.2000.310 Lemma 3.3. *If A_1, $A_2 \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ and B is a bounded operator on $L^2(\mathbb{R}^n)$ then $A_1BA_2 \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$.*

Proof. The kernel of the composite $C = A_1BA_2$ can be written as a distributional pairing

1.2.2000.321
$$C(x,y) = \int_{\mathbb{R}^{2n}} B(x',y')A_1(x,x')A_2(y',y)dx'dy' = (B, A_1(x,\cdot)A_2(\cdot,y)) \in \mathcal{S}(\mathbb{R}^{2n}). \tag{3.21}$$

Thus the result follows from the continuity of the exterior product, $\mathcal{S}(\mathbb{R}^{2n}) \times \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{4n})$. □

In fact the same conclusion, with essentially the same proof, holds for any continuous linear operator B from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

S.Isotropic.ring

3.4. The residual isotropic ring

Recall that a bounded operator is said to have finite rank if its range is finite dimensional. If we consider a bounded operator B on $L^2(\mathbb{R}^n)$ which is of finite rank then we may choose an orthonormal basis f_j, $j = 1, \dots, N$ of the range $BL^2(\mathbb{R}^n)$. The functionals $u \longmapsto \langle Bu, f_j\rangle$ are continuous and so define non-vanishing elements $g_j \in L^2(\mathbb{R}^n)$. It follows that the Schwartz kernel of B is

1.2.2000.311
$$B = \sum_{j=1}^{N} f_j(x)\overline{g_j(y)}. \tag{3.22}$$

If $B \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then the range must lie in $\mathcal{S}(\mathbb{R}^n)$ and similarly for the range of the adjoint, so the functions f_j and g_j are also in $\mathcal{S}(\mathbb{R}^n)$. Clearly the finite rank elements of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ form an ideal in $\Psi^{\infty}_{\infty-\text{iso}}(\mathbb{R}^n)$.

1.2.2000.312 Proposition 3.4. *If $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then $\operatorname{Id}+A$ has, as an operator on $L^2(\mathbb{R}^n)$, finite dimensional null space and closed range which is the orthocomplement of the null space of $\operatorname{Id}+A^*$. There is an element $B \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ such that*

1.2.2000.322
$$(\operatorname{Id}+A)(\operatorname{Id}+B) = \operatorname{Id}-\Pi_1,\ (\operatorname{Id}+B)(\operatorname{Id}+A) = \operatorname{Id}-\Pi_0 \tag{3.23}$$

where $\Pi_0,\ \Pi_1 \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ are the orthogonal projections onto the null spaces of $\operatorname{Id}+A$ and $\operatorname{Id}+A^*$ and furthermore, there is an element $A' \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ of rank equal to the dimension of the null space such that $\operatorname{Id}+A+sA'$ is an invertible operator on $L^2(\mathbb{R}^n)$ for all $s \neq 0$.

Proof. Most of these properties are a direct consequence of the fact that A is compact as an operator on $L^2(\mathbb{R}^n)$; nevertheless we give brief proofs.

We have shown, in Proposition 3.3 that each $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is compact. It follows that

1.2.2000.327
$$N_0 = \operatorname{Nul}(\operatorname{Id}+A) \subset L^2(\mathbb{R}^n) \tag{3.24}$$

has compact unit ball. Indeed the unit ball, $B = \{u \operatorname{Nul}(\operatorname{Id}+A)\}$ satisfies $B = A(B)$, since $u = -Au$ on B. Thus B is closed and precompact. Any Hilbert space with a compact unit ball is finite dimensional, so $\operatorname{Nul}(\operatorname{Id}+A)$ is finite dimensional.

Now, let $R_1 = \operatorname{Ran}(\operatorname{Id}+A)$ be the range of $\operatorname{Id}+A$; we wish to show that this is a closed subspace of $L^2(\mathbb{R}^n)$. Let $f_k \to f$ be a sequence in R_1, converging in $L^2(\mathbb{R}^n)$. For each k there exists a unique $u_k \in L^2(\mathbb{R}^n)$ with $u_k \perp N_0$ and $(\operatorname{Id}+A)u_k = f_k$. We wish to show that $u_k \to u$. First we show that $\|u_k\|$ is bounded. If not, then along a subsequent $v_j = u_{k(j)}$, $\|v_j\| \to \infty$. Set $w_j = v_j/\|v_j\|$. Using the compactness of A, $w_j = -Aw_j + f_{k(j)}/\|v_j\|$ must have a convergent subsequence, $w_j \to w$. Then $(\operatorname{Id}+A)w = 0$ but $w \perp N_0$ and $\|w\| = 1$ which are contradictory. Thus the sequence u_k is bounded in $L^2(\mathbb{R}^n)$. Then again $u_k = -Au_k + f_k$ has a convergent subsequence with limit u which is a solution of $(\operatorname{Id}+A)u = f$; hence R_1 is closed. The orthocomplement of the range of a bounded operator is always the null space of its adjoint, so R_1 has a finite-dimensional complement $N_1 = \operatorname{Nul}(\operatorname{Id}+A^*)$. The same argument applies to $\operatorname{Id}+A^*$ so gives the orthogonal decompositions

1.2.2000.328
$$\begin{gathered} L^2(\mathbb{R}^n) = N_0 \oplus R_0,\ N_0 = \operatorname{Nul}(\operatorname{Id}+A),\ R_0 = \operatorname{Ran}(\operatorname{Id}+A^*) \\ L^2(\mathbb{R}^n) = N_1 \oplus R_1,\ N_1 = \operatorname{Nul}(\operatorname{Id}+A^*),\ R_1 = \operatorname{Ran}(\operatorname{Id}+A). \end{gathered} \tag{3.25}$$

Thus we have shown that $\operatorname{Id}+A$ induces a continuous bijection $\tilde{A} : R_0 \longrightarrow R_1$. From the closed graph theorem the inverse is a bounded operator $\tilde{B} : R_1 \longrightarrow R_0$. In this case continuity also follows from the argument above.[1] Thus $\tilde{B}$ is the generalized inverse of $\operatorname{Id}+A$ in the sense that $B = \tilde{B} - \operatorname{Id}$ satisfies (3.23). It only remains to show that $B \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. This follows from (3.23), the identities in which show that

1.2.2000.329
$$\begin{aligned} B = -A - AB - \Pi_1,\ -B = A + BA + \Pi_0 \\ \Longrightarrow B = -A + A^2 + ABA - \Pi_1 + A\Pi_0. \end{aligned} \tag{3.26}$$

[1] We need to show that $\|\tilde{B}f\|$ is bounded when $f \in R_1$ and $\|f\| = 1$. This is just the boundedness of $u \in R_0$ when $f = (\operatorname{Id}+A)u$ is bounded in R_1.

All terms here are in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$; for ABA this follows from Proposition 3.3.

It remains to show the existence of the finite rank perturbation A'. This is equivalent to the vanishing of the index, that is

1.2.2000.323
$$\operatorname{Ind}(\mathrm{Id}+A) = \dim \operatorname{Nul}(\mathrm{Id}+A) - \dim \operatorname{Nul}(\mathrm{Id}+A^*) = 0. \tag{3.27}$$

Indeed, let f_j and g_j, $j = 1, \dots, N$, be respective bases of the two finite dimensional spaces $\operatorname{Nul}(\mathrm{Id}+A)$ and $\operatorname{Nul}(\mathrm{Id}+A^*)$. Then

1.2.2000.324
$$A' = \sum_{j=1}^{N} g_j(x)\overline{f_j(y)} \tag{3.28}$$

is an isomorphism of N_0 onto N_1 which vanishes on R_0. Thus $\mathrm{Id}+A+sA'$ is the direct sum of $\mathrm{Id}+A$ as an operator from R_0 to R_1 and sA' as an operator from N_0 to N_1, invertible when $s \neq 0$.

There is a very simple proof[2] of the equality (3.27) if we use the trace functional discussed in Section 3.14 below; this however is logically suspect as we use (although not crucially) approximation by finite rank operators in the discussion of the trace and this in turn might appear to use the present result via the discussion of ellipticity and the harmonic oscillator. Even though this is not really the case we give a clearly independent, but less elegant proof.

Consider the one-parameter family of operators $\mathrm{Id}+tA$, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. We shall see that the index, the difference in dimension between $\operatorname{Nul}(\mathrm{Id}+tA)$ and $\operatorname{Nul}(\mathrm{Id}+tA^*)$ is locally constant. To see this it is enough to consider a general A near the point $t = 1$. Consider the pieces of A with respect to the decompositions $L^2(\mathbb{R}^n) = N_i \oplus R_i$, $i = 0, 1$, of domain and range. Thus A is the sum of four terms which we write as a 2×2 matrix

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}.$$

Since $\mathrm{Id}+A$ has only one term in such a decomposition, $\tilde{A}$ in the lower right, the solution of the equation $(\mathrm{Id}+tA)u = f$ can be written

1.2.2000.325
$$(t-1)A_{00}u_0 + (t-1)A_{01}u_\perp = f_1,\ (t-1)A_{10}u_0 + (A' + (t-1)A_{11})u_\perp = f_\perp \tag{3.29}$$

Since $\tilde{A}$ is invertible, for $t-1$ small enough the second equation can be solved uniquely for $u_\perp$. Inserted into the first equation this gives

1.2.2000.326
$$\begin{gathered} G(t)u_0 = f_1 + H(t)f_\perp, \\ G(t) = (t-1)A_{00} - (t-1)^2 A_{01}(A' + (t-1)A_{11})^{-1}A_{10}, \\ H(t) = -(t-1)A_{01}(A' + (t-1)A_{11})^{-1}. \end{gathered} \tag{3.30}$$

The null space is therefore isomorphic to the null space of $G(t)$ and a complement to the range is isomorphic to a complement to the range of $G(t)$. Since $G(t)$ is a finite rank operator acting from N_0 to N_1 the difference of these dimension is constant in t, namely equal to $\dim N_0 - \dim N_1$, near $t = 1$ where it is defined.

[2] Namely the trace of a finite rank projection, such as either Π_0 or Π_1, is its rank, hence the dimension of the space onto which it projects. From the identity satisfied by the generalized inverse we see that

$$\operatorname{Ind}(\mathrm{Id}+A) = \operatorname{Tr}(\Pi_0) - \operatorname{Tr}(\Pi_1) = \operatorname{Tr}\left((\mathrm{Id}+B)(\mathrm{Id}+A) - (\mathrm{Id}+A)(\mathrm{Id}+B)\right) = \operatorname{Tr}([B, A]) = 0$$

from the basic property of the trace.

This argument can be applied to tA so the index is actually constant in $t \in [0,1]$ and since it certainly vanishes at $t = 0$ it vanishes for all t. In fact, as we shall note below, $\mathrm{Id} + tA$ is invertible outside a discrete set of $t \in \mathbb{C}$. □

1.2.2000.330 Corollary 3.1. *If* $\mathrm{Id} + A$, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *is injective or surjective on* $L^2(\mathbb{R}^n)$, *in particular if it is invertible as a bounded operator, then it has an inverse in the ring* $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

1.2.2000.333 Corollary 3.2. *If* $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *then as an operator on* $\mathcal{S}(\mathbb{R}^n)$ *or* $\mathcal{S}'(\mathbb{R}^n)$, $\mathrm{Id} + A$ *is Fredholm in the sense that its null space is finite dimensional and its range is closed with a finite dimensional complement.*

Proof. This follows from the existence of the generalized inverse of the form $\mathrm{Id} + B$, $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. □

3.5. Exponential and logarithm

1.2.2000.350 Proposition 3.5. *The exponential*

1.2.2000.351
$$\exp(A) = \sum_j \frac{1}{j!} A^j : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.31}$$

is a globally defined, entire, function with range containing a neighbourhood of the identity and with inverse on such a neighbourhood given by the analytic function

1.2.2000.352
$$\log(\mathrm{Id} + A) = \sum_j \frac{(-1)^j}{j} A^j, \ \ A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n), \ \ \|A\|_{L^2} < 1 \tag{3.32}$$

S.Fredholm.property

3.6. Fredholm property

An element $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is said to be elliptic (of order m in the isotropic calculus) if its left-reduced symbol is elliptic in $S^m_\infty(\mathbb{R}^{2n})$.

21.2.1998.106 Theorem 3.3. *Each elliptic element* $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *has a two-sided parametrix* $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *in the sense that*

1.2.2000.365
$$A \circ B - \mathrm{Id}, \ B \circ A - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.33}$$

and it follows that any $u \in \mathcal{S}'(\mathbb{R}^n)$ *satisfying* $Au \in \mathcal{S}(\mathbb{R}^n)$ *is an element of* $\mathcal{S}(\mathbb{R}^n)$; *if* $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ *is elliptic then its parametrix is in* $\Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$.

Proof. This is just the inductive argument used to prove Lemma 5.31 2.7. Nevertheless we repeat it here.

The existence of a right inverse for $\sigma_k(A)$ means that the equation $\sigma_k(A)c = d$ always has a solution $c \in \mathcal{A}_{j-k}$ for given $d \in \mathcal{A}_j$, namely $c = bd$. This in turn means that given $C_j \in \Psi^j_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ there always exists $B_j \in \Psi^{j-k}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ such that $AB_j - D_j \in \Psi^{j-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Choosing $B_0 \in \Psi^{-k}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ to have $\sigma_{-k}(B_0) = b$ we can define $C_1 = \mathrm{Id} - AB_0 \in \Psi^{-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Then, proceeding inductively we may assume that B_j for $j < l$ have been chosen such that $A(B_0 + \cdots + B_{l-1}) - \mathrm{Id} = -C_l \in \Psi^{-l}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Then using the solvability we may choose B_l so that $AB_l - C_l = -C_{l+1} \in \Psi^{-l-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ which completes the induction, since $A(B_0 + \cdots + B_l) - \mathrm{Id} = AB_l - C_l = -C_{l+1}$. Finally by the asymptotic completeness we may choose $B \sim B_0 + B_1 + \dots$ which is a right parametrix.

The argument showing the existence of a left parametrix for a left-elliptic operator is completely analogous. □

Combining the earlier symbolic discussion and these analytic results we can see that elliptic operators in these calculi are Fredholm.

21.3.1998.169 PROPOSITION 3.6. *If* $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $A \in \Psi^{l,m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)$*) is elliptic then it has a generalized inverse* $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $B \in \Psi^{-l,-m}_{\infty-\mathrm{sc}}(\mathbb{R}^n)$*) satisfying*

21.3.1998.170
$$AB - \mathrm{Id} = \Pi_1,\ BA - \mathrm{Id} = \Pi_0 \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.34}$$

where Π_1 *and* Π_0 *are the finite rank orthogonal (in* $L^2(\mathbb{R}^n)$*) projections onto the null spaces of* A^* *and* A.

PROOF. In the case of an elliptic isotropic operator or order m we know that it has a parametrix $B' \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ modulo $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus

$$AB' = \mathrm{Id} - E_R,\ E_R \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),$$
$$B'A = \mathrm{Id} - E_L,\ E_L \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Using Proposition 3.4 it follows that the null space of A is contained in the null space of $B'A = \mathrm{Id} - E_L$, hence is finite dimensional. Similarly, the range of A contains the range of $AB' = \mathrm{Id} - E_R$ so is closed with a finite codimensional complement. Defining B as the linear map which vanishes on $\mathrm{Nul}(A^*)$, and inverts A on $\mathrm{Ran}(A)$ with values in $\mathrm{Ran}(A^*) = \mathrm{Nul}(A)^\perp$ gives (3.34). Furthermore these identities show that $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ since applying B' gives

1.2.2000.331
$$\begin{gathered} B - E_L B = B'AB = B' - B'\Pi_1,\ B - BE_R = BAB' = B' - \Pi_0 B' \Longrightarrow \\ B = B' - B'\Pi_1 + E_L B' + E_L B E_R - E_L \Pi_0 B' \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n). \end{gathered} \tag{3.35}$$

The proof in the scattering case is essentially the same. □

1.2.2000.332 COROLLARY 3.3. *If* $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ *is elliptic then its generalized inverse lies in* $\Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ *and similarly if* $A \in \Psi^{l,m}_{\mathrm{sc}}(\mathbb{R}^n)$ *is elliptic then its generalized inverse lies in* $\Psi^{-l,-m}_{\mathrm{sc}}(\mathbb{R}^n)$.

S.Harmonic.oscillator

3.7. The harmonic oscillator

The harmonic oscillator is the differential operator on $\mathbb{R}^n$

$$H = \sum_{j=1}^n (D_j^2 + x_j^2) = \Delta + |x|^2 .$$

This is an elliptic element of $\Psi^2_{\mathrm{iso}}(\mathbb{R}^n)$. The main immediate interest is in the spectral decomposition of H. The ellipticity of $H - \lambda$, $\lambda \in \mathbb{C}$, shows that

eq:HO.1
$$(H - \lambda)u = 0,\ u \in \mathcal{S}'(\mathbb{R}^n) \Longrightarrow u \in \mathcal{S}(\mathbb{R}^n) . \tag{3.36}$$

Since H is (formally) self-adjoint, i.e., $H^* = H$, there are no non-trivial tempered solutions of $(H - \lambda)u = 0$, $\lambda \notin \mathbb{R}$. Indeed if $(H - \lambda)u = 0$,

eq:HO.2
$$0 = \langle Hu, u\rangle - \langle u, Hu\rangle = (\lambda - \overline{\lambda})\langle u, u\rangle \Longrightarrow u = 0. \tag{3.37}$$

As we shall see below in more generality, the spectrum of H is a discrete subset of $\mathbb{R}$. In this case we can compute it explicitly.

The direct computation of eigenvalues and eigenfunctions is based on the properties of the creation and annihilation operators

eq:HO.3
$$C_j = D_j + ix_j, \quad C_j^* = A_j = D_j - ix_j,\ j = 1, \dots, n. \tag{3.38}$$

These satisfy the elementary identities

eq:HO.4 (3.39)
$$
\begin{gathered}
[A_j, A_k] = [C_j, C_k] = 0,\ [A_j, C_k] = 2\delta_{jk},\ j,k = 1,\dots,n \\
H = \sum_{j=1}^{n} C_j A_j + n, \quad [C_j, H] = -2C_j, \quad [A_j, H] = 2A_j\,.
\end{gathered}
$$

Now, if λ is an eigenvalue, $Hu = \lambda u$, then

eq:HO.5 (3.40)
$$
\begin{aligned}
H(C_j u) &= C_j(Hu + 2u) = (\lambda + 2)C_j u, \\
H(A_j u) &= A_j(Hu - 2u) = (\lambda - 2)A_j u.
\end{aligned}
$$

prop:HO.6 Proposition 3.7. *The eigenvalues of H are*

eq:HO.7 (3.41)
$$\sigma(H) = \{n, n+2, n+4, \dots\}.$$

Proof. We already know that eigenvalues must be real and from the decomposition of H in (eq:HO.5 3.40) it follows that, for $u \in \mathcal{S}(\mathbb{R}^n)$,

eq:HO.9 (3.42)
$$\langle Hu, u\rangle = \sum_j \|A_j u\|^2 + n\|u\|^2\,.$$

Thus if $\lambda \in \sigma(H)$ is an eigenvalue then $\lambda \geq n$.

By direct computation we see that n is an eigenvalue with a 1-dimensional eigenspace. Indeed, from (eq:HO.9 3.42), $Hu = nu$ iff $A_j u = 0$ for $j = 1, \dots, n$. In each variable separately

$$A_j u(x_j) = 0 \Leftrightarrow u(x_j) = c_1 \exp\left(-\frac{x_j^2}{2}\right).$$

Thus the only tempered solutions of $A_j u = 0$, $i = 1, \dots, n$ are the constant multiples of

eq:HO.10 (3.43)
$$u_0 = \exp\left(-\frac{|x|^2}{2}\right),$$

which is often called the *ground state*.

Now, if λ is an eigenvalue with eigenfunction $u \in \mathcal{S}(\mathbb{R}^n)$ it follows from (eq:HO.5 3.40) that $\lambda - 2$ is an eigenvalue with eigenfunction $A_j u$. Since all the $A_j u$ cannot vanish unless u is the ground state, it follows that the eigenvalues are contained in the set in (eq:HO.7 3.41). We can use the same argument to show that if u is an eigenfunction with eigenvalue λ then $C_j u$ is an eigenfunction with eigenvalue $\lambda + 2$. Moreover, $C_j u \equiv 0$ would imply $u \equiv 0$ since $C_j v = 0$ has no non-trivial tempered solutions, the solution in each variable being $\exp(x_j^2/2)$. □

Using the creation operators we can parameterize the eigenspaces quite explicitly.

prop:HO.11 Proposition 3.8. *For each $k \in \mathbb{N}_0$ there is an isomorphism*

eq:HO.12 (3.44)
$$
\begin{aligned}
&\{\textit{Polynomials, homogeneous of degree } k \textit{ on } \mathbb{R}^n\} \ni p \\
&\qquad\qquad \longmapsto p(C)\exp\left(-\frac{|x|^2}{2}\right) \in E_k
\end{aligned}
$$

where E_k is the eigenspace of H with eigenvalue $n + 2k$.

PROOF. Notice that the C_j, $j = 1, \dots, n$ are commuting operators, so $p(C)$ is well-defined. By iteration from (3.40),

eq:HO.13 (3.45)
$$HC^\alpha u_0 = C^\alpha(H + 2|\alpha|)u_0 = (n + 2|\alpha|)C^\alpha u_0\,.$$

Thus (3.44) is a linear map into the eigenspace as indicated.

To see that (3.44) is an isomorphism consider the action of the annihilation operators. Again from (3.40)

eq:HO.14 (3.46)
$$|\beta| = |\alpha| \Longrightarrow A^\beta C^\alpha u_0 = \begin{cases} 0 & \beta \neq \alpha \\ 2^{|\alpha|}\alpha! u_0 & \beta = \alpha. \end{cases}$$

This allows us to recover the coefficients of p from $p(C)u_0$, so (3.44) is injective. Conversely if $v \in E_k \subset \mathcal{S}(\mathbb{R}^n)$ is orthogonal to all the $C^\alpha u_0$ then

eq:HO.15 (3.47)
$$\langle A^\alpha v, u_0\rangle = \langle v, C^\alpha u_0\rangle = 0 \ \forall\ |\alpha| = k\,.$$

From (3.40), the $A^\alpha v$ are all eigenfunctions of H with eigenvalue n, so (3.47) implies that $A^\alpha v = 0$ for all $|\alpha| = k$. Proceeding inductively in k we see that $A^{\alpha'}A_j v = 0$ for all $|\alpha'| = k-1$ and $A_j v \in E_{k-1}$ implies $A_j v = 0$, $j = 1, \dots, n$. Since $v \in E_k$, $k > 0$, this implies $v = 0$ so Proposition 3.8 is proved. □

Thus H has eigenspaces as described in (3.44). The same argument shows that for any integer p, positive or negative, the eigenvalues of H^p are precisely $(n+2k)^p$ with the same eigenspaces E_k. For $p < 0$, H^p is a compact operator on $L^2(\mathbb{R}^n)$; this is obvious for large negative p. For example, if $p \le -n-1$ then

eq:HO.16 (3.48)
$$x_i^\beta D_j^\alpha H \in \Psi^0_{\text{iso}}(\mathbb{R}^n),\ \ |\alpha| \le n+1, |\beta| \le n+1$$

are all bounded on L^2. If $S \subset L^2(\mathbb{R}^n)$ is bounded this implies that $H^{-n-1}(S)$ is bounded in $\langle x\rangle^{n+1}C^1_\infty(\mathbb{R}^n)$, so compact in $\langle x\rangle^n C^0_\infty(\mathbb{R}^n)$ and hence in $L^2(\mathbb{R}^n)$. It is a general fact that for compact self-adjoint operators, such as H^{-n-2}, the eigenfunctions span $L^2(\mathbb{R}^n)$. We give a brief proof of this for the sake of 'completeness'.

lem:HO.17 LEMMA 3.4. *The eigenfunction of H, $u_\alpha = \pi^{-\frac{n}{4}}(2^{|\alpha|}\alpha!)^{-1/2}C^\alpha u_0$ form an orthonormal basis of $L^2(\mathbb{R}^n)$.*

PROOF. Let $V \subset L^2(\mathbb{R}^n)$ be the closed subspace consisting of the orthocomplements of all the u_α's. Certainly H^{-n-2} acts on it as a compact self-adjoint operator. Since we have found all the eigenvalues of H, and hence of H^{-n-1}, it has no eigenvalue in V. We wish to conclude that $V = \{0\}$. Set

$$\tau = \|H^{-n-1}\|_V = \sup\{\|H^{-n-1}\varphi\|; \varphi \in V, \|\varphi\| = 1\}\,.$$

Then there is a weakly convergent sequence $\varphi_j \rightharpoonup \varphi$, $\|\varphi_j\| = 1$, so $\|\varphi\| \le 1$, with $\|H^{-n-1}\varphi_j\| \to \tau$. The compactness of H^{-n-2} allows a subsequence to be chosen such that $H^{-n-1}\varphi_j \to \psi$ in $L^2(\mathbb{R}^n)$. So, by the continuity of H^{-n-1}, $H^{-n-1}\varphi = \psi$ and $\|H^{-n-1}\varphi\| = \tau$, $\|\varphi\| = 1$. If $\varphi' \in V$, $\varphi' \perp \varphi$, $\|\varphi'\| = 1$ then

$$\tau^2 \ge \|H^{-n-2}\left(\frac{\varphi + t\varphi'}{\sqrt{1+t^2}}\right)\|^2 = \tau^2 + 2t\langle H^{-2n-2}\varphi, \varphi'\rangle + 0(t^2)$$
$$\Longrightarrow \langle H^{-2n-2}\varphi, \varphi'\rangle = 0 \Longrightarrow H^{-2n-2}\varphi = \tau^2\varphi\,.$$

This contradicts the fact that H^{-2n-2} has no eigenvalues in V, so $V = \{0\}$ and the eigenbasis is complete. □

Thus, if $u \in L^2(\mathbb{R}^n)$ then

eq:HO.18 (3.49)
$$u = \sum_\alpha c_\alpha u_\alpha, \quad c_\alpha = \langle u, u_\alpha \rangle \,. \tag{3.49}$$

lem:HO.19 LEMMA 3.5. *If $u \in \mathcal{S}(\mathbb{R}^n)$ the convergence in (3.49) is rapid, i.e., $|C_\alpha| \le C_N(1+|\alpha|)^{-N}$ for all N and the series converges in $\mathcal{S}(\mathbb{R}^n)$.*

PROOF. Since $u \in \mathcal{S}(\mathbb{R}^n)$ implies $H^N u \in L^2(\mathbb{R}^n)$ we see that

$$C_N \ge |\langle H^N u, u_\alpha \rangle| = |\langle u, H^N u_\alpha \rangle| = (n + 2|\alpha|)^N |c_\alpha| \;\forall\; \alpha\,.$$

Furthermore, $2ix_j = C_j - A_j$ and $2D_j = C_j + A_j$ so the polynomial derivatives of the u_α can be estimated (using the Sobolev embedding theorem) by polynomials in α; this implies that the series converges in $\mathcal{S}(\mathbb{R}^n)$. □

1.2.2000.407 COROLLARY 3.4. *Finite rank elements are dense in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ in the topology of $\mathcal{S}(\mathbb{R}^{2n})$.*

PROOF. Consider the approximation (3.49) to the kernel A of an element of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ as an element of $\mathcal{S}(\mathbb{R}^{2n})$. In this case the ground state is

$$U_0 = \exp\left(-\frac{|x|^2}{2} - \frac{|y|^2}{2}\right) = \exp\left(-\frac{|x|^2}{2}\right)\exp\left(-\frac{|y|^2}{2}\right)$$

and so has rank one as an operator. The higher eigenfunctions

$$C^\alpha U_0 = Q_\alpha(x,y) U_0$$

are products of U_0 and a polynomial, so are also of finite rank. □

3.8. L^2 boundedness and compactness

Recall that $\Psi^0_{\infty-\text{iso}}(\mathbb{R}^n) \subset \Psi^0_\infty(\mathbb{R}^n)$ so, by Proposition 2.6, these operators are bounded on $L^2(\mathbb{R}^n)$. Using the same argument the bound on the L^2 norm can be related to the norm of the principal symbol and an $N \times N$ matrix.

1.2.2000.357 PROPOSITION 3.9. *If $A \in \Psi^0_{\text{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ has principal symbol*

$$a = \sigma_L(A)\big|_{\mathbb{S}^{2n-1}} \in \mathcal{C}^\infty(\mathbb{S}^{2n-1}; M(N,\mathbb{C}))$$

then

1.2.2000.358 (3.50)
$$\inf_{E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n;\mathbb{C}^N)} \|A + E\|_{\mathcal{B}(L^2(\mathbb{R}^n;\mathbb{C}^N)} \le \sup_{p \in \mathbb{S}^{2n-1}} \|a(p)\|. \tag{3.50}$$

PROOF. It suffices to prove (3.50) for all single operators $A \in \Psi^0_{\text{iso}}(\mathbb{R}^n)$. Indeed if $j_v(z) = zv$ is the linear map from $\mathbb{C}$ to $\mathbb{C}^N$ defined by $v \in \mathbb{C}^N$ then

1.2.2000.363 (3.51)
$$\|A\|_{\mathcal{B}(L^2(\mathbb{R};\mathbb{C}^N))} = \sup_{\{v,w \in \mathbb{C}^N; \|v\| = \|w\| = 1\}} \|j^*_w A j_v\|_{\mathcal{B}(L^2(\mathbb{R}))}. \tag{3.51}$$

Since the symbol of $j^*_w A j_v$ is just $j^*_w \sigma(A) j_v$, (3.50) follows from the corresponding equality for a single operator:

1.2.2000.364 (3.52)
$$\inf_{E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)} \|A + E\|_{\mathcal{B}(L^2(\mathbb{R}^n)} \le \sup_{p \in \mathbb{S}^{2n-1}} |a(p)|,\ a = \sigma_L(A)\big|_{\mathbb{S}^{2n-1}}. \tag{3.52}$$

The construction of the approximate square-root of $C - A^*A$ in Proposition 2.7 only depends on the existence of a positive smooth square-root for $C - |a|^2$, so can be carried out for any

1.2.2000.359 (3.53)
$$C > \sup_{p \in \mathbb{S}^{2n-1}} |a(p)|^2. \tag{3.53}$$

Thus we conclude that with such a value of C

$$\|Au\|^2 \le C\|u\|^2 + \|\langle Gu, u\rangle| \ \forall\ u \in L^2(\mathbb{R}^n),$$

where $G \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Since G is an isotropic smoothing operator, for any $\delta > 0$ there is a finite dimensional subspace $W \subset \mathcal{S}(\mathbb{R}^n)$ such that

1.2.2000.360
$$\|\langle Gu, u\rangle\| \le \delta\|u\|^2 \ \forall\ u \in W^{\perp}. \tag{3.54}$$

Thus if we replace A by $A(\mathrm{Id} - \Pi_W) = A + E$ where E is a (finite rank) smoothing operator we see that

$$\|(A+E)u\|^2 \le (C+\delta)\|Gu\|^2 \ \forall\ u \in L^2(\mathbb{R}^n) \Longrightarrow \|(A+E)\| \le (C+\delta)^{\frac{1}{2}}.$$

This proves half of the desired estimate (3.51), namely

1.2.2000.362
$$\inf_{E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)} \|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \sup_{p \in \mathbb{S}^{2n-1}} |a(p)|. \tag{3.55}$$

To prove the opposite inequality, leading to (3.55), it is enough to arrive at a contradiction by supposing to the contrary that there is some $A \in \Psi^0_{\text{iso}}(\mathbb{R}^n)$ satisfying the strict inequality

$$\|A\|_{\mathcal{B}(L^2(\mathbb{R}^n))} < \sup_{p \in \mathbb{S}^{2n-1}} |a(p)|.$$

From this it follows that we may choose $c > 0$ such that $c = |a(p)|^2$ for some $p \in \mathbb{S}^{2n-1}$ and yet $A' = A^*A - c$ has a bounded inverse, B. By making an arbitrariy small perturbation of the full symbol of A' we may assume that it vanishes identically near p. By (3.55) we may choose $G \in \Psi^0_{\text{iso}}(\mathbb{R}^n)$ with arbitrariy small L^2 such that $\tilde{A} = A' + B$ has left symbol rapidly vanishing near p. When the norm of the perturbation is small enough, $\tilde{A}$ will still be invertible, with inverse $\tilde{B} \in \mathcal{B}(L^2(\mathbb{R}^n)$. Now choose an element $G \in \Psi^0_{\text{iso}}(\mathbb{R}^n)$ with left symbol supported sufficiently near p, so that $G \circ \tilde{A} \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ but yet the principal symbol of G should not vanish at p. Thus

$$G = G \circ \tilde{A} \circ \tilde{B} : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),$$
$$G* = G = \tilde{B}^* \circ \tilde{A}^* \circ G^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

It follows that $G^*G : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is an isotropic smoothing operator. This is the expected contradiction, since G, and hence G^*G, may be chosen to have non-vanishing principal symbol at p. Thus we have proved (3.55) and hence the Proposition. □

It is then easy to characterize the compact operators amongst the polyhomogeneous isotropic operators as those of negative.

1.2.2000.366 Lemma 3.6. *If $A \in \Psi^0_{\text{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ then, as an operator on $L^2(\mathbb{R}^n; \mathbb{C}^N)$, A is compact if and only if it has negative order.*

Proof. The necessity of vanishing of the principal symbol for a compact operator follows from Proposition 3.9 and the sufficiency follows from the density of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ in $\Psi^{-1}_{\text{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ in the topology of $\Psi^{-\frac{1}{2}}_{\infty-\text{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ and hence in the topology of bounded operators. Thus, such an operator is the norm limit of compact operators so itself is compact. □

Also as a consequence of Proposition 3.9 we can see the necessity of the assumption of ellipticity in Proposition 3.6.

1.2.2000.367 Corollary 3.5. *If $A \in \Psi^0_{\text{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then A is Fredholm as an operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ if and only if it is elliptic.*

3.9. Sobolev spaces

The space of square-integrable functions plays a basic rôle in the theory of distributions; one reason for this is that it is associated with the embedding of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$. We know that pseudodifferential operators of order 0 are bounded on $L^2(\mathbb{R}^n)$. There is also a natural collection of Sobolev spaces associated to the isotropic calculus, and another associated to the scattering calculus. The isotropic Sobolev space of order m may be defined as the collection of distributions mapped in $L^2(\mathbb{R}^n)$ by any one elliptic operator of order $-m$. Correspondingly the scattering Sobolev spaces have two orders.

Note that a differential operator $P(x, D_x)$ on $\mathbb{R}^n$ is an isotropic pseudodifferential operator if and only if its coefficients are polynomials. The fundamental symmetry between coefficients and differentiation suggest that the isotropic Sobolev spaces of non-negative integral order be defined by

1.2.2000.337
$$H^k_{\text{iso}}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); x^\alpha D^\beta_x u \in L^2(\mathbb{R}^n) \text{ if } |\alpha| + |\beta| \le k\},\ k \in \mathbb{N}. \tag{3.56}$$

The norms

1.2.2000.339
$$\|u\|^2_{k,\text{iso}} = \sum_{|\alpha|+|\beta|\le k} \int_{\mathbb{R}^n} |x^\alpha D^\beta_x u|^2 \, dx \tag{3.57}$$

turn these into Hilbert spaces. For negative integral orders we identify the isotropic Sobolev spaces with the duals of these spaces

1.2.2000.340
$$H^k_{\text{iso}}(\mathbb{R}^n) = (H^{-k}_{\text{iso}}(\mathbb{R}^n))' \hookrightarrow \mathcal{S}'(\mathbb{R}^n),\ k \in -\mathbb{N}. \tag{3.58}$$

The (continuous) injection into tempered distributions here arises from the density of the image of the inclusion $\mathcal{S}(\mathbb{R}^n) \longrightarrow H^k_{\text{iso}}(\mathbb{R}^n)$.

1.2.2000.338 Lemma 3.7. *For any $k \in \mathbb{Z}$,*

1.2.2000.341
$$\begin{aligned} H^k_{\text{iso}}(\mathbb{R}^n) &= \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n)\ \forall\ A \in \Psi^{-k}_{\text{iso}}\right\} \\ &= \left\{u \in \mathcal{S}'(\mathbb{R}^n); \exists\ A \in \Psi^{-k}_{\text{iso}} \textit{ elliptic and such that } Au \in L^2(\mathbb{R}^n)\right\} \end{aligned} \tag{3.59}$$

and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^k_{\text{iso}}(\mathbb{R}^n)$ is dense for each $k \in \mathbb{Z}$.

Proof. [3] For $k \in \mathbb{N}$, the functions $x^\alpha \xi^\beta$ for $|\alpha| + |\beta| = k$ are 'collectively elliptic' in the sense that

1.2.2000.344
$$q_k(x,\xi) = \sum_{|\alpha|+|\beta|=k} (x^\alpha \xi^\beta)^2 \ge c(|x|^2 + |\xi|^2)^k,\ c > 0. \tag{3.60}$$

Thus $Q_k = \sum\limits_{|\alpha|+|\beta|\le k} (D^\beta x^\alpha x^\alpha D^\beta) \in \Psi^{2k}_{\text{iso}}(\mathbb{R}^n)$, which has principal reduced symbol q_k, has a left parameterix $A_k \in \Psi^{-2k}_{\text{iso}}(\mathbb{R}^n)$. This gives the identity

1.2.2000.345
$$\sum_{|\alpha|+|\beta|\le k} R_{\alpha,\beta} x^\alpha D^\beta = A_k Q_k = \operatorname{Id} + E, \text{ where} \tag{3.61}$$
$$R_{\alpha,\beta} = A_k D^\beta x^\alpha \in \Psi^{-2k+|\alpha|+|\beta|}_{\text{iso}}(\mathbb{R}^n),\ E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n).$$

[3] This is an essentially microlocal proof.

Thus if $A \in \Psi^k_{\text{iso}}(\mathbb{R}^n)$

$$Au = -AEu + \sum_{|\alpha|+|\beta|\le k} AR_{\alpha,\beta}x^\alpha D^\beta u.$$

If $u \in H^k_{\text{iso}}(\mathbb{R}^n)$ then by definition $x^\alpha D^\beta u \in L^2(\mathbb{R}^n)$. By the boundedness of operators of order 0 on L^2, all terms on the right are in $L^2(\mathbb{R}^n)$ and we have shown the inclusion of $H^k_{\text{iso}}(\mathbb{R}^n)$ in the first space space on the right in (3.59). The converse is immediate, so this proves the first equality in (3.59) for $k > 0$. Certainly the third space in (3.59) contains in the second. The existence of elliptic parametrix B for the ellipic operator A proves the converse since any isotropic pseudodifferential operator of order A' of order k can be effectively factorized as

$$A' = A'(BA+E) = B'A + E',\ B' \in \Psi^0_{\infty-\text{iso}}(\mathbb{R}^n),\ E' \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n).$$

Thus, $Au \in L^2(\mathbb{R}^n)$ implies that $A'u \in L^2(\mathbb{R}^n)$.

It also follows from second identification that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^k_{\text{loc}}(\mathbb{R}^n)$. Thus, if $Au \in L^2(\mathbb{R}^n)$ and we choose $f_n \in \mathcal{S}(\mathbb{R}^n)$ with $f_n \to Au$ in $L^2(\mathbb{R}^n)$ then, with B a parametrix for A, $u'_n = Bf_n \to BAu = u + Eu$. Thus $u_n = u'_n - Eu \in \mathcal{S}(\mathbb{R}^n) \to u$ in $L^2(\mathbb{R}^n)$ and $Au_n \to u$ in $L^2(\mathbb{R}^n)$ proving the density.

The Riesz representation theorem shows that $v\mathcal{S}'(\mathbb{R}^n)$ is in the dual space, $H^{-k}_{\text{iso}}(\mathbb{R}^n)$, if and only if there exists $v' \in H^k_{\text{iso}}(\mathbb{R}^n)$ such that

1.2.2000.343 (3.62) $$v(u) = \langle u, v'\rangle_{k,\text{iso}} = \langle u, Q_{2k}v'\rangle_{L^2},\ \forall\ u \in \mathcal{S}(\mathbb{R}^n) \hookrightarrow H^k_{\text{iso}}(\mathbb{R}^n)$$
$$\text{with } Q_{2k} = \sum_{|\alpha|+|\beta|\le k} D^\beta x^{2\alpha} D^\beta.$$

This shows that Q_{2k} is an isomorphism of $H^k_{\text{iso}}(\mathbb{R}^n)$ onto $H^{-k}_{\text{iso}}(\mathbb{R}^n)$ as subspaces of $\mathcal{S}'(\mathbb{R}^n)$. Notice that $Q_{2k} \in \Psi^{2k}_{\text{iso}}(\mathbb{R}^n)$ is elliptic, self-adjoint and invertible, since it is strictly positive. This now gives the same identification (3.59) for $k < 0$.

The case $k = 0$ follows directly from the L^2 boundedness of operators of order 0 so the proof is complete. □

In view of this identification we define the isotropic Sobolev spaces or any real order the same way

1.2.2000.347 (3.63) $$H^s_{\text{iso}}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n)\ \forall\ A \in \Psi^{-s}_{\text{iso}}\right\},\ s \in \mathbb{R}.$$

These are Hilbertable spaces, with the Hilbert norm being given by $\|Au\|_{L^2(\mathbb{R}^n}$ for any $A \in \Psi^s_{\text{iso}}(\mathbb{R}^n)$ which is elliptic and invertible.

1.2.2000.346 Proposition 3.10. *Any element $A \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$, $m \in \mathbb{R}$, defines a bounded linear operator*

1.2.2000.348 (3.64) $$A : H^s_{\text{iso}}(\mathbb{R}^n) \longrightarrow H^{s-m}_{\text{iso}}(\mathbb{R}^n),\ \forall\ s \in \mathbb{R}.$$

This operator is Fredholm if and only if A is elliptic. For any $s \in \mathbb{R}$, $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^s_{\text{iso}}(\mathbb{R}^n)$ is dense and $H^{-s}_{\text{iso}}(\mathbb{R}^n)$ may be identified as the dual of $H^s_{\text{iso}}(\mathbb{R}^n)$ with respect to the continuous extension of the L^2 pairing.

Proof. A straightforward application of the calculus, with the exception of the necessity of ellipticity for an isotropic pseudodifferential operator to be Fredholm. This is discussed in the problems beginning at Problem 3.10. □

sec:TG

3.10. The residual group

By definition, $\mathcal{G}^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is the set (if you want to be concrete you can think of them as operators on $L^2(\mathbb{R}^n)$) of invertible elements of the ring $\text{Id} + \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. If we identify this topologically with $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then, as follows from Corollary 3.1, $\mathcal{G}^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is open. We will think of it as an infinite-dimensional manifold modeled, of course, on the linear space $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n})$. Since I have no desire to get too deeply into the general theory of such Fréchet manifolds I will keep the discussion as elementary as possible. 1.2.2000.330

The dual space of $\mathcal{S}(\mathbb{R}^p)$ is $\mathcal{S}'(\mathbb{R}^p)$. If we want to think of $\mathcal{S}(\mathbb{R}^p)$ as a manifold we need to consider smooth functions and forms on it. In the finite-dimensional case, the exterior bundles are the antisymmetric parts of the tensor powers of the dual. Since we are in infinite dimensions the tensor power needs to be completed and the usual choice is the 'projective' tensor product. In our case this is something quite simple namely the k-fold completed tensor power of $\mathcal{S}'(\mathbb{R}^p)$ is just $\mathcal{S}'(\mathbb{R}^{kp})$. Thus we set

TG.1
$$\Lambda^k\mathcal{S}(\mathbb{R}^p) = \{u \in \mathcal{S}'(\mathbb{R}^{kp});\ \text{for any permutation}\ e,\ u(x_{e(1)}, \dots x_{e(h)}) = \operatorname{sgn}(e)u(x_1, \dots x_k)\}\,. \tag{3.65}$$

In view of this it is enough for us to consider smooth functions on open sets $F \subset \mathcal{S}(\mathbb{R}^p)$ with values in $\mathcal{S}'(\mathbb{R}^p)$ for general p. Thus

TG.2
$$v : F \longrightarrow \mathcal{S}'(\mathbb{R}^p),\ F \subset \mathcal{S}(\mathbb{R}^n) \text{ open} \tag{3.66}$$

is continuously differentiable on F if there exists a continuous map

$$v' : F \longrightarrow \mathcal{S}'(\mathbb{R}^{n+p}) \text{ and each } u \in F \text{ has a neighbourhood } U \text{ such that for each } N\ \exists\ M \text{ with}$$
$$\|v(u+u') - v(u) - v'(u;u')\|_N \le C\|u'\|^2_M,\ \forall\ u, u+u' \in U.$$

Then, as usual we define smoothness as infinite differentiability by iterating this definition. The smoothness of v in this sense certainly implies that if $f : X \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is smooth then $v \circ F$ is smooth.

Thus we define the notion of a smooth *form* on $F \subset \mathcal{S}(\mathbb{R}^n)$, an open set, as a smooth map

TG.3
$$\alpha : F \to \Lambda^k\mathcal{S}(\mathbb{R}^p) \subset \mathcal{S}'(\mathbb{R}^{kp})\,. \tag{3.67}$$

In particular we know what smooth forms are on $\mathcal{G}^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

The de Rham differential acts on forms as usual. If $v : F \to \mathbb{C}$ is a function then its differential at $f \in F$ is $dv : F \longrightarrow \mathcal{S}'(\mathbb{R}^n) = \Lambda^1\mathcal{S}(\mathbb{R}^n)$, just the derivative. As in the finite-dimensional case d extends to forms by enforcing the condition that $dv = 0$ for constant forms and the identity distribution over exterior products

TG.5
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge d\beta\,. \tag{3.68}$$

S.Representations

3.11. Representations

In §1.9 (Sect.radial.compactification) the compactification of Euclidean space to a ball, or half-sphere, is described. We make the following definition, recalling that $\rho \in \mathcal{C}^\infty(\mathbb{S}^{n,+})$ is a boundary defining function.

28.2.1998.153 Definition 3.2. *The space of of 'Laurent functions' on the half-sphere is*

28.2.1998.154
$$\mathcal{L}(\mathbb{S}^{n,+}) = \bigcup_{k\in\mathbb{N}_0} \rho^{-k}\mathcal{C}^\infty(\mathbb{S}^{n,+}), \quad \rho^{-k}\mathcal{C}^\infty(\mathbb{S}^{n,+}) = \{u \in \mathcal{C}^\infty(\operatorname{int}(\mathbb{S}^{n,+})); \rho^k u \in \mathcal{C}^\infty(\mathbb{S}^{n,+}).\tag{3.69}$$

More generally if $m \in \mathbb{R}$ we denote by $\rho^m\mathcal{C}^\infty(\mathbb{S}^{n,+})$ the space of functions which can be written as products $u = \rho^m v$, with $v \in \mathcal{C}^\infty(\mathbb{S}^{n,+})$; again it can be identified with a subspace of the space of $\mathcal{C}^\infty$ functions on the open half-sphere.

28.2.1998.155 Proposition 3.11. *The compactification map (1.94) extends from (1.96) to give, for each $m \in \mathbb{R}$, an identification of $\rho^{-m}\mathcal{C}^\infty(\mathbb{S}^{n,+})$ and $S^m_{\mathrm{cl}}(\mathbb{R}^n)$.*

Thus, the fact that the $\Psi^{\mathbb{Z}}_{\mathrm{iso,cl}}(\mathbb{R}^n)$ form an order-filtered $*$-algebra means that $\rho^{\mathbb{Z}}\mathcal{C}^\infty(\mathbb{S}^{2n,+})$ has a non-commutative product defined on it, with $\mathcal{C}^\infty(\mathbb{S}^{2n,+})$ a subalgebra, using the left symbol isomorphism, followed by compactification.

S.Symplectic.invariance

3.12. Symplectic invariance of the isotropic product

The composition law for the isotropic calculus, and in particular for it smoothing part, is derived from its identification as a subalgebra of the (weighted) spaces of pseudodifferential operator on $\mathbb{R}^n$. There is a much more invariant formulation of the product which puts into evidence more of the invariance properties.

Let W be a real symplectic vector space. Thus, W is a vector space equipped with a real, antisymmetic and non-degenerate bilinear form

23.3.1998.174
$$\omega : W \times W \longrightarrow \mathbb{R},\ \omega(w_1, w_2) + \omega(w_2, w_1) = 0\ \forall\ w_1, w_2 \in W, \quad \omega(w_1, w) = 0\ \forall\ w \in W \Longrightarrow w_1 = 0.\tag{3.70}$$

A Lagrangian subspace of W is a vector space $V \subset W$ such that ω vanishes when restricted to V and such that $2\dim V = \dim W$.

23.3.1998.175 Lemma 3.8. *Every symplectic vector space has a Lagrangian subspace and for any choice of Lagrangian subspace U_1 there is a second Lagrangian subspace U_2 such that $W = U_1 \oplus U_2$ is a* Lagrangian decomposition.

Proof. First we show that there is a Lagrangian subspace. If $\dim W > 0$ then the antisymmetry of ω shows that any 1-dimensional vector subspace is *isotropic*, that is ω vanishes when restricted to it. Let V be a maximal isotropic subspace, that is an isotropic subspace of maximal dimension amongst isotropic subspaces. Let U be a complement to V in W. Then

23.3.1998.176
$$\omega : V \times U \longrightarrow \mathbb{R}\tag{3.71}$$

is a non-degenerate pairing. Indeed $u \in U$ and $\omega(v, u) = 0$ for all $v \in V$ then $V + \mathbb{R}\{u\}$ is also isotropic, so $u = 0$ by the assumed maximality. Similarly if $v \in V$ and $\omega(v, u) = 0$ for all $u \in U$ then, recalling that ω vanishes on V, $\omega(v, w) = 0$ for all $w \in W$ so $v = 0$. The pairing (3.71) therefore identifies U with V', the dual of V. In particular $\dim w = 2\dim V$.

Now, choose any Lagrangian subspace U_1. We proceed to show that there is a complementary Lagrangian subspace. Certainly there is a 1-dimensional subspace which does not meet U_1. Let V be an isotropic subspace which does not meet U_1 and is of maximal dimension amongst such subspaces. Suppose that $\dim V < \dim U_1$. Choose $w \in W$ with $w \notin V \oplus U_1$. Then $V \ni v \longrightarrow \omega(w, v)$ is a linear functional on U_1. Since U_1 can be completed to a complement, any such linear functional can be written $\omega(u_1, v)$ for some $u_1 \in U_1$. It follows that $\omega(w - u_1, v) = 0$ for all $v \in V$. Thus $V \oplus \mathbb{R}\{w - u_1\}$ a non-trivial isotropic extension of V, contradicting the assumed maximality. Thus $V = U_2$ is a complement of U_1. $\square$

Given such a Lagrangian decomposition of the symplectic vector space W, let $X_1, \dots X_n$ be a basis for the dual of U_1, and let $\Xi_1, \dots, \Xi_n$ be the dual basis, of U_1 itself. The pairing (3.71) with $U = U_1$ and $V = U_2$ identifies $U_2 = U_1'$ so the Ξ_i can also be regarded as a basis of the dual of U_2. Thus $X_1 \dots X_n, \Xi_1, \dots, \Xi_n$ gives a basis of $W' = U_1' \oplus U_2'$. The symplectic form can then be written

23.3.1998.178 (3.72)
$$\omega(w_1, w_2) = \sum_{i=1}^{n} (\Xi_i(w_1) X_i(w_2) - \Xi_i(w_2) X_i(w_1)).$$

This is the *Darboux* form of ω. If the X_i, Ξ_i are thought of as linear functions x_i, ξ_i on W, now considered as a manifold then these are *Darboux coordinates* in which(3.72) becomes

23.3.1998.179 (3.73)
$$\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i.$$

The symplectic form ω defines a volume form on W, namely the n-fold wedge product ω^n. In Darboux coordinates this is just, up to sign, the Lebesgue form $d\xi dx$.

23.3.1998.180 Proposition 3.12. *On any symplectic vector space, W, the bilinear map on $\mathcal{S}(W)$,*

23.3.1998.181 (3.74)
$$a\#b(w) = (2\pi)^{-2n} \int_{W^2} e^{i\omega(w_1, w_2)} a(w + w_1) b(w + w_2) \omega^n(w_1) \omega^2(w_2), \ \dim W = 2n$$

defines an associative product isomorphic to the composition of $\Psi^{-\infty}_{\text{iso}}(U_1)$ for any Lagrangian decomposition $W = U_1 \oplus U_2$.

23.3.1998.182 Corollary 3.6. *Extended by continuity in the symbol space (3.74) defines a filtered product on $S^\infty(W)$ which is isomorphic to the isotropic algebra on $\mathbb{R}^{2n}$ and is invariant under symplectic linear transformation of W.*

Proof. Written in the form (3.74) the symplectic invariance is immediate. That is, if F is a linear transformation of W which preserves the symplectic form, $\omega(Fw_1, Fw_2) = \omega(w_1, w_2)$ then

23.3.1998.183 (3.75)
$$F^*(a\#b) = (F^*a)\#(F^*b) \ \forall \ a, b \in \mathcal{S}(W).$$

The same result holds for general symbols once the continuity is established.

Let us start from the Weyl quatization of the isotropic algebra. As usual for computations we may assume that the amplitudes are of order $-\infty$. Thus, $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ may be written

23.3.1998.184 (3.76)
$$Au(x) = \int A(x, y) u(y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(\frac{1}{2}(x+y), \xi) u(y) dy d\xi.$$

Both the kernel $A(x,y)$ and the amplitude $a(x,\xi)$ are elements of $\mathcal{S}(\mathbb{R}^{2n})$. The relationship (3.76) and its inverse may be written

23.3.1998.185 (3.77)
$$A(s+\frac{t}{2},s-\frac{t}{2})=(2\pi)^{-n}\int e^{it\cdot\xi}a(s,\xi)d\xi,$$
$$a(x,\xi)=\int e^{-it\cdot\xi}A(x+\frac{t}{2},x-\frac{t}{2})dt.$$

If A has Weyl symbol a and B has Weyl symbol b let c be the Weyl symbol of the composite $A\circ B$. Using (3.77) and (3.76)

$$c(s,\zeta)=\int e^{-it\cdot\zeta}A(s+\frac{t}{2},z)B(z,s-\frac{t}{2})dt$$
$$=(2\pi)^{-2n}\int\int\int dt\,dz\,d\xi\,d\eta e^{i\Phi}a(\frac{s}{2}+\frac{t}{4}+\frac{z}{2},\xi)a(\frac{z}{2}+\frac{s}{2}-\frac{t}{4},\eta)$$
$$\text{where }\Phi=-t\cdot\zeta+(s+\frac{t}{2}-z)\cdot\xi+(z-s+\frac{t}{2})\cdot\eta.$$

Changing variables of integration to $X=\frac{z}{2}+\frac{t}{4}-\frac{s}{2}$, $Y=\frac{z}{2}-\frac{t}{4}-\frac{s}{2}$, $\Xi=\xi-\zeta$ and $H=\eta-\zeta$ this becomes

$$c(s,\zeta)=(2\pi)^{-2n}4^n\int\int\int dY\,dX\,d\Xi\,dH$$
$$e^{2i(X\cdot H-Y\cdot\Xi)}a(X+s,\Xi+\zeta)a(Y+s,H+\zeta).$$

This reduces to (3.74), written out in Darboux coordinates, after the change of variable $H'=2H$, $\Xi'=2\Xi$ and $\zeta'=2\zeta$. Thus the precise isomorphism with the product in Weyl form is given by

23.3.1998.186 (3.78)
$$A(x,y)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a_\omega(\frac{1}{2}(x+y),2\xi)u(y)dyd\xi$$

so that composition of kernels reduces to (3.74). □

Discuss metaplectic group here.

S.Complex.order

3.13. Complex order

The identification of polyhomogeneous symbols of order zero on $\mathbb{R}^{2n}$ with the smooth functions on the radial compactification allows us to define the isotropic operators of a given complex order $z\in\mathbb{C}$. Namely, we use the left quantization map to identify

1.2.2000.334 (3.79)
$$\Psi^z_{\text{iso}}(\mathbb{R}^n)=\rho^{-z}\mathcal{C}^\infty(\mathbb{S}^{2n,1})\subset\Psi^{\Re z}_{\infty-\text{iso}}(\mathbb{R}^n).$$

Here, $\rho\in\mathcal{C}^\infty(\mathbb{S}^{2n,1})$ is a boundary defining function. Any other boundary defining function is of the form $a\rho$ with $0<a\in\mathcal{C}^\infty(\mathbb{S}^{2n,1})$. It follows that the definition is independent of the choice of ρ since $a^z\in\mathcal{C}^\infty(\mathbb{S}^{2n,1})$ for any $z\in\mathbb{Z}$.

In fact it is even more useful to consider holomorphic families. Thus if $\Omega\subset\mathbb{C}$ is an open set and $h:\Omega\longrightarrow\mathbb{C}$ is holomorphic then we may consider holomorphic families of order h as elements of

1.2.2000.335 (3.80)
$$\Psi^{h(z)}_{\text{iso}}(\mathbb{R}^{2n})=\big\{A:\Omega\longrightarrow\Psi^\infty_{\infty-\text{iso}}(\mathbb{R}^{2n});$$
$$\Omega\ni z\longmapsto\rho^{h(z)}A(z)\in\mathcal{C}^\infty(\mathbb{S}^{2n,1})\text{ is holomorphic.}\big\}$$

Note that a map from $\Omega \subset \mathbb{C}$ into $\mathcal{C}^\infty(\mathbb{S}^{2n1,})$ is said to be holomorphic it is defines an element of $\mathcal{C}^\infty(\Omega \times \mathbb{S}^{2n,1})$ which satisies the Cauchy-Riemann equation in the first variable.

1.2.2000.336

Proposition 3.13. *If h and g are holomorphic functions on an open set $\Omega \subset \mathbb{C}$ and $A(z)$, $B(z)$ are holomorphic familes of isotropic operators of orders $h(z)$ and $g(z)$ then the composite family $A(z) \circ B(z)$ is holomorphic of order $h(z) + g(z)$.*

Proof. It suffices to consider an arbitrary open subset $\Omega' \subset \Omega$ with compact closure inside Ω. Then h and g have bounded real parts, so $A(z), B(z) \in \Psi^M_{\infty-\mathrm{iso}}(\mathbb{R}^{2n})$ for $z \in \Omega'$ for some fixed M. It follows that the composite $A(z) \circ B(z) \in \Psi^{2M}_{\infty-\mathrm{iso}}(\mathbb{R}^{2n})$. The symbol is given by the usual formula. Furthermore

□

S.Traces.residual

3.14. Traces on the residual algebra

The algebras we are studying are topological algebras, so it makes sense to consider continuous linear functionals on them. The most important of these is the *trace*. To remind you what it is we consider first its properties for matrix algebras.

Let $M(N;\mathbb{C})$ denote the algebra of $N \times N$ complex matrices. We can simply define

eq:1

$$\mathrm{Tr} : M(N;\mathbb{C}) \to \mathbb{C}, \quad \mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii} \tag{3.81}$$

as the sum of the diagonal entries. The fundamental property of this functional is that

eq:2

$$\mathrm{Tr}([A,B]) = 0 \ \forall \ A, B \in M(N;\mathbb{C})\,. \tag{3.82}$$

To check this it is only necessary to write down the definition of the composition in the algebra. Thus

$$(AB)_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj}\,.$$

It follows that

$$\begin{aligned} \mathrm{Tr}(AB) &= \sum_{i=1}^{N} (AB)_{ii} = \sum_{i,k=1}^{N} A_{ik} B_{ki} \\ &= \sum_{k=1}^{N} \sum_{i=1}^{N} B_{ki} A_{ik} = \sum_{k=1}^{N} (BA)_{kk} = \mathrm{Tr}(BA) \end{aligned}$$

which is just (3.82).

Of course any multiple of Tr has the same property (3.82) but the normalization condition

eq:3

$$\mathrm{Tr}(\mathrm{Id}) = N \tag{3.83}$$

distinguishes it from its multiples. In fact (3.82) and (3.83) together distinguish $\mathrm{Tr} \in M(N;\mathbb{C})'$ as a point in the N^2 dimensional linear space which is the dual of $M(N;\mathbb{C})$.

lem:trace

Lemma 3.9. *If $F : M(N;\mathbb{C}) \to \mathbb{C}$ is a linear functional satisfying (3.82) and $B \in M(N;\mathbb{C})$ is any matrix such that $F(B) \neq 0$ then $F(A) = \frac{F(B)}{\mathrm{Tr}(B)} \mathrm{Tr}(A)$.*

Proof. Consider the basis of $M(N;\mathbb{C})$ given by the elementary matrices E_{jk}, where E_{jk} has jk-th entry 1 and all others zero. Thus

$$E_{jk}E_{pq} = \delta_{kp}E_{jq}.$$

If $j \neq k$ it follows that

$$E_{jj}E_{jk} = E_{jk},\ E_{jk}E_{jj} = 0.$$

Thus

$$F([E_{jj}, E_{jk}]) = F(E_{jk}) = 0 \text{ if } j \neq k.$$

On the other hand, for any i and j

$$E_{ji}E_{ij} = E_{jj},\ E_{ij}E_{ji} = E_{ii}$$

so

$$F(E_{jj}) = F(E_{11})\ \forall\ j.$$

Since the E_{jk} are a basis,

$$\begin{aligned} F(A) &= F(\sum_{j,k=1}^{N} A_{ij}E_{ij}) \\ &= \sum_{j,l=1}^{N} A_{jj}F(E_{ij}) \\ &= F(E_{11})\sum_{j=1}^{N} A_{jj} = F(E_{11})\,\mathrm{Tr}(A). \end{aligned}$$

This proves the lemma. □

For the isotropic smoothing algebra we have a similar result.

isotropic trace

Proposition 3.14. *If* $F : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathbb{C}$ *is a continuous linear functional satisfying*

eq:4 (3.84)
$$F([A,B]) = 0\ \forall\ A, B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$$

then $F([A,B]) = 0$ *for all* $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *and* $B \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *and* F *is a constant multiple of the functional*

eq:5 (3.85)
$$\mathrm{Tr}(A) = \int_{\mathbb{R}^n} A(x,x)\,dx.$$

Proof. Recall that $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is an ideal so $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ implies that $AB,\ BA \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and it follows that the equality $F(AB) = F(BA)$, or $F([A,B]) = 0$, is meaningful. To see that it holds we just use the continuity of F. We know that if $B \in \Psi^{\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then there is a sequence $B_n \to B$ in the topology of $\Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n)$ for some m. Since this implies $AB_n \to AB$, $B_nA \to BA$ in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ we see that

$$F([A,B]) = \lim_{n\to\infty} F([A,B_n]) = 0\,.$$

We use this identity to prove (3.85) [eq:5]. Take $B = x_j$ or D_j, $j = 1, \dots, n$. Thus for any $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$

$$F([A, x_j]) = F([A, D_j]) = 0\,.$$

Now consider F as a distribution acting on the kernel $A \in \mathcal{S}(\mathbb{R}^{2n})$. Since the kernel of $[A, x_j]$ is $A(x, y)(y_j - x_j)$ and the kernel of (A, D_j) is $-(D_{y_j} + D_{x_j})A(x, y)$ we conclude that, as an element of $\mathcal{S}'(\mathbb{R}^{2n})$, F satisfies

$$(x_j - y_j)F(x, y) = 0,\ (D_{x_j} + D_{y_j})F(x, y) = 0.$$

If we make the linear change of variables to $p_i = \frac{x_i+y_i}{2}$, $q_i = x_i - y_i$ and set $\tilde{F}(p, q) = F(x, y)$ these conditions become

$$D_{q_i}\tilde{F} = 0,\ p_i\tilde{F} = 0,\ i = 1, \dots, N.$$

As we know from Lemmas 1.2 and 1.3, this implies that $\tilde{F} = c\delta(p)$ so

$$F(x, y) = c\delta(x - y)$$

as a distribution. Clearly $\delta(x - y)$ gives the functional Tr defined by (3.85), so the proposition is proved. □

We still need to justify the use of the same notation, Tr, for these two functionals. However, if $L \subset \mathcal{S}(\mathbb{R}^n)$ is any finite dimensional subspace we may choose an orthonal basis $\varphi_i \in L$, $i = 1, \dots, l$,

$$\int_{\mathbb{R}^n} |\varphi_i(x)|^2\, dx = 0,\ \int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x)\, dx = 0,\ i \neq j\,.$$

Then if a_{ij} is an $l \times l$ matrix,

$$A = \sum_{i,j=1}^{\ell} a_{ij}\varphi_i(x)\overline{\varphi_j(y)} \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)\,.$$

From (3.85) we see that

$$\begin{aligned}\operatorname{Tr}(A) &= \sum_{ij} a_{ij} \operatorname{Tr}(\varphi_i\bar{\varphi}_j)\\ &= \sum_{ij} a_{ij} \int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x)\, dx\\ &= \sum_{i=1}^{n} a_{ii} = \operatorname{Tr}(a)\,.\end{aligned}$$

Thus the two notions of trace coincide. In any case this already follows, up to a constant, from the uniqueness in Lemma 3.9.

3.15. Fredholm determinant

For $N \times N$ matrices, the linear space of which we denote $M(N; \mathbb{C})$, the determinant is a multiplicative polynomial map

1.2.2000.404

$$\det : M(N; \mathbb{C}) \longrightarrow \mathbb{C},\ \det(AB) = \det(A)\det(B),\ \det(\mathrm{Id}) = 1. \tag{3.86}$$

It is not quite determined by these conditions, since $\det(A)^k$ also satisfies then. The fundamental property of the determinant is that it defines the group of invertible elements

1.2.2000.405

$$\mathrm{GL}(N, \mathbb{C}) = \{A \in M(N; \mathbb{C}); \det(A) \neq 0\}. \tag{3.87}$$

A reminder of a direct definition is given in Problem 3.7.

The Fredholm determinant is an extension of this definition to a function on the ring $\mathrm{Id} + \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$, or even further to $\mathrm{Id} + \Psi^{-2n-\epsilon}_{\infty-\text{iso}}(\mathbb{R}^n)$ for $\epsilon > 0$. This can be done in several ways using the density of finite rank operators, as shown in Corollary 3.4. We proceed by generalizing the formula relating the determinant to the trace. Thus, for any smooth curve with values in $\mathrm{GL}(N;\mathbb{C})$ for any N,

$$\frac{d}{ds}\det(A_s)=\det(A_s)\operatorname{tr}(A_s^{-1}\frac{A_s}{ds}). \tag{3.88}$$

In particular if (3.86) is augmented by the normalization condition

$$\frac{d}{ds}\det(\mathrm{Id}+sA)\big|_{s=0}=\operatorname{tr}(A)\ \forall\ A\in M(N;\mathbb{C}) \tag{3.89}$$

then it is determined.

A branch of the logarithm can be introduced along any curve, smoothly in the parameter, and then (3.88) can be rewritten

$$d\log\det(A)=\operatorname{tr}(A^{-1}dA). \tag{3.90}$$

Here $\mathrm{GL}(N;\mathbb{C})$ is regarded as a subset of the linear space $M(N;\mathbb{C})$ and dA is the canonical identification, at the point A, of the tangent space to $M(N,\mathbb{C})$ with $M(N,\mathbb{C})$ itself. This just arises from the fact that $M(N,\mathbb{C})$ is a linear space. Thus $dA(\frac{d}{ds}(A+sB)\big|_{s=0}=B$. This allows the expression on the right in (3.90) to be interpreted as a smooth 1-form on the manifold $\mathrm{GL}(N;\mathbb{C})$. Note that it is independent of the local choice of logarithm.

To define the Fredholm determinant we shall extend the 1-form

$$\alpha=\operatorname{Tr}(A^{-1}dA) \tag{3.91}$$

to the group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)\hookrightarrow\mathrm{Id}+\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Here dA has essentially the same meaning as before, given that Id is fixed. Thus at any point $A=\mathrm{Id}+B\in\mathrm{Id}+\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ it is the identification of the tangent space with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ using the linear structure:

$$dA(\frac{d}{ds}(\mathrm{Id}+B+sE)\big|_{s=0})=E,\ E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Since dA takes values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, the trace functional in (3.91) is well defined.

The 1-form α is closed. In the finite-dimensional case this follows from (3.90). For (3.91) we can compute directly. Since $d(dA)=0$, essentially by definition, and

$$dA^{-1}=-A^{-1}dAA^{-1} \tag{3.92}$$

we see that

$$d\alpha=-\operatorname{Tr}(A^{-1}(dA)A^{-1}(dA))=0. \tag{3.93}$$

Here we have used the trace identity, and the antisymmetry of the implicit wedge product in (3.93), to conlcude that $d\alpha=0$. For a more detailed discussion of this point see Problem 3.8.

From the fact that $d\alpha=0$ we can be confident that there is, locally near any point of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, a function f such that $df=\alpha$; then we will define the Fredholm determinant by $\det_{\mathrm{Fr}}(A)=\exp(f)$. To define $\det_{\mathrm{Fr}}$ globally we need to see that this is well defined.

Lemma 3.10. *For any smooth closed curve $\gamma:\mathbb{S}^1\longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ the integral*

$$\int_\gamma\alpha=\int_{\mathbb{S}^1}\gamma^*\alpha\in2\pi i\mathbb{Z}. \tag{3.94}$$

That is, α defines an integral cohomology class, $[\frac{\alpha}{2\pi i}]\in H^1(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n);\mathbb{Z})$.

Proof. This is where we use the approximability by finite rank operators. If π_N is the orthogonal projection onto the span of the eigenspaces of the smallest N eigenvalues of the harmonic oscillator then we know from Section 3.7 that $\pi_N E \pi_N \to E$ in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ for any element. In fact it follows that for the smooth curve that $\gamma(s) = \mathrm{Id} + E(s)$ and $E_N(s) = \pi_N E(s) \pi_N$ converges uniformly with all s derivatives. Thus, for some N_0 and all $N > N_0$, $\mathrm{Id} + E_N(s)$ is a smooth curve in $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and hence $\gamma_N(s) = \mathrm{Id}_N + E_N(s)$ is a smooth curve in $\mathrm{GL}(N;\mathbb{C})$. Clearly

iml.11
$$\int_{\gamma_N} \alpha \longrightarrow \int_{\gamma} \alpha \text{ as } N \to \infty, \tag{3.95}$$

and for finite N it follows from the identity of the trace with the matrix trace (see Section 3.14) that $\int_N \gamma_N^* \alpha$ is the variation of $\arg \log \det(\gamma_N)$ around the curve. This gives (3.94). □

Now, once we have (3.94) and the connectedness of $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ we may define

iml.12
$$\det{}_{\mathrm{Fr}}(A) = \exp(\int_{\gamma} \alpha),\ \gamma : [0,1] \longrightarrow G^{-\infty}_{\text{iso}}(\mathbb{R}^n),\ \gamma(0) = \mathrm{Id},\ \gamma(1) = A. \tag{3.96}$$

Indeed, Lemma 3.10 shows that this is independent of the path chosen from the identity to A. Notice that the connectedness of $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ follows from the connectedness of the $\mathrm{GL}(N,\mathbb{C})$ and the density argument above.

The same arguments and results apply to $G^{-2n-\epsilon}_{\infty-\text{iso}}(\mathbb{R}^n)$ using the fact that the trace functional extends continuously to $\Psi^{-2n-\epsilon}_{\infty-\text{iso}}(\mathbb{R}^n)$ for any $\epsilon > 0$.

iml.13 Proposition 3.15. *The Fredholm determinant, defined by* (3.96) *on* $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ *(or* $G^{-2n-\epsilon}_{\text{iso}}(\mathbb{R}^n)$ *for* $\epsilon > 0$*) and to be zero on the complement in* $\mathrm{Id} + \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ *(or* $\mathrm{Id} + \Psi^{-2n-\epsilon}_{\text{iso}}(\mathbb{R}^n)$*) is an entire function satisfying*

iml.14
$$\det{}_{\mathrm{Fr}}(AB) = \det{}_{\mathrm{Fr}}(A) \det{}_{\mathrm{Fr}}(B),\ A, B \in \mathrm{Id} + \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \\ (\textit{or}\ \mathrm{Id} + \Psi^{-2n-\epsilon}_{\text{iso}}(\mathbb{R}^n)),\ \det{}_{\mathrm{Fr}}(\mathrm{Id}) = 1. \tag{3.97}$$

Proof. We start with the multiplicative property of $\det_{\mathrm{Fr}}$ on $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus is $\gamma_1(s)$ is a smooth curve from Id to A_1 and $\gamma_2(s)$ is a smooth curve from Id to A_2 then $\gamma(s) = \gamma_1(s)\gamma_2(s)$ is a smooth curve from Id to $A_1 A_2$. Consider the differential on this curve. Since

$$\frac{d(A_1(s)A_2(s))}{ds} = \frac{dA_1(s)}{ds} A_2(s) + A_1(s) \frac{dA_2(s)}{ds}$$

the 1-form becomes

iml.15
$$\gamma^*(s)\alpha(s) = \mathrm{Tr}(A_2(s)^{-1} \frac{dA_2(s)}{ds}) + \mathrm{Tr}(A_2(s)^{-1} A_1(s)^{-1} \frac{dA_2(s)}{ds} A_2(s)). \tag{3.98}$$

In the second term on the right we can use the trace identity, since $\mathrm{Tr}(GA) = \mathrm{Tr}(AG)$ if $G \in \Psi^{\mathbb{Z}}_{\text{iso}}(\mathbb{R}^n)$ and $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus (3.98) becomes

$$\gamma^*(s)\alpha(s) = \gamma_1^* \alpha + \gamma_2^* \alpha.$$

Inserting this into the definition of $\det_{\mathrm{Fr}}$ gives (3.97) when both factors are in $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Of course if either factor is not invertible, then so is the product and hence both $\det_{\mathrm{Fr}}(AB)$ and at least one of $\det_{\mathrm{Fr}}(A)$ and $\det_{\mathrm{Fr}}(B)$ vanishes. Thus (3.97) holds in general when $\det_{\mathrm{Fr}}$ is extended to be zero on the non-invertible elements.

Thus it remains to establish the smoothness. That $\det_{\mathrm{Fr}}(A)$ is smooth in any real parameters in which $A \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ depends, or indeed is holomorphic in holomorphic parameters, follows from the definition since α clearly depends smoothly, or holomorphically, on parameters. In fact the same follows if holomorphy is examined as a function of E, $A = \mathrm{Id} + E$, for $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus it is only smoothness across the non-invertibles that is at issue. To prove this we use the multiplicativity just established.

If $A = \mathrm{Id} + E$ is not invertible, $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then it has a generalized inverse $\mathrm{Id} + E'$ as in Proposition 3.6. Since A has index zero, we may actually replace E' by $E' + E''$, where E'' is an invertible linear map from the orthocomplement of the range of A to its null space. Then $\mathrm{Id} + E' + E'' \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and $(\mathrm{Id} + E' + E'')A = \mathrm{Id} - \Pi_0$. To prove the smoothness of $\det_{\mathrm{Fr}}$ on a neighbourhood of A it is enough to prove the smoothness on a neighbourhood of $\mathrm{Id} - \Pi_0$ since $\mathrm{Id} + E' + E''$ maps a neighbourhood of the first to a neighbourhood of the second and $\det_{\mathrm{Fr}}$ is multiplicative. Thus consider $\det_{\mathrm{Fr}}$ on a set $\mathrm{Id} - \Pi_0 + E$ where E is near 0 in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, in particular we may assume that $\mathrm{Id} + E \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus

$$\det{}_{\mathrm{Fr}}(\mathrm{Id} + E - \Pi_0) = \det(\mathrm{Id} + E)\det(\mathrm{Id} - \Pi_0 + (G_E - \mathrm{Id})\Pi_0)$$

were $G_E = (\mathrm{Id} + E)^{-1}$ depends holomorphically on E. Thus it suffices to prove the smoothness of $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H\Pi_0)$ where $H \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$

Consider the deformation $H_s = \Pi_0 H \Pi_0 + s(\mathrm{Id} - \Pi_0)H\Pi_0$, $s \in [0,1]$. If $\mathrm{Id} - \Pi_0 + H_s$ is invertible for one value of s it is invertible for all, since its range is always the range of $\mathrm{Id} - \Pi_0$ plus the range of $\Pi_0 H \Pi_0$. It follows that $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)$ is smooth in s; in fact it is constant. If the family is not invertible this follows immediately and if it is invertible then

$$\frac{d\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)}{ds} = \det{}_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + H_s)\,\mathrm{Tr}\left((\mathrm{Id} - \Pi_0 + H_s)^{-1}(\mathrm{Id} - Pi_0)H\Pi_0)\right) = 0$$

since the argument of the trace is finite rank and off-diagonal with respect to the decomposition by Π_0.

Thus finally it is enough to consider the smoothness of $\det_{\mathrm{Fr}}(\mathrm{Id} - \Pi_0 + \Pi_0 H \Pi_0)$ as a function of $H \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Since this is just $\det(\Pi_0 H \Pi_0)$, interpreted as a finite rank map on the range of Π_0 the result follows from the finite dimensional case. □

iml.18 LEMMA 3.11. *If $A \in G^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ and $B \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then $ABA^{-1} \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and*

iml.19 (3.99)
$$\det{}_{\mathrm{Fr}}(ABA^{-1}) = \det{}_{\mathrm{Fr}}(B).$$

PROOF. If ABA^{-1} is not invertible then neither is B so both sides of (3.99) vanish. Thus we may assume that $B = \mathrm{Id} + E$ is invertible and let $\mathrm{Id} + E(s)$ be a smooth curve in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ connecting it to the identity. Consider the function

iml.20 (3.100)
$$\det{}_{\mathrm{Fr}}(A(\mathrm{Id} + E(s))A^{-1}) = \det{}_{\mathrm{Fr}}(\mathrm{Id} + AE(s)A^{-1}).$$

This is certainly smooth and non-vanishing and its logarithm has derivative

$$\operatorname{Tr}(A(\operatorname{Id}+E(s))^{-1}A^{-1}\frac{d\left(A(\operatorname{Id}+E(s)A^{-1}\right)}{ds})$$
$$= \operatorname{Tr}(A(\operatorname{Id}+E(s))^{-1}\frac{E(s)}{ds}A^{-1}) = \operatorname{Tr}((\operatorname{Id}+E(s))^{-1}\frac{E(s)}{ds}).$$

This is also the derivative of the logarithm of $\det_{\mathrm{Fr}}(\operatorname{Id}+E(s))$ so the result follows. □

S.Fredholm.alternative

3.16. Fredholm alternative

Since we have shown that $\det_{\mathrm{Fr}} : \operatorname{Id}+\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is an entire function, we see that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the complement of a (singular) holomorphic hypersurface, namely the surface $\{\operatorname{Id}+E; \det_{\mathrm{Fr}}(\operatorname{Id}+E) = 0\}$. This has the following consequence, which is sometimes call the 'Fredholm alternative' and also part of 'analytic Fredholm theory'.

iml.21 Lemma 3.12. *If $\Omega \subset \mathbb{C}$ is an open, connected set and $A : \Omega \longrightarrow \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic function then either $\operatorname{Id}+A(z)$ is invertible of all but a discrete subset of Ω and $(\operatorname{Id}+A(z))$ is meromorphic on Ω with all residues of finite rank, or else it is invertible at no point of Ω.*

Proof. Of course the point here is that $\det_{\mathrm{Fr}}(\operatorname{Id}+A(z))$ is a holomorphic function on Ω. Thus, either $\det_{\mathrm{Fr}}(A(z)) = 0$ is a discrete set, $D \subset \Omega$ or else $\det_{\mathrm{Fr}}(\operatorname{Id}+A(z)) \equiv 0$ on Ω; this uses the connectedness of Ω. Since this corresponds exactly to the invertibility of $\operatorname{Id}+A(z)$ the main part of the lemma is proved. It remains only to show that, in the former case, $(\operatorname{Id}+A(z))^{-1}$ is meromorphic. Thus consider a point $p \in D$. Thus the claim is that near p

iml.22

$$(\operatorname{Id}+A(z))^{-1} = \operatorname{Id}+E(z) + \sum_{j=1}^{N} z^{-j}E_j,\ E_j \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \text{ of finite rank} \tag{3.101}$$

and where $E(z)$ is locally holomorphic with values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

If N is sufficiently large and Π_N is the projection onto the first N eigenspaces of the harmonic oscillator then $B(z) = \operatorname{Id}+E(z) - \Pi_N E(z)\Pi_N$ is invertible near p with the inverse being of the form $\operatorname{Id}+F(z)$ with $F(z)$ locally holomorphic. Now

$$(\operatorname{Id}+F(z))(\operatorname{Id}+E(z)) = \operatorname{Id}+(\operatorname{Id}+F(z))\Pi_N E(z)\Pi_N$$
$$= (\operatorname{Id}-\Pi_N) + \Pi_N M(z)\Pi_N + (\operatorname{Id}-\Pi_N)M'(z)\Pi_N.$$

It follows that this is invertible if and only if $M(z)$ is invertible as a matrix on the range of Π_N. Since it must be invertible near, but not at, p, its inverse is a meromorphic matrix $K(z)$. It follows that the inverse of the product above can be written

iml.23

$$\operatorname{Id}-\Pi_N + \Pi_N K(z)\Pi_N - (\operatorname{Id}-\Pi_N)M'(z)\Pi_N K(z)\Pi_N. \tag{3.102}$$

This is meromorphic and has finite rank residues, so it follows that the same is true of $A(z)^{-1}$. □

This result for the smoothing operators, which really follows from the corresponding result for matrices, gives a similar result for a holomorphic family of elliptic operators.

iml.24 PROPOSITION 3.16. *Suppose* $\Omega \subset \mathbb{C}$ *is open and connected,* $h(z)$ *is holomorphic on* Ω *and* $A(z) \in \Psi^{h(z)}_{\mathrm{iso}}(\mathbb{R}^n)$ *is a holomorphic and elliptic family. Then if* $A(z)$ *is invertible at one point it is invertible for all but a discrete set and its inverse is locally of the form* $B(z) + H(z)$ *where* $B(z) \in \Psi^{-h(z)}_{\mathrm{iso}}(\mathbb{R}^n)$ *is a locally holomorphic family and* $H(z)$ *is meromorphic with values in* $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *and has all residues of finite rank.*

PROOF. By the results of Section S.Complex.order 3.13 we can choose, at least locally near any point, a holomorphic parametrix $B(z)$ for the family in $\Psi^{-h(z)}_{\mathrm{iso}}(\mathbb{R}^n)$. Suppose $p \in \Omega$ is a point at which $A(z)$ is invertible. Then the parametrix differs from the inverse by a smoothing operator, so modifying $B(z)$ by a constant smoothing operator if follows that $B(z)A(z) = \mathrm{Id} + E(z)$ is a holomorphic family in $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ with $E(p) = 0$. It follows from the Fredholm alternative that $\mathrm{Id} + E(z)$ is invertible with holomorphic inverse near p. Thus, $A(z)$ is invertible for z in an open set around p; the set at which it is invertible is therefore open. Let $Z \subset \Omega$ be the closed set at which $A(z)$ is not invertible. If p is a boundary point of Z then it follows, using the notation above for a parametrix of $A(z)$ near p, that $\mathrm{Id} + E(z)$ is invertible near p and hence p is an isolated point of non-invertibility. Thus all boundary points of Z are isolated. A closed subset of an open connected set in Euclidean space is either discrete or has a non-isolated boundary point, so this shows that the set of non-invertibility it is discrete. The local structure of the inverse follows from this discussion. □

3.17. Resolvent and spectrum

One direct application of analytic Fredholm theory is to the resolvent of an elliptic operator of positive order. For simplicity we assume that $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ with $m \in \mathbb{N}$, although the case of non-integral positive order is only slightly more complicated.

iml.25 PROPOSITION 3.17. *If* $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$, $m \in \mathbb{N}$, *and there exists one point* $\lambda' \in \mathbb{C}$ *such that* $A - \lambda'$ *and* $A^* - \overline{\lambda'}$ *both have trivial null space, then*

iml.26
$$(A - \lambda)^{-1} \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \tag{3.103}$$

is a meromorphic family with all residues finite rank smoothing operators; the span of the ranges of the residues at any $\tilde{\lambda}$ *is the linear space of generalized eigenvalues, the solutions of*

iml.27
$$(A - \tilde{\lambda})^p u = 0 \text{ for some } p \in \mathbb{N}. \tag{3.104}$$

PROOF. Since A is elliptic and of positive integral order, m, $A - \lambda \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ is and entire elliptic family. By hypothesis, its inverse exists for some $\lambda' \in \mathbb{C}$. Thus, by Proposition iml.24 3.16 $(A - \lambda)^{-1} \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ is a meromorphic family in the complex plane, with all residues finite rank smoothing operators.

Let $\tilde{\lambda}$ be a pole of $A - \lambda$. Since we can replace A by $A - \tilde{\lambda}$ we may suppose without loss of generality that $\tilde{\lambda} = 0$. Thus, for some k the product $\lambda^k (A - \lambda)^{-1}$ is holomorphic near $\lambda = 0$. Differentiating the identities

$$(A - \lambda)[\lambda^k (A - \lambda)^{-1}] = \lambda^k \,\mathrm{Id} = [\lambda^k (A - \lambda)^{-1}](A - \lambda)$$

up to k times gives the relations

iml.28

$$(3.105)\quad \begin{gathered} A\circ R_{k-j}=R_{k-j}\circ A=R_{k-j+1},\ j=0,\cdots,k-1,\\ A\circ R_0=R_0\circ A=\mathrm{Id}+R_1,\ \text{where}\\ (A-\lambda)^{-1}=R_k\lambda^{-k}+R_{k-1}\lambda^{-k+1}+\cdots+R_0+\cdots,\ R_{k+1}=0.\end{gathered}$$

Thus $A^p\circ R_{k-p+1}=0=R_{k-p+1}\circ A^p$ for $0<p\le k$, which shows that all the residues, R_j, $1\le j\le k$, have ranges in the generalized eigenfunctions. □

Notice also from (3.105) that the range of R_{k-j+1} is contained in the range of R_{k-j} for each $j=0,\dots,k-1$, and conversely for the null spaces

$$\begin{gathered}\mathrm{Ran}(R_k)\subset\mathrm{Ran}(R_{k-1})\subset\cdots\subset\mathrm{Ran}(R_1)\\ \mathrm{Nul}(R_k)\supset\mathrm{Nul}(R_{k-1})\supset\cdots\supset\mathrm{Nul}(R_1).\end{gathered}$$

Thus,

iml.29

$$(3.106)\qquad u\in\mathrm{Ran}(R_p),\ p\ge1\Longleftrightarrow\exists\, u_1\in\mathrm{Ran}(R_1)\ \text{s.t.}\ A^{p-1}u_1=u.$$

S.Residue.trace

3.18. Residue trace

We have shown, in Proposition 3.14, the existence of a unique trace functional on the residual algebra $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. We now follow ideas originating with Seeley, [**11**], and developed by Guillemin [**5**], [**6**] and Wodzicki [**14**], [**13**] to investigate the traces on the full algebra $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ of polyhomogeneous operators of integral order. We will prove the existence of a trace but defer until later the proof of its uniqueness.

Observe that for $A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ the kernel can be written

$$A(x,y)=(2\pi)^{-n}\int e^{i(x-y)\xi}a_L(x,\xi)\,d\xi$$

and hence the trace, from (3.85), becomes

Feb.17.2000.eq:1

$$(3.107)\qquad \mathrm{Tr}(A)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}a_L(x,\xi)\,dx\,d\xi\,,$$

just the integral of the left-reduced symbol. In fact this is true for *any* amplitude (of order $-\infty$) representing A :

Feb.17.2000.eq:2

(3.108)

$$A=(2\pi)^{-n}\int e^{i(x-y)}a(x,y,\xi)\,d\xi\Longrightarrow\mathrm{Tr}(A)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}a(x,x,\xi)\,dx\,d\xi\,.$$

The integral in (3.107) extends by continuity to $a_L\in S^m_\infty(\mathbb{R}^{2n})$ provided $m<-2n$. Thus, as a functional,

Feb.17.2000.eq:3

$$(3.109)\qquad \mathrm{Tr}:\Psi^{-2n-\epsilon}_{\infty,\mathrm{iso}}(\mathbb{R}^n)\to\mathbb{C},\ \text{for any}\ \epsilon>0.$$

To extend it further we need somehow to *regularize* the resultant divergent integral in (3.107) (and to pay the price in terms of properties). One elegant way to do this is to use a holomorphic family as discussed in Section 3.13. Notice that we are passing from the algebra-with-bounds in (3.109) to polyhomogeneous operators.

Lemma 3.13. *If $A(z)\in\Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family then $f(z)=\mathrm{Tr}(A(z))$, defined by (3.108) when $\Re(z)<-2n$, extends to a meromorphic function of z with at most simple poles on the divisor*

$$\{-2n,-2n+1,\dots,-1,0,1,\dots\}\subset\mathbb{C}\,.$$

Proof. We know that $A(z) \in \Psi^z_{\text{iso}}(\mathbb{R}^n)$ is a holomorphic family if and only if its left-reduced symbol is of the form

$$\sigma_L(A(z)) = (1 + |x|^2 + |z|^2)^{z/2} a(z; x, \xi)$$

where $a(z; x, y)$ is an entire function with values in $S^0_{\text{phg}}(\mathbb{R}^n)$. For $\Re z < -2n$ the trace of $A(z)$ is

$$f(z) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} (1 + |x|^2 + |\xi|^2)^{z/2} a(z; x, \xi)\, dx\, d\xi\,.$$

Consider the part of this integral on the ball

$$f_1(z) = (2\pi)^{-n} \int_{|x|^2 + |\xi|^2 \le 1} (1 + |x|^2 + |\xi|^2)^{z/2} a(z, x, y)\, dx\, d\xi\,.$$

This is clearly an entire function of z, since the integrand is entire and the domain compact.

To analyze the remaining part $f_2(z) = f(z) - f_1(z)$ let us introduce polar coordinates

$$r = (|x|^2 + |\xi|^2)^{1/2},\ \theta = \frac{(x, \xi)}{r} \in \mathbb{S}^{2n-1}\,.$$

The integral, convergent in $\Re z < -2n$, becomes

$$f_2(z) = (2\pi)^{-n} \int_1^\infty \int_{\mathbb{S}^{2n-1}} (1 + r^2)^{z/2} \tilde{a}(z; r, \theta)\, d\theta r^{2n-1}\, dr.$$

Let us now pass to the radical compactification of $\mathbb{R}^{2n}$ or more prosaically, introduce $t = 1/r \in [0, 1]$ as variable of integration, so

$$f_2(z) = (2\pi)^{-n} \int_0^1 \int_{\mathbb{S}^{2n-1}} t^{-z} (1 + t^2)^{z/2} \tilde{a}(z; \frac{1}{t}, \theta)\, d\theta t^{-2n} \frac{dt}{t}\,.$$

Now the definition of $S^0_{\text{phg}}(\mathbb{R}^{2n})$ reduces to the statement that

Feb.17.2000.eq:4

$$b(z; t, \theta) = (1 + t^2)^{z/2} \tilde{a}(z; \frac{1}{t}, \theta) \in \mathcal{C}^\infty(\mathbb{C} \times [0, 1] \times \mathbb{S}^{2n-1}) \tag{3.110}$$

is holomorphic in z.

If we replace b by its Taylor series at $t = 0$ to high order,

Feb.17.2000.eq:5

$$b(z; t, \theta) = \sum_{j=0}^{k} \frac{t^j}{j!} b_j(z; \theta) + t^{k+1} b_{(k)}(z; t, \theta)\,, \tag{3.111}$$

where $b_{(k)}(z; t, \theta)$ has the same regularity (3.110), then $f_2(z)$ is decomposed as

Feb.17.2000.eq:6

$$f_2(z) = (2\pi)^{-n} \sum_{j=0}^{k} \int_0^1 \int_{\mathbb{S}^{2n-1}} \frac{t^{-z+j}}{j!} b_j(z; \theta) t^{-2n} \frac{dt}{t} + f_2^{(k)}(z). \tag{3.112}$$

The presence of this factor t^k in the remainder in (3.111) shows that $f_2^{(k)}(z)$ is holomorphic in $\Re z < -2n + k$. On the other hand the individual terms in the sum in (3.112) can be computed (for $\Re z < -2n$) as

$$(2\pi)^{-n} \left[\frac{t^{-z+j-2n}}{(-z + j - 2n)} \right]_0^1 \int_{\mathbb{S}^{2n-1}} b_j(z, \theta) \frac{d\theta}{j!}$$
$$= (2\pi)^{-n} \frac{1}{(z - j + 2n)} \int_{\mathbb{S}^{2n-1}} b_j(z, \theta) \frac{d\theta}{j!}.$$

Each of these terms extends to be meromorphic in the entire complex plane, with a simple pole (at most) at $z = -2n + j$. This shows that $f(z)$ has a meromorphic continuation as claimed. □

By this argument we have actually computed the residues of the analytic continuation of $\mathrm{Tr}(A(z))$ as

1.2.2000.283

$$\lim_{z\to -2n+j}(z - j + 2n)\,\mathrm{Tr}(A(z)) = (2\pi)^{-n}\int_{\mathbb{S}^{2n-1}} a_j(\theta)\,d\theta \tag{3.113}$$

when $a_j(\theta) \in \mathcal{C}^\infty(S^{2n-1})$ is the function occurring in the asymptotic expansion of the left symbol of $A(z)$:

Feb.17.2000.eq:7

$$\sigma_L(A(z)) \sim \sum_{j=0}^{\infty}(|x|^2 + |\xi|^2)^{z/2-j}\tilde{a}_j(z,\theta) \tag{3.114}$$
$$|x|^2 + |\xi|^2 \to \infty,\ \theta = \frac{(x,\xi)}{(|x|^2+|\xi|^2)^{1/2}},\ a_j(\theta) = \tilde{a}_j(-2n+j,\theta).$$

More generally, if $m \in \mathbb{Z}$ and $A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family then

$$\mathrm{Tr}(A(z)) \text{ is meromorphic with at most}$$
$$\text{simple poles at } -2n - m + \mathbb{N}_0\,.$$

Indeed this just follows by considering the family $A(z - m)$.

We are especially interested in the behavior at $z = 0$. Since the residue there is an integral of the term of order $-2n$, we know that

Feb.24.2000.eq:2

$$\begin{gathered} A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n) \text{ holomorphic with } A(0) = 0 \\ \Longrightarrow \mathrm{Tr}(A(z)) \text{ is regular at } z = 0\,. \end{gathered} \tag{3.115}$$

This allows us to make the following definition:

$$\mathrm{Tr}_{\mathrm{Res}}(A) = \lim_{z\to 0} z\,\mathrm{Tr}(A(z)) \text{ if}$$
$$A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n) \text{ is holomorphic with } A(0) = A\,.$$

We know that such a holomorphic family exists, since we showed in Section 3.13 [S.Complex.order] the existence of a holomorphic family $F(z) \in \Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ with $F(0) = \mathrm{Id}$; $A(z) = AF(z)$ is therefore an example. Similarly we know that $\mathrm{Tr}_{\mathrm{Res}}(A)$ is independent of the choice of holomorphic family $A(z)$ because of (3.115) [Feb.24.2000.eq:2] applied to the difference, which vanishes at zero.

LEMMA 3.14. *The residue functional* $\mathrm{Tr}_{\mathrm{Res}}(A)$, $A \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$, *is a trace:*

Feb.24.2000.eq:4

$$\mathrm{Tr}_{\mathrm{Res}}([A,B]) = 0\ \forall\ A, B \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.116}$$

which vanishes on $\Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n)$ *and is given explicitly by*

Feb.24.2000.eq:5

$$\mathrm{Tr}_{\mathrm{Res}}(A) = (2\pi)^{-n}\int_{\mathcal{S}^{2n-1}} a_{-2n}(\theta)\,d\theta \tag{3.117}$$

where $a_{-2n}(\theta)$ *is the term of order* $-2n$ *in the expansion of the left (or right) symbol of* a.

Proof. We have already shown that $\mathrm{Tr}_{\mathrm{Res}}(A)$ is well-defined and (3.117) follows from (3.113) with $a_{-2n}(\theta)$ the term of order $-2n$ in the left-reduced symbol of $A = A(0)$. On the other hand, the same argument applies for the right-reduced symbol.

To see (3.116) just note that if $A(z)$ and $B(z)$ are holomorphic families with $A(0) = A$, and $B(0) = B$ then $C(z) = [A(z), B(z)]$ is a holomorphic family with $C(0) = [A, B]$. On the other hand, $\mathrm{Tr}(C(z)) = 0$ when $\Re z \gg 0$ so the analytic continuation of $\mathrm{Tr}(C(z))$ vanishes identically and (3.116) follows. □

As we shall see below, $\mathrm{Tr}_{\mathrm{Res}}$ is the unique trace (up to a multiple of course) on $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$.

3.19. Exterior derivation

Let $A(z) \in \Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ be a holomorphic family with $A(0) = \mathrm{Id}$. Then

$$G(z) = A(z) \cdot A(-z) \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$$

is a holomorphic family of fixed order with $G(0) = \mathrm{Id}$. By analytic Fredholm theory

1.2.2000.284 (3.118) $$G^{-1}(z) \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n) \text{ is a meromorphic family with finite rank poles.}$$

It follows that $A^{-1}(z) = A(-z)G^{-1}(z)$ is a meromorphic family of order $-z$ with at most finite rank poles and regular near 0. Set

1.2.2000.286 (3.119) $$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \ni B \mapsto A(z)BA^{-1}(z) = B(z)\,.$$

Thus $B(z)$ is a meromorphic family of order m with $B(0) = B$. The derivative gives a linear map.

1.2.2000.285 (3.120) $$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \ni B \mapsto D_A B = \frac{d}{dz} A(z)BA^{-1}(z)|_{z=0} \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)\,.$$

Proposition 3.18. *For any holomorphic family of order z, with $A(0) = \mathrm{Id}$, the map (3.120), defined through (3.119), is a derivation and for two choices of $A(z)$ the derivations differ by an inner derivation.*

Proof. Since

$$A(z)B_1B_2A^{-1}(z) = A(z)B_1A^{-1}(z)A(z)B_2A^{-1}(z)$$

it follows that

$$\frac{d}{dz} A(z)B_1B_2A^{-1}(z)|_{z=0} = (D_AB_1) \circ B_2 + B_1 \circ (D_AB_2)\,.$$

If $A_1(z)$ and $A_2(z)$ are two holomorphic families of order z with $A_1(0) = A_2(0) = \mathrm{Id}$ then

$$A_2(z) = A_1(z)G(z)$$

when $G(z) \in \Psi^\infty_{\mathrm{iso}}(\mathbb{R}^n)$ is a meromorphic family, with finite rank poles. Thus

$$\begin{aligned} A_2(z)BA_2^{-1}(z) &= A_1(z)G(z)BG^{-1}(z)A_1^{-1}(z) \\ &= A_1(z)BA^{-1}(z) + zA_1(z)H(z)A_1^{-1}(z)\,. \end{aligned}$$

Here $H(z) = (G(z)BG^{-1}(z) - B)/z$ is a holomorphic family of degree m with $H(0) = G'(0)B - BG'(0)$. Thus

$$\frac{d}{dz} A_2(z)BA_2^{-1}(z)|_{z=0} = \frac{d}{dz} A_1(z)BA^{-1}(z)|_{z=0} + [G'(0), B]$$

which shows that the two derivations differ by an inner derivation, which is to say commutation with an element of $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$. □

Note that in fact

$$D_A : \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \to \Psi^{m-1}_{\mathrm{iso}}(\mathbb{R}^n)\ \forall\ m$$

since the symbol of $A(z)BA^{-1}(z)$ is equal to the principal symbol of B for all z.

For the specific choice of $A(z) = H(z)$ given by

$$\sigma_L(H(z)) = (1 + |x|^2 + |\xi|^2)^{z/2}$$

we shall set

$$D_A B = D_H B\,.$$

Observe that $\frac{1}{2}\log(1 + |x|^2 + |\xi|^2) \in S^{\epsilon}_{\infty}(\mathbb{R}^{2n})\ \forall\ \epsilon > 0$. Thus $\log(1 + |x|^2 + |\xi|^2)$, defined by Weyl quantization, is an element of $\Psi^{-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for all $\epsilon > 0$. By differentiation the symbols satisfy

$$D_H B = [\frac{1}{2}\log(1 + |x|^2 + |D|^2), B] + [G, B]$$

where $G \in \Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus D_H is *not* itself an interior derivation. It is therefore an *exterior* derivation.

S.Regularized.trace

3.20. Regularized trace

In Section S.Residue.trace 3.18 we defined the residue trace of B as the residue at $z = 0$ of the analytic continuation of $\mathrm{Tr}(BA(z))$, where $A(z)$ is a holomorphic family of order z with $A(0) = \mathrm{Id}$. Next we consider the functional

Feb.24.2000.eq:E

$$\overline{\mathrm{Tr}}_{\mathrm{A}}(B) = \lim_{z=0}(\mathrm{Tr}(BA(z)) - \frac{1}{z}\,\mathrm{Tr}_{\mathrm{Res}}(B))\,. \tag{3.121}$$

In contrast to the residue trace, $\overline{\mathrm{Tr}}_{\mathrm{A}}(z)$ *does* depend on the choice of analytic family $A(z)$.

Lemma 3.15. *If $A_i(z)$, $i = 1, 2$, are two holomorphic families of order z with $A_i(0) = \mathrm{Id}$ and $G'(0) = \frac{d}{dz}A_2(z)A_1^{-2}(z)|_{z=0}$ then*

Feb.24.2000.eq:F

$$\overline{\mathrm{Tr}}_{\mathrm{A}_2}(B) - \overline{\mathrm{Tr}}_{\mathrm{A}_1}(B) = \mathrm{Tr}_{\mathrm{Res}}(BG'(0))\,. \tag{3.122}$$

Proof. Writing $G(z) = A_2(z)A_1^{-1}(z)$, which is a meromorphic family of order 0 with $G(0) = \mathrm{Id}$,

$$\begin{aligned}\mathrm{Tr}(BA_2(z)) &= \mathrm{Tr}(BG(z)A_1(z))\\ &= \mathrm{Tr}(BA_1(z)) + z\,\mathrm{Tr}(BG'(0)A_1(z)) + z^2\,\mathrm{Tr}(H(z)A_1(z))\end{aligned}$$

where $H(z) = \frac{B}{z^2}(G(z) - \mathrm{Id} - zG'(0))$ is then meromorphic with only finite rank poles and is regular near $z = 0$. Thus the analytic continuation of $z^2\,\mathrm{Tr}(H(z)A(z))$ vanishes at zero from which (Feb.24.2000.eq:F 3.122) follows. □

This regularized trace $\overline{\mathrm{Tr}}_{\mathrm{A}}(B)$ therefore only depends on the first order, in z, term in $A(z)$ at $z = 0$. It is important to note that it is *not* itself a trace.

Lemma 3.16. *If B_1, $B_2 \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ then*

Feb.24.2000.eq:G

$$\overline{\mathrm{Tr}}_{\mathrm{A}}([B_1, B_2]) = \mathrm{Tr}_{\mathrm{Res}}(B_2 D_A B_1)\,. \tag{3.123}$$

Proof. Since $\overline{\mathrm{Tr}}_{\mathrm{A}}([B_1, B_2])$ is the regularized value at 0 of the analytic continuation of the trace of

Feb.24.2000.eq:H
$$B_1B_2A(z) - B_2B_1A(z) = B_2[A(z), B_1] + [B_1, B_2A(z)] \\ = B_2([A(z), B]A^{-1}(z))A_1(z) + [B_1B_2A(z)]\,. \tag{3.124}$$

The second term on the right in (Feb.24.2000.eq:H 3.124) has zero trace before analytic continuation. Thus $\overline{\mathrm{Tr}}_A([B_1, B_2])$ is the regularized value of the analytic continuation of the trace of $Q(z)A(z)$ where

$$Q(z) = B_2[A(z), B_1]A^{-1}(z) = zD_AB_1 + z^2L(z)$$

with $L(z)$ meromorphic of fixed order and regular at $z = 0$. Thus (Feb.24.2000.eq:G 3.123) follows. □

Note that

Feb.24.2000.eq:H1
$$\mathrm{Tr}_{\mathrm{Res}}(D_AB) = 0 \;\forall\; B \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.125}$$

and any family A. Indeed the residue trace is the residue of $z = 0$ of the analytic continuation of $\mathrm{Tr}(H(z)A(z))$ when $A(z)$ is any meromorphic family of fixed order with $H(0) = D_AB$. In particular we can take

$$H(z) = \frac{1}{z}(A(z)BA^{-1}(z) - B)\,.$$

Then $H(z)A(z) = \frac{1}{z}[A(z), B]$ so the trace vanishes before analytic continuation.

3.21. Projections

3.22. Complex powers

.Index.and.invertibility

3.23. Index and invertibility

We have already seen that the elliptic elements

1.2.2000.369
$$E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \subset \Psi^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \hookrightarrow \mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N)) \tag{3.126}$$

define Fredholm operators. The *index* of such an operator

1.2.2000.370
$$\mathrm{Ind}(A) = \dim \mathrm{Nul}(A) - \dim \mathrm{Nul}(A^*) \tag{3.127}$$

is a measure of its non-invertibility. Set

1.2.2000.371
$$E^0_{\mathrm{iso},k}(\mathbb{R}^n; \mathbb{C}^N) = \left\{A \in E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N); \mathrm{Ind}(A) = k\right\},\ k \in \mathbb{Z}. \tag{3.128}$$

1.2.2000.372

Proposition 3.19. *If* $A \in E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ *and* $\mathrm{Ind}(A) = 0$ *then there exists* $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ *such that* $A + E$ *is invertible in* $\mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N))$ *and the inverse then lies in* $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$.

Proof. Let B be the generalized inverse of A, assumed to be elliptic. The assumption that $\mathrm{Ind}(A) = 0$ means that $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$ have the same dimension. Let $e_1, \cdots, e_p \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ and $f_1, \cdots, f_p \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ be bases of $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$. Then consider

1.2.2000.375
$$E = \sum_{j=1}^{p} f_j(x)\overline{e_j(y)} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N). \tag{3.129}$$

By construction E is an isomorphism (in fact an arbitrary one) between $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*)$. Thus $A+E$ is continuous, injective and surjective, hence has an inverse in $\mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N))$. Indeed this inverse is $B + E^{-1}$ where E^{-1} is the inverse of E as a map from $\mathrm{Nul}(A)$ to $\mathrm{Nul}(A^*)$. This shows that A can be perturbed by a smoothing operator to be invertible. □

Let

1.2.2000.374 $$G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \subset E^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N) \subset E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \subset \Psi 0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \tag{3.130}$$

denote the group of the invertible elements (invertibility being either in $\mathcal{B}(L^2(\mathbb{R};\mathbb{C}^N)$ or in $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$) in the ring of elliptic elements of index 0.

1.2.2000.373 Corollary 3.7. *The first inclusion in* (3.130) *is dense in the topology of* $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$.

Proof. This follows from the proof of Proposition 3.19, since $A + sE$ is invertible for all $s \neq 0$. □

We next derive some simple formulæ for the index of an element of $E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. First observe that the trace of a finite dimensional projection is its rank, the dimension of its range. Thus

1.2.2000.376 $$\mathrm{Ind}(A) = \mathrm{Tr}(\Pi_{\mathrm{Nul}(A)}) - \mathrm{Tr}(\Pi_{\mathrm{Nul}(A^*)}) \tag{3.131}$$

where the trace may be reinterpreted as the trace on smoothing operators. The identities, (3.34), satisfied by the generalized inverse of A shows that this can be rewritten

1.2.2000.377 $$\mathrm{Ind}(A) = -\,\mathrm{Tr}(BA - \mathrm{Id}) + \mathrm{Tr}(AB - \mathrm{Id}) = \mathrm{Tr}([A,B]). \tag{3.132}$$

Here $[A,B] = \Pi_{\mathrm{Nul}(A)} - \Pi_{\mathrm{Nul}(A^*)}$ is a smoothing operator, even though both A and B are elliptic of order 0.

1.2.2000.378 Lemma 3.17. *If* $A \in E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ *the identity* (3.132), *which may be rewritten*

1.2.2000.379 $$\mathrm{Ind}(A) = \mathrm{Tr}([A,B]), \tag{3.133}$$

holds for any parametrix B.

Proof. If B' is a parametrix and B is the generalized inverse then $B' - B = E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. Thus

$$[A,B'] = [A,B] + [A,E].$$

Since $\mathrm{Tr}([A,E] = 0$, one of the arguments being a smoothing operator, (3.133) follows in general from the particular case (3.132). □

Note that it follows from (3.133) that $\mathrm{Ind}(A) = \mathrm{Ind}(A+E)$ if E is smoothing. In fact the index is even more stable than this as we shall see, since it is locally constant on $E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. In any case this shows that

1.2.2000.381 $$\begin{gathered}\mathrm{Ind} : \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z},\ \mathrm{Ind}(a) = \mathrm{Ind}(A) \text{ if } a = [A],\\ \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) = E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\\ \subset \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) = \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\end{gathered} \tag{3.134}$$

is well-defined.

The argument of the trace functional in (3.133) is a smoothing operator, but we may still rewrite the formula in terms of the regularized trace, with respect to the standard regularizer $H(z)$ with left symbol $(1+|x|^2+|\xi|^2)^{\frac{z}{2}}$. The advantage of doing so is that we can then use the trace defect formula (Feb.24.2000.eq:G 3.123). Thus for any elliptic isotropic operator of order 0

1.2.2000.380 (3.135)
$$\operatorname{Ind}(A) = \operatorname{Tr}_{\mathrm{Res}}(BD_H A).$$

Here B is a parametrix for A. The residue trace is actually a functional

$$\operatorname{Tr}_{\mathrm{Res}} : \mathcal{A}^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{C},$$

so if we write a^{-1} for the inverse of a in the ring $\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then

1.2.2000.382 (3.136)
$$\operatorname{Ind}(a) = \operatorname{Tr}_{\mathrm{Res}}(a^{-1}D_H a),\ \ D_H : \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$$

being the induced derivation (since D_H clearly preserves the ideal $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$.

From this simple formula we can easily deduce two elementary properties of elliptic operators. These actually hold in general for Fredholm operators, although the proofs here are not valid in that generality. Namely

1.2.2000.383 (3.137)
$$\operatorname{Ind} : \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z} \text{ is locally constant and}$$

1.2.2000.384 (3.138)
$$\operatorname{Ind}(a_1a_2) = \operatorname{Ind}(a_1) + \operatorname{Ind}(a_2)\ \forall\ a_1,\ a_2 \in \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N).$$

The first of these follows the continuity of the formula (1.2.2000.382 3.136) since under deformation of a in $\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ the inverse a^{-1} varies continuously, so Ind is continuous and integer-valued, hence locally constant. Similarly the second, logarithmic additivity, property follows from the fact that D_H is a derivation, so

$$D_H(a_1a_2) = (D_Ha_1)a_2 + a_1D_Ha_2$$

and the the trace property of $\operatorname{Tr}_{\mathrm{Res}}$ which shows that

1.2.2000.385 (3.139)
$$\begin{aligned}\operatorname{Ind}(a_1a_2) = \operatorname{Tr}_{\mathrm{Res}}((a_1a_2)^{-1}D_H(a_1a_2) &= \operatorname{Tr}(a_2^{-1}a_1^{-1}\left((D_Ha_1)a_2 + a_1D_Ha_2\right)\\ &= \operatorname{Tr}(a_2^{-1}a_1^{-1}(D_Ha_1)a_2) + \operatorname{Tr}(a_2^{-1}D_Ha_2) = \operatorname{Ind}(a_1) + \operatorname{Ind}(a_2).\end{aligned}$$

3.24. Variation 1-form

In the previous section we have seen that the index

1.2.2000.386 (3.140)
$$\operatorname{Ind} : E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z}$$

is a multiplicative map which is the obstruction to perturbative invertibility. In the next two sections we will derive a closely related obstruction to the perturbative invertibility of a family of elliptic operators. Thus, suppose

1.2.2000.387 (3.141)
$$Y \ni y \longmapsto A_y \in E^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N)$$

is a family of elliptic operators depending smoothly on a parameter in the compact manifold Y. We are interested in the *families perturbative invertibility question.* That is, does there exist a smooth family

1.2.2000.388 (3.142)
$$Y \ni y \longmapsto E_y \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \text{ such that } (A_y + E_y) \in G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\ \forall\ y.$$

We have assumed that the operators have index zero since this is necessary (and sufficient) for E_y to exist for any one $y \in Y$. Thus the issue is the smoothness (really just the continuity) of the perturbation E_y.

We shall start by essentially writing down such a putative obstruction directly and then subsequently we shall investigate its topological origins.

1.2.2000.389 PROPOSITION 3.20. *If a smooth family* (1.2.2000.387 3.141)*, parameterized by a compact manifold Y, is perturbatively invariant in the sense that there is a smooth family as in* (1.2.2000.388 3.142)*, then the closed 2-form on Y*

1.2.2000.390
$$\beta = \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \cdot a_y^{-1}D_H a_y) \in \mathcal{C}^\infty(Y;\Lambda^2),\quad a_y = [A_y] \in \mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N), \tag{3.143}$$

is exact.

PROOF. Note first that β is indeed a smooth form, since the full symbolic inverse depends smoothly on parameters. Next we show that β is always closed. The 1-forms $a_y^{-1}d_y a_y a_y^{-1}$ and da_y are exact so differentiating directly gives

1.2.2000.391
$$\begin{aligned} d\beta &= \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge d(a_y^{-1}D_H a_y)) \\ &= -\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y^{-1}D_H a_y)) \\ &\quad +\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}D_H(da_y)) \\ &= \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge D_H(a_y^{-1}da_y)). \end{aligned} \tag{3.144}$$

Using the trace property and the commutativity of a 2-form with other forms the last expression can be written

1.2.2000.392
$$\frac{1}{6}\operatorname{Tr}_{\mathrm{Res}}(D_H(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}da_y)) = 0 \tag{3.145}$$

by property (Feb.24.2000.eq:H1 3.125) of the residue trace.

Now, suppose that a smooth perturbation as in (1.2.2000.388 3.142) does exist. We can replace A_y by $A_y + E_y$ without affecting β, since the residue trace vanishes on the ideal of smoothing operators. Thus we can assume that A_y itself is invertible. Then consider the 1-form defined using the regularized trace

1.2.2000.393
$$\overline{\alpha} = \overline{\operatorname{Tr}}_{\mathrm{H}}(A_y^{-1}d_y A_y). \tag{3.146}$$

This is an extension of the 1-form $d\log\det_F$ on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. The extension is not in general closed, because the regularized trace does not satisfy the trace condition. Using the stanadard formula for the variation of the inverse, $dA_y^{-1} = -A_y^{-1}dA_y A_y^{-1}$, the exterior derivative is the 2-form

1.2.2000.394
$$d\overline{\alpha} = -\overline{\operatorname{Tr}}_{\mathrm{H}}(A_y^{-1}(d_y A)A_y^{-1}d_y A_y). \tag{3.147}$$

The 2-form argument is a commutator. Indeed, in terms of local coordinates we can write

$$\begin{aligned} A_y^{-1}(d_y A)A_y^{-1}d_y A_y &= \sum_{j,k=1}^{p} A_y^{-1}(\frac{\partial A}{\partial y_j})A_y^{-1}(\frac{\partial A}{\partial y_k})dy_j\wedge dy_k \\ &= \frac{1}{2}\sum_{j,k=1}^{p}\left(A_y^{-1}(\frac{\partial A}{\partial y_j})A_y^{-1}(\frac{\partial A}{\partial y_k}) - A_y^{-1}(\frac{\partial A}{\partial y_k})A_y^{-1}(\frac{\partial A}{\partial y_j})\right)dy_j\wedge dy_k \\ &= \frac{1}{2}\sum_{j,k=1}^{p}[A_y^{-1}(\frac{\partial A}{\partial y_j}), A_y^{-1}(\frac{\partial A}{\partial y_k})]dy_j\wedge dy_k \end{aligned}$$

Applying the trace defect formula (3.123) shows that

1.2.2000.395
$$d\overline{\alpha} = -\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge D_H(A_y^{-1}d_yA_y)\right), \tag{3.148}$$

locally and hence globally.

Expanding the action of the derivation D_H gives

1.2.2000.396
$$d\overline{\alpha} = \beta - \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge A_y^{-1}d_y(D_HA_y)\right) = \beta - d\gamma, \text{ where}$$
$$\gamma = \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge .A_y^{-1}D_HA_y\right). \tag{3.149}$$

We conclude that if A_y has an invertible lift then β is exact. □

Note that the form γ in (3.149) is well-defined as a form on $\mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N)$, and is independent of the perturbation. Thus the cohomology class which we have constructed as the obstruction to perturbative invertibility can be written

1.2.2000.397
$$[\beta] = [\beta - d\gamma] \in H^2(\mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N)). \tag{3.150}$$

3.25. Determinant bundle

To better explain the topological origin of the cohomology class (3.150) we construct the determinant bundle. This was originally introduced for families of Dirac operators by Quillen [**10**]. Recall that the Fredholm determinant is a character

1.2.2000.398
$$\det{}_{\mathrm{Fr}} : \operatorname{Id} + \Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{C},$$
$$\det{}_{\mathrm{Fr}}(AB) = \det{}_{\mathrm{Fr}}(A)\det{}_{\mathrm{Fr}}(B)\forall\ A,\ B \in \operatorname{Id} + \Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N). \tag{3.151}$$

As we shall see, it is not possible to extend the Fredholm determinant as a multiplicative function to $G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$, essentially because of the non-extendibility of the trace.

However in trying to extend the determinant we can consider the possible values it would take on a point $A \in G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ as the set of pairs (A, z), $z \in \mathbb{C}$. Thus we simple consider the product

1.2.2000.399
$$D^0 = G^0 \times \mathbb{C}, \tag{3.152}$$

where from now on we simplify the notation and write $G^0 = G_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ etc. Although it is not reasonable to expect full multiplicative of the determinant, it is more reasonable to expect the determinant of $A(\operatorname{Id} + B)$, $B \in \Psi^{-2n-1}$ to be related to the product of determinants. Thus it is natural to identify pairs in D^0,

1.2.2000.400
$$(A, z) \sim_p (A', z') \text{ if}$$
$$A, A' \in G^0,\ A' = A(\operatorname{Id} + B),\ z' = \det{}_{\mathrm{Fr}}(\operatorname{Id} + B)z,\ B \in \Psi^p,\ p < -2n. \tag{3.153}$$

The equivalence relations here are slightly different, depending on p. In all cases the action of the determinant is linear, so the quotient is a line bundle.

1.2.2000.401 Lemma 3.18. *For any integer $p < -2n$, and also $p = -\infty$, the quotient*

1.2.2000.402
$$\mathcal{D}^0_p = D^0 / \sim_p \tag{3.154}$$

is a smooth line bundle over $\mathcal{G}^0_p = G^0/G^p$.

PROOF. The projection is just the quotient in the first factor and this clearly defines a commutative square

1.2.2000.403 (3.155)
$$\begin{array}{ccc} D^0 & \xrightarrow{[\sim_p]} & \mathcal{D}^0_p \\ \downarrow \pi & & \downarrow \pi \\ G^0 & \xrightarrow{/G^p} & \mathcal{G}^0_p. \end{array}$$

□

3.26. Index bundle

3.27. Index formulæ

susceptible

3.28. Isotropic essential support

3.29. Isotropic wavefront set

3.30. Isotropic FBI transform

S.Problems.3

3.31. Problems

21.2.1998.121 PROBLEM 3.1. Define the *isotropic* Sobolev spaces of integral order by

21.2.1998.122 (3.156)
$$H^k_{\mathrm{iso}}(\mathbb{R}^n) = \begin{cases} \left\{u \in L^2(\mathbb{R}^n); x^\alpha D^\beta_x u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| + |\beta| \le k\right\} & k \in \mathbb{N} \\ \left\{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum\limits_{|\alpha|+|\beta|\le -k} x^\alpha D^\beta_x u_{\alpha,\beta},\ u_{\alpha,\beta} \in L^2(\mathbb{R}^n)\right\} & k \in -\mathbb{N}. \end{cases}$$

Show that if $A \in \Psi^p_{\mathrm{iso}}(\mathbb{R}^n)$ with p an integer, then $A : H^k_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{k-p}_{\mathrm{iso}}(\mathbb{R}^n)$ for any integral k. Deduce (using the properties of elliptic isotropic operators) that the general definition

21.2.1998.123 (3.157)
$$H^m_{\mathrm{iso}}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n),\ \forall\ A \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)\right\},\ m \in \mathbb{R}$$

is consistent with (3.156) [21.2.1998.122] and has the properties

21.2.1998.124 (3.158)
$$A \in \Psi^M_{\mathrm{iso}}(\mathbb{R}^n) \Longrightarrow A : H^m_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{m-M}_{\mathrm{iso}}(\mathbb{R}^n),$$

21.2.1998.126 (3.159)
$$\bigcap_m H^m_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n),\ \bigcup_m H^m_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)$$

21.2.1998.125 (3.160)
$$A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n),\ u \in \mathcal{S}'(\mathbb{R}^n),\ Au \in H^{m'}(\mathbb{R}^n) \Longrightarrow u \in H^{m'-m}(\mathbb{R}^n),$$

21.2.1998.127 PROBLEM 3.2. Show that if $\epsilon > 0$ then

$$H^\epsilon_{\mathrm{iso}}(\mathbb{R}^n) \subsetneq (1 + |x|)^{-\epsilon} L^2(\mathbb{R}^n) \cap H^\epsilon(\mathbb{R}^n)$$

Deduce that $H^\epsilon_{\mathrm{iso}}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is a compact inclusion (i.e. the image of a bounded set is precompact).

21.2.1998.128 PROBLEM 3.3. Using Problem 3.2 [21.2.1998.127], or otherwise, show that each element of $\Psi^{-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$, $\epsilon > 0$, defines a compact operator on $L^2(\mathbb{R}^n)$.

21.2.1998.129 Problem 3.4. Show that if $E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then there exists $F \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ such that

$$(\mathrm{Id} + E)(\mathrm{Id} + F) = \mathrm{Id}_G \text{ with } G \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \text{ of finite rank,}$$

that is, $G \cdot \mathcal{S}(\mathbb{R}^n)$ is finite dimensional.

21.2.1998.130 Problem 3.5. Using Problem 3.4 show that an elliptic element $A \in \Psi^m_{\text{iso}}(\mathbb{R}^n)$ has a parametrix $B \in \Psi^{-m}_{\text{iso}}(\mathbb{R}^n)$ *up to finite rank error*; that is, such that $A \circ B - \mathrm{Id}$ and $B \circ A - \mathrm{Id}$ are finite rank elements of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Deduce that such an elliptic A defines a Fredholm operator

$$A : H^M_{\text{iso}}(\mathbb{R}^n) \longrightarrow H^{M-m}_{\text{iso}}(\mathbb{R}^n)$$

for any M. [The requirements for an operator A between Hilbert spaces to be Fredholm are that it be bounded, have finite-dimensional null space and closed range with a finite-dimensional complement.]

21.2.1998.111 Problem 3.6. [The harmonic oscillator] Show that the 'harmonic oscillator'

$$H = |D|^2 + |x|^2,\ Hu = \sum_{j=1}^n D_j^2 u + |x|^2 u,$$

is an elliptic element of $\Psi^2_{\text{iso}}(\mathbb{R}^n)$. Consider the 'creation' and 'annihilation' operators

21.2.1998.131
$$C_j = D_j + ix_j,\ A_j = D_j - ix_j = C_j^*, \tag{3.161}$$

and show that

21.2.1998.132
$$H = \sum_{j=1}^n C_j A_j + n = \sum_{j=1}^n A_j C_j - n, \tag{3.162}$$
$$[A_j, H] = 2A_j,\ [C_j, H] = -2C_j,\ [C_l, C_j] = 0,\ [A_l, A_j] = 0,\ [A_l, C_j] = 2\delta_{lk}\,\mathrm{Id},$$

where $[A, B] = A \circ B - B \circ A$ is the commutator bracket and δ_{lk} is the Kronecker symbol. Knowing that $(H - \lambda)u = 0$, for $\lambda \in \mathbb{C}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ implies $u \in \mathcal{S}(\mathbb{R}^n)$ (why?) show that

21.2.1998.133
$$E_\lambda = \{u \in \mathcal{S}'(\mathbb{R}^n); (H - \lambda)u = 0\} \neq \{0\} \Longleftrightarrow \lambda \in n + 2\mathbb{N}_0 \tag{3.163}$$

21.2.1998.134
$$\text{and } E_{-n+2k} = \left\{ \sum_{|\alpha|=k} c_\alpha C^\alpha \exp(-|x|^2/2),\ c_\alpha \in \mathbb{C} \right\},\ k \in \mathbb{N}_0. \tag{3.164}$$

1.2.2000.406 Problem 3.7. [Definition of determinant of matrices.]

iml.6 Problem 3.8. [Proof that $d\alpha = 0$ in (3.93).] To prove that the 1-form is closed it suffices to show that it is closed when restricted to any 2-dimensional submanifold. Thus we may suppose that $A = A(s,t)$ depends on 2 parameters. In terms of these parameters

iml.7
$$\alpha = \mathrm{Tr}(A(s,t)^{-1} \frac{dA(s,t)}{ds}) ds + \mathrm{Tr}(A(s,t)^{-1} \frac{dA(s,t)}{dt}) dt. \tag{3.165}$$

Show that the exterior derivative can be written

iml.8
$$d\alpha = \mathrm{Tr}([A(s,t)^{-1} \frac{dA(s,t)}{dt}, A(s,t)^{-1} \frac{dA(s,t)}{ds}]) ds \wedge dt \tag{3.166}$$

and hence that it vanishes.

21.2.1998.137 Problem 3.9. If E and F are vector spaces, show that the space of operators $\Psi^m_{\text{iso}}(\mathbb{R}^n; E, F)$ from $\mathcal{S}'(\mathbb{R}^n; E)$ to $\mathcal{S}'(\mathbb{R}^n; F)$ is well-defined as the matrices with entries in $\Psi^m_{\text{iso}}(\mathbb{R}^n)$ for any choice of bases of E and F.

1.2.2000.349 Problem 3.10. Necessity of ellipticity for a psuedodifferential operator to be Fredholm on the isotropic Sobolev spaces.

(1) Reduce to the case of operators of order 0.
(2) Construct a sequence in L^2 such that $\|u_n\| = 1$, $u_n \to 0$ weakly and $Au_n \to 0$ strongly in L^2.

21.2.1998.135 Problem 3.11. [Koszul complex] Consider the form bundles over $\mathbb{R}^n$. That is $\Lambda^k\mathbb{R}^n$ is the vector space of dimension $\binom{n}{k}$ consisting of the totally antisymmetric k-linear forms on $\mathbb{R}^n$. If $e_1, e_2, \dots, e_n$ is the standard basis for $\mathbb{R}^n$ then for a k-tuple α e^α defined on basis elements by

$$e^\alpha(e_{i_1}, \dots, e_{i_k}) = \prod_{j=1}^{k} \delta_{1_j\alpha_j}$$

extends uniquely to a k-linear map. Elements $dx^\alpha \in \Lambda^k\mathbb{R}^n$ are defined by the total antisymmetrization of the e^α. Explicitly,

$$dx^\alpha(v_1, \dots, v_k) = \sum_\pi \operatorname{sgn} \pi e^\alpha(v_{\pi_1}, \dots, v_{\pi_n})$$

where the sum is over permutations π of $\{1, \dots, n\}$ and $\operatorname{sgn}\pi$ is the parity of π. The dx^α for strictly increasing k-tuples α of elements of $\{1, \dots, n\}$ give a basis for $\Lambda^k\mathbb{R}^n$. The wedge product is defined by $dx^\alpha \wedge dx^\beta = dx^{\alpha,\beta}$.

Now let $\mathcal{S}'(\mathbb{R}^n; \Lambda^k)$ be the tensor product, that is $u \in \mathcal{S}'(\mathbb{R}^n; \Lambda^k)$ is a finite sum

21.2.1998.136 (3.167) $$u = \sum_\alpha u_\alpha dx^\alpha.$$

The annihilation operators in (3.161) [21.2.1998.131] define an operator, for each k,

$$D : \mathcal{S}'(\mathbb{R}^n; \Lambda^k) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \Lambda^{k+1}),\ Du = \sum_{j=1}^{n} A_j u_\alpha dx^j \wedge dx^\alpha.$$

Show that $D^2 = 0$. Define inner products on the $\Lambda^k\mathbb{R}^n$ by declaring the basis introduced above to be orthonormal. Show that the adjoint of D, defined with respect to these inner products and the L^2 pairing is

$$D^* : \mathcal{S}'(\mathbb{R}^n; \Lambda^k) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \Lambda^{k-1}),\ D^*u = \sum_{j=1}^{n} C_j u_\alpha \iota_j dx^\alpha.$$

Here, ι_j is 'contraction with e_j;' it is the adjoint of $dx^j \wedge$. Show that $D + D^*$ is an elliptic element of $\Psi^1_{\text{iso}}(\mathbb{R}^n; \Lambda^*)$. Maybe using Problem 3.6 [21.2.1998.111] show that the null space of $D + D^*$ on $\mathcal{S}'(\mathbb{R}^n; \Lambda^*\mathbb{R}^n)$ is 1-dimensional. Deduce that

21.2.1998.138 (3.168) $$\{u \in \mathcal{S}'(\mathbb{R}^n); Du = 0\} = \mathbb{C}\exp(-|x|^2/2),$$
$$\{u \in \mathcal{S}'(\mathbb{R}^n; \Lambda^k); Du = 0\} = (\mathcal{S}'(\mathbb{R}^n; \Lambda^{k-1}),\ k \ge 1.$$

Observe that, as an operator from $\mathcal{S}'(\mathbb{R}^n; \Lambda^{\text{odd}})$ to $\mathcal{S}'(\mathbb{R}^n; \Lambda^{\text{even}})$, $D + D^*$ is an elliptic element of $\Psi^1_{\text{iso}}(\mathbb{R}^n; \Lambda^{\text{odd}}, \Lambda^{\text{even}})$ and has index 1.

22.2.1998.141 Problem 3.12. [Isotropic essential support] For an element of $S^m(\mathbb{R}^n)$ define (isotropic) essential support, or operator wavefront set, of $A \in \Psi^m_{\text{iso}}(\mathbb{R}^n)$ by

22.2.1998.145
$$\operatorname{WF}_{\text{iso}}(A) = \operatorname{cone}\operatorname{supp}(\sigma_L(A)) \subset \mathbb{R}^{2n} \setminus \{0\}. \tag{3.169}$$

Show that $\operatorname{WF}_{\text{iso}}(A) = \operatorname{cone}\operatorname{supp}(\sigma_L(A))$ and check the following

22.2.1998.149
$$\operatorname{WF}'_{\text{iso}}(A + B) \cup \operatorname{WF}'_{\text{iso}}(A \circ B) \subset \operatorname{WF}'_{\text{iso}}(A) \cap \operatorname{WF}'_{\text{iso}}(B), \tag{3.170}$$

22.2.1998.146
$$\operatorname{WF}'_{\text{iso}}(A) = \emptyset \Longleftrightarrow A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n). \tag{3.171}$$

22.2.1998.150 Problem 3.13. [Isotropic partition of unity] Show that if $U_i \subset \mathbb{S}^{n-1}$ is an open cover of the unit sphere and $\tilde{U}_i = \{Z \in \mathbb{R}^{2n} \setminus \{0\}; \frac{Z}{|Z|} \in U_i\}$ is the corresponding *conic* open cover of $\mathbb{R}^{2n} \setminus \{0\}$ then there exist (finitely many) operators $A_i \in \Psi^0_{\text{iso}}(\mathbb{R}^n)$ with $\operatorname{WF}'_{\text{iso}}(A_i) \subset \tilde{U}_i$, such that

22.2.1998.151
$$\operatorname{Id} - \sum_i A_i \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n). \tag{3.172}$$

22.2.1998.152 Problem 3.14. Suppose $A \in \Psi^m_{\text{iso}}(\mathbb{R}^n)$, is elliptic and has index zero as an operator on $\mathcal{S}'(\mathbb{R}^n)$. Show that there exists $E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ such that $A + E$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$.

22.2.1998.147 Problem 3.15. [Isotropic wave front set] For $u \in \mathcal{S}'(\mathbb{R}^n)$ define

22.2.1998.148
$$\operatorname{WF}_{\text{iso}}(u) = \bigcap \left\{ \operatorname{WF}'_{\text{iso}}(A); A \in \Psi^0_{\text{iso}}(\mathbb{R}^n),\ Au \in \mathcal{S}(\mathbb{R}^n) \right\}. \tag{3.173}$$

CHAPTER 4

C.Microlocalization

Microlocalization

4.1. Calculus of supports

Recall that we have already defined the support of a tempered distribution in the slightly round-about way:

7.1 (4.1) $$\text{if } u \in \mathcal{S}'(\mathbb{R}^n),\ \operatorname{supp}(u) = \{x \in \mathbb{R}^n;\ \exists\ \phi \in \mathcal{S}(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^{\complement}.$$

Now if $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is any continuous linear operator we can consider the support of the kernel:

7.2 (4.2) $$\operatorname{supp}(A) = \operatorname{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

We write out the space as a product here to point to the fact that *any* subset of the product defines (is) a *relation* i.e. a map on subsets:

7.3 (4.3) $$\begin{gathered} G \subset \mathbb{R}^n \times \mathbb{R}^n, \quad S \subset \mathbb{R}^n \Longrightarrow \\ G \circ S = \big\{x \in \mathbb{R}^n;\ \exists\ y \in S \text{ s.t. } (x,y) \in G\big\}. \end{gathered}$$

One can write this much more geometrically in terms of the two projection maps

7.4 (4.4) $$\begin{array}{ccc} & \mathbb{R}^{2n} & \\ {}^{\pi_L}\swarrow & & \searrow^{\pi_R} \\ \mathbb{R}^n & & \mathbb{R}^n. \end{array}$$

Thus $\pi_R(x,y) = y$, $\pi_L(x,y) = x$. Then (4.3) can be written in terms of the action of maps on sets as

7.5 (4.5) $$G \circ S = \pi_L\left(\pi_R^{-1}(S) \cap G\right).$$

From this it follows that if S is compact and G is closed, then $G \circ S$ is closed, since its intersection with any compact set is the image of a compact set under a continuous map, hence compact. Now, by the *calculus of supports* we mean the 'trivial' result.

7.6 Proposition 4.1. *If $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is a continuous linear map then*

7.7 (4.6) $$\operatorname{supp}(A\phi) \subset \operatorname{supp}(A) \circ \operatorname{supp}(\phi)\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n).$$

Proof. Since we want to *bound* $\operatorname{supp}(A\phi)$ we can use (4.1) directly, i.e. show that

7.8 (4.7) $$x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi) \Longrightarrow x \notin \operatorname{supp}(A\phi).$$

Since we know $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$ to be closed, the assumption that x is outside this set means that there exists $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with

$$\psi(x) \neq 0 \text{ and } \operatorname{supp}(\psi) \cap \operatorname{supp}(A) \circ \operatorname{supp}(\phi) = \emptyset.$$

From (4.3) or (4.5) this means

7.9 (4.8) $$\operatorname{supp}(A) \cap (\operatorname{supp}(\psi) \times \operatorname{supp}(\phi)) = \emptyset \text{ in } \mathbb{R}^{2n}.$$

But this certainly implies that

7.10 (4.9) $$K_A(x,y)\psi(x)\phi(y) = 0 \\ \Longrightarrow \psi A(\phi) = \int K_A(x,y)\psi(x)\phi(y)dy = 0.$$

Thus we have proved (4.6) and the lemma. □

Diff ops.

4.2. Singular supports

As well as the support of a tempered distribution we can consider the singular support:

7.11 (4.10) $$\operatorname{sing\,supp}(u) = \left\{x \in \mathbb{R}^n; \exists\ \phi \in \mathcal{S}(\mathbb{R}^n), \phi(x) \neq 0, \phi u \in \mathcal{S}(\mathbb{R}^n)\right\}^{\complement}.$$

Again this is a closed set since $x \notin \operatorname{sing\,supp}(u) \Longrightarrow \exists\ \phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi u \in \mathcal{S}(\mathbb{R}^n)$ and $\phi(x) \neq 0$ so $\phi(x') \neq 0$ for $|x - x'| < \epsilon$, some $\epsilon > 0$ and hence $x' \notin \operatorname{sing\,supp}(u)$ i.e. the complement of $\operatorname{sing\,supp}(u)$ is open.

Directly from the definition we have

7.12 (4.11) $$\operatorname{sing\,supp}(u) \subset \operatorname{supp}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n) \text{ and}$$

7.13 (4.12) $$\operatorname{sing\,supp}(u) = \emptyset \Longleftrightarrow u \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Examples

4.3. Pseudolocality

We would like to have a result like (4.6) for singular support, and indeed we can get one for pseudodifferential operators. First let us work out the singular support of the kernels of pseudodifferential operators.

7.14 Proposition 4.2. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ then*

7.15 (4.13) $$\operatorname{sing\,supp}(A) = \operatorname{sing\,supp}(K_A) \subset \left\{(x,y) \in \mathbb{R}^{2n}; x = y\right\}.$$

Proof. The kernel is defined by an oscillatory integral

(4.14) $$I(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi) d\xi.$$

If the order m is $< -n$ we can show by integration by parts that

7.16 (4.15) $$(x-y)^\alpha I(a) = I\left((-D_\xi)^\alpha a\right),$$

and then this must hold by continuity for all orders. If a is of order m and $|\alpha| > m + n$ then $(-D_\xi)^\alpha a$ is of order less than $-n$, so

7.17 (4.16) $$(x-y)^\alpha I(a) \in \mathcal{C}^0_\infty(\mathbb{R}^n), |\alpha| > m + n.$$

In fact we can also differentiate under the integral sign:

7.18 (4.17) $$D^\beta_x D^\gamma_y (x-y)^\alpha I(a) = I\left(D^\beta_x D^\gamma_y (-D_\xi)^\alpha a\right)$$

so generalizing (4.16) to

7.19 (4.18) $$(x-y)^\alpha I(a) \in \mathcal{C}^k_\infty(\mathbb{R}^n) \text{ if } |\alpha| > m + n + k.$$

This implies that $I(A)$ is $\mathcal{C}^\infty$ on the complement of the diagonal, $\{x = y\}$. This proves (4.13). □

An operator is said to be *pseudolocal* if it satisfies the condition

7.20 (4.19)
$$\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u)\ \forall\ u \in \mathcal{C}^{-\infty}(\mathbb{R}^n).$$

7.21 PROPOSITION 4.3. *Pseudodifferential operators are pseudolocal.*

PROOF. Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ has compact support and $\overline{x} \notin \operatorname{sing\,supp}(u)$. Then we can choose $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi \equiv 1$ near $\overline{x}$ and $\phi u \in \mathcal{S}(\mathbb{R}^n)$ (by definition). Thus

7.22 (4.20)
$$u = u_1 + u_2,\ u_1 = (1-\phi)u, \quad u_2 \in \mathcal{S}(\mathbb{R}^n).$$

Since $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, $Au_2 \in \mathcal{S}(\mathbb{R}^n)$ so

7.23 (4.21)
$$\operatorname{sing\,supp}(Au) = \operatorname{sing\,supp}(Au_1) \text{ and } \overline{x} \notin \operatorname{supp}(u_1).$$

Choose $\psi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $\psi(\overline{x}) = 1$ and

7.24 (4.22)
$$\operatorname{supp}(\psi) \cap \operatorname{supp}(1-\phi) = \emptyset.$$

Thus

7.25 (4.23)
$$\psi A u_1 = \psi A(1-\phi)u = \tilde{A}u$$

where

(4.24)
$$K_{\tilde{A}}(x,y) = \psi(x) K_A(x,y)(1-\phi(y)).$$

Combining (4.22) and (4.13) shows that $K_{\tilde{A}} \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ so, by Lemma 2.8, $\tilde{A}u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\overline{x} \notin \operatorname{sing\,supp}(Au)$ by (4.13)(?). This proves the proposition. □

Sect.CooInv 4.4. Coordinate invariance

If $\Omega \subset \mathbb{R}^n$ is an open set, put

7.26 (4.25)
$$\mathcal{C}_c^\infty(\Omega) = \left\{u \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\right\}$$
$$\mathcal{C}_c^{-\infty}(\Omega) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\right\}$$

respectively the space of $\mathcal{C}^\infty$ functions of compact support in Ω and of distributions of compact support in Ω. Here $K \Subset \Omega$ indicates that K is a compact subset of Ω. Notice that if $u \in \mathcal{C}_c^{-\infty}(\Omega)$ then u defines a continuous linear functional

7.27 (4.26)
$$\mathcal{C}^\infty(\Omega) \ni \phi \longmapsto u(\phi) = u(\psi\phi) \in \mathbb{C}$$

where if $\psi \in \mathcal{C}_c^\infty(\Omega)$ is chosen to be identically one near $\operatorname{supp}(u)$ then (4.26) is independent of ψ. [Think about what continuity means here!]

Now suppose

7.28 (4.27)
$$F : \Omega \longrightarrow \Omega'$$

is a diffeomorphism between open sets of $\mathbb{R}^n$. The pull-back operation is

7.29 (4.28)
$$F^* :\ \mathcal{C}_c^\infty(\Omega') \longleftrightarrow \mathcal{C}_c^\infty(\Omega),\ F^*\phi = \phi \circ F.$$

7.30 LEMMA 4.1. *If F is a diffeomorphism, (4.27), between open sets of $\mathbb{R}^n$ then there is an extension by continuity of (4.28) to*

7.31 (4.29)
$$F^* : \mathcal{C}_c^{-\infty}(\Omega') \longleftrightarrow \mathcal{C}_c^{-\infty}(\Omega).$$

Proof. The density of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{C}_c^{-\infty}(\Omega)$, in the weak topology given by the seminorms from (4.26), can be proved in the same way as the density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ (see Problem 4.5). Thus, we only need to show continuity of (4.29) in this sense. Suppose $u \in \mathcal{C}_c^\infty(\Omega)$ and $\phi \in \mathcal{C}_c^\infty(\Omega')$ then

7.32
$$(F^*u)(\phi) = \int u(F(x))\phi(x)dx = \int u(y)\phi(G(y))|J_G(y)|dy \tag{4.30}$$

where $J_G(y) = \left(\frac{\partial G(y)}{\partial y}\right)$ is the Jacobian of G, the inverse of F. Thus (4.28) can be written

7.33
$$F^*u(\phi) = (|J_G|u)\,(G^*\phi) \tag{4.31}$$

and since $G^* : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega')$ is continuous (!) we conclude that F^* is continuous as desired. □

Now suppose that

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

has

7.34
$$\operatorname{supp}(A) \Subset \Omega \times \Omega \subset \mathbb{R}^{2n}. \tag{4.32}$$

Then

7.35
$$A : \mathcal{C}_c^\infty(\Omega) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega) \tag{4.33}$$

by Proposition 4.1. Applying a diffeomorphism, F, as in (4.27) set

7.36
$$A_F : \mathcal{C}_c^\infty(\Omega') \longrightarrow \mathcal{C}_c^{-\infty}(\Omega'),\ A_F = G^* \circ A \circ F^*. \tag{4.34}$$

7.37 Lemma 4.2. *If A satisfies (4.32) and F is a diffeomorphism (4.27) then*

7.38
$$K_{A_F}(x,y) = (G \times G)^* K \cdot |J_G(y)| \text{ on } \Omega' \times \Omega' \tag{4.35}$$

has compact support in $\Omega' \times \Omega'$.

Proof. Essentially the same as that of (4.30). □

7.39 Proposition 4.4. *Suppose $A \in \Psi_\infty^m(\mathbb{R}^n)$ has kernel satisfying (4.32) and F is a diffeomorphism as in (4.27) then A_F, defined by (4.34), is an element of $\Psi_\infty^m(\mathbb{R}^n)$.*

Proof. Since $A \in \Psi_\infty^m(\mathbb{R}^n)$,

7.40
$$K_A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,\xi)d\xi \tag{4.36}$$

for some $a \in S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)$. Now choose $\psi \in \mathcal{C}_c^\infty(\Omega)$ such that $\psi(x)\psi(y) = 1$ on $\operatorname{supp}(K_A)$, possible by (4.32). Then

7.41
$$K_A = I\left(\psi(x)\psi(y)a(x,\xi)\right). \tag{4.37}$$

In fact suppose $\mu_\epsilon(x,y) \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ and $\mu \equiv 1$ in $|x-y| < \epsilon$ for $\epsilon > 0$, $\mu(x,y) = 0$ in $|x-y| > 2\epsilon$. Then if

7.42
$$K_{A_\epsilon} = I\left(\mu_\epsilon(x,y)\psi(x)\psi(y)a(x,\xi)\right) \tag{4.38}$$

we know that if

7.43
$$A'_\epsilon = A - A_\epsilon \text{ then } K_{A'_\epsilon} = \left(1 - \mu_\epsilon(x,y)\right)K_A \in \Psi_\infty^{-\infty}(\mathbb{R}^n), \tag{4.39}$$

by (4.13). Certainly A'_ϵ also satisfies (4.32) and from (4.35)

7.44 (4.40) $$(A'_\epsilon)_F \in \Psi^{-\infty}_\infty(\mathbb{R}^n).$$

So we only need to consider A_ϵ defined by (4.38). Again using (4.35) (assuming $m < -n$)

7.45 (4.41) $$K_{(A_\epsilon)_F}(x,y) = (2\pi)^{-n}\int e^{i(G(x)-G(y))\cdot\xi} b(G(x),G(y),\xi)\Big|\frac{\partial G}{\partial y}\Big| d\xi$$

where $b(x,y,\xi) = \mu_\epsilon(x-y)\psi(x)\psi(y)a(x,\xi)$. Applying Taylor's formula,

7.46 (4.42) $$G(x) - G(y) = (x-y)\cdot T(x,y)$$

where $T(x,y)$ is an invertible $\mathcal{C}^\infty$ matrix on $K \times K \cap \{|x-y| < \epsilon\}$ for $\epsilon < \epsilon(K)$, where $\epsilon(K) > 0$ depends on the compact set $K \Subset \Omega'$. Thus we can set(??)

(4.43) $$\eta = T^t(x,y)\cdot\xi$$

and rewrite (4.41) as

7.47 (4.44) $$\begin{gathered} K_{(A_\epsilon)_F}(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\eta}c(x,y,\eta)d\eta \\ c(x,y,\eta) = b\left(G(x),G(y),(T^t)^{-1}(x,y)\eta\right)\Big|\frac{\partial G}{\partial y}\Big| \cdot \Big|\det T(x,y)\Big|^{-1}. \end{gathered}$$

So it only remains to show that $c \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ and the proof is complete. We can drop all the $\mathcal{C}^\infty$ factors, given by $\big|\partial G/\partial y\big|$ etc. and proceed to show that

(4.45) $$\Big|D^\alpha_x D^\beta_y D^\gamma_\xi a\left(G(x),G(y),S(x,y)\xi\right)\Big| \le C(1+|\xi|)^{m-|\gamma|} \quad \text{on } K\times K\times\mathbb{R}^n$$

where $K \subset\subset \Omega'$ and S is $\mathcal{C}^\infty$ with $|\det S| \ge \epsilon$. The estimates with $\alpha = \beta = 0$ follow easily and the general case by induction:

$$D^\alpha_x D^\beta_y D^\gamma_\xi a\left(G(x),G(y),S(x,y)\xi\right) = \sum_{\substack{|\mu|\le|\alpha|+|\beta|+|\gamma| \\ |\alpha'|\le|\alpha|,|\beta|\le|\beta| \\ |\nu|+|\gamma|\le|\mu|}} M^{\alpha',\rho',\mu'}_{\alpha,\beta,\gamma,\nu}(x,y)\xi^\nu\left(D^{\alpha'}D^{\beta'}D^\mu a\right)(G(x),G(y),S\xi)$$

where the coefficients are $\mathcal{C}^\infty$ and the main point is that $|\nu| \le |\mu|$. □

P.7.1

4.5. Problems

P5.1 Problem 4.1. Show that *Weyl quantization*

(4.46) $$S^\infty_\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto q_W(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(\frac{x+y}{2},\xi)d\xi$$

is well-defined by continuity from $S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ and induces an isomorphism

(4.47) $$S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \Psi^m_\infty(\mathbb{R}^n) \ \forall\ m \in \mathbb{R}.$$

Find an asymptotic formula relating $q_W(A)$ to $q_L(A)$ for any $A \in \Psi^m_\infty(\mathbb{R}^n)$.

P5.2 Problem 4.2. Show that if $A \in \Psi^m_\infty(\mathbb{R}^n)$ then $A^* = A$ if and only if $\sigma_W(A)$ is real-valued.

P5.3 Problem 4.3. Is it true that every $E \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ defines a map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$?

5.46 PROBLEM 4.4. Show that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ by proving that if $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ has compact support and is identically equal to 1 near the origin then

$$u_n(x) = (2\pi)^{-n}\phi(\frac{x}{n})\int e^{ix\cdot\xi}\phi(\xi/n)\hat{u}(\xi)d\xi \in \mathcal{S}(\mathbb{R}^n) \text{ if } u \in L^2(\mathbb{R}^n) \tag{4.48}$$

and $u_n \to u$ in $L^2(\mathbb{R}^n)$. Can you see any relation to pseudodifferential operators here?

5.47 PROBLEM 4.5. Check carefully that with the definition

$$H^k(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{|\alpha|\le -k} D^\alpha u_\alpha,\ u_\alpha \in L^2(\mathbb{R}^n) \right\} \tag{4.49}$$

for $-k \in \mathbb{N}$ one does have

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n) \tag{4.50}$$

as claimed in the text.

5.48 PROBLEM 4.6. Suppose that $a(x) \in \mathcal{C}^\infty_\infty(\mathbb{R}^n)$ and that $a(x) \ge 0$. Show that the operator

$$A = \sum_{j=1}^n D^2_{x_j} + a(x) \tag{4.51}$$

can have no solution which is in $L^2(\mathbb{R}^n)$.

5.49 PROBLEM 4.7. Show that for any open set $\Omega \subset \mathbb{R}^n$, $\mathcal{C}^\infty_c(\Omega)$ is dense in $\mathcal{C}^{-\infty}_c(\Omega)$ in the weak topology.

5.53 PROBLEM 4.8. Use formula (4.44) to find the principal symbol of A_F; more precisely show that if $F^* : T^*\Omega' \longrightarrow T^*\omega$ is the (co)-differential of F then

$$\sigma_m(A_F) = \sigma_m(A) \circ F^*.$$

We have now studied special distributions, the Schwartz kernels of pseudodifferential operators. We shall now apply this knowledge to the study of general distributions. In particular we shall examine the wavefront set, a refinement of singular support, of general distributions. This notion is fundamental to the general idea of 'microlocalization.'

4.6. Characteristic variety

If $A \in \Psi^m_\infty(\mathbb{R}^n)$, the left-reduced symbol is elliptic at $(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ if there exists $\epsilon > 0$ such that

8.1
$$\begin{gathered} \left|\sigma_L(A)(x,\xi)\right| \ge \epsilon|\xi|^m \quad \text{in} \\ \left\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}); |x-\overline{x}| \le \epsilon,\ \left|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\xi|}\right| \le \epsilon,\ |\xi| \ge \frac{1}{\epsilon}\right\}. \end{gathered} \tag{4.52}$$

Directly from the definition, ellipticity at $(\overline{x}, \overline{\xi})$ is actually a property of the principal symbol, $\sigma_m(A)$ and if A is elliptic at $(\overline{x}, \overline{\xi})$ then it is elliptic at $(\overline{x}, t\overline{\xi})$ for any $t > 0$. Clearly

$$\left\{(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\});\ A \text{ is elliptic (of order } m\text{) at } (\overline{x}, \overline{\xi})\right\}$$

is an *open* cone in $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$. The complement

8.2 (4.53) $\Sigma_m(A) = \big\{(\overline{x},\overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})\,;\ A \text{ is } \textit{not} \text{ elliptic of order } m \text{ at } (\overline{x},\overline{\xi})\big\}$

is therefore a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$; it is the characteristic set (or variety) of A. Since the product of two symbols is only elliptic at $(\overline{x},\overline{\xi})$ if they are both elliptic there, if follows from the composition properties of pseudodifferential operators that

8.3 (4.54) $$\Sigma_{m+m'}(A \circ B) = \Sigma_m(A) \cup \Sigma_{m'}(B).$$

4.7. Wavefront set

We adopt the following bald definition:

8.4 (4.55) $$\begin{gathered}\text{If } u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n);\ \operatorname{supp}(u) \Subset \mathbb{R}^n\big\} \text{ then}\\ \operatorname{WF}(u) = \bigcap\big\{\Sigma_0(A);\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ and } Au \in \mathcal{C}^\infty(\mathbb{R}^n)\big\}.\end{gathered}$$

Thus $\operatorname{WF}(u) \subset \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ is always a closed conic set, being the intersection of such sets. The first thing we wish to show is that $\operatorname{WF}(u)$ is a refinement of $\operatorname{sing\,supp}(u)$. Let

8.5 (4.56) $$\pi:\ \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}) \ni (x,\xi) \longmapsto x \in \mathbb{R}^n$$

be projection onto the first factor.

8.6 Proposition 4.5. *If* $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *then*

8.7 (4.57) $$\pi(\operatorname{WF}(u)) = \operatorname{sing\,supp}(u).$$

Proof. The inclusion $\pi(\operatorname{WF}(u)) \subset \operatorname{sing\,supp}(w)$ is straightforward. Indeed, if $\overline{x} \notin \operatorname{sing\,supp}(u)$ then there exists $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi(\overline{x}) \neq 0$ such that $\phi u \in \mathcal{C}^\infty(\mathbb{R}^n)$. Of course as a multiplication operator, $\phi \in \Psi^0_\infty(\mathbb{R}^n)$ and $\Sigma_0(\phi) \not\ni (\overline{x},\overline{\xi})$ for any $\overline{\xi} \neq 0$. Thus the definition (4.55) shows that $(\overline{x},\overline{\xi}) \notin \operatorname{WF}(u)$ for all $\overline{\xi} \in \mathbb{R}^n\diagdown 0$ proving the inclusion.

Using the calculus of pseudodifferential operators, the opposite inclusion,

(4.58) $$\pi(\operatorname{WF}(u)) \supset \operatorname{sing\,supp}(u)$$

is only a little more complicated. Thus we have to show that if $(\overline{x},\overline{\xi}) \notin \operatorname{WF}(u)$ for all $\overline{\xi} \in \mathbb{R}^n\diagdown 0$ then $\overline{x} \notin \operatorname{sing\,supp}(u)$. The hypothesis is that for each $(\overline{x},\overline{\xi}), \overline{\xi} \in \mathbb{R}^n\diagdown 0$, there exists $A \in \Psi^0_\infty(\mathbb{R}^n)$ such that A is elliptic at $(\overline{x},\overline{\xi})$ and $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. The set of elliptic points is open so there exists $\epsilon = \epsilon(\overline{\xi}) > 0$ such that A is elliptic on

8.8 (4.59) $$\big\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\diagdown 0); |x - \overline{x}| < \epsilon, \Big|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\Big| < \epsilon\big\}.$$

Let B_j, $j = 1,\dots,N$ be a finite set of such operators associated to $\overline{\xi}_j$ and such that the corresponding sets in (4.59) cover $\{\overline{x}\} \times (\mathbb{R}^n\diagdown 0)$; the finiteness follows from the compactness of the sphere. Then consider

$$B = \sum_{j=1}^N B_j^* B_j \Longrightarrow Bu \in \mathcal{C}^\infty(\mathbb{R}^n).$$

This operator B is elliptic at $(\overline{x},\xi)$, for all $\xi\neq 0$. Thus if $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$, $0\le\phi(x)\le 1$, has support sufficiently close to $\overline{x}$, $\phi(x)=1$ in $|x-\overline{x}|<\epsilon/2$ then, since B has non-negative principal symbol

$$B+(1-\phi)\in\Psi^0_\infty(\mathbb{R}^n) \tag{4.60}$$

is globally elliptic. Thus, by Lemma 2.7, there exists $G\in\Psi^0_\infty(\mathbb{R}^n)$ which is a parametrix for $B+(1-\phi)$:

8.9 $$\mathrm{Id}\equiv G\circ B+G(1-\phi)\quad \mathrm{mod}\ \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{4.61}$$

Let $\psi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ be such that $\operatorname{supp}(\psi)\subset\{\phi=1\}$ and $\psi(\overline{x})\neq 0$. Then, from the reduction formula

$$\psi\circ G\circ(1-\phi)\in\Psi^{-\infty}_\infty(\mathbb{R}^n).$$

Thus from (4.61) we find

$$\psi u=\psi G\circ Bu+\psi G(1-\phi)u\in\mathcal{C}^\infty(\mathbb{R}^n).$$

Thus $\overline{x}\notin\operatorname{sing\,supp}(u)$ and the proposition is proved. □

We extend the definition to general tempered distributions by setting

8.10 $$\mathrm{WF}(u)=\bigcup_{\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)}\mathrm{WF}(\phi u),\ \ u\in\mathcal{S}'(\mathbb{R}^n). \tag{4.62}$$

Then (4.57) holds for every $u\in\mathcal{S}'(\mathbb{R}^n)$.

4.8. Essential support

Next we shall consider the notion of the essential support of a pseudodifferential operator. If $a\in S^m_\infty(\mathbb{R}^N;\mathbb{R}^n)$ we define the cone support of a by

8.11 $$\begin{aligned}\operatorname{cone\,supp}(a)=\big\{&(\overline{x},\overline{\xi})\in\mathbb{R}^N\times(\mathbb{R}^n\setminus 0);\exists\ \epsilon>0 \text{ and } \forall\ M\in\mathbb{R},\exists\ C_M \text{ s.t.}\\ &|a(x,\xi)|\le C_M\langle\xi\rangle^{-M}\text{ if }|x-\overline{x}|\le\epsilon,\Big|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\Big|\le\epsilon\big\}^{\complement}.\end{aligned} \tag{4.63}$$

This is clearly a closed conic set in $\mathbb{R}^N\times(\mathbb{R}^n\setminus 0)$. By definition the symbol decays rapidly outside this cone, in fact even more is true.

8.29 Lemma 4.3. *If $a\in S^\infty_\infty(\mathbb{R}^N;\mathbb{R}^n)$ then*

8.15 $$\begin{gathered}(\overline{x},\overline{\eta})\notin\operatorname{cone\,supp}(a)\Longrightarrow\\ \exists\ \epsilon>0\ \textit{s.t.}\ \forall\ M,\alpha,\beta\ \exists\ C_M\ \textit{with}\\ \big|D^\alpha_x D^\beta_\xi a(x,\eta)\big|\le C_M\langle\eta\rangle^{-M}\ \textit{if}\ |x-\overline{x}|<\epsilon,\Big|\frac{\eta}{|\eta|}-\frac{\overline{\eta}}{|\overline{\eta}|}\Big|<\epsilon.\end{gathered} \tag{4.64}$$

Proof. To prove (4.64) it suffices to show it to be valid for $D_{x_j}a$, $D_{\xi_k}a$ and then use an inductive argument, i.e. to show that

8.17 $$\operatorname{cone\,supp}(D_{x_j}a),\operatorname{cone\,supp}(D_{\xi_k}a)\subset\operatorname{cone\,supp}(a). \tag{4.65}$$

Arguing by contradiction suppose that $D_{x_\ell}a$ does not decay to order M in any cone around $(\overline{x},\overline{\xi})\notin\operatorname{cone\,supp}$. Then there exists a sequence (x_j,ξ_j) with

8.18 $$\begin{cases}x_j\longrightarrow\overline{x},\ \Big|\frac{\xi_j}{|\xi_j|}-\frac{\overline{\xi}}{|\overline{\xi}|}\Big|\longrightarrow 0,\ |\xi_j|\longrightarrow\infty\\ \text{and }\big|D_{x_\ell}a(x_j,\xi_j)\big|>j\langle\xi_j\rangle^M.\end{cases} \tag{4.66}$$

We can assume that $M < m$, since $a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^N\right)$. Applying Taylor's formula with remainder, and using the symbol bounds on $D^2_{x_j}a$, gives

8.19 $$a(x_j + te_\ell, \xi_j) = a(x_j, \xi_j) + it(D_{x_j}a)(x_j, \xi_j) + O\left(t^2\langle\xi_j\rangle^m\right),\ (e_\ell)_j = \delta_{\ell j} \tag{4.67}$$

providing $|t| < 1$. Taking $t = \langle\xi_j\rangle^{M-m} \longrightarrow 0$ as $j \longrightarrow \infty$, the first and third terms on the right in (4.67) are small compared to the second, so

$$\left|a\left(x_j + \langle\xi_j\rangle^{\frac{M-m}{2}}, \xi_j\right)\right| > \langle\xi_j\rangle^{2M-m}, \tag{4.68}$$

contradicting the assumption that $(\overline{x}, \overline{\xi}) \notin \operatorname{cone\,supp}(a)$. A similar argument applies to $D_{\xi_\ell}a$ so (4.64), and hence the lemma, is proved. $\square$

For a pseudodifferential operator we define the *essential support* by

8.12 $$\mathrm{WF}'(A) = \operatorname{cone\,supp}(\sigma_L(A)) \subset \mathbb{R}^n \times (\mathbb{R}^n\setminus 0). \tag{4.69}$$

8.13 LEMMA 4.4. *For every $A \in \Psi^m_\infty(\mathbb{R}^n)$*

8.14 $$\mathrm{WF}'(A) = \operatorname{cone\,supp}(\sigma_R(A)). \tag{4.70}$$

PROOF. Using (4.64) and the formula relating $\sigma_R(A)$ to $\sigma_L(A)$ we conclude that

8.16 $$\operatorname{cone\,supp}(\sigma_L(A)) = \operatorname{cone\,supp}(\sigma_R(A)), \tag{4.71}$$

from which (4.70) follows. $\square$

A similar argument shows that

8.20 $$\mathrm{WF}'(A \circ B) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B). \tag{4.72}$$

Indeed the asymptotic formula for $\sigma_L(A \circ B)$ in terms of $\sigma_L(A)$ and $\sigma_L(B)$ shows that

8.21 $$\operatorname{cone\,supp}(\sigma_L(A \circ B)) \subset \operatorname{cone\,supp}(\sigma_L(A)) \cap \operatorname{cone\,supp}(\sigma_L(B)) \tag{4.73}$$

which is the same thing.

Sect.MicPar 4.9. Microlocal parametrices

The concept of essential support allows us to refine the notion of a parametrix for an elliptic operator to that of a *microlocal parametrix.*

9.1 LEMMA 4.5. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $z \notin \Sigma_m(A)$ then there exists a microlocal parametrix at z, $B \in \Psi^{-m}_\infty(\mathbb{R}^n)$ such that*

9.2 $$z \notin \mathrm{WF}'(\mathrm{Id} - AB) \text{ and } z \notin \mathrm{WF}'(\mathrm{Id} - BA). \tag{4.74}$$

PROOF. If $z = (\overline{x}, \overline{\xi})$, $\overline{\xi} \neq 0$, consider the symbol

$$\gamma_\epsilon(x, \xi) = \phi\left(\frac{x - \overline{x}}{\epsilon}\right)(1 - \phi)(\epsilon\xi)\phi\left(\left(\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\right)/\epsilon\right) \tag{4.75}$$

where as usual $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\phi(\zeta) = 1$ in $|\zeta| \le \frac{1}{2}$, $\phi(\zeta) = 0$ in $|\zeta| \ge 1$. Thus $\gamma_\epsilon \in S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ has support in

9.3 $$|x - \overline{x}| \le \epsilon,\ |\xi| \ge \frac{1}{2\epsilon},\ \left|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\right| \le \epsilon \tag{4.76}$$

and is identically equal to one, and hence elliptic, on a similar smaller set

9.4 (4.77)
$$|x-\overline{x}|<\frac{\epsilon}{2}, |\xi|\ge\frac{1}{\epsilon},\ \left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\le\frac{\epsilon}{2}.$$

Define $L_\epsilon\in\Psi^0_\infty(\mathbb{R}^n)$ by $\sigma_L(L_\epsilon)=\gamma_\epsilon$. Thus, for any $\epsilon>0$,

8.33 (4.78)
$$z\notin \mathrm{WF}'(\mathrm{Id}-L_\epsilon),\ \mathrm{WF}'(L_\epsilon)\subset\left\{(x,\xi);|x-\overline{x}|\le\epsilon \text{ and } \left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\le\epsilon\right\}.$$

Let $G_{2m}\in\Psi^{2m}_\infty(\mathbb{R}^n)$ be a globally elliptic operator with positive principal symbol. For example take $\sigma_L(G_{2m})=(1+|\xi|^2)^m$, so $G_s\circ G_t=G_{s+t}$ for any s, $t\in\mathbb{R}$. Now consider the operator

(4.79)
$$J=(\mathrm{Id}-L_\epsilon)\circ G_{2m}+A^*A\in\Psi^{2m}_\infty(\mathbb{R}^n).$$

The principal symbol of J is $(1-\gamma_\epsilon)(1+|\xi|^2)^m+|\sigma_m(A)|^2$ which is globally elliptic if $\epsilon>0$ is small enough (so that $\sigma_m(A)$ is elliptic on the set (4.76)). According to Lemma 2.75, J has a global parametrix $H\in\Psi^{-2m}_\infty(\mathbb{R}^n)$. Then

8.34 (4.80)
$$B=H\circ A^*\in\Psi^{-m}_\infty(\mathbb{R}^n)$$

is a microlocal right parametrix for A in the sense that $B\circ A-\mathrm{Id}=R_R$ with $z\notin\mathrm{WF}'(R_R)$ since

8.35 (4.81)
$$\begin{aligned}R_R=B\circ A-\mathrm{Id}&=H\circ A^*\circ A-\mathrm{Id}\\&=(H\circ J-\mathrm{Id})+H\circ(\mathrm{Id}-L_\epsilon)G_{2m}\circ A\end{aligned}$$

and the first term on the right is in $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ whilst z is not in the operator wavefront set of $(\mathrm{Id}-L_\epsilon)$ and hence not in the operator wavefront set of the second term.

By a completely analogous construction we can find a left microlocal parametrix. Namely $(\mathrm{Id}-L_\epsilon)\circ G_{2m}+A\circ A^*$ is also globally elliptic with parametrix H' and then $B'=A^*\circ H'$ satisfies

(4.82)
$$B'\circ A-\mathrm{Id}=R_L,\ z\notin\mathrm{WF}'(R_L).$$

Then, as usual,

(4.83)
$$B=(B'\circ A-R_L)\,B=B'\,(A\circ B)-R_LB=B'+B'R_R-R_LB$$

so $z\notin\mathrm{WF}'(B-B')$, which implies that B is both a left and right microlocal parametrix. □

In fact this argument shows that such a left parametrix is essentially unique. See Problem 4.25.

4.10. Microlocality

Now we can consider the relationship between these two notions of wavefront set.

8.22 Proposition 4.6. *Pseudodifferential operators are* microlocal *in the sense that*

8.23 (4.84)
$$\mathrm{WF}(Au)\subset\mathrm{WF}'(A)\cap\mathrm{WF}(u)\quad\forall\ A\in\Psi^\infty_\infty(\mathbb{R}^n),\ u\in\mathcal{C}^{-\infty}_c(\mathbb{R}^n).$$

Proof. We need to show that

8.24 (4.85) $$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \text{ and } \mathrm{WF}(Au) \subset \mathrm{WF}(u).$$

the second being the usual definition of microlocality. The first inclusion is easy. Suppose $(\overline{x}, \overline{\xi}) \notin \operatorname{cone\,supp} \sigma_L(A)$. If we choose $B \in \Psi^0_\infty(\mathbb{R}^n)$ with $\sigma_L(B)$ supported in a small cone around $(\overline{x}, \overline{\xi})$ then we can arrange

(4.86) $$(\overline{x}, \overline{\xi}) \notin \Sigma_0(B),\ \mathrm{WF}'(B) \cap \mathrm{WF}'(A) = \emptyset.$$

Then from (4.72), $\mathrm{WF}'(BA) = \emptyset$ so $BA \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ and $BAu \in \mathcal{C}^\infty(\mathbb{R}^n)$. Thus $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(Au)$.

Similarly suppose $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$. Then there exists $G \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\overline{x}, \overline{\xi})$ with $Gu \in \mathcal{C}^\infty(\mathbb{R}^n)$. Let B be a microlocal parametrix for G at $(\overline{x}, \overline{\xi})$ as in Lemma 4.5. Thus

(4.87) $$u = BGu + Su,\ (\overline{x}, \overline{\xi}) \notin \mathrm{WF}'(S).$$

Now apply A to this identity. Since, by assumption, $Gu \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ the first term on the right in

8.31 (4.88) $$Au = ABGu + ASu$$

is smooth. Since, by (4.72), $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}'(AS)$ it follows from the first part of the argument above that $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(ASu)$ and hence $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(Au)$. □

We can deduce from the existence of microlocal parametrices at elliptic points a partial converse of (8.24).

8.32 Proposition 4.7. *For any $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and any $A \in \Psi^m_\infty(\mathbb{R}^n)$*

(4.89) $$\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \Sigma_m(A).$$

Proof. If $(\overline{x}, \overline{\xi}) \notin \Sigma_m(A)$ then, by definition, A is elliptic at $(\overline{x}, \overline{\xi})$. Thus, by Lemma 4.5, A has a microlocal parametrix B, so

(4.90) $$u = BAu + Su,\ (\overline{x}, \overline{\xi}) \notin \mathrm{WF}'(S).$$

It follows that $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(Au)$ implies that $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$ proving the Proposition. □

4.11. Explicit formulations

From this discussion of $\mathrm{WF}'(A)$ we can easily find a 'local coordinate' formulations of $\mathrm{WF}(u)$ in general.

8.25 Lemma 4.6. *If $(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ then $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$ if and only if there exists $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(\overline{x}) \neq 0$ such that for some $\epsilon > 0$, and for all M there exists C_M with*

8.26 (4.91) $$\left|\widehat{\phi u}(\xi)\right| \le C_M \langle \xi \rangle^M \text{ in } \left|\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}\right| < \epsilon.$$

Proof. If $\zeta \in \mathcal{C}^\infty(\mathbb{R})$, $\zeta(\xi) \equiv 1$ in $|\xi| < \frac{\epsilon}{2}$ and $\operatorname{supp}(\zeta) \subset \left[\frac{-3\epsilon}{4}, \frac{3\epsilon}{4}\right]$ then

(4.92) $$\gamma(\xi) = (1 - \zeta)(\xi) \cdot \zeta\left(\frac{\xi}{|\xi|} - \frac{\overline{x}}{|\overline{x}|}\right) \in S^0_\infty(\mathbb{R}^n)$$

is elliptic at $\bar{\xi}$ and from (4.91)

(4.93) $$\gamma(\xi) \cdot \widehat{\phi u}(\xi) \in \mathcal{S}(\mathbb{R}^n).$$

Thus if $\sigma_R(A) = \phi_1(x)\gamma(\xi)$ then $A(\phi_2 u) \in \mathcal{C}^\infty$ where $\phi_1\phi_2 = \phi$, $\phi_1(\overline{x}), \phi_2(\overline{x}) \neq 0$, $\phi_1, \phi_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Thus $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$. Conversely if $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$ and A is chosen as above then $A(\phi_1 u) \in \mathcal{S}(\mathbb{R}^n)$ and Lemma 4.6 holds. □

4.12. Wavefront set of K_A

At this stage, a natural thing to look at is the wavefront set of the kernel of a pseudodifferential operator, since these kernels are certainly an interesting class of distributions.

8.27 Proposition 4.8. *If $A \in \Psi_\infty^m(\mathbb{R}^n)$ then*

8.28 (4.94)
$$\mathrm{WF}(K_A) = \Big\{ (x, y, \xi, \eta) \in \mathbb{R}^{2n} \times \big(\mathbb{R}^{2n} \setminus 0\big)\,; \\ x = y,\ \xi + \eta = 0 \text{ and } (x, \xi) \in \mathrm{WF}'(A) \Big\}.$$

In particular this shows that $\mathrm{WF}'(A)$ determines $\mathrm{WF}(K_A)$ and conversely.

Proof. Using Proposition 4.5 we know that $\pi(\mathrm{WF}(K_A)) \subset \big\{(x,x)\big\}$ so
$$\mathrm{WF}(K_A) \subset \big\{(x, x; \xi, \eta)\big\}.$$
To find the wave front set more precisely consider the kernel
$$K_A(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} b(x, \xi) d\xi$$
where we can assume $|x - y| < 1$ on $\mathrm{supp}(K_A)$. Thus is $\phi \in \mathcal{C}_c^\infty(X)$ then
$$g(x, y) = K_A(x, y) \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$$
and
$$\begin{aligned}\hat{g}(\zeta, \eta) &= (2\pi)^{-n} \int e^{-i\zeta x - i\eta y} e^{i(x-y)\cdot\zeta} (\phi b)(x, \xi) d\zeta dx dy \\ &= \int e^{-i(\zeta+\eta)\cdot x} (\phi b)(x, -\eta) dx \\ &= \widehat{\phi b}(\zeta + \eta, -\eta).\end{aligned}$$
The fact that ϕb is a symbol of compact support in x means that for every M
$$\left|\widehat{\phi b}(\zeta + \eta, -\eta)\right| \leq C_M \left(\langle \zeta + \eta \rangle\right)^{-M} \langle \eta \rangle^m.$$
This is rapidly decreasing if $\zeta \neq -\eta$, so
$$\mathrm{WF}(K_A) \subset \big\{(x, x, \eta, -\eta)\big\} \text{ as claimed.}$$
Moreover if $(\overline{x}, \overline{\eta}) \notin \mathrm{WF}'(A)$ then choosing ϕ to have small support near $\overline{x}$ makes $\widehat{\phi b}$ rapidly decreasing near $-\overline{\eta}$ for all ζ. This proves Proposition 4.8. □

4.13. Elementary calculus of wavefront sets

We want to achieve a reasonable understanding, in terms of wavefront sets, of three fundamental operations. These are

pb (4.95) Pull-back: $F^* u$

proof (4.96) Push-forward: $F_* u$ and

mult (4.97) Multiplication: $u_1 \cdot u_2$.

In order to begin to analyze these three operations we shall first introduce and discuss some other more “elementary” operations:

pair
$$\text{Pairing: } (u,v) \longrightarrow \langle u,v\rangle = \int u(x)\overline{v(x)}dx \tag{4.98}$$

proj
$$\text{Projection: } u(x,y) \longmapsto \int u(x,y)dy \tag{4.99}$$

rest
$$\text{Restriction: } u(x,y) \longmapsto u(x,0) \tag{4.100}$$

expr
$$\text{Exterior product: } (u,v) \longmapsto (u \boxtimes v)(x,y) = u(x)v(y) \tag{4.101}$$

inv
$$\text{Invariance: } F^*u, \text{ for } F \text{ a diffeomorphism.} \tag{4.102}$$

Here (4.100) and (4.102) are special cases of (4.95), (4.99) of (4.96) and (4.101) is a combination of (4.97) and (4.95). Conversely the three fundamental operations can be expressed in terms of these elementary ones. We can give direct definitions of the latter which we then use to analyze the former. We shall start with the pairing in (4.98).

4.14. Pairing

We know how to ‘pair’ a distribution and a $\mathcal{C}^\infty$ function. If both are $\mathcal{C}^\infty$ and have compact supports then

9.7
$$\langle u_1,u_2\rangle = \int u_1(x)\overline{u_2(x)}dx \tag{4.103}$$

and in general this pairing extends by continuity to either $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)\times\mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathcal{C}^\infty(\mathbb{R}^n)\times\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ Suppose *both* u_1 and u_2 are distributions, when can we pair them?

9.8 Proposition 4.9. *Suppose* $u_1,\ u_2 \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ *satisfy*

9.9
$$\mathrm{WF}(u_1)\cap\mathrm{WF}(u_2)=\emptyset \tag{4.104}$$

then if $A\in\Psi^0_\infty(\mathbb{R}^n)$ *has*

9.10
$$\mathrm{WF}(u_1)\cap\mathrm{WF}'(A)=\emptyset,\ \mathrm{WF}(u_2)\cap\mathrm{WF}'(\mathrm{Id}-A^*)=\emptyset \tag{4.105}$$

the bilinear form

9.11
$$\langle u_1,u_2\rangle = \langle Au_1,u_2\rangle + \langle u_1,(\mathrm{Id}-A^*)u_2\rangle \tag{4.106}$$

is independent of the choice of A.

Notice that A satisfying (4.105) does indeed exist, just choose $a\in S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ to be identically 1 on $\mathrm{WF}(u_2)$, but to have $\operatorname{cone\,supp}(a)\cap\mathrm{WF}(u_1)=\emptyset$, possible because of (4.104), and set $A=q_L(a)$.

Proof. Of course (4.106) makes sense because $Au_1,\ (\mathrm{Id}-A^*)u_2\in\mathcal{C}^\infty(\mathbb{R}^n)$. To prove that this definition is independent of the choice of A, suppose A' also satisfies (4.105). Set

$$\langle u_1,u_2\rangle' = \langle A'u_1,u_2\rangle + \langle u_1,(\mathrm{Id}-A')^*u_2\rangle. \tag{4.107}$$

Then

$$\mathrm{WF}'(A-A')\cap\mathrm{WF}(u_1)=\mathrm{WF}'((A-A')^*)\cap\mathrm{WF}(u_2)=\emptyset. \tag{4.108}$$

The difference can be written

9.12
$$\langle u_1,u_2\rangle' - \langle u_1,u_2\rangle = \langle (A-A')u_1,u_2\rangle - \langle u_1,(A-A')^*u_2\rangle. \tag{4.109}$$

Naturally we expect this to be zero, but this is not quite obvious since u_1 and u_2 are both distributions. We need an approximation argument to finish the proof.

Choose $B \in \Psi^0_\infty(\mathbb{R}^n)$ with

9.13 (4.110)
$$\begin{gathered}\mathrm{WF}'(B) \cap \mathrm{WF}(u_1) = \mathrm{WF}'(B) \cap \mathrm{WF}(u_2) = \emptyset \\ \mathrm{WF}'(\mathrm{Id} - B) \cap \mathrm{WF}(A - A') = \emptyset\end{gathered} \tag{4.110}$$

If $v_n \longrightarrow u_2$, in $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$, $v_n \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ then

$$w_n = \phi\big[(\mathrm{Id} - B)\, v_n + Bu_2\big] \longrightarrow u_2 \tag{4.111}$$

if $\phi \equiv 1$ in a neighbourhood of $\mathrm{supp}(u_2)$, $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. Here $Bu_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$, so

$$(A - A')w_n = (A - A')\phi(\mathrm{Id} - B) \cdot v_n + (A - A')\phi B u_2 \longrightarrow (A - A')u_2 \text{ in } \mathcal{C}^\infty(\mathbb{R}^n), \tag{4.112}$$

since $(A - A')\phi(\mathrm{Id} - B) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Thus

$$\begin{gathered}\langle (A - A')\, u_1, u_2\rangle \longrightarrow \langle (A - A')\, u_1, u_2\rangle \\ \langle u_1, (A - A')^*\, w_n\rangle \longrightarrow \langle u_1, (A - A')^*\, u_2\rangle,\end{gathered}$$

since $w_n \longrightarrow u_2$ in $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and $(A - A')^* w_n \longrightarrow (A - A')^*\, u_2$ in $\mathcal{C}^\infty(\mathbb{R}^n)$. Thus

$$\langle u_1, u_2\rangle' - \langle u_1, u_2\rangle = \lim_{n\to\infty}\big[\langle (A - A')\, u_1, w_n\rangle - \langle u_1, (A - A')^*\, w_n\big] = 0. \tag{4.113}$$

□

Here we are using the *complex* pairing. If we define the real pairing by

9.15 (4.114)
$$(u_1, u_2) = \langle u_1, \overline{u}_2\rangle \tag{4.114}$$

then we find

9.16 Proposition 4.10. *If u_1, $u_2 \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ satisfy*

9.17 (4.115)
$$(x, \xi) \in \mathrm{WF}(u_1) \Longrightarrow (x, -\xi) \notin \mathrm{WF}(u_2) \tag{4.115}$$

then the real pairing, defined by

9.18 (4.116)
$$(u_1, u_2) = (Au_1, u_2) + (u_1, A^t u_2), \tag{4.116}$$

where A satisfies (4.105)[9.10]*, is independent of A.*

Proof. Notice that

9.19 (4.117)
$$\mathrm{WF}(\overline{u}) = \big\{(x, -\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \smallsetminus 0); (x, \xi) \in \mathrm{WF}(u)\big\}. \tag{4.117}$$

We can write (4.115)[9.17], using (4.114)[9.15], as

$$(u_1, u_2) = \langle Au_1, \overline{u}_2\rangle + \langle u_1, \overline{A^t u_2}\rangle. \tag{4.118}$$

Since, by definition, $\overline{A^t u_2} = A^* \overline{u}_2$,

$$(u_1, u_2) = \langle Au_1, \overline{u}_2\rangle + \langle u_1, A^* \overline{u}_2\rangle = \langle u_1, \overline{u}_2\rangle \tag{4.119}$$

is defined by (4.106)[9.11], since (4.115)[9.17] translates to (4.104)[9.9]. □

4.15. Multiplication of distributions

The pairing result (4.116) can be used to define the *product* of two distributions under the same hypotheses, (4.115).

9.20 Proposition 4.11. *If $u_1,\ u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfy*

9.21 (4.120) $$(x,\xi) \in \operatorname{WF}(u_1) \Longrightarrow (x,-\xi) \notin \operatorname{WF}(u_2)$$

then the product of u_1 and $u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ is well-defined by

9.22 (4.121) $$u_1u_2(\phi) = (u_1, \phi u_2) = (\phi u_1, u_2)\ \forall\ \phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n)$$

using (4.116).

Proof. We only need to observe that if $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and $A \in \Psi_\infty^m(\mathbb{R}^n)$ has $\operatorname{WF}'(A) \cap \operatorname{WF}(u) = \emptyset$ then for any fixed $\psi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n)$

9.23 (4.122) $$\|\psi A \phi u\|_{C^k} \le C\|\phi\|_{C^p} \quad p = k + N$$

for some N, depending on m. This implies the continuity of $\phi \longmapsto u_1u_2(\phi)$ defined by (4.121). □

4.16. Projection

Here we write $\mathbb{R}^n_z = \mathbb{R}^p_x \times \mathbb{R}^k_y$ and define a continuous linear map, which we write rather formally as an integral

10.1 (4.123) $$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \ni u \longmapsto \int u(x,y)dy \in \mathcal{C}_c^{-\infty}(\mathbb{R}^p)$$

by pairing. If $\phi \in \mathcal{C}^\infty(\mathbb{R}^p)$ then

10.2 (4.124) $$\pi_1^*\phi \in \mathcal{C}^\infty(\mathbb{R}^n), \quad \pi_1 :\ \mathbb{R}^n \ni (x,y) \longmapsto x \in \mathbb{R}^p$$

and for $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ we define the formal 'integral' in (4.123) by

10.3 (4.125) $$(\int u(x,y)dy, \phi) = ((\pi_1)_* u, \phi) := u(\pi_1^*\phi).$$

In this sense we see that the projection is dual to pull-back (on functions) under π_1, so is "push-forward under π_1," a special case of (4.96). The support of the projection satisfies

10.4 (4.126) $$\operatorname{supp}((\pi_1)_* u) \subset \pi_1(\operatorname{supp}(u))\ \ \forall\ u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),$$

as follows by duality from

10.5 (4.127) $$\operatorname{supp}(\pi_1^*\phi) \subset \pi_1^{-1}(\operatorname{supp}\phi)\,.$$

10.6 Proposition 4.12. *Let $\pi_1 : \mathbb{R}^{p+k} \longrightarrow \mathbb{R}^p$ be projection, then for every $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{p+k})$*

10.7 (4.128) $$\begin{aligned}\operatorname{WF}((\pi_1)_* u) \subset \big\{&(x,\xi) \in \mathbb{R}^p \times (\mathbb{R}^p\backslash 0)\,;\\ &\exists\ y \in \mathbb{R}^k \text{ with } (x,y,\xi,0) \in \operatorname{WF}(u)\big\}.\end{aligned}$$

Proof. First notice that

10.8 (4.129) $$(\pi_1)_* :\ \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}^p).$$

Combining this with (4.126) we see that

10.9 (4.130) $$\operatorname{sing\,supp}((\pi_1)_* u) \subset \pi_1(\operatorname{sing\,supp} u)$$

which is at least consistent with Proposition 4.12. To prove the proposition in full let me restate the local characterization of the wavefront set, in terms of the Fourier transform:

10.10 Lemma 4.7. *Suppose* $K \subset\subset \mathbb{R}^n$ *and* $\Gamma \subset \mathbb{R}^n \setminus 0$ *is a* closed *cone, then*

10.11 (4.131)
$$u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),\ \mathrm{WF}(u) \cap (K \times \Gamma) = \emptyset,\ A \in \Psi_\infty^m(\mathbb{R}^n),\ \mathrm{WF}'(A) \subset K \times \Gamma \\ \Longrightarrow Au \in \mathcal{S}(\mathbb{R}^n).$$

In particular

10.12 (4.132)
$$u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),\ \mathrm{WF}(u) \cap (K \times \Gamma) = \emptyset,\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n), \mathrm{supp}(\phi) \subset K \\ \Longrightarrow \widehat{\phi u}(\xi) \text{ is rapidly decreasing in } \Gamma.$$

Conversely suppose $\Gamma \subset \mathbb{R}^n \setminus 0$ *is a closed cone and* $u \in \mathcal{S}'(\mathbb{R}^n)$ *is such that for some* $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

10.13 (4.133)
$$\widehat{\phi u}(\xi) \text{ is rapidly decreasing in } \Gamma$$

then

10.14 (4.134)
$$\mathrm{WF}(u) \cap \big\{x \in \mathbb{R}^n; \phi(x) \neq 0\big\} \times \mathrm{int}(\Gamma) = \emptyset.$$

With these local tools at our disposal, let us attack (4.128). We need to show that

10.15 (4.135)
$$(\overline{x}, \overline{\xi}) \in \mathbb{R}^p \times (\mathbb{R}^p \backslash 0) \text{ s.t. } (\overline{x}, y, \overline{\xi}, 0) \notin \mathrm{WF}(u)\ \forall\ y \in \mathbb{R}^n \\ \Longrightarrow (\overline{x}, \overline{\xi}) \notin \mathrm{WF}\left((\pi_1)_* u\right).$$

Notice that, $\mathrm{WF}(u)$ being conic and $\pi(\mathrm{WF}(u))$ being compact, $\mathrm{WF}(u) \cap (\mathbb{R}^n \times S^{n-1})$ is compact. The hypothesis (4.135) is the statement that

(4.136)
$$\{\overline{x}\} \times \mathbb{R}^k \times \mathbb{S}^{n-1} \times \{0\} \cap \mathrm{WF}(u) = \emptyset.$$

Thus $\overline{x}$ has an open neighbourhood, W, in $\mathbb{R}^p$, and $(\overline{\xi}, 0)$ a conic neighbourhood γ_1 in $(\mathbb{R}^n \backslash 0)$ such that

(4.137)
$$(W \times \mathbb{R}^k \times \gamma_1) \cap \mathrm{WF}(u) = \emptyset.$$

Now if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^p)$ is chosen to have support in W

10.16 (4.138)
$$\widehat{(\pi_1^*\phi)u}(\xi, \eta) \text{ is rapidly decreasing in } \gamma_1.$$

Set $v = \phi(\pi_1)_* u$. From the definition of projection and the identity

(4.139)
$$v = \phi(\pi_1)_* u = (\pi_1)_*[(\pi_1^*\phi)u],$$

we have

10.17 (4.140)
$$\widehat{v}(\xi) = v(e^{-ix\cdot\xi}) = (\widehat{(\pi_1^*\phi)u})(\xi, 0).$$

Now (4.138) shows that $\widehat{v}(\xi)$ is rapidly decreasing in $\gamma_1 \cap (\mathbb{R}^p \times \{0\})$, which is a cone around $\overline{\xi}$ in $\mathbb{R}^p$. Since $v = \phi(\pi_1)_* u$ this shows that $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}\left((\pi_1)_* u\right)$, as claimed. □

Before going on to talk about the other operations, let me note a corollary of this which is useful and, even more, helps to explain what is going on:

10.18 Corollary 4.1. *If* $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *and*

(4.141)
$$\mathrm{WF}(u) \cap \big\{(x, y, \xi, 0); x \in \mathbb{R}^p, y \in \mathbb{R}^k, \xi \in \mathbb{R}^p \backslash 0\big\} = \emptyset$$

then $(\pi_1)_*(u) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

PROOF. Indeed, (4.128) says $\mathrm{WF}\left((\pi_1)_* u\right) = \emptyset$. □

Here, the vectors $(x, y, \xi, 0)$ are the ones "normal" (as we shall see, really *conormal*) to the surfaces over which we are integrating. Thus Lemma 4.7 and Corollary 4.1 both state that the only singularities that survive integration are the ones which are conormal to the surface along which we integrating; the ones even partially in the direction of integration are wiped out. This in particular fits with the fact that if we integrate in *all variables* then there are no singularities left.

4.17. Restriction

Next we wish to consider the *restriction* of a distribution to a subspace

10.19 (4.142) $$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \ni u \longmapsto u \restriction \{y = 0\} \in \mathcal{C}_c^{-\infty}(\mathbb{R}^p).$$

This is *not* always defined, i.e. no reasonable map (4.142) exists for all distributions. However under an appropriate condition on the wavefront set we can interpret (4.142) in terms of pairing, using our definition of products. Thus let

(4.143) $$\iota : \mathbb{R}^p \ni x \longmapsto (x, 0) \in \mathbb{R}^n$$

be the *inclusion* map. We want to think of $u \restriction \{y = 0\}$ as $\iota^* u$. If $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then for any $\phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ the identity

10.35 (4.144) $$\iota^* u(\iota^* \phi') = u\left(\phi' \delta(y)\right)$$

holds.

The restriction map $\iota^* : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}^p)$ is surjective. If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfies the condition

10.21 (4.145) $$\mathrm{WF}(u) \cap \left\{(x, 0, 0, \eta); x \in \mathbb{R}^p, \eta \in \mathbb{R}^{n-p}\right\} = \emptyset$$

then we can interpret the pairing

10.20 (4.146) $$\begin{gathered} \iota^* u(\phi) = u\left(\phi' \delta(y)\right) \ \forall \ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^p) \\ \text{where } \phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ and } \iota^* \phi' = \phi \end{gathered}$$

to define $\iota^* u$. Indeed, the right side makes sense by Proposition 4.11.

Thus we have directly proved the first part of

10.22 PROPOSITION 4.13. *Set* $\mathcal{R} = \left\{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); (4.145) \textit{ holds}\right\}$ *then* (4.146) *defines a linear* restriction *map* $\iota^* : \mathcal{R} \longrightarrow \mathcal{C}_c^{-\infty}(\mathbb{R}^p)$ *and*

10.23 (4.147) $$\mathrm{WF}(\iota^* u) \subset \left\{(x, \xi) \in \mathbb{R}^p \times (\mathbb{R}^p \setminus 0);\ \exists\ \eta \in \mathbb{R}^n \textit{ with } (x, 0, \xi, \eta) \in \mathrm{WF}(u)\right\}.$$

PROOF. First note that (4.145) means *precisely* that

10.24 (4.148) $$\hat{u}(\xi, \eta) \text{ is rapidly decreasing in a cone around } \{0\} \times \mathbb{R}^k \setminus 0.$$

When $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ taking Fourier transforms in (4.144) gives

10.25 (4.149) $$\widehat{\iota^* u}(\xi) = \frac{1}{(2\pi)^k} \int \hat{u}(\xi, \eta) d\eta.$$

In general (4.148) ensures that the integral in (4.149) converges, it will then hold by continuity.

We actually apply (4.149) to a localized version of u; if $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^p)$ then

10.26 (4.150) $$\widehat{\psi \iota^*(u)}(\xi) = (2\pi)^{-k} \int \widehat{\psi}(\xi) \widehat{u}(\xi, \eta) d\eta.$$

Thus suppose $(\overline{x},\overline{\xi}) \in \mathbb{R}^p \times (\mathbb{R}^p \backslash 0)$ is such that $(\overline{x},0,\overline{\xi},\eta) \notin \mathrm{WF}(u)$ for *any* η. If ψ has support close to $\overline{x}$ and $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^{n-p})$ has support close to 0 this means

10.27 (4.151) $$\widehat{\psi\zeta u}(\xi,\eta) \text{ is rapidly decreasing in a cone around each } (\overline{\xi},\eta).$$

We also have rapid decrease around $(0,\eta)$ from ([10.24]4.148) (make sure you understand this point) as

(4.152) $$\widehat{\psi\zeta u}(\xi,\eta) \text{ is rapidly decreasing in } \gamma \times \mathbb{R}^p$$

for a cone, γ, around $\overline{\xi}$. From ([10.25]4.149)

(4.153) $$\widehat{\psi\iota^*(\zeta u)}(\xi) \text{ is rapidly decreasing in } \gamma.$$

Thus $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(\iota^*(\zeta u))$. Of course if we choose $\zeta(y) = 1$ near 0, $\iota^*(\zeta u) = \iota^*(u)$ so $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$, provided $(\overline{x},0,\overline{\xi},\eta) \notin \mathrm{WF}(u)$, for all η. This is what ([10.23]4.147) says. □

Try to picture what is going on here. We can restate the main conclusion of Proposition [10.22]4.13 as follows.

Take $\mathrm{WF}(u) \cap \left\{(x,0,\xi,\eta) \in \mathbb{R}^p \times \{0\} \times (\mathbb{R}^n \backslash 0)\right\}$ and let Z denote projection *off* the η variable:

10.28 (4.154) $$\mathbb{R}^p \times \{0\} \times \mathbb{R}^p \times \mathbb{R}^k \xrightarrow{Z} \mathbb{R}^p \times \mathbb{R}^p$$

then

(4.155) $$\mathrm{WF}(\iota^* u) \subset Z(\mathrm{WF}(u) \cap \{y=0\}).$$

We will want to think more about these operations later.

4.18. Exterior product

This is maybe the easiest of the elementary operators. It is always defined

10.29 (4.156) $$(u_1 \boxtimes u_2)(\phi) = u_1\left(u_2(\phi(x,\cdot)\right) = u_2(u_1(\phi(\cdot,y)).$$

Moreover we can easily compute the Fourier transform:

10.30 (4.157) $$\widehat{u_1 \boxtimes u_2}(\xi,\eta) = \hat{u}_1(\xi)\hat{u}_2(\eta).$$

10.31 PROPOSITION 4.14. *The (exterior) product*

10.32 (4.158) $$\mathcal{C}_c^{-\infty}(\mathbb{R}^p) \times \mathcal{C}_c^{-\infty}(\mathbb{R}^k) \longleftarrow \mathcal{C}_c^{-\infty}(\mathbb{R}^{p+k})$$

is a bilinear map such that

10.33 (4.159) $$\begin{aligned}\mathrm{WF}(u_1 \boxtimes u_2) \subset {} & [(\mathrm{supp}(u_1) \times \{0\}) \times \mathrm{WF}(u_2)] \\ & \cup [\mathrm{WF}(u_1) \times (\mathrm{supp}(u_2) \times \{0\})] \cup [\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)].\end{aligned}$$

PROOF. We can localize near any point $(\overline{x},\overline{y})$ with $\phi_1(x)\phi_2(y)$, where ϕ_1 is supported near $\overline{x}$ and ϕ_2 is supported near $\overline{y}$. Thus we only need examine the decay of

10.34 (4.160) $$\phi_1 \widehat{u_1 \boxtimes \phi_2} u_2 = \widehat{\phi_1 u_1}(\xi) \cdot \widehat{\phi_2 u_2}(\eta).$$

Notice that if $\widehat{\phi_1 u_1}(\xi)$ is rapidly decreasing around $\overline{\xi} \neq 0$ then the product is rapidly decreasing around *any* $(\overline{\xi},\eta)$. This gives ([10.33]4.159). □

4.19. Diffeomorphisms

We next turn to the question of the extension of F^*, where $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map, from $\mathcal{C}^\infty(\Omega_2)$ to some elements of $\mathcal{C}^{-\infty}(\Omega_2)$. The simplest example of pull-back is that of transformation by a diffeomorphism.

We have already noted how pseudodifferential operators behave under a diffeomorphism: $F : \ \Omega_1 \longrightarrow \Omega_2$ between open sets of $\mathbb{R}^n$. Suppose $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel of *compact* support in $\Omega_1 \times \Omega_1$ then we define

11.1 (4.161) $$A_F : \mathcal{C}^\infty_c(\Omega_2) \longrightarrow \mathcal{C}^\infty_c(\Omega_2)$$

by $A_F = G^* \cdot A \cdot F^*$, $G = F^{-1}$. In § Sect.CooInv 4.4 we showed that $A_F \in \Psi^m_\infty(\mathbb{R}^n)$. In fact we showed much more, namely we computed a (very complicated) formula for the full symbols. Recall the definition of the *cotangent bundle* of $\mathbb{R}^n$

11.2 (4.162) $$T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$$

identified as pairs of points $(\overline{x}, \overline{\xi})$, where $\overline{x} \in \mathbb{R}^n$ and

11.3 (4.163) $$\overline{\xi} = df(\overline{x}) \text{ for some } f \in \mathcal{C}^\infty(\mathbb{R}^n).$$

The *differential* $df(\overline{x})$ of f at $\overline{x} \in \mathbb{R}^n$ is just the equivalence class of $f(x) - f(\overline{x}) \in \mathcal{I}_{\overline{x}}$ modulo $\mathcal{I}^2_{\overline{x}}$. Here

11.4 (4.164) $$\begin{cases} \mathcal{I}_{\overline{x}} = \left\{ g \in \mathcal{C}^\infty(\mathbb{R}^n);\ g(\overline{x}) = 0 \right\} \\ \mathcal{I}^2_{\overline{x}} = \Big\{ \sum\limits_{\text{finite}} g_i h_i,\ g_i, h_i \in \mathcal{I}_{\overline{x}} \Big\}. \end{cases}$$

The identification of $\overline{\xi}$, given by (11.2 4.162) and (11.3 4.163), with a point in $\mathbb{R}^n$ is obtained using Taylor's formula. Thus if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$

11.5 (4.165) $$f(x) = f(\overline{x}) + \sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x})(x - \overline{x})_j + \sum_{i,j=1} g_{ij}(x) x_i x_j.$$

The double sum here is in $\mathcal{I}^2_{\overline{x}}$, so the residue class of $f(x) - f(\overline{x})$ in $\mathcal{I}_{\overline{x}} / \mathcal{I}^2_{\overline{x}}$ is the same as that of

(4.166) $$\sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x})(x - \overline{x})_j.$$

That is, $d(x - \overline{x})_j = dx_j$, $j = 1, \dots, n$ form a basis for $T^*_{\overline{x}}\mathbb{R}^n$ and in terms of this basis

(4.167) $$df(\overline{x}) = \sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x}) dx_j.$$

Thus the entries of $\overline{\xi}$ are just $\left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n} \right)$ for some f. Another way of saying this is that the *linear functions* $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2 \cdots \xi_n x_n$ have differentials spanning $T^*_x\mathbb{R}^n$.

So suppose $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map. Then

(4.168) $$F^* : T^*_{\overline{y}}\Omega_2 \longrightarrow T^*_{\overline{x}}\Omega_1,\ \overline{y} = F(\overline{x})$$

is defined by $F^* df(\overline{y}) = d(F^* f)(\overline{x})$ since $F^* : \mathcal{I}_{\overline{y}} \longrightarrow \mathcal{I}_{\overline{x}}$, $F^* : \mathcal{I}^2_{\overline{y}} \longrightarrow \mathcal{I}^2_{\overline{x}}$. In coordinates $F(x) = y \Longrightarrow$

$$\frac{\partial}{\partial x_j}(F^* f(x)) = \frac{\partial}{\partial y} f(F(x)) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(y) \frac{\partial F_k}{\partial x_j} \tag{4.169}$$

i.e. $F^*(\eta \cdot dy) = \xi \cdot dx$ if

11.6
$$\xi_j = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_j}(x) \cdot \eta_k. \tag{4.170}$$

Of course if F is a diffeomorphism then the Jacobian matrix $\frac{\partial F}{\partial x}$ is invertible and (4.170) is a linear isomorphism. In this case

11.7
$$\begin{gathered} F^* : T^*_{\Omega_2}\mathbb{R}^n \longleftrightarrow T^*_{\Omega_1}\mathbb{R}^n \\ (x, \xi) \longleftrightarrow (F(x), \eta) \end{gathered} \tag{4.171}$$

with ξ and η connected by (4.170). Thus $(F^*)^* : \mathcal{C}^\infty(T^*\Omega_1) \longrightarrow \mathcal{C}^\infty(T^*\Omega_2)$.

11.8 Proposition 4.15. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism of open sets of $\mathbb{R}^n$ and $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel with compact support in $\Omega_1 \times \Omega_2$ then*

11.9
$$\sigma_m(A_F) = (F^*)^* \sigma_m(A) \tag{4.172}$$

and

11.10
$$F^*\big(\mathrm{WF}'(A_F)\big) = \mathrm{WF}'(A). \tag{4.173}$$

It follows that symbol $\sigma_m(A)$ of A is well-defined as an element of $S^{m-[1]}_\infty(T^*\mathbb{R}^n)$ independent of coordinates and $\mathrm{WF}'(A) \subset T^*\mathbb{R}^n \backslash 0$ is a well-defined closed conic set, independent of coordinates. The elliptic set and the characteristic set Σ_m are therefore also well-defined complementary conic subsets of $T^*\Omega\backslash 0$.

Proof. Look at the formulae. □

The main use we make of this invariance result is the freedom it gives us to *choose* local coordinates adapted to a particular problem. It also suggests that there should be neater ways to write various formulae, e.g. the wavefront sets of push-forward and pull-backs.

11.12 Proposition 4.16. *If $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ has $\mathrm{supp}(u) \subset \Omega_2$ and $F : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism then*

11.13 (4.174)

$$\mathrm{WF}(F^* u) \subset \big\{(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash 0); (F(x), \eta) \in \mathrm{WF}(u), \eta_j = \sum_i \frac{\partial F_i}{\partial x_j}(x) \xi_i\big\}.$$

Proof. Just use the standard definition

$$\mathrm{WF}(F^* u) = \bigcap \big\{\Sigma(A);\ A(F^* u) \in \mathcal{C}^\infty\big\}. \tag{4.175}$$

To test the wavefront set of $F^* u$ it suffices to consider A's with kernels supported in $\Omega_1 \times \Omega_1$ since $\mathrm{supp}(F^* u \Subset \Omega_1$ and for a general pseudodifferential operator A'

there exists A with kernel supported in Ω_1 such that $A'u - Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then $AF^*u \in \mathcal{C}_c^\infty(\Omega_1) \Longleftrightarrow A_F u \in \mathcal{C}_c^\infty(\Omega_2)$. Thus

$$\mathrm{WF}(F^*u) = \bigcap \big\{\Sigma(A);\ A_F u \in \mathcal{C}^\infty\big\} \tag{4.176}$$

$$= \bigcap \big\{F^*(\Sigma(A_F)); A_F u \in \mathcal{C}^\infty\big\} \tag{4.177}$$

$$= F^* \,\mathrm{WF}(u) \tag{4.178}$$

since, for u, it is enough to consider operators with kernels supported in $\Omega_2 \times \Omega_2$. □

4.20. Products

Although we have discussed the definition of the product of two distributions we have not yet analyzed the wavefront set of the result.

11.14 Proposition 4.17. *If $u_1, u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ are such that*

$$(x, \xi) \in \mathrm{WF}(u_1) \Longrightarrow (x, -\xi) \notin \mathrm{WF}(u_2) \tag{4.179}$$

then the product $u_1 u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$, defined by Proposition 9.20 *4.11 satisfies*

11.35
$$\begin{aligned}\mathrm{WF}(u_1u_2) \subset \big\{(x,\xi); x \in \mathrm{supp}(u_1) \text{ and } (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ \cup \big\{(x,\xi); x \in \mathrm{supp}(u_2) \text{ and } (x,\xi) \in \mathrm{WF}(u_1)\big\} \\ \cup \big\{(x,\xi); \xi = \eta_1 + \eta_2, (x,\eta_i) \in \mathrm{WF}(u_i), i = 1, 2\big\}.\end{aligned} \tag{4.180}$$

Proof. We can represent the product in terms of three 'elementary' operations.

11.15
$$u_1u_2(x) = \iota^*\big[F^*(u_1 \boxtimes u_2)\big] \tag{4.181}$$

where $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is the linear transformation

$$F(x, y) = (x + y, x - y) \tag{4.182}$$

and $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$ is inclusion as the first factor. Thus (11.15 4.181) corresponds to the 'informal' notation

11.16
$$u_1u_2(x) = u_1(x + y)u_2(x - y) \restriction \{y = 0\} \tag{4.183}$$

and will follow by continuity once we analyse the wavefront set properties.

We know from Proposition 10.31 4.14 that

$$\begin{aligned}\mathrm{WF}\,(u_1 \boxtimes u_2) \subset \big\{(X, Y, \Xi, H)\,; X \in \mathrm{supp}(u_1), \Xi = 0, (Y, H) \in \mathrm{WF}(u_2)\big\} \\ \cup \big\{(X, Y, \Xi, H)\,;\ (X, \Xi) \in \mathrm{WF}(u_1),\ Y \in \mathrm{supp}(u_2), H = 0\big\} \\ \cup \big\{(X, Y, \Xi, H)\,;\ (X, \Xi) \in \mathrm{WF}(u_1),\ (Y, H) \in \mathrm{WF}(u_2)\big\}.\end{aligned} \tag{4.184}$$

Since F is a diffeomorphism, by Proposition 11.12 4.16,

$$\begin{aligned}\mathrm{WF}(F^*(u_1 \boxtimes u_2)) = \big\{(x, y, \xi, \eta); (F^t(x, y), \Xi, H) \in \mathrm{WF}(u_1 \boxtimes u_2), \\ (\xi, \eta) = A^t(\Xi, H)\big\}.\end{aligned}$$

where F^t is the transpose of F as a linear map. In fact $F^t = F$, so

$$\begin{aligned}&\mathrm{WF}(F^*(u_1 \boxtimes u_2)) \subset \\ &\quad \big\{(x, y, \xi, \eta); x + y \in \mathrm{supp}(u_1),\ \xi + \eta = 0, (x - y, \tfrac{1}{2}(\xi - \eta)) \in \mathrm{WF}(u_2)\big\} \\ &\quad \cup \big\{(x, y, \xi, \eta); (x + y, \tfrac{1}{2}(\xi + \eta)) \in \mathrm{WF}(u_1),\ (x - y, \tfrac{1}{2}(\xi - \eta)) \in \mathrm{WF}(u_2)\big\}\end{aligned}$$

and so using Proposition 4.13

$$\begin{aligned}\mathrm{WF}(F^*(u_1 \boxtimes u_2)) \restriction \{y=0\} \\ \subset \big\{(x,0,\xi,-\xi); x \in \mathrm{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ \cup \big\{(x,0,\xi,\eta); (x \in \mathrm{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ \cup \big\{(x,0,\xi,\eta); (x, \frac{1}{2}(\xi+\eta)) \in \mathrm{WF}(u_2), (x, \frac{1}{2}(\xi-\eta)) \in \mathrm{WF}(u_1)\big\}\end{aligned}$$

Notice that
(4.185)

$$(x,0,0,\eta) \in \mathrm{WF}\left(F^*(u_1 \boxtimes u_2)\right) \Longrightarrow (x, \frac{1}{2}\eta) \in \mathrm{WF}(u_1) \text{ and } (x, \frac{1}{2}\eta)\,\mathrm{WF}(u_2)$$

which introduces the assumption under which $u_1 u_2$ is defined. Finally then we see that

11.17 (4.186)

$$\begin{aligned}\mathrm{WF}(u_1 u_2) \subset \big\{(x,\xi); x \in \mathrm{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ \cup \big\{(x,\xi); x \in \mathrm{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_1)\big\} \\ \cup \big\{(x,\xi); (x,\eta_1) \in \mathrm{WF}(u_1), (x,\eta_2) \in \mathrm{WF}(u_2) \text{ and } \xi = \eta_1 + \eta_2\big\}.\end{aligned}$$

which is another way of writing the conclusion of Proposition 4.17. □

4.21. Pull-back

Now let us consider a general $\mathcal{C}^\infty$ map

11.18 (4.187)
$$F : \Omega_1 \longrightarrow \Omega_2,\ \Omega_1 \subset \mathbb{R}^n, \Omega_2 \subset \mathbb{R}^m.$$

Thus even the dimension of domain and range spaces can be different. When can we *define* F^*u, for $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$ and what can we say about $\mathrm{WF}(F^*u)$? For a general map F it is not possible to give a sensible, i.e. consistent, definition of F^*u for all distributions $u \in \mathcal{C}^{-\infty}(\Omega_2)$.

For smooth functions we have defined

(4.188)
$$F^* : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_1)$$

but in general $F^*\phi$ does not have compact support, even if ϕ does. We therefore impose the condition that F be *proper*

11.19 (4.189)
$$F^{-1}(K) \Subset \Omega_2 \ \forall\ K \Subset \Omega_2,$$

(mostly just for convenience). In fact if we want to understand F^*u near $\overline{x}_1 \in \Omega_1$ we only need to consider u near $F(\overline{x}_1) \in \Omega_2$.

The problem is that the map (4.187) may be rather complicated. However *any* smooth map can be decomposed into a product of simpler maps, which we can analyze locally. Set

(4.190)
$$\mathrm{graph}(F) = \big\{(x,y) \in \Omega_1 \times \Omega_2;\ y = F(x)\big\} \xrightarrow{\iota_F} \Omega_1 \times \Omega_2.$$

This is *always* an embedded submanifold of $\Omega_1 \times \Omega_2$ the functions $y_i - F_i(x)$, $i = 1, \dots, N$ are independent defining functions for $\mathrm{graph}(F)$ and $x_1, \dots, x_n$ are coordinates on it. Now we can write

11.20 (4.191)
$$F = \pi_2 \circ \iota_F \circ g$$

where $g : \Omega_1 \longleftrightarrow \operatorname{graph}(F)$ is the diffeomorphism onto its range $x \longmapsto (x, F(x))$. This decomposes F as a projection, an inclusion and a diffeomorphism. Now consider

11.21 (4.192) $$F^*\phi = g^* \cdot \iota_F^* \cdot \pi_2^*\phi$$

i.e. $F^*\phi$ is obtained by pulling ϕ back from Ω_2 to $\Omega_1 \times \Omega_2$, restricting to graph(F) and then introducing the x_i as coordinates. We have directly discussed $(\pi_2^*\phi)$ but we can actually write it as

11.22 (4.193) $$\pi_2^*\phi = 1 \boxtimes \phi(y),$$

so the result we have proved can be applied to it. So let us see what writing (4.192) as

11.23 (4.194) $$F^*\phi = g^* \circ \iota_F^*(1 \boxtimes \phi)$$

tells us. If $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$ then

11.24 (4.195) $$\operatorname{WF}(1 \boxtimes u) \subset \big\{(x, y, 0, \eta); (y, \eta) \in \operatorname{WF}(u)\big\}$$

by Proposition 4.14. So we have to discuss $\iota_F^*(1 \boxtimes u)$, i.e. restriction to $y = F(x)$. We can do this by making a diffeomorphism:

11.25 (4.196) $$T_F(x, y) = (x, y + F(x))$$

so that $T_F^{-1}(\operatorname{graph}(F)) = \{(x, 0)\}$. Notice that $g \circ T_F = \pi_1$, so

11.26 (4.197) $$F^*\phi = \iota_{\{y=0\}}^* \left(T_F^*(1 \boxtimes u)\right).$$

Now from Proposition 4.16 we know that

11.27 (4.198) $$\begin{aligned}\operatorname{WF}(T_F^*(1 \boxtimes u)) &= T_F^*(\operatorname{WF}(1 \boxtimes u)) \\ &= \big\{(X, Y, \Xi, H); (X, Y + F(X), \xi, \eta) \in \operatorname{WF}(1 \boxtimes u), \\ &\qquad \eta = H, \xi_i = \Xi_i + \Sigma \frac{\partial F_j}{\partial x_i} H_j\big\}\end{aligned}$$

i.e.

11.28 (4.199) $$\operatorname{WF}(T_F^*(1 \boxtimes u)) = \big\{(x, y, \xi, \eta); \xi_i = \sum_j \frac{\partial F_j}{\partial x_j}(x)\eta_j, (F(x), \eta) \in \operatorname{WF}(u)\big\}.$$

So consider our existence condition for restriction to $y = 0$, that $\xi \neq 0$ on $\operatorname{WF}(T_F^*(1 \boxtimes u))$ i.e.

11.29 (4.200) $$(F(x), \eta) \in \operatorname{WF}(u) \Longrightarrow \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j \neq 0.$$

If (4.200) holds then, from (4.198) and Proposition 4.13

11.30 (4.201) $$\operatorname{WF}(F^*u) \subset \big\{(x, \xi);\ \exists\ (F(x), \eta) \in \operatorname{WF}(u) \text{ and } \xi_j = \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j\big\}.$$

We can reinterpret (4.200) and (4.201) more geometrically. The differential of F gives a map

(4.202) $$\begin{gathered}F^* : T_{F(x)}^*\Omega_2 \longrightarrow T_x^*\Omega_1\ \forall\ x \in \Omega_1 \\ (F(x), \eta) \longmapsto (x, \xi) \text{ where } \xi_i = \Sigma \frac{\partial F_j}{\partial x_i}\eta_j.\end{gathered}$$

Thus (4.200) can be restated as:

11.31 (4.203)
$$\forall\ x \in \Omega_1,\ \text{the null space of } F_x^* : T^*_{F(x)}\Omega_2 \longrightarrow T^*_x\Omega_1 \text{ does not meet } \mathrm{WF}(u)$$

and then (4.201) becomes

11.32 (4.204)
$$\mathrm{WF}(F^*u) \subset \bigcup_{x\in\Omega_1} F_x^*[\mathrm{WF}(u) \cap T^*_{F(x)}\Omega_2] = F^*(\mathrm{WF}(u))$$

(proved we are a little careful in that F^* is *not* a map; it is a *relation* between $T^*\Omega_2$ and $T^*\Omega_1$) and in this sense (4.203) holds. Notice that (4.201) is a sensible "consequence" of (4.203), since otherwise $\mathrm{WF}(F^*u)$ would contain some zero directions.

11.33 Proposition 4.18. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a proper $\mathcal{C}^\infty$ map then F^* extends (by continuity) from $\mathcal{C}^\infty_c(\Omega_2)$ to*

11.34 (4.205)
$$\big\{u \in \mathcal{C}^{-\infty}_c(\Omega_2); F^*(\mathrm{WF}(u)) \cap (\Omega_1 \times 0) = \emptyset \text{ in } T^*\Omega_1\big\}$$

and (4.204) holds.

4.22. The operation F_*

Next we will look at the dual operation, that of push-forward. Notice the basic properties of pull-back:

12.1 (4.206)
$$\text{Maps } \mathcal{C}^\infty_c \text{ to } \mathcal{C}^\infty_c (\text{if } F \text{ is proper})$$

12.2 (4.207)
$$\textit{Not} \text{ always defined on distributions.}$$

Dually we find

12.3 Proposition 4.19. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map of an open subset of $\mathbb{R}^n$ into an open subset of $\mathbb{R}^n$ then for any $u \in \mathcal{C}^{-\infty}_c(\Omega_1)$*

12.4 (4.208)
$$F_*(u)(\phi) = u(F^*\phi)$$

is a distribution of compact support and

12.5 (4.209)
$$F_* :\ \mathcal{C}^{-\infty}_c(\Omega_1) \longrightarrow \mathcal{C}^{-\infty}_c(\Omega_2)$$

has the property:

12.6 (4.210)
$$\mathrm{WF}(F_*u) \subset \big\{(y,\eta); y \in F(\mathrm{supp}(u)), y = F(x), F_x^*\eta = 0\big\} \cup \big\{(y,\eta); y = F(x), (x, F_x^*\eta) \in \mathrm{WF}(u)\big\}.$$

Proof. Notice that the 'opposite ' of (4.206) and (4.207) hold, i.e. F_* is *always* defined but even if $u \in \mathcal{C}^\infty_c(\Omega_1)$ in general $F_*u \notin \mathcal{C}^\infty_c(\Omega_2)$. All we really have to prove is (4.210). As usual we look for a formula in terms of elementary operations. So suppose $u \in \mathcal{C}^\infty_c(\Omega_1)$

12.7 (4.211)
$$\begin{aligned} F_*u(\phi) &= u(F^*\phi) \quad \phi \in \mathcal{C}^\infty_c(\Omega_2) \\ &= \int u(x)\ \phi(F(x))\ dx \\ &= \int u(x)\delta(y - F(x))\ \phi(y)\ dydx. \end{aligned}$$

Thus, we see that

12.8 (4.212)
$$F_*u = \pi_* H^*(u \boxtimes \delta)$$

where $\delta = \delta(y) \in \mathcal{C}_c^{-\infty}(\mathbb{R}^m)$, H is the diffeomorphism

$$H(x,y) = (x, y - F(x)) \tag{4.213}$$

and $\pi : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ is projection *off* the first factor.

Thus (4.212) is the desired decomposition into elementary operations, since $u \boxtimes \delta \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{n+m})$, $\pi_* H^*(u \boxtimes \delta)$ is always defined and indeed the map is continuous, which actually proves (4.212).

So all we need to do is estimate the wavefront set using our earlier results. From Proposition 4.14 it follows that

$$\begin{aligned} \mathrm{WF}(u \boxtimes \delta) \subset \big\{(x,0,\xi,\eta); x \in \mathrm{supp}(u), \xi = 0\big\} &\cup \big\{(x,0,\xi,0); (x,\xi) \in \mathrm{WF}(u)\big\} \\ \cup \big\{(x,0,\xi,\eta); (x,\xi) \in \mathrm{WF}(u)\big\} \\ = \big\{(x,0,\xi,\eta); x \in \mathrm{supp}(u), \xi = 0\big\} &\cup \big\{(x,0,\xi,\eta); (x,\xi) \in \mathrm{WF}(u)\big\}. \end{aligned} \tag{4.214}$$

Then consider what happens under H^*. This is a diffeomorphism so the wavefront set transforms under the pull-back:

$$\begin{aligned} \mathrm{WF}(H^*(u \boxtimes \delta)) &= \mathrm{WF}(u(x)\delta(y - F(x)) \\ &= \big\{(x, F(x), \Xi, \eta); \Xi_i = \xi_i - \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j, (x,0,\xi,\eta) \in \mathrm{WF}(u \boxtimes \delta)\big\} \\ &= \big\{(x, F(x), \Xi, \eta); x \in \mathrm{supp}(u), \Xi_i = -\sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j)\big\} \\ &\cup \big\{(x, F(x), \Xi, \eta); \eta \in \mathbb{R}^m, (x,\xi) \in \mathrm{WF}(u)), \Xi_i = \xi_i - \sum_j \frac{\partial F_i}{\partial x_j}\eta_j\big\}. \end{aligned} \tag{4.215}$$

Finally recall the behaviour of wavefront sets under projection, to see that

$$\begin{aligned} \mathrm{WF}(F_* u) &\subset \big\{(y,\eta);\ \exists\ (x,y,0,\eta) \in \mathrm{WF}(H^*(u \boxtimes \delta))\big\} \\ = \big\{(y,\eta); y = F(x) &\text{ for some } x \in \mathrm{supp}(u) \text{ and} \\ &\sum_j \frac{\partial F_j}{\partial x_i}\eta_j = 0,\ i = 1, \dots, n\big\} \\ \cup \big\{(y,\eta); y = F(x) &\text{ for some } (x,\xi) \in \mathrm{WF}(u) \text{ and} \\ &\xi_i = \sum_j \frac{\partial F_i}{\partial x_i}\eta_j, i = 1, \dots, n)\big\}. \end{aligned}$$

This says

$$\mathrm{WF}(F_* u) \subset \big\{(y,\eta); y \in F(\mathrm{supp}(u)) \text{ and } F_x^*(\eta) = 0\big\} \tag{4.216}$$

$$\cup \big\{(y,\eta); y = F(x) \text{ with } (x, F_x^*\eta) \in \mathrm{WF}(u)\big\} \tag{4.217}$$

which is just (4.210). □

As usual one should note that the two terms here are "really the same".

Now let us look at F_* as a linear map,

12.9
$$F_* :\ \mathcal{C}_c^{\infty}(\Omega_1) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_2). \tag{4.218}$$

As such it has a Schwartz kernel, indeed (4.212) is just the usual formula for an operator in terms of its kernel:

12.10 (4.219) $$F_*u(y) = \int K(y,x)u(x)dx,\ K(y,x) = \delta(y - F(x)).$$

So consider the wavefront set of the kernel:

12.11 (4.220) $$\mathrm{WF}(\delta(y - F(x)) = \mathrm{WF}(H^*\delta(y)) = \big\{(y,x;\eta,\xi); y = F(x), \xi = F_x^*\eta\big\}.$$

Now changing the order of the factors we can regard this as a subset

12.12 (4.221) $$\mathrm{WF}'(K) = \big\{((y,\eta),(x,\xi)); y = F(x), \xi = F^*\eta\big\} \subset (\Omega_2 \times \mathbb{R}^m) \times (\Omega_1 \times \mathbb{R}^n).$$

As a subset of the product we can regard $\mathrm{WF}'(K)$ as a *relation*: if $\Gamma \subset \Omega_2 \times (\mathbb{R}^n\backslash 0)$ set

$$\mathrm{WF}'(K) \circ \Gamma = \big\{(y,\eta) \in \Omega_2 \times (\mathbb{R}^m\backslash 0); \exists\ ((y,\eta)),(x,\xi)) \in \mathrm{WF}'(K) \text{ and } (x,\xi) \in \Gamma\big\}$$

Indeed with this definition

12.14 (4.222) $$\mathrm{WF}(F_*u) \subset \mathrm{WF}'(K) \circ \mathrm{WF}(u), \quad K = \text{ kernel of } F_*.$$

4.23. Wavefront relation

One serious application of our results to date is:

12.15 Theorem 4.1. *Suppose $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ are open and $A \in \mathcal{C}^{-\infty}(\Omega_1 \times \Omega_2)$ has proper support, in the sense that the two projections*

12.16 (4.223) $$\begin{array}{ccc} & \mathrm{supp}(A) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \Omega_1 & & \Omega_2 \end{array}$$

are proper, then A defines a linear map

12.17 (4.224) $$A : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1)$$

and extends by continuity to a linear map

(4.225) $$A : \big\{u \in \mathcal{C}_c^{-\infty}(X); \mathrm{WF}(u) \cap \big\{(y,\eta) \in \Omega_2 \times (\mathbb{R}^n\backslash 0);$$

12.18 (4.226) $$\exists\ (x, 0, y, -\eta) \in \mathrm{WF}(K)\big\} = \emptyset\big\} \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1)$$

for which

12.19 (4.227) $$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(u),$$

where

12.20 (4.228) $$\mathrm{WF}'(A) = \big\{((x,\xi),(y,\eta)) \in (\Omega_1 \times \mathbb{R}^n) \times (\Omega_2 \times \mathbb{R}^m); (\xi,\eta) \neq 0 \\ \text{and } (x,y,\xi,-\eta) \in \mathrm{WF}(K)\big\}.$$

Proof. The action of the map A can be written in terms of its Schwartz kernel as

12.21 (4.229) $$Au(x) = \int K(x,y)u(y)dy = (\pi_1)_*(K \cdot (1 \boxtimes u)).$$

Here $1 \boxtimes u$ is always defined and

(4.230) $$\mathrm{WF}(1 \boxtimes u) \subset \big\{(x,y,0,\eta); (y,\eta) \in \mathrm{WF}(u)\big\}.$$

So the main question is, when is the product defined? Our sufficient condition for this is:

$$(x,y,\xi,\eta)\in \mathrm{WF}(K)\Longrightarrow (x,y,-\xi,-\eta)\notin \mathrm{WF}(1\boxtimes u) \tag{4.231}$$

which is

$$(x,y,0,\eta)\in \mathrm{WF}(K)\Longrightarrow (x,y,0,-\eta)\notin \mathrm{WF}(1\boxtimes u) \tag{4.232}$$

$$\text{i.e. } (y,-\eta)\notin \mathrm{WF}(u) \tag{4.233}$$

This of course is ([12.18] 4.226):

$$Au \text{ is defined (by continuity) if} \tag{4.234}$$

$$\big\{(y,\eta)\in \mathrm{WF}(u);\ \exists\ (x,0,y,-\eta)\in \mathrm{WF}(A)\big\}=\emptyset. \tag{4.235}$$

Then from our bound on the wavefront set of a product

$$\begin{aligned}\mathrm{WF}\,(K\cdot(1\boxtimes u))\subset&\\ \big\{(x,y,\xi,\eta);(\xi,\eta)=&(\xi',\eta')+(0,\eta'')\text{ with}\\ &(x,y,\xi',\eta')\in\mathrm{WF}(K)\text{ and }(x,\eta'')\in\mathrm{WF}(u)\big\}\\ \cup\big\{(x,y,\xi,\eta);(x,&y,\xi,\eta)\in\mathrm{WF}(K),\ y\in\mathrm{supp}(u)\big\}\\ \cup\big\{(x,y,0,\eta);&(x,y)\in\mathrm{supp}(A)(y,\eta)\in\mathrm{WF}(u)\big\}.\end{aligned} \tag{4.236}$$

This gives the bound

$$\mathrm{WF}\,(\pi_*(K\cdot(1\boxtimes u)))\subset\big\{(x,\xi);(x,y,\xi,0)\in\mathrm{WF}(K\cdot(1\boxtimes u))\text{ for some }y\big\} \tag{4.237}$$

$$\subset \mathrm{WF}'(A)\circ\mathrm{WF}(u). \tag{4.238}$$

□

4.24. Applications

Having proved this rather general theorem, let us note some examples and applications.

First, for pseudodifferential operators we know that

$$\mathrm{WF}'(A)\subset\{(x,x,\xi,\xi)\} \tag{4.239}$$

i.e. corresponds to the identity relation (which is a map). Then ([12.19] 4.227) is the microlocality of pseudodifferential operators. The next result also applies to all pseudodifferential operators.

12.22 Corollary 4.2. *If $K\in\mathcal{C}^{-\infty}(\Omega_1\times\Omega_2)$ has proper support and*

12.23 $$\mathrm{WF}'(K)\cap\{(x,y,\xi,0)\}=\emptyset \tag{4.240}$$

then the operator with Scwartz kernel K defines a continuous linear map

12.24 $$A:\mathcal{C}^\infty_c(\Omega_2)\longrightarrow\mathcal{C}^\infty_c(\Omega_1). \tag{4.241}$$

If

12.25 $$\mathrm{WF}'(K)\cap\{(x,y,0,\eta)\}=\emptyset \tag{4.242}$$

then A extends by continuity to

12.26 $$A:\mathcal{C}^{-\infty}_c(\Omega_2)\longrightarrow\mathcal{C}^{-\infty}_c(\Omega_1). \tag{4.243}$$

Proof. Immediate from ([12.17] 4.224)-([12.26] 4.243). □

P.9.1 4.25. Problems

12.27 Problem 4.9. Show that the general definition (4.62) reduces to

$$\mathrm{WF}(u) = \bigcap \left\{ \Sigma_0(A);\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ and } Au \in \mathcal{C}^\infty(\mathbb{R}^n) \right\},\ u \in \mathcal{S}'(\mathbb{R}^n) \tag{4.244}$$

and prove the basic result of 'microlocal elliptic regularity:'

8.30
$$\begin{gathered} \text{If } u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } A \in \Psi^m_\infty(\mathbb{R}^n) \text{ then} \\ \mathrm{WF}(u) \subset \Sigma(A) \cup \mathrm{WF}(Au). \end{gathered} \tag{4.245}$$

12.28 Problem 4.10. Compute the wavefront set of the following distributions:

$$\begin{gathered} \delta(x) \in \mathcal{S}'(\mathbb{R}^n),\ |x| \in \mathcal{S}'(\mathbb{R}^n) \text{ and} \\ \chi_{\mathbb{B}^n}(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1. \end{cases} \end{gathered} \tag{4.246}$$

12.29 Problem 4.11. Let $\Gamma \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ be an open cone and define

$$\mathcal{C}^{-\infty}_{c,\Gamma}(\mathbb{R}^n) = \left\{ u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n); Au \in \mathcal{C}^\infty(\mathbb{R}^n) \right. \tag{4.247}$$

$$\left. \forall\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ with } \mathrm{WF}'(A) \cap \Gamma = \emptyset \right\}. \tag{4.248}$$

Describe a complete topology on this space with respect to which $\mathcal{C}^\infty_c(\mathbb{R}^n)$ is a dense subspace.

12.30 Problem 4.12. Show that, for any pseudodifferential operator $A \in \Psi^m_\infty(\mathbb{R}^n)$, $\mathrm{WF}'(A) = \mathrm{WF}'(A^*)$.

12.31 Problem 4.13. Give an alternative proof to Lemma 4.5 along the following lines (rather than using Lemma 2.75). If $\sigma_L(A)$ is the left reduced symbol then for $\epsilon > 0$ small enough

$$b_0 = \gamma_\epsilon / \sigma_L(A) \in S^{-m}_\infty(\mathbb{R}^n; \mathbb{R}^n). \tag{4.249}$$

If we choose $B_0 \in \Psi^{-m}_\infty(\mathbb{R}^n)$ with $\sigma_L(B_0) = b_0$ then

9.5
$$\mathrm{Id} - A \circ B_0 = G \in \Psi^0_\infty(\mathbb{R}^n) \tag{4.250}$$

has principal symbol

$$\sigma_0(G) = 1 - \sigma_L(A) \cdot b_0. \tag{4.251}$$

From (4.77)

$$\gamma_{\epsilon/4} \sigma_0(G) = \gamma_{\epsilon/4}. \tag{4.252}$$

Thus we conclude that if $\sigma_L(C) = \gamma_{\epsilon/4}$ then

$$G = (\mathrm{Id} - C)G + CG \text{ with } CG \in \Psi^{-1}_\infty(\mathbb{R}^n). \tag{4.253}$$

Thus (4.250) becomes

9.6
$$\mathrm{Id} - AB_0 = CG + R_1 \quad \mathrm{WF}'(R_1) \not\ni z. \tag{4.254}$$

Let $B_1 \sim \sum\limits_{j \ge 1} (CG)^j$, $B_1 \in \Psi^{-1}$ and set

$$B = B_0 (\mathrm{Id} + B_1) \in \Psi^{-m}_\infty(\mathbb{R}^n). \tag{4.255}$$

From (9.6 4.254)

$$AB = AB_0(I + B_1) \tag{4.256}$$

$$= (\mathrm{Id} - CG)\,(I + B_1) - R_1\,(\mathrm{Id} + B_1) \tag{4.257}$$

$$= \mathrm{Id} + R_2, \quad \mathrm{WF}'(R_2) \not\ni z. \tag{4.258}$$

Thus B is a right microlocal parametrix as desired. Write out the construction of a left parametrix using the same method, or by finding a right parametrix for the adjoint of A and then taking adjoints using Problem 12.30 4.12.

12.32 Problem 4.14. Essential uniqueness of left and right parametrices.

12.33 Problem 4.15. If $(\bar{x}, \bar{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is a given point, construct a distribution $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ which has

$$\mathrm{WF}(u) = \left\{(\bar{x}, t\bar{\xi}); t > 0\right\} \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0). \tag{4.259}$$

12.34 Problem 4.16. Suppose that $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel of compact support. If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ use the four 'elementary operations' (and an earlier result on the wavefront set of kernels) to investigate under what conditions

$$\kappa(x, y) = K_A(x, y)u(y) \text{ and then } \gamma(x) = (\pi_1)_*\kappa \tag{4.260}$$

make sense. What can you say about $\mathrm{WF}(\gamma)$?

12.35 Problem 4.17. Consider the projection operation under $\pi_1 : \mathbb{R}^p \times \mathbb{R}^k \longrightarrow \mathbb{R}^p$. Show that $(\pi_1)_*$ can be extended to some distributions which do not have compact support, for example

$$\left\{u \in \mathcal{S}'(\mathbb{R}^n); \mathrm{supp}(u) \cap K \times \mathbb{R}^k \text{ is compact for each } K \subset\subset \mathbb{R}^n\right\}. \tag{4.261}$$

CHAPTER 5

Pseudodifferential operators on manifolds

In this chapter the notion of a pseudodifferential on a manifold is discussed. Some preliminary material on manifolds is therefore necessary. However the discussion of the basic properties of differentiable manifolds is kept to a bare minimum. For a more leisurely treatement the reader might well consult XX or YY. Our main aims here are first, to be able to prove the Hodge theorem (given the deRham theorem). Then we describe some global object which are very useful in applications, namely a global quantization map, the structure of complex powers and the zeta function.

5.1. $\mathcal{C}^\infty$ structures

Let X be a paracompact Hausdorff topological space. A $\mathcal{C}^\infty$ structure on X is a subspace

13.1 (5.1) $$\mathcal{F} \subset C^0(X) = \{u : X \longrightarrow \mathbb{R} \ \text{ continuous } \}$$

with the following property:

For each $\overline{x} \in X$ there exists elements $f_1, \dots, f_n \in \mathcal{F}$ such that for some open neighbourhood $\Omega \ni \overline{x}$

13.2 (5.2) $$F : \Omega \ni x \longmapsto (f_1(x), \dots, f_n(x)) \in \mathbb{R}^n$$

is a homeomorphism onto an open subset of $\mathbb{R}^n$ and *every* $f \in \mathcal{F}$ satisfies

13.3 (5.3) $$f \restriction \Omega = g \circ F \quad \text{ for some } g \in \mathcal{C}^\infty(\mathbb{R}^n).$$

The map (5.2) is a coordinate system near $\overline{x}$. Two $\mathcal{C}^\infty$ structures $\mathcal{F}_1$ and $\mathcal{F}_2$ are 'compatible' if $\mathcal{F}_1 \cup \mathcal{F}_2$ is also a $\mathcal{C}^\infty$ structure. Compatibility in this sense is an equivalence relation on $\mathcal{C}^\infty$ structures. It therefore makes sense to say that:

13.4 Definition 5.1. *A $\mathcal{C}^\infty$ manifold is a (connected) paracompact Hausdorff topological space with a* maximal *$\mathcal{C}^\infty$ structure.*

The maximal $\mathcal{C}^\infty$ structure is conventionally denoted

(5.4) $$\mathcal{C}^\infty(X) \subset \mathcal{C}^0(X).$$

It is necessarily an algebra. If we let $\mathcal{C}^\infty_c(W) \subset \mathcal{C}^\infty(X)$ denote the subspace of functions which vanish outside a compact subset of W then any local coordinates (5.2) have the property

13.5 (5.5) $$F^* : \mathcal{C}^\infty_c(F(\Omega)) \longleftrightarrow \big\{u \in \mathcal{C}^\infty(X);\ u = 0 \ \text{ on } X \backslash K, K \subset\subset \Omega\big\}.$$

Futhermore $\mathcal{C}^\infty(X)$ is *local*:

13.6 (5.6) $$\begin{gathered} u : X \longrightarrow \mathbb{R} \text{ and } \ \forall\, \overline{x} \in X \ \exists\ \Omega_{\overline{x}} \text{ open, } \Omega_{\overline{x}} \ni \overline{x}, \\ \text{s.t. } u - f_{\overline{x}} = 0 \text{ on } \Omega_{\overline{x}} \text{ for some } f_{\overline{x}} \in \mathcal{C}^\infty(X) \Longrightarrow u \in \mathcal{C}^\infty(X). \end{gathered}$$

A map $G : X \longrightarrow Y$ between $\mathcal{C}^\infty$ manifolds X and Y is $\mathcal{C}^\infty$ if

$$G^* : \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(X) \tag{5.7}$$

i.e. $G \circ u \in \mathcal{C}^\infty(X)$ for all $u \in \mathcal{C}^\infty(Y)$.

5.2. Form bundles

A vector bundle is a triple $\pi : V \longrightarrow X$ consisting of two manifolds, X and V, and a surjective $\mathcal{C}^\infty$ map π with each

$$V_x = \pi^{-1}(x) \tag{5.8}$$

having a *linear structure* such that

$$\mathcal{F} = \left\{u : V \longrightarrow \mathbb{R}, u \text{ is linear on each } V_x\right\} \tag{5.9}$$

is a $\mathcal{C}^\infty$ structure on V compatible with $\mathcal{C}^\infty(V)$ (i.e. contained in it, since it is maximal).

The basic example is the *cotangent bundle* which we defined before for open sets in $\mathbb{R}^n$. The same definition works here. Namely for each $\overline{x} \in X$ set

$$\begin{gathered} \mathcal{I}_{\overline{x}} = \left\{u \in \mathcal{C}^\infty(X); u(\overline{x}) = 0\right\} \\ \mathcal{I}^2_{\overline{x}} = \left\{u = \sum_{\text{finite}} u_i u_i';\ u_i, u_i' \in \mathcal{I}_{\overline{x}}\right\} \\ T^*_{\overline{x}}X = \mathcal{I}_{\overline{x}}/\mathcal{I}^2_{\overline{x}},\ T^*X = \bigcup_{\overline{x}\in X} T^*_{\overline{x}}X. \end{gathered} \tag{5.10}$$

So $\pi : T^*X \longrightarrow X$ just maps each $T^*_{\overline{x}}X$ to $\overline{x}$. We need to give T^*X a $\mathcal{C}^\infty$ structure so that "it" (meaning $\pi : T^*X \longrightarrow X$) becomes a vector bundle. To do so note that the *differential* of any $f \in \mathcal{C}^\infty(X)$

$$df : X \longrightarrow T^*X \quad df(\overline{x}) = [f - f(\overline{x})] \in T^*_{\overline{x}}X \tag{5.11}$$

is a section ($\pi \circ df = \mathrm{Id}$). Put

$$\mathcal{F} = \left\{u : T^*X \longrightarrow \mathbb{R}; u \circ df : X \longrightarrow \mathbb{R} \text{ is } \mathcal{C}^\infty\ \forall\ f \in \mathcal{C}^\infty(X)\right\}. \tag{5.12}$$

Then $\mathcal{F} = \mathcal{C}^\infty(T^*X)$ is a maximal $\mathcal{C}^\infty$ structure on T^*X and

$$\mathcal{F}_{\text{lin}} = \left\{u : T^*X \longrightarrow \mathbb{R},\ \text{linear on each } T^*_{\overline{x}}X; u \in \mathcal{F}\right\}$$

is therefore compatible with it. Clearly df is $\mathcal{C}^\infty$.

Any (functorial) operation on finite dimensional vector spaces can be easily seen to generate new vectors bundles from old. Thus *duality*, *tensor product*, *exterior powers* all lead to new vector bundles:

$$T_xX = \left(T^*_xX\right)^*,\ TX = \bigcup_{x\in X} T_xX \tag{5.13}$$

is the tangent bundle

$$\Lambda^k_xX = \left\{u : \overbrace{T_xX \times \cdots \times T_xX}^{k \text{ factors}} \longrightarrow \mathbb{R}; u \text{ is multilinear and antisymmetric }\right\}$$

leads to the k-form bundle

$$\Lambda^kX = \bigcup_{x\in X} \Lambda^k_xX,\ \Lambda^1X \simeq T^*X$$

where equivalence means there exists (in this case a *natural*) $\mathcal{C}^\infty$ diffeomorphism mapping fibres to fibres linearly (and in this case projecting to the identity on X).

A similar construction leads to the *density bundles*

$$\Omega^\alpha_x X = \big\{u : \overset{n=\dim X \quad \text{factors}}{T_x X \wedge \cdots \wedge T_x X} \longrightarrow \mathbb{R};\ \text{absolutely homogeneous of degree } \alpha\big\}$$

that is

$$u(tv_1 \wedge \dots v_n) = |t|^\alpha u(v_1 \wedge \cdots \wedge v_n).$$

These are important because of integration. In general if $\pi : V \longrightarrow X$ is a vector bundle then

$$\mathcal{C}^\infty(X; V) = \big\{u :\ X \longrightarrow V;\ \pi \circ u = \mathrm{Id}\big\}$$

is the space of sections. It has a natural linear structure. Suppose $W \subset X$ is a coordinate neighbourhood and $u \in \mathcal{C}^\infty(X; \Omega)$, $\Omega = \Omega^1 X$, has compact support in W. Then the coordinate map gives an identification

$$\Omega^*_x X \longleftrightarrow \Omega^*_{F(x)}\mathbb{R}^n \quad \forall\ \alpha$$

and

$$\int u = \int_{\mathbb{R}^n} g_u(x), \quad u = g_u(x)|dx| \tag{5.14}$$

is defined independent of coordiantes. That is the integral

$$\int : \mathcal{C}^\infty_c(X; \Omega) \longrightarrow \mathbb{R} \tag{5.15}$$

is well-defined.

5.3. Pseudodifferential operators

Let X be a $\mathcal{C}^\infty$ manifold, and let $\mathcal{C}^\infty_c(X) \subset \mathcal{C}^\infty(X)$ be the space of $\mathcal{C}^\infty$ functions of compact support. Then, for any $m \in \mathbb{R}$, $\Psi^m(X)$ is the space of linear operators

13.7 $$A : \mathcal{C}^\infty_c(X) \longrightarrow \mathcal{C}^\infty(X) \tag{5.16}$$

with the following properties. First,

13.8 $$\begin{gathered}\text{if } \phi,\ \psi \in \mathcal{C}^\infty(X) \text{ have disjoint supports then } \exists\ K \in \mathcal{C}^\infty(X \times X; \Omega_R) \\ \text{such that } \forall\ u \in \mathcal{C}^\infty_c(X)\ \phi A \psi u = \int_X K(x, y) u(y),\end{gathered} \tag{5.17}$$

and secondly if $F : W \longrightarrow \mathbb{R}^n$ is a coordinate system in X and $\psi \in \mathcal{C}^\infty_c(X)$ has support in W then

$$\begin{gathered}\exists\ B \in \Psi^m_\infty(\mathbb{R}^n),\ \mathrm{supp}(B) \subset W \times W \text{ s.t.} \\ \psi A \psi u \restriction W = F^*(B((F^{-1})^*(\psi u)))\ \forall\ u \in \mathcal{C}^\infty_c(X).\end{gathered}$$

This is a pretty horrible definition, since it requires us to check *every* coordinate system, at least in principle. In practice the coordinate-invariance we proved earlier means that this is *not* necessary and also that there are plenty of examples!

Any open cover of a $\mathcal{C}^\infty$ manifold has a partition of unity subordinate to it, i.e. if $A_r \subset X$ are open sets for $r \in R$ and

$$X = \bigcup_{r \in R} A_r \tag{5.18}$$

there exists $\phi_i \in \mathcal{C}^\infty_c(X)$, all non-negative with locally finite support:

$$\forall\ i\ \ \mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j) \neq \emptyset \text{ for a finite set of indices } j, \tag{5.19}$$

where each $\operatorname{supp}(\phi_i) \subset A_r$ for some $r = r(i)$ and

$$\sum_i \phi_i(x) = 1 \quad \forall\ x. \tag{5.20}$$

In fact one can do slightly better than this, for a covering by coordinate neighbourhoods.

13.9 LEMMA 5.1. *There exists a partition of unity on X (a $\mathcal{C}^\infty$ manifold) ϕ_i s.t. for every i*

$$\bigcup_j \big\{\operatorname{supp}(\phi_j); \operatorname{supp}(\phi_j) \cap \operatorname{supp}(\phi_i) \neq 0\big\} \tag{5.21}$$

is contained in a coordinate neighbourhood!

Using such a partition of unity we see that every element of $\Psi^m(X)$ can be written in the form

$$A = \sum_i \sum_j \phi_j F^* K_A \phi_i$$

where the terms have smooth kernels if the supports of ϕ_i and ϕ_j do not meet, or else are pseudodifferential operators in any local coordinates in a patch containing both supports. Below we shall use this to prove:

13.10 THEOREM 5.1. *Let X be a compact $\mathcal{C}^\infty$ manifold then the pseudodifferential operators $\Psi^*(X)$ form a symbol-filtered ring.*

If X is a $\mathcal{C}^\infty$ manifold we have defined the space $\Psi^m(X)$ as consisting of those linear operators

14.1
$$A : \mathcal{C}^\infty_c(X) \longrightarrow \mathcal{C}^\infty(X) \tag{5.22}$$

which are given locally by pseudodifferential operators of order m on X, in a precise sense. Let us recast this definition in terms of the Schwartz kernel theorem. Over the product, X^2, consider the right density bundle, $\Omega_R = \pi_R^* \Omega$. Here we use the pull-back operation on vector bundles:

14.2 THEOREM 5.2. *If $W \longrightarrow Y$ is a $\mathcal{C}^\infty$ vector bundle and $F : X \longrightarrow Y$ is a $\mathcal{C}^\infty$ map then $F^*W \longrightarrow X$ is a well-defined $\mathcal{C}^\infty$ vector bundle over X with total space*

14.3
$$F^*W = \bigcup_{x \in X} W_{F(x)}; \tag{5.23}$$

if $\phi \in \mathcal{C}^\infty(Y; W)$ then $F^*\phi$ is a section of F^*W and $\mathcal{C}^\infty(X; F^*W)$ is spanned by $\mathcal{C}^\infty(X) \cdot F^*\mathcal{C}^\infty(Y; W)$.

Distributional sections of any $\mathcal{C}^\infty$ vector bundle can be defined in two equivalent ways:

14.4
$$\text{“Algebraically”}\ \mathcal{C}^{-\infty}(X; W) = \mathcal{C}^{-\infty}(X) \bigotimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X; W) \tag{5.24}$$

or

14.5
$$\text{“Analytically”}\ \mathcal{C}^{-\infty}(X; W) = \big[\mathcal{C}^\infty_c(X; \Omega \otimes W')\big] \tag{5.25}$$

where W' is the dual bundle and Ω the density bundle over X. In order to use (5.25) we need to define a topology on $\mathcal{C}^\infty_c(X; U)$ for any vector bundle U over X. One can do this by reference to local coordinates.

If $W \longrightarrow X$ is a vector bundle the spaces $S^m(W)$ of symbols on W is well-defined for each $m \in \mathbb{R}$.

14.6 Proposition 5.1. *A pseudodifferential operator $A \in \Psi^m(X)$ can be written in terms of its kernel*

$$K_A \in \mathcal{C}^\infty(X^2; \Omega_R) \tag{5.26}$$

where

14.7
$$K_A \text{ is } \mathcal{C}^\infty \text{ in } X^2 \backslash \Delta \tag{5.27}$$

and if $X_i \subset X$ is a coordinate partition, $\rho_i \in \mathcal{C}^\infty_c(X)$ has support in X_i then in terms of the same coordinates $x_i = F_i^*(x_i)$ and $y_i = F^*(x_i)$ in the two factors of X

14.8
$$\rho_i(x)\rho(y)K_A = K_i(x,y)|dy|,\ K_i \in \Psi^m(\mathbb{R}^n). \tag{5.28}$$

Suppose ρ_i^2 is a partition of unity of X, subordinate to a coordinate covering. For each i the symbol of K_i in (5.28) is an equivalence class on $\mathbb{R}^n \times (\mathbb{R}^n \backslash 0)$, with support in $\operatorname{supp}(\rho_i) * (\mathbb{R}^2 \backslash 0)$. Set

14.9
$$\sigma_m(\tau) = \sum_{\{i, \pi(\tau) \in \operatorname{supp}(\rho_i)\}} a_i(x^{(i)}, \xi^{(i)}) \tag{5.29}$$

where

$$\tau = F_i^*(\sum_j \xi_j^{(i)} \cdot dx_j) \quad \xi^{(i)} \cdot dx \in T^*_{x^{(i)}}\mathbb{R}^n,\ x^{(i)} = F_i(\pi(\tau)) \tag{5.30}$$

and the a_i are representations of the symbols of the K_i. This defines a function on $T^*X \backslash 0$, in fact the equivalence class

14.10
$$\sigma_m(A) \in S^{m-[1]}(T^*X) \tag{5.31}$$

is well-defined.

14.11 Proposition 5.2. *The principal symbol map in (5.31), defined as in (5.29), gives a short exact sequence:*

14.12
$$0 \hookrightarrow \Psi^{m-1}(X) \hookrightarrow \Psi^m(X) \xrightarrow{\sigma_m} S^{m-[1]}(T^*X) \longrightarrow 0. \tag{5.32}$$

Proof. First we need to check that $\sigma_m(A)$ is indeed well-defined. This involves checking what happens under a change of coordinate covering and a change of partition of unity subordinate to it. First, under a change of partition of unity, subordinate to a fixed covering note that

14.13
$$\begin{aligned} \rho_j'(x)\rho_j'(y)K_A &= \sum_i \rho_j'(x)\rho_i^2(x)\rho_j'(y)K_A \\ &\equiv \sum_i \rho_j'(x)\rho_i(x)\rho_i(y)\rho_j'(y)K_A \\ &\equiv (\rho_j')^2 \sum_i \rho_i(x)K_A\rho_j(y). \end{aligned} \tag{5.33}$$

where equality is modulo Ψ^{m-1}, since $[\phi, \Psi^m] \subset \Psi^{m-1}$ for any $\mathcal{C}^\infty$ function ϕ.

It follows from (5.32) that the principal symbols, defined by (5.29), for the two partitions of unity agree.

For a change of coordinate covering it suffices to use the transformation law for the principal symbol under a diffeomorphism of $\mathbb{R}^n$ and the freedom, just established, to choose the partition of unity to be subordinates to *both* coordinate coverings. Thus σ_m is well-defined.

Certainly if $A \in \Psi^{m-1}(X)$ then $\sigma_m(N) \equiv 0$. Moreover if $A \in \Psi^m(X)$ and $\sigma_m(A) \equiv 0$ then all the operators $\rho_i(x)K\rho_i(y)$ are actually of order $m-1$. Since

14.14 (5.34) $$A \equiv \sum_i \sum_i \rho_i' \rho_j(X) A \rho_i' \rho_j \qquad \text{mod } \Psi^{m-1}(X)$$

for any two partitions of units $\rho_i'^2, (\rho_i')^2$ we can choose the ρ_i' to each have support in a region where $\rho_j \neq 0$ for some j. Then (5.34) shows that $A \in \Psi^{m-1}(X)$.

Thus it only remains to show that the map σ_m is surjective. If $a \in S^m(T^*X)$ choose $A_i \in \Psi^m_\infty(\mathbb{R}^n)$ by

(5.35) $$\sigma_L(A_i) = \rho_i(x)(F^*)^{-1} a_i \rho_i(y) \in S^m_\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

and set

14.15 (5.36) $$A = \sum_i F_i^* A_i G_i^* \qquad G_i = F_i^{-1}.$$

Then, from (5.29) $\sigma_m(A) \equiv a$ by invariance of the principal symbol. □

The other basic properties of the calculus are easily established. For example

14.16 (5.37) $$\sigma_{m+m'}(A \cdot B) = \sigma_m(A) \cdot \sigma_m(B)$$

if $A \in \Psi^m(X), B \in \Psi^{m'}(X), X$ compact. Similarly note that

(5.38) $$AB = \sum_{i,j} \rho_i^2 A \rho_j^2 B = \sum_{i,j} \rho_i A \rho_i \cdot \rho_j B \rho_i \qquad \text{mod } \Psi^{m_m'-1}$$

which gives (5.37).

In § 4.9 we used the symbol calculus to construct a left and right parametrix for an elliptic element of $\Psi^m(X)$, where X is compact, i.e. an element $B \in \Psi^{-m}(X)$, such that

14.17 (5.39) $$AB - \text{Id}, BA - \text{Id} \in \Psi^{-\infty}(X).$$

As a consequence of this construction note that:

14.18 Proposition 5.3. *If $A \in \Psi^m(X)$ is elliptic, and X is compact, then*

14.19 (5.40) $$A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X)$$

is Fredholm, *i.e. has finite dimensional null space and closed range with finite dimensional complement. If ν is a non-vanishing $\mathcal{C}^\infty$ measure on X and a generalized inverse of A is defined by*

14.20 (5.41) $$\begin{aligned} &Gu = f \text{ if } u \in \text{Ran}(A), Af = u, f \perp_\nu \text{Nul}(A) \\ &Gu = 0 \text{ if } u \perp_\nu \text{Ran}(A) \end{aligned}$$

then $G \in \Psi^{-m}(X)$ satisfies

14.21 (5.42) $$\begin{aligned} GA &= \text{Id} - \pi_N \\ AG &= \text{Id} - \pi_R \end{aligned}$$

where π_N and π_R are ν-orthogonal projections onto the null space of A and the ν-orthocomplement of the range of A respectively.

Proof. The main point to note is that $E \in \Psi^{-\infty}(X)$ is smoothing,

14.22 (5.43) $$E : \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^{\infty}(X) \quad \forall\ E \in \Psi^{-\infty}(X).$$

Such a map is *compact* on $L^2(X)$, i.e. maps bounded sets into precompact sets by the theorem of Ascoli and Arzela. The second thing to recall is that a Hilbert space with a compact unit ball is finite dimensional. Then

14.23 (5.44) $$\mathrm{Nul}(A) = \{u \in \mathcal{C}^{\infty}(X); Au = 0\} = \{u \in L^2(X); Au = 0\}$$

since, from (5.44) $Au = 0 \Longrightarrow (BA - \mathrm{Id})u = -Eu$, $E \in \Psi^{\infty}(X)$, so $Au = 0$, $u \in \mathcal{C}^{-\infty}(X) \Longrightarrow u \in \mathcal{C}^{\infty}(X)$. Then

(5.45) $$\mathrm{Nul}(A) = \{u \in L^2(X); Au = 0 \int |u|^2\nu = 1\} \subset L^2(X)$$

is compact since it is closed (A is continuous) and so $\mathrm{Nul}(A) = E(\mathrm{Nul}(A))$ is *precompact*. Thus $\mathrm{Nul}(A)$ is finite dimensional.

Next let us show that $\mathrm{Ran}(A)$ is closed. Suppose $f_j = Au_j \longrightarrow f$ in $\mathcal{C}^{\infty}(X)$, $u_j \in \mathcal{C}^{\infty}(X)$. By what we have just shown we can assume that $u_j \perp_\nu \mathrm{Nul}(A)$. Now if B is the parametrix

(5.46) $$u_j = Bf_j + Eu_j,\ E \in \Psi^{-\infty}(X).$$

Suppose, along some subsequence, $\|u_j\|_\nu \longrightarrow \infty$. Then

(5.47) $$\frac{u_j}{\|u_j\|_\nu} = B\left(\frac{f_j}{\|u_j\|_\nu}\right) + E\left(\frac{u_j}{\|u_j\|_\nu}\right)$$

shows that $\frac{u_j}{\|u_j\|_\nu}$ lies in a precompact subset of L^2, $\frac{u_j}{\|u_j\|_\nu} \longrightarrow u$. This is a contradiction, since $Au = 0$ but $\|u\| = 1$ and $u \perp_\nu \mathrm{Nul}(A)$. Thus the norm sequence $\|u_j\|$ is bounded and therefore the sequence has a weakly convergent subsequence, which we can relabel as u_j. The parametrix shows that $u = Bf_j + Eu_j$ is strongly convergent with limit u, which satisfies $Au = f$.

Finally we have to show that $\mathrm{Ran}(A)$ has a finite dimensional complement. If π_R is orthogonal projection off $\mathrm{Ran}(A)$ then from the second part of (5.39) $f = \pi_R E' f$ for some smoothing operator E. This shows that the orthocomplement has compact unit ball, hence is finite dimensional. □

Notice that it follows that the two projections in (5.42) are both smoothing operators of finite rank.

5.4. Pseudodifferential operators on vector bundles

We have just shown that any elliptic pseudodifferential operator, $A \in \Psi^m(X)$ on a compact manifold X has a generalized inverse $B \in \Psi^{-m}(X)$, meaning

15.1 (5.48) $$\begin{aligned} BA &= \mathrm{Id} - \pi_N \\ AB &= \mathrm{Id} - \pi_R \end{aligned}$$

where π_N and π_R are the orthogonal projections onto the null space of A and the orthocomplement of the range of A with respect to a prescribed $\mathcal{C}^\infty$ positive density ν, both are elements of $\Psi^{-\infty}(X)$ and have finite rank. To use this theorem in geometric situations we need first to make the "trivial" extension to operators on sections of vector bundles.

As usual there are two ways (at least) to approach this extension; the high road and the low road. The "low" road is to go back to the definition of $\Psi^m(X)$ and to generalize to $\Psi^m(X;V,W)$. This just requires to take the definition, following (5.16), but using a covering with respect to which the bundles V, W are *both* locally trivial. The local coordinate representatives of the pseudodifferential operator are then matrices of pseudodifferential opertors. The symbol mapping becomes

15.2 (5.49)
$$\Psi^m(X;V,W) \longrightarrow S^{m-[1]}\,(T^*X;\mathrm{Hom}(V,W))$$

where $\mathrm{Hom}(V,W) \simeq V \otimes W'$ is the bundle of homomorphisms from V to W and the symbol space consists of symbolic sections of the lift of this bundle to T^*X. We leave the detailed description and proof of these results to the enthusiasts.

So what is the "high" road. This involves only a little sheaf-theoretic thought. Namely we want to define the space $\Psi^m(X;V,W)$ using $\Psi^m(X)$ by:

15.3 (5.50)
$$\Psi^m(X;V,W) = \Psi^m(X) \bigotimes_{\mathcal{C}^\infty(X^2)} \mathcal{C}^\infty(X^2;V \boxtimes W').$$

To make sense of this we first note that $\Psi^m(X)$ is a $\mathcal{C}^\infty(X^2)$-module as is the space $\mathcal{C}^\infty(X^2;V\boxtimes W')$ where $V \boxtimes W'$ is the "exterior" product:

15.4 (5.51)
$$(V \boxtimes W')_{(x,y)} = V_x \otimes W'_y.$$

The tensor product in (5.50) means that

15.5 (5.52)
$$A \in \Psi^m(X;V,W) \text{ is of the form } A = \sum_i A_i \cdot G_i$$

where $A_i \in \Psi^m(X)$, $G_i \in \mathcal{C}^\infty(X^2;V\boxtimes W')$ and equality is fixed by the relation

15.6 (5.53)
$$\phi A \cdot G - A \cdot \phi G \equiv 0.$$

Now what we really need to note is:

15.7 PROPOSITION 5.4. *For any compact $\mathcal{C}^\infty$ manifold Y and any vector bundle U over Y*

15.8 (5.54)
$$\mathcal{C}^{-\infty}(Y;U) \equiv \mathcal{C}^{-\infty}(Y) \bigotimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(Y;U).$$

PROOF. $\mathcal{C}^{-\infty}(Y;U) = (\mathcal{C}^\infty(Y;\Omega\otimes U'))'$ is the definition. Clearly we have a mapping

(5.55)
$$\mathcal{C}^{-\infty}(Y) \bigotimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(Y;U) \ni \sum_i A_i \cdot g_i \longrightarrow \mathcal{C}^{-\infty}(Y;U)$$

given by

15.9 (5.56)
$$\sum_i u_i \cdot g_i(\psi) = \sum_i u_i(g_i \cdot \psi)$$

since $g_i\psi \in \mathcal{C}^\infty(Y;\Omega)$ and linearity shows that the map descends to the tensor product. To prove that the map is an isomorphism we construct an inverse. Since Y is compact we can find a finite number of sections $g_i \in \mathcal{C}^\infty(Y;U)$ such that *any* $u \in \mathcal{C}^\infty(Y;U)$ can be written

(5.57)
$$u = \sum_i h_i g_i \quad h_i \in \mathcal{C}^\infty(Y).$$

By reference to local coordinates the same is true of distributional sections with

(5.58)
$$h_i = u \cdot q_i \quad q_i \in \mathcal{C}^\infty(Y;U').$$

This gives a left and right inverse. □

15.10 THEOREM 5.3. *The calculus extends to operators sections of vector bundles over any compact $\mathcal{C}^\infty$ manifolds.*

5.5. Laplacian on forms

Since this is more differential geometry than differential analysis I will be brief. We already defined the exterior derivative

15.11 (5.59)
$$d : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X; T^*X)$$

as part of the definition of T^*X, i.e.

(5.60)
$$df(\overline{x}) = [f(x - f(\overline{x})] \in T^*_{\overline{x}}X \ \forall \ \overline{x} \in X.$$

The importance of the form bundles is tha they give a *resolution* of (5.59).

[Actually I didn't mean to be this brief, but was interupted while writing the lecture!]

5.6. Hodge theorem

S.pseudo.proj

5.7. Pseudodifferential projections

5.6.1998.228 PROPOSITION 5.5. *If $P \in \Psi^0(M; E)$ is such that $P^2 - P \in \Psi^{-\infty}(M; E)$ then there exists $\Pi \in \Psi^0(M; E)$ such that $\Pi^2 = \Pi$ and $\Pi - P \in \Psi^{-\infty}(M; E)$.*

4.6.1998.227 PROPOSITION 5.6. *If $P \in \Psi^0(M; E)$ is such that $P^2 - P \in \Psi^{-\infty}(M; E)$ and $F \subset H^s(M; E)$ is a closed subspace corresponding to which there are smoothing operators $A, B \in \Psi^{-\infty}(M; E)$ with $\mathrm{Id} - P = A$ on F and $(P + B)L^2(M; E) \subset F$ then there is a smoothing operator $B' \in \Psi^{-\infty}(M : E)$ such that $F = \mathrm{Ran}(P + B')$ and $(P + B')^2 = P + B'$.*

PROOF. Assume first that $s = 0$, so F is a closed subspace of $L^2(X; E)$. Applying Proposition 5.5 to P we may assume that it is a projection, without affecting the other conditions. Consider the intersection $E = F \cap \mathrm{Ran}(\mathrm{Id} - \Pi)$. This is a closed subspace of $L^2(M; E)$. With A as in the statement of the proposition, $E \subset \mathrm{Nul}(\mathrm{Id} - A)$. Indeed P vanishes on $\mathrm{Ran}(\mathrm{Id} - P)$ and hence on E and by hypothesis $\mathrm{Id} - P - A$ vanishes on F and hence on E. From the properties of smoothing operators, E is contained in a finite dimensional subspace of $\mathcal{C}^\infty(M; E)$, so is itself such a space. We may modify P by adding a smoothing projection onto E to it, and so assume that $F \cap \mathrm{Ran}(\mathrm{Id} - P) = \{0\}$.

Consider the sum $G = F + \mathrm{Ran}(\mathrm{Id} - P)$. Consider the operator $\mathrm{Id} + B = (P + B) + (\mathrm{Id} - P)$, with B as in the statement of the Proposition. The range of $\mathrm{Id} + B$ is contained in G. Thus G must be a closed subspace of $L^2(M; E)$ with a finite dimensional complement in $\mathcal{C}^\infty(M; E)$. Adding a smoothing projection onto such a complement we can, again by altering P by smoothing term, arrange that

5.6.1998.229 (5.61)
$$L^2(M; E) = F \oplus \mathrm{Ran}(\mathrm{Id} - P)$$

is a (possibly non-orthogonal) direct sum. Since P has only been altered by a smoothing operator the hypotheses of the Proposition continue to hold. Let Π be the projection with range F and null space equal to the range of $\mathrm{Id} - P$. It follows that $P' = P + (\mathrm{Id} - P)RP$ for some bounded operator R (namely $R = (\mathrm{Id} - P)(P' - P)P$.) Then restricted to F, $P' = \mathrm{Id}$ and $P = \mathrm{Id} + A$ so $R = -A$ on F. In fact $R = AP \in \Psi^{-\infty}(M; E)$, since they are equal on F and both vanish on $\mathrm{Ran}(\mathrm{Id} - P)$. Thus P' differs from P by a smoothing operator.

The case of general s follows by conjugating with a pseudodifferential isomorphism of $H^s(M;E)$ to $L^2(M;E)$ since this preserves both the assumptions and the conclusions. $\square$

5.8. Heat kernel

5.9. Resolvent

5.10. Complex powers

5.11. Problems

6.3.1998.156 PROBLEM 5.1. Show that compatibility in the sense defined before Definition 5.1 is an equivalence relation on $\mathcal{C}^\infty$ structures. Conclude that there is a unique maximal $\mathcal{C}^\infty$ structure containing any give $\mathcal{C}^\infty$ structure.

6.3.1998.158 PROBLEM 5.2. Let $\mathcal{F}$ be a $\mathcal{C}^\infty$ structure on X and let O_a, $a \in A$, be a covering of X by coordinate neighbourhoods, in the sense of (5.2) and (5.3). Show that the maximal $\mathcal{C}^\infty$ structure containing $\mathcal{F}$ consists of ALL functions on X which are of the form (5.3) on each of these coordinate patches. Conclude that the maximal $\mathcal{C}^\infty$ structure is an algebra.

6.3.1998.159 PROBLEM 5.3 (Partitions of unity). Show that any $\mathcal{C}^\infty$ manifold admits partitions of unity. That is, if O_a, $a \in A$, is an open cover of X then there exist elements $\rho_{a,i} \in \mathcal{C}^\infty(X)$, $a \in A$, $i \in \mathbb{N}$, with $0 \le \rho_{a,i} \le 1$, with each $\rho_{a,i}$ vanishing outside a compact subset $K_{a,i} \subset O_a$ such that only finite collections of the $\{K_{a,i}\}$ have non-trivial intersection and for which

$$\sum_{a\in A, i\in\mathbb{N}} \rho_{a,i} = 1.$$

CHAPTER 6

C.Elliptic.boundary

Elliptic boundary problems

Summary

Elliptic boundary problems are discussed, especially for operators of Dirac type. We start with a discussion of manifolds with boundary, including functions spaces and distributions. This leads to the 'jumps formula' for the relationship of the action of a differential operator to the operation of cutting off at the boundary; this is really Green's formula. The idea behind Calderòn's approach to boundary problems is introduced in the restricted context of a dividing hypersurface in a manifold without boundary. This includes the fundamental result on the boundary behaviour of a pseudodifferential operator with a rational symbol. These ideas are then extended to the case of an operator of Dirac type on a compact manifold with boundary with the use of left and right parametrices to define the Calderòn projector. General boundary problems defined by pseudodifferential projections are discussed by reference to the 'Calderòn realization' of the operator. Local boundary conditions, and the corresponding ellipticity conditions, are then discussed and the special case of Hodge theory on a compact manifold with boundary is analysed in detail for absolute and relative boundary conditions.

Introduction

Elliptic boundary problems arise from the fact that elliptic differential operators on compact manifolds with boundary have infinite dimensional null spaces. The main task we carry out below is the parameterization of this null space, in terms of boundary values, of an elliptic differential operator on a manifold with boundary. For simplicity of presentation the discussion of elliptic boundary problems here will be largely confined to the case of first order systems of differential operators of Dirac type. This has the virtue that the principal results can be readily stated.

Status as of 4 August, 1998

S.Manifolds.boundary S.Smooth.functions.MWB

Read through Section 6.1–Section 6.2: It is pretty terse in places! Several vital sections are still missing.

S.Manifolds.boundary

6.1. Manifolds with boundary

Smooth manifolds with boundary can be defined in very much the same was as manifolds without boundary. Thus we start with a paracompact Hausdorff space X and assume that it is covered by 'appropriate' coordinate patches with corresponding transition maps. In this case the 'model space' is $\mathbb{R}^{n,1} = [0, \infty) \times \mathbb{R}^{n-1}$, a Euclidean half-space of fixed dimension, n. As usual it is more convenient to use as models all open subsets of $\mathbb{R}^{n,1}$; of course this means *relatively open*, not open as subsets of $\mathbb{R}^n$. Thus we allow any

$$O = O' \cap \mathbb{R}^{n,1}\,, \quad O' \subset \mathbb{R}^n \text{ open},$$

as local models.

By a smooth map between open sets in this sense we mean a map with a smooth extension. Thus if O_i for $i = 1, 2$ are open in $\mathbb{R}^{n,1}$ then smoothness of a map F means that

1.6.1998.220

$$F : O_1 \to O_2,\ \exists\ O'_i \subset \mathbb{R}^n,\ i = 1, 2,\ \text{open and } \tilde{F} : O'_1 \to O'_2 \quad \text{which is } \mathcal{C}^\infty \text{ with } O_i = O'_i \cap \mathbb{R}^{n,1} \text{ and } F = F'|O_1. \tag{6.1}$$

It is important to note that the smoothness condition is *much* stronger than just smoothness of F on $O \cap (0, \infty) \times \mathbb{R}^{n-1}$.

By a diffeomorphism between such open sets we mean an invertible smooth map with a smooth inverse. Various ways of restating the condition that a map be a diffeomorphism are discussed below.

With this notion of local model we define a *coordinate system (in the sense of manifolds with boundary)* as a homeomorphism of open sets

$$X \supset U \xrightarrow{\Phi} O \subset \mathbb{R}^{n,1}\,, \quad O, U \text{ open}.$$

Thus Φ^{-1} is assumed to exist and both Φ and Φ^{-1} are assumed to be continuous. The *compatibility* of two such coordinate systems (U_1, Φ_1, O_1) and (U_2, Φ_2, O_2) is the requirement that *either* $U_1 \cap U_2 = \phi$ *or* if $U_1 \cap U_2 \neq \phi$ then

$$\Phi_{1,2} = \Phi_2 \circ \Phi_1^{-1} : \Phi_1(U_1 \cap U_2) \to \Phi_2(U_1 \cap U_2)$$

is a diffeomorphism in the sense described above. Notice that both $\Phi_1(U_1 \cap U_2)$ and $\Phi_2(U_1 \cap U_2)$ are open in $\mathbb{R}^{n,1}$. The inverse $\Phi_{1,2}$ is defined analogously.

A $\mathcal{C}^\infty$ manifold with boundary can then be formally defined as a paracompact Hausdorff topological space which has a maximal covering by mutually compatible coordinate systems.

An alternative definition, i.e. characterization, of a manifold with boundary is that there exists a $\mathcal{C}^\infty$ manifold $\tilde{X}$ *without boundary* and a function $f \in \mathcal{C}^\infty(\tilde{X})$ such that $df \neq 0$ on $\{f = 0\} \subset \tilde{X}$ and

$$X = \left\{ p \in \tilde{X}; f(p) \geq 0 \right\},$$

with coordinate systems obtained by restriction from $\tilde{X}$. The doubling construction described below shows that this is in fact an equivalent notion.

S.Smooth.functions.MWB

6.2. Smooth functions

As in the boundaryless case, the space of functions on a compact manifold with boundary is the primary object of interest. There are two basic approaches to defining local smoothness, the one intrinsic and the other extrinsic, in the style of the two definitions of a manifold with boundary above. Thus if $O \subset \mathbb{R}^{n,1}$ is open we can simply set

$$\mathcal{C}^\infty(O) = \{u : O \to \mathbb{C}; \exists\ \tilde{u} \in \mathcal{C}^\infty(O')\,, \quad O' \subset \mathbb{R}^n \text{ open}, \quad O = O' \cap \mathbb{R}^{n,1}\,, u = \tilde{u}|_O\right\}\,.$$

Here the open set in the definition might depend on u. The derivatives of $\tilde{u} \in \mathcal{C}^\infty(O')$ are bounded on all compact subsets, $K \Subset 0$. Thus

eq:F1
$$\sup_{K\cap O^\circ} |D^\alpha u| < \infty\,, \quad O^\circ = O \cap ((0,\infty)\times\mathbb{R}^{n-1})\,. \tag{6.2}$$

The second approach is to use (6.2) as a definition, i.e. to set

eq:F2
$$\mathcal{C}^\infty(O) = \{u : O^\circ \to \mathbb{C}; (6.2) \text{ holds } \forall\ K \Subset O \text{ and all } \alpha\}\,. \tag{6.3}$$

In particular this implies the continuity of $u \in \mathcal{C}^\infty(O)$ up to any point $p \in O \cap (\{0\}\times\mathbb{R}^{n-1})$, the boundary of O as a manifold with boundary.

As the notation here asserts, these two approaches are equivalent. This follows (as does much more) from a result of Seeley:

Proposition 6.1. *If $\mathcal{C}^\infty(O)$ is defined by (6.3) and $O' \subset \mathbb{R}^n$ is open with $O = O' \cap \mathbb{R}^{n,1}$ then there is a linear extension map*

$$E : \mathcal{C}^\infty(O) \to \mathcal{C}^\infty(O')\,, \quad Eu|_{O'} = u$$

which is continuous in the sense that for each $K' \Subset O'$, compact, there is some $K \Subset O$ such that for each α

$$\sup_{K'} |D^\alpha Eu| \le C_{\alpha,K'} \sup_{K\cap O} |D^\alpha u|\,.$$

The existence of such an extension map shows that the definition of a diffeomorphism of open sets O_1, O_2, given above, is equivalent to the condition that the map be invertible and that it, and its inverse, have components which are in $\mathcal{C}^\infty(O_1)$ and $\mathcal{C}^\infty(O_2)$ respectively.

Given the local definition of smoothness, the global definition should be evident. Namely, if X is a $\mathcal{C}^\infty$ manifold with boundary then

$$\mathcal{C}^\infty(X) = \left\{u : X \to \mathbb{C}; (\Phi^{-1})^*(u|_U) \in \mathcal{C}^\infty(O)\ \forall \text{ coordinate systems}\right\}\,.$$

This is also equivalent to demanding that local regularity, i.e. the regularity of $(\Phi^{-1})^*(u|_O)$, hold for any one covering by admissible coordinate systems.

As is the case of manifolds without boundary, $\mathcal{C}^\infty(X)$ admits partitions of unity. In fact the proof of Lemma 5.1 applies verbatim; see also Problem 5.3.

The topology of $\mathcal{C}^\infty(X)$ is given by the supremum norms of the derivatives in local coordinates. Thus a seminorm

$$\sup_{K\Subset O} \left|D^\alpha(\Phi^{-1})^*(u|_U)\right|$$

arises for each compact subset of each coordinate patch. In fact there is a countable set of norms giving the same topology. If X is compact, $\mathcal{C}^\infty(X)$ is a Fréchet space, if it is not compact it is an inductive limit of Fréchet spaces (an LF space).

The boundary of X, ∂X, is the union of the $\Phi^{-1}(O \cap (\{0\}\times\mathbb{R}^{n-1}))$ over coordinate systems. It is a manifold without boundary. It is compact if X is compact. Furthermore, ∂X has a global defining function $\rho \in \mathcal{C}^\infty(X)$; that is $\rho \ge 0$, $\partial X = \{\rho = 0\}$ and $d\rho \ne 0$ at ∂X. Moreover if ∂X is compact then any such boundary defining function can be extended to a product decomposition of X near ∂X:

11.6.1998.250
$$\exists\, C \supset \partial X\,, \text{ open in } X\ \epsilon > 0 \text{ and a diffeomorphism}\ \varphi : C \simeq [0,\epsilon)_\rho \times \partial X. \tag{6.4}$$

If ∂X is not compact this is still possible for an appropriate choice of ρ. For an outline of proofs see Problem 6.1.

11.6.1998.249 Lemma 6.1. *If X is a manifold with compact boundary then for any boundary defining function $\rho \in \mathcal{C}^\infty(X)$ there exists $\epsilon > 0$ and a diffeomorphism* (11.6.1998.250 6.4).

11.6.1998.251 Problem 6.1.

The existence of such a product decomposition near the boundary (which might have several components) allows the doubling construction mentioned above to be carried through. Namely, let

eq:F4
$$\tilde{X} = (X \cup X)/\partial X \tag{6.5}$$

be the disjoint union of two copies of X with boundary points identified. Then consider

eq:F5
$$\begin{aligned} \mathcal{C}^\infty(\tilde{X}) = \{(u_1, u_2) \in \mathcal{C}^\infty(X) \oplus \mathcal{C}^\infty(X);& \\ (\varphi^{-1})^*(u_1|_C) = f(\rho, \cdot)\,,\ (\varphi^{-1})^*(u_2|_C) = f(-\rho, \cdot)\,,& \\ f \in \mathcal{C}^\infty((-1,1) \times \partial X)\}&\ . \end{aligned} \tag{6.6}$$

This is a $\mathcal{C}^\infty$ structure on $\tilde{X}$ such that $X \hookrightarrow \tilde{X}$, as the first term in (eq:F4 6.5), is an embedding as a submanifold with boundary, so

$$\mathcal{C}^\infty(X) = \mathcal{C}^\infty(\tilde{X})|_X\,.$$

In view of this possibility of extending X to $\tilde{X}$, we shall not pause to discuss all the usual 'natural' constructions of tensor bundles, density bundles, bundles of differential operators, etc. They can simply be realized by restriction from $\tilde{X}$. In practice it is probably preferable to use intrinsic definitions.

The definition of $\mathcal{C}^\infty(X)$ implies that there is a well-defined restriction map

$$\mathcal{C}^\infty(X) \ni u \longmapsto u|_{\partial X} \in \mathcal{C}^\infty(\partial X).$$

It is always surjective. Indeed the existence of a product decomposition shows that any smooth function on ∂X can be extended locally to be independent of the chosen normal variable, and then cut off near the boundary.

There are important points to observe in the description of functions near the boundary. We may think of $\mathcal{C}^\infty(X) \subset \mathcal{C}^\infty(X^\circ)$ as a subspace of the smooth functions on the interior of X which describes the 'completion' (compactification if X is compact!) of the interior to a manifold with boundary. It is in this sense that the action of a differential operator $P \in \operatorname{Diff}^m(X)$

$$P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

should be understood. Thus P is just a differential operator on the interior of X with 'coefficients smooth up to the boundary.'

Once this action is understood, there is an obvious definition of the space of $\mathcal{C}^\infty$ functions which vanish to all orders at the boundary,

$$\dot{\mathcal{C}}^\infty(X) = \{u \in \mathcal{C}^\infty(X); Pu|_{\partial X} = 0\ \forall\ P \in \operatorname{Diff}^*(X)\}\ .$$

Having chosen a product decomposition near the boundary, Taylor's theorem gives us an isomorphism

$$\mathcal{C}^\infty(X)/\dot{\mathcal{C}}^\infty(X) \cong \bigoplus_{k \geq 0} \mathcal{C}^\infty(\partial X) \cdot [d\rho|_{\partial X}]^k.$$

S.Distributions.MWB

6.3. Distributions

It is somewhat confusing that there are *three* (though really only *two*) spaces of distributions immediately apparent on a compact manifold with boundary. Understanding the relationship between them is important to the approach to boundary problems used here.

The *coarsest* (as it is a little dangerous to say *largest*) space is $\mathcal{C}^{-\infty}(X^\circ)$, the dual of $\mathcal{C}^\infty_c(X^\circ;\Omega)$, just the space of distributions on the interior of X. The elements of $\mathcal{C}^{-\infty}(X^\circ)$ may have unconstrained growth, and unconstrained order of singularity, approaching the boundary. They are not of much practical value here and appear for conceptual reasons.

Probably the most natural space of distributions to consider is the dual of $\mathcal{C}^\infty(X;\Omega)$ since this is arguably the direct analogue of the boundaryless case. We shall denote this space

eq:D1
$$\dot{\mathcal{C}}^{-\infty}(X) = (\mathcal{C}^\infty(X;\Omega))' \tag{6.7}$$

and call it the space of *supported distributions*. The 'dot' is supposed to indicate this support property, which we proceed to describe.

If $\tilde X$ is any compact extension of X (for example the double) then, as already noted, the restriction map $\mathcal{C}^\infty(\tilde X;\Omega) \to \mathcal{C}^\infty(X;\Omega)$ is continuous and surjective. Thus, by duality, we get an injective 'extension' map

eq:D2
$$\dot{\mathcal{C}}^{-\infty}(X) \ni u \mapsto \tilde u \in \mathcal{C}^{-\infty}(\tilde X),\ \tilde u(\varphi) = u(\varphi|_X). \tag{6.8}$$

We shall regard this injection as an identification $\dot{\mathcal{C}}^{-\infty}(X) \hookrightarrow \mathcal{C}^{-\infty}(\tilde X)$; its range is easily characterized.

prop:D4

Proposition 6.2. *The range of the map* (eq:D2 6.8) *is the subspace consisting of those* $\tilde u \in \mathcal{C}^{-\infty}(\tilde X)$ *with* $\operatorname{supp}\tilde u \subset X$.

The proof is given below. This proposition is the justification for calling $\dot{\mathcal{C}}^{-\infty}(X)$ the space of supported distributions; the dot is support to indicate that this is the subspace of the 'same' space for $\tilde X$, i.e. $\mathcal{C}^{-\infty}(\tilde X)$, of elements with support in X.

This notation is consistent with $\dot{\mathcal{C}}^\infty(X) \subset \mathcal{C}^\infty(\tilde X)$ being the subspace (by extension as zero) of elements with support in X. The same observation applies to sections of any vector bundle, so

$$\dot{\mathcal{C}}^\infty(X;\Omega) \subset \mathcal{C}^\infty(\tilde X;\Omega)$$

is a well-defined closed subspace. We set

eq:D5
$$\mathcal{C}^{-\infty}(X) = (\dot{\mathcal{C}}^\infty(X;\Omega))' \tag{6.9}$$

and call this the space of *extendible distributions* on X. The inclusion map for the test functions gives by duality a restriction map:

eq:D6
$$R_X : \mathcal{C}^{-\infty}(\tilde X) \to \mathcal{C}^{-\infty}(X),\quad R_X u(\varphi) = u(\varphi)\ \forall\ \varphi \in \dot{\mathcal{C}}^\infty(X;\Omega) \hookrightarrow \mathcal{C}^\infty(\tilde X;\Omega)\,. \tag{6.10}$$

We write, at least sometimes, R_X for the map since it has a large null space so should *not* be regarded as an identification. In fact

eq:D7
$$\operatorname{Nul}(R_X) = \left\{v \in \mathcal{C}^{-\infty}(\tilde X); \operatorname{supp}(v) \cap X^\circ = \phi\right\} = \dot{\mathcal{C}}^{-\infty}(\tilde X \backslash X^\circ)\,, \tag{6.11}$$

is just the space of distributions supported 'on the other side of the boundary'. The primary justification for calling $\mathcal{C}^{-\infty}(X)$ the space of extendible distributions is:

prop:D8 PROPOSITION 6.3. *If X is a compact manifold with boundary, then the space $\mathcal{C}_c^\infty(X^\circ;\Omega)$ is dense in $\dot{\mathcal{C}}^\infty(X;\Omega)$ and hence the restriction map*

eq:D9
$$\mathcal{C}^{-\infty}(X) \hookrightarrow \mathcal{C}^{-\infty}(X^\circ) \tag{6.12}$$

is injective, whereas the restriction map from (6.10), $R_X : \dot{\mathcal{C}}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X)$, is surjective.

PROOF. If V is a real vector field on $\tilde{X}$ which is inward-pointing across the boundary then

$$\exp(sV) : \tilde{X} \to \tilde{X}$$

is a diffeomorphism with $F_s(X) \subset X^\circ$ for $s > 0$. Furthermore if $\varphi \in \mathcal{C}^\infty(\tilde{X})$ then $F_s^*\varphi \to \varphi$ in $\mathcal{C}^\infty(\tilde{X})$ as $s \to 0$. The support property shows that $F_s^*\varphi \in \mathcal{C}_c^\infty(X^\circ)$ if $s < 0$ and $\varphi \in \dot{\mathcal{C}}^\infty(X)$. This shows the density of $\mathcal{C}_c^\infty(X^\circ)$ in $\dot{\mathcal{C}}^\infty(X)$. Since all topologies are uniform convergence of all derivatives in open sets. The same argument applies to densities. The injectivity of (6.12) follows by duality.

On the other hand the surjectivity of (6.10) follows directly from the Hahn-Banach theorem. □

PROOF OF PROPOSITION 6.2. For $\tilde{u} \in \mathcal{C}^{-\infty}(\tilde{X})$ the condition that $\operatorname{supp} \tilde{u} \subset X$ is just

eq:D10
$$\tilde{u}(\varphi) = 0 \ \forall \ \varphi \in \mathcal{C}_c^\infty \subset (\tilde{X}\backslash X;\Omega) \subset \mathcal{C}^\infty(\tilde{X};\Omega)\,. \tag{6.13}$$

Certainly (6.13) holds if $u \in \dot{\mathcal{C}}^{-\infty}(X)$ since $\varphi|_X = 0$. Conversely, if (6.13) holds, then by continuity and the density of $\mathcal{C}_c^\infty(\tilde{X}\backslash X;\Omega)$ in $\mathcal{C}^\infty(\tilde{X}\backslash X^\circ;\,\Omega)$, what follows from Proposition 6.3, u vanishes on $\dot{\mathcal{C}}^\infty(X\backslash X^\circ)$. □

It is sometimes useful to consider topologies on the spaces of distributions $\mathcal{C}^{-\infty}(X)$ and $\dot{\mathcal{C}}^{-\infty}(X)$. For example we may consider the *weak topology*. This is given by all the seminorms $u \mapsto \|\langle u, \phi\rangle\|$, where ϕ is a test function.

11.6.1998.252 LEMMA 6.2. *With respect to the weak topology, the subspace $\mathcal{C}_c^\infty(X^\circ)$ is dense in both $\dot{\mathcal{C}}^{-\infty}(X)$ and $\mathcal{C}^{-\infty}(X)$.*

S.Boundary.terms

6.4. Boundary Terms

To examine the precise relationship between the supported and extendible distributions consider the space of 'boundary terms'.

BT1
$$\dot{\mathcal{C}}_{\partial X}^{-\infty}(X) = \left\{u \in \dot{\mathcal{C}}^{-\infty}(X)\,;\, \operatorname{supp}(u) \subset \partial X\right\}. \tag{6.14}$$

Here the support may be computed with respect to any extension, or intrinsically on X. We may also define a map 'cutting off' at the boundary:

BT2
$$\mathcal{C}^\infty(X) \ni u \mapsto u_c \in \dot{\mathcal{C}}^{-\infty}(X)\,,\; u_c(\varphi) = \int_X u\varphi \ \forall \ \varphi \in \mathcal{C}^\infty(X\,;\,\Omega)\,. \tag{6.15}$$

BT3 Proposition 6.4. *If X is a compact manifold with boundary then there is a commutative diagram*

BT4 (6.16)
$$\begin{array}{ccccccccc} & & & & & & \dot{\mathcal{C}}^\infty(X) & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathcal{C}^\infty(X) & & \\ & & & & & \overset{()_c}{\swarrow} & \downarrow & & \\ 0 & \longrightarrow & \dot{\mathcal{C}}^{-\infty}_{\partial X}(X) & \longrightarrow & \dot{\mathcal{C}}^{-\infty}(X) & \longrightarrow & \mathcal{C}^{-\infty}(X) & \longrightarrow & 0 \end{array}$$

with the horizontal sequence exact.

Proof. The commutativity of the triangle follows directly from the definitions. The exactness of the horizontal sequence follows from the density of $\mathcal{C}^\infty_c(X^\circ;\Omega)$ in $\dot{\mathcal{C}}^\infty(X;\Omega)$. Indeed, this shows that $v\in\dot{\mathcal{C}}^{-\infty}_{\partial X}(X)$ maps to 0 in $\mathcal{C}^{-\infty}(X)$ since $v(\varphi)=0\ \forall\ \varphi\in\mathcal{C}^\infty_c(X^\circ;\Omega)$. Similarly, if $u\in\dot{\mathcal{C}}^{-\infty}(X)$ maps to zero in $\mathcal{C}^{-\infty}(X)$ then $u(\varphi)=0$ for all $\varphi\in\mathcal{C}^\infty_c(X^\circ;\Omega)$, so $\operatorname{supp}(u)\cap X^\circ=\emptyset$. □

Note that both maps in (6.16) from $\mathcal{C}^\infty(X)$ into supported and extendible distributions are injective. We regard the map into $\mathcal{C}^{-\infty}(X)$ as an identification. In particular this is consistent with the action of differential operators. Thus $P\in\operatorname{Diff}^m(X)$ acts on $\mathcal{C}^\infty(X)$ and then the smoothness of the coefficients of P amount to the fact that it preserves $\mathcal{C}^\infty(X)$, as a subspace. The formal adjoint P^* with respect to the sesquilinear pairing for some smooth positive density, ν

BT5 (6.17)
$$\langle\varphi,\psi\rangle=\int_X\varphi\overline{\psi}\nu\quad\forall\ \varphi,\psi\in\mathcal{C}^\infty(X)$$

acts on $\dot{\mathcal{C}}^\infty(X)$:

BT6 (6.18)
$$\langle P^*\varphi,\psi\rangle=\langle\varphi P\psi\rangle\quad\forall\ \varphi\in\dot{\mathcal{C}}^\infty(X)\,,\psi\in\mathcal{C}^\infty(X),\ P^*:\dot{\mathcal{C}}^\infty(X)\longrightarrow\dot{\mathcal{C}}^\infty(X).$$

However, $P^*\in\operatorname{Diff}^m(X)$ is fixed by its action over X°. Thus we do have

BT7 (6.19)
$$\langle P^*\varphi,\psi\rangle=\langle\varphi,P\psi\rangle\quad\forall\ \varphi\in\mathcal{C}^\infty(X)\,,\ \psi\in\dot{\mathcal{C}}^\infty(X)\,.$$

We define the action of P by duality. In view of the possibility of confusion, we denote P the action on $\mathcal{C}^{-\infty}(X)$ and by $\dot{P}$ the action on $\dot{\mathcal{C}}^\infty(X)$.

BT8 (6.20)
$$\begin{gathered}\langle Pu\,,\,\varphi\rangle=\langle u,P^*\varphi\rangle\quad\forall\ u\in\mathcal{C}^{-\infty}(X)\,,\,\varphi\in\dot{\mathcal{C}}^\infty(X),\ P:\mathcal{C}^{-\infty}(X)\longrightarrow\mathcal{C}^{-\infty}(X)\\ \langle\dot{P}u\,,\,\varphi\rangle=\langle u,P^*\varphi\rangle\quad\forall\ u\in\dot{\mathcal{C}}^{-\infty}(X)\,,\,\varphi\in\mathcal{C}^\infty(X),\ \dot{P}:\dot{\mathcal{C}}^{-\infty}(X)\longrightarrow\dot{\mathcal{C}}^{-\infty}(X).\end{gathered}$$

It is of fundamental importance that (6.19) does not hold for all $\varphi,\psi\in\mathcal{C}^\infty(X)$. This failure is reflected in Green's formula for the boundary terms, which appears below as the 'Jump formula'. This is a distributional formula for the difference

BT9 (6.21)
$$\dot{P}u_c-(Pu)_c\in\dot{\mathcal{C}}^{-\infty}_{\partial X}\,,\ u\in\mathcal{C}^\infty(X)\ P\in\operatorname{Diff}^m(X)\,.$$

Recall that a product decomposition of $C\subset X$ near ∂X is fixed by an inward pointing vector field V. Let $x\in\mathcal{C}^\infty(X)$ be a corresponding boundary defining function, with $Vx=0$ near ∂X, with $\chi_V:C\to\partial X$ the projection onto the boundary from the product neighborhood C. Then Taylor's formula for $u \in \mathcal{C}^\infty(X)$ becomes

BT10 (6.22)
$$u \sim \sum_k \frac{1}{k!} \chi_V^*(V^k u|_{\partial x}) x^k \,.$$

It has the property that a finite sum

$$u_N = \varphi u - \varphi \sum_{k=0}^{N} \frac{1}{k!} \chi_V^*(V^k u|_{\partial X}) x^k$$

where $\varphi \equiv 1$ near ∂X, $\operatorname{supp} \varphi \subset C$, satisfies

BT11 (6.23)
$$\dot{P}(u_N)_c = (P u_N)_c \,,\; P \in \operatorname{Diff}^m(X) \,,\; m < N \,.$$

Since $(1-\varphi)u \in \dot{\mathcal{C}}^\infty(X)$ also satisfies this identity, the difference in (BT9 6.21) can (of course) only depend on the $V^k u|_{\partial X}$ for $k \le m$, in fact only for $k < m$.

Consider the Heaviside function $1_c \in \dot{\mathcal{C}}^{-\infty}(X)$, detained by cutting off the identity function of the boundary. We define distributions

BT12 (6.24)
$$\delta^{(j)}(x) = V^{j+1} 1_c \in \dot{\mathcal{C}}^{-\infty}_{\partial X} \,,\; j \ge 0 \,.$$

Thus, $\delta^{(0)}(x) = \delta(x)$ is a 'Dirac delta function' at the boundary. Clearly $\operatorname{supp} \delta(x) \subset \partial X$, so the same is true of $\delta^{(j)}(x)$ for every j. If $\psi \in \mathcal{C}^\infty(\partial X)$ we define

BT13 (6.25)
$$\psi \cdot \delta^{(j)}(x) = \varphi(X_V^* \psi) \cdot \delta^{(j)}(x) \in \dot{\mathcal{C}}^{-\infty}_{\partial X}(X) \,.$$

This, by the support property of $\delta^{(j)}$, is independent of the cut off φ used to define it.

BT14 Proposition 6.5. *For each $P \in \operatorname{Diff}^m(X)$ there are differential operators on the boundary $P_{ij} \in \operatorname{Diff}^{m-i-j-1}(\partial X)$, $i + j < m$, $i, j \ge 0$, such that*

BT15 (6.26)
$$\dot{P} u_c - (Pu)_c = \sum_{i,j} (P_{ij}(V_u^j|_{\partial X}) \cdot \delta^{(j)}(x), \;\; \forall \; u \in \mathcal{C}^\infty(X),$$

and $P_{0m-1} = i^{-m} \sigma(P, dx) \in \mathcal{C}^\infty(\partial X)$.

Proof. In the local product neighborhood C,

BT16 (6.27)
$$P = \sum_{0 \le l \le m} P_l V^l$$

where P_l is a differential operator of the order at most $m-l$, on X be depending on x as a parameter. Thus the basic cases we need to analyze arise from the application of V to powers of x :

BT17 (6.28)
$$x^l \left(V^{j+1}(x^p)_c - (V^{j+1} x^p)_c \right) = \begin{cases} 0 & l+p > j \\ \frac{p!(j-p)!}{(j-p-l)!} \;\; (-1)^{l \delta^{(j-p-l)}} & l+p \le j \,. \end{cases}$$

Taking the Taylor sense of the P_l,

$$P_l \sim \sum_r x^r P_{l,r}$$

and applying P to (BT10 6.22) gives

BT18 (6.29)
$$P u_c - (Pu)_c = \sum_{r+k<l} (-1)^r (P_{l,r}(V^k u|_{\partial x})) \quad \delta^{(l-1-r-k)}(x) \,.$$

This is of the form (BT15 6.26). The only term with $l-1-r-k=m-1$ arises from $l-m$, $k=r=0$ so is the operator P_m at $x=0$. This is just $i^{-m}\sigma(P,dx)$. □

S.Sobolev.boun

6.5. Sobolev spaces

As with $\mathcal{C}^\infty$ functions we may define the standard (extendible) Sobolev spaces by restriction or intrinsically. Thus, if $\tilde{X}$ is an extension of a compact manifold with boundary, X, the we can define

11.6.1998.253

$$H^m(X) = H^m_c(\tilde{X})|X,\ \forall\ m \in \mathbb{R}; H^m(X) \subset \mathcal{C}^{-\infty}(X). \tag{6.30}$$

That this is independent of the choice of $\tilde{X}$ follows from the standard properties of the Sobolev spaces, particularly their localizability and invariance under diffeomorphisms. The norm in $H^m(X)$ can be taken to be

11.6.1998.254

$$\|u\|_m = \inf\left\{\|\tilde{u}\|_{H^m(\tilde{X})}; \tilde{u} \in H^m(\tilde{X}),\ u = \tilde{u}_X\right\}. \tag{6.31}$$

A more intrinsic defintion of these spaces is discussed in the problems.

There are also *supported* Sobolev spaces,

11.6.1998.255

$$\dot{H}^m(X) = \left\{u \in H^m(\tilde{X}); \mathrm{supp}(u) \subset X\right\} \subset \dot{\mathcal{C}}^{-\infty}(X). \tag{6.32}$$

Sobolev space of sections of any vector bundle can be defined similarly.

11.6.1998.256

Proposition 6.6. *For any $m \in \mathbb{R}$ and any compact manifold with boundary X, $H^m(X)$ is the dual of $\dot{H}^{-m}(X;\Omega)$ with respect to the continuous extension of the densely defined bilinear pairing*

$$(u,v) = \int_X uv,\ u \in \mathcal{C}^\infty(X),\ v \in \dot{\mathcal{C}}^\infty(X;\Omega).$$

Both $H^m(X)$ and $\dot{H}^m(X)$ are $\mathcal{C}^\infty(X)$-modules and for any vector bundle over X, $H^m(X;E) \equiv H^m(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;E)$ and $\dot{H}^m(X;E) \equiv \dot{H}^m(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;E)$.

Essentially from the definition of the Sobolev spaces, any $P \in \mathrm{Diff}^k(X;E_1,E_2)$ defines a continuous linear map

11.6.1998.257

$$P : H^m(X;E_1) \longrightarrow H^{m-k}(X;E_2). \tag{6.33}$$

We write the dual (to P^* of course) action

11.6.1998.258

$$\dot{P} : \dot{H}^m(X;E_1) \longrightarrow \dot{H}^{m-k}(X;E_2). \tag{6.34}$$

These actions on Sobolev spaces are consistent with the corresponding actions on distributions. Thus

$$\mathcal{C}^{-\infty}(X;E) = \bigcup_m H^m(X),\ \mathcal{C}^\infty(X;E) = \bigcap_m H^m(X),$$
$$\dot{\mathcal{C}}^{-\infty}(X;E) = \bigcup_m \dot{H}^m(X),\ \dot{\mathcal{C}}^\infty(X;E) = \bigcap_m \dot{H}^m(X).$$

S.Dividing.hypersurfaces

6.6. Dividing hypersurfaces

As already noted, the point of view we adopt for boundary problems is that they provide a parametrization of the space of solutions of a differential operator on a space with boundary. In order to clearly indicate the method pioneered by Calderòn, we shall initially consider the restrictive context of an operator of Dirac type on a compact manifold without boundary with an embedded separating hypersurface.

Thus, suppose initially that D is an elliptic first order differential operator acting between sections of two (complex) vector bundles V_1 and V_2 over a compact manifold without boundary, M. Suppose further that $H \subset M$ is a dividing hypersurface. That is, H is an embedded hypersuface with oriented (i.e. trivial) normal bundle and that $M = M_+ \cup M_-$ where $M_\pm$ are compact manifolds with boundary which intersect in their common boundary, H. The convention here is that M_+ is on the positive side of H with respect to the orientation.

In fact we shall make a further analytic assumption, that

11.4.1998.195

$$D : \mathcal{C}^\infty(M; V_1) \longrightarrow \mathcal{C}^\infty(M; V_2) \text{ is an isomorphism.} \tag{6.35}$$

As we already know, D is always Fredholm, so this implies the topological condition that the index vanish. However we only assume (6.35) to simplify the initial discussion.

Our objective is to study the space of solutions on M_+. Thus consider the map

11.4.1998.196

$$\left\{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0 \text{ in } M_+^\circ\right\} \xrightarrow{b_H} \mathcal{C}^\infty(H; V_1),\ b_H u = u_{|\partial M_+}. \tag{6.36}$$

The idea is to use the boundary values to parameterize the solutions and we can see immediately that this is possible.

11.4.1998.197

LEMMA 6.3. *The assumption (6.35) imples that map b_H in (6.36) is injective.*

PROOF. Consider the form of D in local coordinates near a point of H. Let the coordinates be $x, y_1, \dots, y_{n-1}$ where x is a local defining function for H and assume that the coordinate patch is so small that V_1 and V_2 are trivial over it. Then

$$D = A_0 D_x + \sum_{j=1}^{n-1} A_j D_{y_j} + A'$$

where the A_j and A' are local smooth bundle maps from V_1 to V_2. In fact the ellipticity of D implies that each of the A_j's is invertible. Thus the equation can be written locally

$$D_x u = Bu,\ B = -\sum_{j=1}^{n-1} A_0^{-1} D_{y_j} - A_0^{-1} A'.$$

The differential operator B is tangent to H. By assumption u vanishes when restricted to H so it follows that $D_x u$ also vanishes at H. Differentiating the equation with respect to x, it follows that all derivatives of u vanish at H. This in turn implies that the global section of V_1 over M

$$\tilde{u} = \begin{cases} u & \text{in } M_+ \\ 0 & \text{in } M_- \end{cases}$$

is smooth and satisfies $D\tilde{u} = 0$. Then assumption (6.35) implies that $\tilde{u} = 0$, so $u = 0$ in M_+ and b_H is injective. □

In the proof of this Lemma we have used the strong assumption (6.35). As we show below, if it is assumed instead that D is of Dirac type then the Lemma remains true without assuming (6.35). Now we can state the basic result in this setting.

29.3.1998.187 Theorem 6.1. *If $M = M_+ \cup M_-$ is a compact manifold without boundary with separating hypersurface H as described above and $D \in \mathrm{Diff}^1(M; V_1, V_2)$ is a generalized Dirac operator then there is an element $\Pi_C \in \Psi^0(H; V)$, $V = V_1|H$, satisfying $\Pi_C^2 = \Pi_C$ and such that*

29.3.1998.193 (6.37)
$$b_H : \{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0\} \longrightarrow \Pi_C \mathcal{C}^\infty(H; V)$$

is an isomorphism. The projection Π_C can be chosen so that

11.4.1998.198 (6.38)
$$b_H : \{u \in \mathcal{C}^\infty(M_-; V_1); Du = 0\} \longrightarrow (\mathrm{Id} - \Pi_C)\mathcal{C}^\infty(H; V)$$

then Π_C is uniquely determined and is called the Calderòn projection.

This result remains true for a general elliptic operator of first order if (6.35) is assumed, and even in a slightly weakened form without (6.35). Appropriate modifications to the proofs below are consigned to problems.

For first order operators the jump formula discussed above takes the following form.

11.4.1998.202 Lemma 6.4. *Let D be an elliptic differential operator of first order on M, acting between vector bundles V_1 and V_2. If $u \in \mathcal{C}^\infty(M_+; V_1)$ satisfies $Du = 0$ in M_+° then*

11.4.1998.203 (6.39)
$$Du_c = \frac{1}{i}\sigma_1(D)(dx)(b_H u) \cdot \delta(x) \in \mathcal{C}^{-\infty}(M; V_2).$$

Since the same result is true for M_-, with an obvious change of sign, D defines a linear operator

11.4.1998.205 (6.40)
$$D : \left\{u \in L^1(M; V_1); u_\pm = u|M_\pm \in \mathcal{C}^\infty(M_\pm; V_1),\ Du_\pm = 0 \text{ in } M_\pm^\circ\right\} \longrightarrow \frac{1}{i}\sigma(D)(dx)(b_H u_+ - b_H u_-) \cdot \delta(x) \in \mathcal{C}^\infty(H; V_2) \cdot \delta(x).$$

To define the Calderòn projection we shall use the 'inverse' of this result.

11.4.1998.204 Proposition 6.7. *If $D \in \mathrm{Diff}^1(M; V_1, V_2)$ is elliptic and satisfies (6.35) then (6.40) is an isomorphism, with inverse I_D, and*

11.4.1998.206 (6.41)
$$\Pi_C v = b_H\left(I_D \frac{1}{i}\sigma(D)(dx) v \cdot \delta(x)\right)_+,\ v \in \mathcal{C}^\infty(H; V_1),$$

satisfies the conditions of Theorem 6.1.

Proof. Observe that the map (6.40) is injective, since its null space consists of solutions of $Du = 0$ globally on M; such a solution must be smooth by elliptic regularity and hence must vanish by the assumed invertibility of D. Thus the main task is to show that D in (6.40) is surjective.

Since D is elliptic and, by assumption, an isomorphism on $\mathcal{C}^\infty$ sections over M it is also an isomorphism on distributional sections. Thus the inverse of (6.40) must be given by D^{-1}. To prove the surjectivity it is enough to show that

13.4.1998.207 (6.42)
$$D^{-1}(w \cdot \delta(x))|M_\pm \in \mathcal{C}^\infty(M_\pm; V_1)\ \forall\ w \in \mathcal{C}^\infty(H; V_2).$$

There can be no singular terms supported on H since $w \cdot \delta(x) \in H^{-1}(M; V_2)$ implies that $u = D^{-1}(w \cdot \delta(x)) \in L^2(M; V_1)$.

Now, recalling that $D^{-1} \in \Psi^{-1}(M; V_2, V_1)$, certainly u is $\mathcal{C}^\infty$ away from H. At any point of H outside the support of w, u is also smooth. Since we may decompose w using a partition of unity, it suffices to suppose that w has support in a small coordinate patch, over which both V_1 and V_2 are trivial and to show that u is smooth 'up to H from both sides' in the local coordinate patch. Discarding smoothing terms from D^{-1} we may therefore replace D^{-1} by any local parametrix Q for D and work in local coordinates and with components:

13.4.1998.208 (6.43)

$$Q_{ij}(w_j(y)\cdot\delta(x)) = (2\pi)^{-n}\int e^{i(x-x')\xi+i(y-y')\cdot\eta} q_{ij}(x,y,\xi,\eta)w(y')\delta(x')dx'dy'd\xi d\eta.$$

For a general pseudodifferential operator, even of order -1, the result we are seeking is not true. We must use special properties of the symbol of Q, that is D^{-1}.

S.Rational.symbols 6.7. Rational symbols

13.4.1998.209 Lemma 6.5. *The left-reduced symbol of any local parametrix for a generalized Dirac operator has an expansion of the form*

13.4.1998.210 (6.44)

$$q_{ij}(z,\zeta) = \sum_{l=1}^{\infty} g(z,\zeta)^{-2l+1} p_{ij,l}(z,\zeta) \text{ with } p_{ij,l} \text{ a polynomial of degree } 3l-2 \text{ in } \zeta;$$

here $g(z,\zeta)$ is the metric in local coordinates; each of the terms in (6.44) is therefore a symbol of order $-l$.

Proof. This follows by an inductive arument, of a now familiar type. First, the assumption that D is a generalized Dirac operator means that its symbol $\sigma_1(D)(z,\zeta)$ has inverse $g(z,\zeta)^{-1}\sigma_1(D)^*(z,\zeta)$; this is the princiapl symbol of Q. Using Leibniz' formula one concludes that for any polynomial r_l of degree j

$$\partial_{\zeta_i}\left(g(z,\zeta)^{-2l+1}r_j(z,\zeta)\right) = g(z,\zeta)^{-2l}r'_{j+1}(z,\zeta)$$

where r_{j+1} has degree (at most) $j+1$. Using this result repeatedly, and proceeding by induction, we may suppose that $q = q'_k + q''_{k+1}$ where q'_k has an expansion up to order k, and so may be taken to be such a sum, and q''_{k+1} is of order at most $-k-1$. The composition formula for left-reduced symbols then shows that

$$\sigma_1(D)q''_{k+1} \equiv g^{-2k}q_{k+1} \mod S^{-k-1}$$

where q_{k+1} is a polynomial of degree at most $3k$. Inverting $\sigma_1(D)(\zeta)$ as at the initial step then shows that q''_{k+1} is of the desired form, $g^{-2k-1}r_{k+1}$ with r_{k+1} of degree $3k+1 = 3(k+1)-2$, modulo terms of lower order. This completes the proof of the lemma. □

With this form for the symbol of Q we proceed to the proof of Proposition 6.7. That is, we consider (6.43). Since we only need to consider each term, we shall drop the indicies. A term of low order in the amplitude q_N gives an operator with kernel in $\mathcal{C}^{N-d}$. Such a kernel gives an operator

$$\mathcal{C}^\infty(H;V_2) \longrightarrow \mathcal{C}^{N-d}(M;V_1)$$

with kernel in $\mathcal{C}^{N-d}$. The result we want will therefore follow if we show that each term in the expansion of the symbol q gives an operator as in (6.42).

To be more precise, we can assume that the amplitude q is of the form

$$q = (1-\phi)g^{-2l}q'$$

where q' is a polynomial of degree $3l-2$ and $\phi = \phi(\xi,\eta)$ is a function of compact support which is identically one near the origin. The cutoff function is to remove the singularity at $\zeta = (\xi,\eta) = 0$. Using continuity in the symbol topology the integrals in x' and y' can be carried out. By assumption $w \in \mathcal{C}^\infty_c(\mathbb{R}^{n-1})$, so the resulting integral is absolutely convergent in η. If $l > 1$ it is absolutely convergent in ξ as well, so becomes

$$Q(w(y)\cdot\delta(x)) = (2\pi)^{-n}\int e^{ix\xi+iy\cdot\eta}q(x,y,\xi,\eta)\hat{w}(\eta)d\xi d\eta.$$

In $|\xi| > 1$ the amplitude is a rational function of ξ, decaying quadratically as $\xi \to \infty$. If we assume that $x > 0$ then the exponential factor is bounded in the half plane $\Im\xi \geq 0$. This means that the limit as $R \to \infty$ over the integral in $\Im\xi \geq 0$ over the semicircle $|\xi| = R$ tends to zero, and does so with uniform rapid decrease in η. Cauchy's theorem shows that, for $R > 1$ the real integral in ξ can be replaced by the contour integral over $\gamma(R)$, which is, for$R >> |\eta|$ given by the real interval $[-R,R]$ together with the semicircle of radius R in the upper half plane. If $|\eta| > 1$ the integrand is meromorphic in the upper half plane with a possible pole at the singular point $g(x,y,\xi,\eta) = 0$; this is at the point $\xi = ir^{\frac{1}{2}}(x,y,\eta)$ where $r(x,y,\eta)$ is a positive-definite quadratic form in η. Again applying Cauchy's theorem

$$Q(w(y)\delta(x) = (2\pi)^{-n+1}i\int e^{xr^{\frac{1}{2}}(x,y,\eta)+iy\cdot\eta}q'(x,y,\eta)\hat{w}(\eta)d\eta$$

where q' is a symbol of order $-k+1$ in η.

The product $e^{xr^{\frac{1}{2}}(x,y,\eta)}q'(x,y,\eta)$ is uniformly a symbol of order $-k+1$ in $x > 1$, with x derivatives of order p being uniformly symbols of order $-k+1+p$. It follows from the properties of pseudodifferential operators that $Q(w\cdot\delta(x))$ is a smooth function in $x > 0$ with all derivatives locally uniformly bounded as $x \downarrow 0$.

S.Proofs.204.187

6.8. Proofs of Proposition 6.7 and Theorem 6.1

This completes the proof of (6.42), since a similar argument applies in $x < 0$, with contour deformation into the lower half plane. Thus we have shown that (6.40) is an isomorphism which is the first half of the statement of Proposition (6.7). Furthermore we see that the limiting value from above is a pseudodifferential operator on H :

13.4.1998.211 (6.45)
$$Q_0 w = \lim_{x\downarrow 0} D^{-1}(w\cdot\delta(x)),\ Q_0 \in \Psi^0(H;V_2,V_1).$$

This in turn implies that Π_C, defined by (6.41) is an element of $\Psi^0(H;V_1)$, since it is $Q_0 \circ \frac{1}{i}\sigma(D)(dx)$.

Next we check that Π_C is a projection, i.e. that $\Pi_C^2 = \Pi_C$. If $w = \Pi_C v$, $v \in \mathcal{C}^\infty(H;V_1)$, then $w = b_H u$, $u = I_D\frac{1}{i}\sigma(D)(dx)v|_{M_+}$, so $u \in \mathcal{C}^\infty(M_+;V_1)$ satisfies $Du = 0$ in M_+°. In particular, by (6.39), $Pu_c = \frac{1}{i}\sigma_1(D)(dx)w\cdot\delta(x)$, which means that $w = \Pi_C w$ so $\Pi_C^2 = \Pi_C$. This also shows that the range of Π_C is precisely the range of b_H as stated in (6.37). The same argument shows that this choice of the projection gives (6.38). □

S.Operators

6.9. Inverses

Still for the case of a generalized Dirac operator on a compact manifold with dividing hypersurface, consider what we have shown. The operator D defines a map in (6.39) with inverse

3.6.1998.221 (6.46)
$$I_D : \{v \in \mathcal{C}^\infty(H; V_1); \Pi_C v = v\} \longrightarrow \{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0 \text{ in } M_+\}.$$

This operator is the 'Poisson' operator for the canonical boundary condition given by the Calderòn operator, that is $u = I_D v$ is the unique solution of

3.6.1998.222 (6.47)
$$Du = 0 \text{ in } M_+,\ u \in \mathcal{C}^\infty(M_+; V_1),\ \Pi_C b_H u = v.$$

We could discuss the regularity properties of I_D but we shall postpone this until after we have treated the 'one-sided' case of a genuine boundary problem.

As well as I_D we have a natural right inverse for the operator D as a map from $\mathcal{C}^\infty(M_+; V_1)$ to $\mathcal{C}^\infty(M_-; V_2)$. Namely

3.6.1998.223 Lemma 6.6. *If $f \in \mathcal{C}^\infty(M_+; V_2)$ then $u = D^{-1}(f_c)|_{M_+} \in \mathcal{C}^\infty(M_+; V_1)$ and the map $R_D : f \longmapsto u$ is a right inverse for D, i.e. $D \circ R_D = \mathrm{Id}$.*

Proof. Certainly $D(D^{-1}(f_c) = f_c$, so $u = D^{-1}(f_c)|_{M_+} \in \mathcal{C}^{-\infty}(M_+; V_1)$ satifies $Du = f$ in the sense of extendible distributions. Since $f \in \mathcal{C}^\infty(M_+; V_2)$ we can solve the problem $Du \equiv f$ in the sense of Taylor series at H, with the constant term freely prescibable. Using Borel's lemma, let $u' \in \mathcal{C}^\infty(M_+; V_1)$ have the appropriate Taylor series, with $b_H u' = 0$.. Then $D(u'_c) - f_c = g \in \dot{\mathcal{C}}^\infty(M +_; V_2)$. Thus $u'' = D^{-1} g \in \mathcal{C}^\infty(M; V_1)$. Since $D(u' - u'') = f_c$, the uniqueness of solutions implies that $u = (u' - u'')|_{M_+} \in \mathcal{C}^\infty(M_+; V_1)$. □

Of course R_D cannot be a two-sided inverse to D since it has a large null space, described by I_D.

3.6.1998.226 Problem 6.2. Show that, for D as in Theorem 6.1 if $f \in \mathcal{C}^\infty(M_+; V_2)$ and $v \in \mathcal{C}^\infty(H; V_1)$ there exists a unique $u \in \mathcal{C}^\infty(M_+; V_2)$ such that $Du = f$ in $\mathcal{C}^\infty(M_+; V_2)$ and $b_H u = \Pi_C v$.

S.Smoothing.operators

6.10. Smoothing operators

The properties of smoothing operators on a compact manifold with boundary are essentially the same as in the boundaryless case. Rather than simply point to the earlier discussion we briefly repeat it here, but in an abstract setting.

Let $\mathcal{H}$ be a separable Hilbert space. In the present case this would be $L^2(X)$ or $L^2(X; E)$ for some vector bundle over X, or some space $H^m(X; E)$ of Sobolev sections. Let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the ideal of compact operators. Where necessary the norm on $\mathcal{B}$ will be written $\| \ \|_{\mathcal{B}}$; $\mathcal{K}$ is a closed subspace of $\mathcal{B}$ which is the closure of the ideal $\mathcal{F} = \mathcal{F}(\mathcal{H})$ of finite rank bounded operators.

We will consider a subspace $\mathcal{J} = \mathcal{J}(\mathcal{H}) \subset \mathcal{B}$ with a stronger topology. Thus we suppose that $\mathcal{J}$ is a Fréchet algebra. That is, it is a Fréchet space with countably many norms $\| \ \|_k$ such that for each k there exists k' and C_k with

SO1 (6.48)
$$\|AB\|_k \le C_k \|A\|_{k'} \|B\|_{k'} \quad \forall\ A, B \in \mathcal{J}\,.$$

In particular of course we are supposing that $\mathcal{J}$ is a subalgebra (but *not* an ideal) in $\mathcal{B}$. To make it a topological *-subalgebra we suppose that

SO2 (6.49)
$$\|A\|_{\mathcal{B}} \le C \|A\|_k \quad \forall\ A \in \mathcal{J},\ * : \mathcal{J} \longrightarrow \mathcal{J}.$$

In fact we may suppose that $k = 0$ by renumbering the norms. The third condition we impose on $\mathcal{J}$ implies that it is a subalgebra of $\mathcal{K}$, namely we insist that

S03 (6.50) $$\mathcal{F} \cap \mathcal{J} \text{ is dense in } \mathcal{J}\,,$$

in the Fréchet topology. Finally, we demand, in place of the ideal property, that $\mathcal{J}$ be a bi-ideal in $\mathcal{B}$ (also called a 'corner') that is,

(6.51) $$A_1, A_2 \in \mathcal{J}\,,\ B \in \mathcal{B} \Longrightarrow A_1 B A_2 \in \mathcal{J}\,,$$
$$\forall\ k\ \exists\ k' \text{ such that } \|A_1 B A_2\|_k \le C\|A_1\|_{k'}\|B\|_{\mathcal{B}}\|A_2\|_{k'}\,.$$

S05 Proposition 6.8. *The space of operators with smooth kernels acting on sections of a vector bundle over a compact manifold satisfies* (6.48)–(6.52) *with* $\mathcal{H} = H^m(X;E)$ *for any vector bundle* E.

Proof. The smoothing operators on sections of a bundle E can be written as integral operators

S06 (6.52) $$Au(x) = \int_E A(x,y)u(y)\,,\ A(x,y) \in \mathcal{C}^\infty(X^2; \mathrm{Hom}(E) \otimes \Omega_R)\,.$$

Thus $\mathcal{J} = \mathcal{C}^\infty(X^2; \mathrm{Hom}(E) \otimes \Omega_R)$ and we make this identification topological. The norms are the C^k norms. If $P_1, \dots, p_{N(m)}$ is a basis, on $\mathcal{C}^\infty(X^2)$, for the differential operators of order m on $\mathrm{Hom}(E) \otimes \Omega_L$ then we may take

S07 (6.53) $$\|A\|_m = \sup_j \|P_j A\|_{L^\infty}$$

for some inner products on the bundles. In fact $\mathrm{Hom}(E) = \pi_L^* E \otimes \pi_R^* E^*$ from it which follows easily that this is a basis $P_j = P_{j,k} \otimes P_{j,R}$ decomposing as products. From this (6.48) follows easily since

S08 (6.54) $$\|AB\|_m = \sup_j \|(P_{jL}A) \cdot (P_{j,R}B)\|_\infty \|AB\|_{L^\infty} \le C\|A\|_{L^\infty}\|B\|_{L^\infty}$$

by the compactnes of X. From this (6.53) follows with $k = 0$.

The density (6.50) is the density of the finite tensor product $\mathcal{C}^\infty(X;E) \otimes \mathcal{C}^\infty(X; E^* \otimes \Omega_L)$ in $\mathcal{C}^\infty(X^2; \mathrm{Hom}(E) \otimes \Omega_L)$. This follows from the boundaryless case by doubling (or directly). Similarly the bi-ideal condition (6.52) can be seen from the regularity of the kernel. A more satisfying argument using distribution theory follows from the next result.

□

S09 Proposition 6.9. *An operator* $A : \dot{\mathcal{C}}^\infty(X;E) \to \mathcal{C}^{-\infty}(X;F)$ *is a smoothing operator if and only if it extends by continuity to* $\dot{\mathcal{C}}^{-\infty}(X;E)$ *and then has range in* $\mathcal{C}^\infty(X;F) \hookrightarrow \mathcal{C}^{-\infty}(X;F)$.

Proof. If A has the stated mapping property then compose with a Seeley extension operator, then $EA = \tilde{A}$ is a continuous linear map

$$\tilde{A} : \dot{\mathcal{C}}^{-\infty}(X;E) \to \mathcal{C}^\infty(\tilde{X};\tilde{F})\,,$$

for an extension of F to $\tilde{F}$ over the double $\tilde{X}$. Localizing in the domain to trivialize E and testing with a moving delta function we recover the kernel of $\tilde{A}$ as

$$\tilde{A}(x,y) = \tilde{A} \cdot \delta_y \in \mathcal{C}^\infty(\tilde{X};\tilde{F})\,.$$

Thus it follows that $\tilde{A} \in \mathcal{C}^\infty(\tilde{X} \times X; \mathrm{Hom}(E,\tilde{F}) \otimes \Omega_R)$. The converse is more obvious.

□

Returning to the general case of a bi-ideal as in (6.48)–(6.52) we may consider the invertibility of $\mathrm{Id} + A$, $A \in \mathcal{J}$.

S010 Proposition 6.10. *If* $A \in \mathcal{J}$, *satisfying* (6.48)–(6.52), *then* $\mathrm{Id} + A$ *has a generalized inverse of the form* $\mathrm{Id} + B$, $B \in \mathcal{J}$, *with*

$$AB = \mathrm{Id} - \pi_R\,,\ BA = \mathrm{Id} - \pi_L \in \mathcal{J} \cap \mathcal{F}$$

both finite rank self-adjoint projections.

Proof. Suppose first that $A \in \mathcal{J}$ and $\|A\|_{\mathcal{B}} < 1$. Then $\mathrm{Id} + A$ is invertible in $\mathcal{B}$ with inverse $Id + B \in \mathcal{B}$,

S011
$$B = \sum_{j \geq 1} (-1)^j A^j\,. \tag{6.55}$$

Not only does this Neumann series converge in $\mathcal{B}$ but also in $\mathcal{J}$ since for each k

S012
$$\|A^j\|_k \leq C_k \|A\|_{k'} \|A^{j-2}\|_{\mathcal{B}} \|A\|_{k'} \leq C'_k \|A\|_{\mathcal{B}}^{j-2}\,,\ j \geq 2\,. \tag{6.56}$$

Thus $B \in \mathcal{J}$, since by assumption $\mathcal{J}$ is complete (being a Fréchet space). In this case $\mathrm{Id} + B \in \mathcal{B}$ is the unique two-sided inverse.

For general $A \in \mathcal{J}$ we use the assumed approximability in (6.50). Then $A = A' + A''$ when $A' \in \mathcal{F} \cap \mathcal{J}$ and $\|A''\|_{\mathcal{B}} \leq C\|A''\|_k < 1$ by appropriate choice. It follows that $\mathrm{Id} + B'' = (\mathrm{Id} + A'')^{-1}$ is the inverse for $\mathrm{Id} + A''$ and hence a parameterix for $\mathrm{Id} + A$:

$$\begin{aligned}(\mathrm{Id} + B'')(\mathrm{Id} + A) &= \mathrm{Id} + A' + B''A' \\ (\mathrm{Id} + A)(\mathrm{Id} + B'') &= \mathrm{Id} + A' + A'B''\end{aligned} \tag{6.57}$$

Unfinished. Lemma on subprojections.

with both ‘error’ terms in $\mathcal{F} \cap \mathcal{J}$. □

6.11. Left and right parametrices

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a bounded linear operator between them. Let $\mathcal{J}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{J}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be bi-ideals as in the previous section. A left parametrix for A, modulo $\mathcal{J}_1$, is a bounded linear map $B_L : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that

8.6.1998.243
$$B_L \circ A = \mathrm{Id} + J_L,\ J_L \in \mathcal{J}_1. \tag{6.58}$$

Similarly a right parametrix for A, modulo $\mathcal{J}_2$ is a bounded linear map $B_R : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that

8.6.1998.244
$$A \circ B_R = \mathrm{Id} + J_R,\ J_R \in \mathcal{J}_2. \tag{6.59}$$

8.6.1998.245 Proposition 6.11. *If a bounded linear operator* $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ *has a left parametrix* B_L *modulo a bi-ideal* $\mathcal{J}_1$, *satisfying* (6.48)–(6.52), *then* A *has closed range, null space of finite dimension and there is a generalized left inverse, differing from the original left parametrix by a term in* $\mathcal{J}_1$, *such that*

8.6.1998.247
$$B_L \circ A = \mathrm{Id} - \pi_L,\ \pi_L \in \mathcal{J}_1 \cap \mathcal{F}, \tag{6.60}$$

with π_L *the self-adjoint projection onto the null space of* A.

Proof. Applying Proposition 6.10, $\mathrm{Id}+J_L$ has a generalized inverse $\mathrm{Id}+J$, $J \in \mathcal{J}_1$, such that $(\mathrm{Id}+J)(\mathrm{Id}+J_L) = (\mathrm{Id}-\pi'_L)$, $\pi'_L \in \mathcal{J}_1 \cap \mathcal{F}$. Replacing B_L by $\tilde{B}_L = (\mathrm{Id}+J)B_L$ gives a new left parametrix with error term $\pi'_L \in \mathcal{J}_1 \cap \mathcal{F}$. The null space of A is contained in the null space of $B'_L \circ A$ and hence in the range of F_L; thus it is finite dimensional. Furthermore the self-dajoint projection π_L onto the null space is a subprojection of π'_L, so is also an element of $\mathcal{J}_1 \cap \mathcal{F}$. The range of A is closed since it has finite codimension in $\mathrm{Ran}(A(\mathrm{Id}-\pi_L))$ and if $f_n \in \mathrm{Ran}(A(\mathrm{Id}-\pi_L)) = Au_n$, $u_n = (\mathrm{Id}-\pi_L)u_n$, converges to $f \in \mathcal{H}_2$, then $u_n = B_L f_n$ converges to $u \in \mathcal{H}_1$ with $A(\mathrm{Id}-\pi_L)u = f$. □

8.6.1998.246 Proposition 6.12. *If a bounded linear operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ has a right parametrix B_R modulo a bi-ideal $\mathcal{J}_2$, satisfying (6.48)–(6.52), then it has closed range of finite codimension and there is a generalized right inverse, differing from the original right parametrix by a term in $\mathcal{J}_2$, such that*

8.6.1998.248 (6.61)
$$A \circ B_R = \mathrm{Id}-\pi_R,\ \pi_R \in \mathcal{J}_2 \cap \mathcal{F},$$

with $\mathrm{Id}-\pi_R$ the self-adjoint projection onto the range space of A.

Proof. The operator $\mathrm{Id}+J_R$ has, by Proposition 6.10, a generalized inverse $\mathrm{Id}+J$ with $J \in \mathcal{J}_1$. Thus $B'_R = B_R \circ (\mathrm{Id}+J)$ is a right parametrix with error term $\mathrm{Id}-\pi'_R$, $\pi'_R \in \mathcal{J}_1 \cap \mathcal{F}$ being a self-adjoint projection. Thus the range of A contains the range of $\mathrm{Id}-\pi'_R$ and is therefore closed with a finite-dimensional complement. Furthemore the self-adjoint projection onto the range of A is of the form $\mathrm{Id}-\pi_R$ where π_R is a subprojection of π'_R, so also in $\mathcal{J}_1 \cap \mathcal{F}$. □

The two cases, of an operator with a right or a left parametrix are sometimes combined in the term 'semi-Fredholm.' Thus an operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is semi-Fredholm if it has closed range and either the null space or the orthocomplement to the range is finite dimensional. The existence of a right or left parametrix, modulo the ideal of compact operators, is a necessary and sufficient condition for an operator to be semi-Fredholm.

S.Right.inverse

6.12. Right inverse

In treating the 'general' case of an elliptic operator on compact manifold with boundary we shall start by constructing an analogue of the right inverse in Lemma 6.6. So now we assume that $D \in \mathrm{Diff}^1(X; V_1, V_2)$ is an operator of Dirac type on a compact manifold with boundary.

To construct a right inverse for D we follow the procedure in the boundaryless case. That is we use the construction of a pseudodifferential parametrix. In order to make this possible we need to extend M and D 'across the boundary.' This is certainly possible for X, since we may double it to a compact manifold without boundary, $2X$. Then there is not obstruction to extending D 'a little way' across the boundary. We shall denote by M an open extension of X (of the same dimension) so $X \subset M$ is a compact subset and by $\tilde{D}$ an extension of Dirac type to M.

The extension of D to $\tilde{D}$, being elliptic, has a parametrix $\tilde{Q}$. Consider the map

28.4.1998.214 (6.62)
$$\tilde{Q}' : L^2(X; V_2) \longrightarrow H^1(X; V_1),\ \tilde{Q}'f = \tilde{Q}f_c|X$$

where f_c is the extension of f to be zero outside X. Then $\tilde{Q}'$ is a right parametrix, $D\tilde{Q}' = \mathrm{Id}+E$ where E is an operator on $L^2(X; V_2)$ with smooth kernel on X^2.

Following Proposition 6.12, D has a generalized right inverse $\tilde{Q}'' = \tilde{Q}'(\mathrm{Id} + E')$ up to finite rank smoothing and

28.4.1998.215

$$D : H^1(X; V_1) \longleftrightarrow L^2(X; V_2) \tag{6.63}$$

has closed range with a finite dimensional complement in $\mathcal{C}^\infty(X; V_2)$.

28.4.1998.216 Proposition 6.13. *The map (6.63) maps $\mathcal{C}^\infty(X; V_2)$ to $\mathcal{C}^\infty(X; V_1)$, it is surjective if and only if the only solution of $D^*u = 0$, $u \in \dot{\mathcal{C}}^\infty(X; V_2)$ is the trivial solution.*

Proof. The regularity statement, that $Q'\mathcal{C}^\infty(X; V) \subset \mathcal{C}^\infty(X; V_1)$ follows as in the proof of Lemma 6.6. Thus Q' maps $\mathcal{C}^\infty(X; V_1)$ to $\mathcal{C}^\infty(X; V_2)$ if and only if any paramatrix $\tilde{Q}'$ does so. Given $f \in \mathcal{C}^\infty(X; V_2)$ we may solve $Du' \equiv f$ in Taylor series at the boundary, with $u' \in \mathcal{C}^\infty(X; V_1)$ satisfying $b_H u' = 0$. Then $D(u')_c - f \in \dot{\mathcal{C}}^\infty(X; V_2)$ so it follows that $Q'(f_c)|_X \in \mathcal{C}^\infty(X; V_1)$.

Certainly any solution of $D^*u = 0$ with $u \in \dot{\mathcal{C}}^\infty(X; V_2)$ is orthogonal to the range of (6.63) so the condition is necessary. So, suppose that (6.63) is not surjective. Let $f \in L^2(X; V_2)$ be in the orthocomplement to the range. Then Green's formula gives the pairing with any smooth section

$$(Dv, f)_X = (D\tilde{v}, f_c)_{\tilde{X}} = (\tilde{v}, D^* f_c)_{\tilde{X}} = 0.$$

This means that $D^* f_c = 0$ in $\tilde{X}$, that is as a supported distribution. Thus, $f \in \dot{\mathcal{C}}^\infty(X; V_2)$ satisfies $D^* f = 0$. □

As noted above we will proceed under the assumption that $D^* f$ has no such non-trivial solutions in $\dot{\mathcal{C}}^\infty(X; V_2)$. This condition is discussed in the next section.

28.4.1998.218 Theorem 6.2. *If unique continuation holds for D^* then D has a right inverse*

28.4.1998.219

$$Q : \mathcal{C}^\infty(X : V_2) \longrightarrow \mathcal{C}^\infty(X; V_1),\ DQ = \mathrm{Id} \tag{6.64}$$

where $Q = \tilde{Q}' + E$, $\tilde{Q}' f = \tilde{Q} f | X$ where $\tilde{Q}$ is a parametrix for an extension of D across the boundary and E is a smoothing operator on X.

Proof. As just noted, unique continuation for D^* implies that D in (6.63) is surjective. Since the parametrix maps $\mathcal{C}^\infty(X; V_2)$ to $\mathcal{C}^\infty(X; V_1)$, D must be surjective as a map from $\mathcal{C}^\infty(X; V_1)$ to $\mathcal{C}^\infty(X; V_2)$. The parametrix modulo finite rank operators can therefore be corrected to a right inverse for D by the addition of a smoothing operator of finite rank. □

S.Boundary.map

6.13. Boundary map

The map b from $\mathcal{C}^\infty(X; E)$ to $\mathcal{C}^\infty(\partial X; E)$ is well defined, and hence is well defined on the space of smooth solutions of D. We wish to show that it has closed range. To do so we shall extend the defintion to the space of square-integrable solutions. For any $s \in \mathbb{R}$ set

6.6.1998.230

$$\mathcal{N}^s(D) = \left\{ u \in H^s(X; E); Du = 0 \right\}. \tag{6.65}$$

Of course the equation $Du = 0$ is to hold in the sense of extendible distributions, which just means in the interior of X. Thus $\mathcal{N}^\infty(D)$ is the space of solutions of D smooth up to the boundary.

6.6.1998.231 LEMMA 6.7. *If $u \in \mathcal{N}^0(D)$ then*

6.6.1998.232
$$\dot{D}u_c = v \cdot \delta(x), \ v \in H^{-\frac{1}{2}}(\partial X; E) \tag{6.66}$$

defines an injective bounded map $\tilde{b} : \mathcal{N}^0(D) \longrightarrow H^{-\frac{1}{2}}(\partial X; E)$ by $\tilde{b}(u) = i\sigma(D)(dx)v$ which is an extension of $b : \mathcal{N}^\infty(D) \longrightarrow \mathcal{C}^\infty(\partial X; E)$ defined by restriction to the boundary.

PROOF. Certainly $\dot{D}u_c \subset \dot{\mathcal{C}}^\infty_{\partial X}(X; E)$ has support in the boundary, so is a sum of products in any product decomposition of X near ∂X,

$$D(u_c) = \sum_j v_j \cdot \delta^{(j)}(x).$$

Since D is a first order operator and $u_c \in L^2(\tilde{X}; E)$, for any local extension, $\dot{D}u_c \in \dot{H}^{-1}(X; E)$. Localizing so that E is trivial and the localized v_j have compact supports this means that

6.6.1998.233
$$(1 + |\eta|^2 + |\xi|^2)^{-\frac{1}{2}} \widehat{v_j}(\eta)\xi^j \in L^2(\mathbb{R}^n). \tag{6.67}$$

If $v_j \neq 0$ for some $j > 0$ this is not true even in some region $|\eta| < C$. Thus $v_j \equiv 0$ for $j > 0$ and (6.66) must hold. Furthermore integration in ξ gives

6.6.1998.234
$$\begin{gathered}\int_{\mathbb{R}} (1 + |\eta|^2 + |\xi|^2)^{-1} d\xi = c(1 + |\eta|^2)^{-\frac{1}{2}}, \ c > 0, \quad \text{so} \\ \int_{\mathbb{R}^{n-1}} (1 + |\eta|^2)^{-\frac{1}{2}} |\hat{v}(\eta)|^2 d\eta < 0.\end{gathered} \tag{6.68}$$

Thus $v \in H^{-\frac{1}{2}}(\partial X; E)$ and $\tilde{b}$ is well defined. The jumps formula shows it to be an extension of b. The injectivity of $\tilde{b}$ follows from the assumed uniqueness of solutions to $\dot{D}u = 0$ in X. □

Notice that (6.68) is actually reversible. That is if $v \in H^{-\frac{1}{2}}(\partial X; E)$ then $v \cdot \delta(x) \in H^{-1}(X; E)$. This is the basis of the construction of a left parametrix for $\tilde{b}$, which then shows its range to be closed.

6.6.1998.235 LEMMA 6.8. *The boundary map $\tilde{b}$ in Lemma 6.7 has a continuous left parametrix $\widetilde{I_D} : H^{-\frac{1}{2}}(\partial X; E) \longrightarrow \mathcal{N}^0(D)$, $I_D \circ \tilde{b} = \mathrm{Id} + G$, where G has smooth kernel on $X \times \partial X$, and the range of $\tilde{b}$ is therefore a closed subspace of $H^{-\frac{1}{2}}(\partial X; E)$.*

PROOF. The parametrix $\widetilde{I_D}$ is given directly by the parametrix $\tilde{Q}$ for $\tilde{D}$, and extension to $\tilde{X}$. Applying $\tilde{Q}$ to (6.66) gives

6.6.1998.239
$$u = \widetilde{I_D}v + Ru, \ \widetilde{I_D} = R_X \circ \tilde{Q} \circ \frac{1}{i}\sigma(D)(dx) \tag{6.69}$$

with R having smooth kernel. Since $\widetilde{I_D}$ is bounded from $H^{-\frac{1}{2}}(\partial X; E)$ to $L^2(X; E)$ and R is smoothing it follows from Proposition 6.11 that the range of $\tilde{b}$ is closed. □

S.Calderon.projector

6.14. Calderòn projector

Having shown that the range of $\tilde{b}$ in Lemma 6.7 is closed in $H^{-\frac{1}{2}}(\partial X; E)$ we now deduce that there is a pseudodifferential projection onto it. The discussion above of the boundary values of the $\tilde{Q}(w \cdot \delta(x))$ is local, and so applies just as well in the present more general case. Since this is just the definition of the map $\widetilde{I_D}$ in Lemma 6.8, we conclude directly that

6.6.1998.240 (6.70) $$Pv = \lim_{X^\circ} \widetilde{I_D} v,\ v \in \mathcal{C}^\infty(\partial X; E)$$

defines $P \in \Psi^0(\partial X; E)$.

6.6.1998.241 Lemma 6.9. *If P is defined by (6.70) then $P^2 - P \in \Psi^{-\infty}(\partial X; E)$ and there exist A, $B \in \Psi^{-\infty}(\partial X; E)$ such that $P - \mathrm{Id} = A$ on $\mathrm{Ran}(\tilde{b})$ and $\mathrm{Ran}(P + B) \subset \mathrm{Ran}(\tilde{b})$.*

Proof needs clarification.

Proof. That $P^2 - P \in \Psi^{-\infty}(\partial X; E)$ follows, as above, from the fact that $\tilde{Q}$ is a two-sided parametrix on distributions supported in X. Similarly we may use the right inverse of D to construct B. If $v \in H^{-\frac{1}{2}}(\partial X; E)$ then by construction,

$$D\widetilde{I_D} v = R'v$$

where R' has a smooth kernel on $X \times \partial X$. Applying the right inverse Q it follows that $u' = \widetilde{I_D} v - (Q \circ R')v \in \mathcal{N}^0(D)$, where $Q \circ R'$ also has smooth kernel on $X \times \partial X$. Thus $\tilde{b}(u') = (P + B)v \in \mathrm{Ran}(\tilde{b})$ where B has kernel arising from the restriction of the kernel of $A \circ R'$ to $\partial X \times \partial X$, so $B \in \Psi^{-\infty}(\partial X; E)$. □

Now we may apply Proposition 5.6 with $F = \mathrm{Ran}(\tilde{b})$ and $s = -\frac{1}{2}$ to show the existence of a Calderòn projector.

6.6.1998.242 Proposition 6.14. *If D is a generalized Dirac operator on X then there is an element $\Pi_C \in \Psi^0(\partial X; E)$ such that $\Pi_C^2 = \Pi_C$, $\mathrm{Ran}(\Pi_C) = \mathrm{Ran}(\tilde{b})$ on $H^{-\frac{1}{2}}(\partial X; E)$, $\Pi_C - P \in \Psi^{-\infty}(\partial X; E)$ where P is defined by (6.70) and $\mathrm{Ran}(\Pi_C) = \mathrm{Ran}(b)$ on $\mathcal{C}^\infty(\partial X; E)$.*

Proof. The existence of psuedodifferential projection, Π_C, differing from P by a smoothing operator and with range $\mathrm{Ran}(\tilde{b})$ is a direct consequence of the application of Proposition 5.6. It follows that $\mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ is dense in $\mathrm{Ran}(\tilde{b})$ in the topology of $H^{-\frac{1}{2}}(\partial X; E)$. Furthermore, if follows that if $v \in \mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ then $u \in \mathcal{N}^0(D)$ such that $\tilde{b}u = v$ is actually in $\mathcal{C}^\infty(X; E)$, i.e. it is in $\mathcal{N}^\infty(D)$. Thus the range of b is just $\mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ so $\mathrm{Ran}(b)$ is the range of Π_C acting on $\mathcal{C}^\infty(\partial X; E)$. □

In particular $\tilde{b}$ is just the continuous extension of b from $\mathcal{N}^\infty(D)$ to $\mathcal{N}^0(D)$, of which it is a dense subset. Thus we no longer distinguish between these two maps and set $\tilde{b} = b$.

S.Poisson.operator

6.15. Poisson operator

S.Unique.continuation

6.16. Unique continuation

S.Boundary.regularity

6.17. Boundary regularity

udodifferential.boundary

6.18. Pseudodifferential boundary conditions

The discussion above shows that for any operator of Dirac type the 'Calderòn realization' of D,

28.4.1998.1 (6.71) $$D_{\mathcal{C}} : \{u \in H^s(X; E_1); \Pi_{\mathcal{C}} bu = 0\} \longrightarrow H^{s-1}(X; E_2),\ s > \frac{1}{2}$$

is an isomorphism.

We may replace the Calderòn projector in (28.4.1998.1 6.71) by a more general projection Π, acting on $\mathcal{C}^\infty(\partial X, V_1)$, and consider the map

28.4.1998.2 (6.72)
$$D_\Pi : \{u \in \mathcal{C}^\infty(X; V_1); \Pi bu = 0\} \longrightarrow \mathcal{C}^\infty(X; V_2).$$

In general this map will not be particularly well-behaved. We will be interested in the case that $\Pi \in \Psi^0(\partial X; V_1)$ is a pseudodifferential projection. Then a condition for the map D_Π to be Fredholm can be given purely in terms of the relationship between Π and the (any) Calderòn projector $\Pi_{\mathcal{C}}$.

29.3.1998.188 THEOREM 6.3. *If $D \in \mathrm{Diff}^1(X; E_1, E_2)$ is of Dirac type and $Pi \in \Psi^0(\partial X; E_1)$ is a projection then the map*

29.3.1998.189 (6.73)
$$D_\Pi : \{u \in \mathcal{C}^\infty(X; E_1); \Pi(u_{\partial X}) = 0\} \xrightarrow{D} \mathcal{C}^\infty(X; E_2)$$

is Fredholm if and only if

29.3.1998.190 (6.74)
$$\Pi \circ \Pi_C : \mathrm{Ran}(\Pi_C) \cap \mathcal{C}^\infty(\partial V_1) \longrightarrow \mathrm{Ran}(\Pi) \cap \mathcal{C}^\infty(\partial E_1) \text{ is Fredholm}$$

and then the index of D_Π is equal to the relative index of $\Pi_{\mathcal{C}}$ and Π, that is the index of (29.3.1998.190 6.74).

Below we give a symbolic condition equivalent which implies the Fredholm condition. If enough regularity conditions are imposed on the generalized inverse to (28.4.1998.1 6.71) then this symbolic is also necessary.

PROOF. The null space of D_Π is easily analysed. Indeed $Du = 0$ implies that $u \in \mathcal{N}$, so the null space is isomorphic to its image under the boundary map:

$$\{u \in \mathcal{N}; \Pi bu = 0\} \xrightarrow{b} \{v \in \mathcal{C}; \Pi v = 0\}.$$

Since $\mathcal{C}$ is the range of $\Pi_{\mathcal{C}}$ this gives the isomorphism

28.4.1998.6 (6.75)
$$\mathrm{Nul}(D_\Pi) \simeq \mathrm{Nul}\left(\Pi \circ \Pi_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Ran}(\Pi)\right).$$

In particular, the null space is finite dimensional if and only if the null space of $\Pi \circ \Pi_{\mathcal{C}}$ is finite dimensional.

Similarly, consider the range of D_Π. We construct a map

28.4.1998.212 (6.76)
$$\tau : \mathcal{C}^\infty(\partial X; V_1) \longrightarrow \mathcal{C}^\infty(X; V_2)/\mathrm{Ran}(D_\Pi).$$

Indeed each $v \in \mathcal{C}^\infty(\partial X; V_1)$ is the boundary value of some $u \in \mathcal{C}^\infty(X : V_1)$, let $\tau(v)$ be he class of DU. This is well-defined since any other extension u' is such that $b(u - u') = 0$, so $D(u - u') \in \mathrm{Ran}(D_\Pi)$. Furthermore, τ is surjective, since $D_{\mathcal{C}}$ is surjective. Consider the null space of τ. This certainly contains the null space of Π. Thus consider the quotient map

$$\tilde{\tau} : \mathrm{Ran}(\Pi) \longrightarrow \mathcal{C}^\infty(X : V_2)/\mathrm{Ran}(D_\Pi),$$

which is still surjective. Then $\tilde{\tau}(v) = 0$ if and only if there exists $v' \in \mathcal{C}$ such that $\Pi(v - v') = 0$. That is, $\tilde{\tau}(v) = 0$ if and only if $\Pi(v) = \Pi \circ \Pi_{\mathcal{C}}$. This shows that the finer quotient map

28.4.1998.213 (6.77)
$$\tau' : \mathrm{Ran}(\Pi)/\mathrm{Ran}(\Pi \circ \Pi_{\mathcal{C}}) \longleftrightarrow \mathcal{C}^\infty(X; V_2)/\mathrm{Ran}(D_\Pi)$$

is an isomorphism. This shows that the range is closed and of finite codimension if $\Pi \circ \Pi_{\mathcal{C}}$ is Fredholm.

The converse follows by reversing these arguments. □

6.19. Gluing

Returning to the case of a compact manifold without boundary, M, with a dividing hypersurface H we can now give a gluing result for the index.

29.3.1998.191 THEOREM 6.4. *If $D \in \operatorname{Diff}^1(M; E_1, E_2)$ is of Dirac type and $M = M_1 \cap M_2$ is the union of two manifolds with boundary intersecting in their common boundary $\partial M_1 \cap \partial M_2 = H$ then*

29.3.1998.192
$$\operatorname{Ind}(D) = \operatorname{Ind}(\Pi_{1,\mathcal{C}}, \operatorname{Id} - \Pi_{2,\mathcal{C}}) = \operatorname{Ind}(\Pi_{2,\mathcal{C}}, \operatorname{Id} - \Pi_{1,\mathcal{C}}) \tag{6.78}$$

where $\Pi_{i,\mathcal{C}}$, $i = 1, 2$, are the Calderòn projections for D acting over M_i.

S.Local.boundary

6.20. Local boundary conditions

S.Absolute.relative

6.21. Absolute and relative Hodge cohomology

S.Transmission.condition

6.22. Transmission condition

C.Scattering calculus

CHAPTER 7

Scattering calculus

CHAPTER 8

C.Wave.kernel

The wave kernel

Let us return to the subject of "good distributions" as exemplified by Dirac delta 'functions' and the Schwartz kernels of pseudodifferential operators. In fact we shall associate a space of "conormal distributions" with any submanifold of a manifold.

Thus let X be a $\mathcal{C}^\infty$ manifold and $Y \subset X$ a closed embedded submanifold – we can easily drop the assumption that Y is closed and even replace embedded by immersed, but let's treat the simplest case first! To say that Y is embedded means that each $\bar{y} \in Y$ has a coordinate neighbourhood U, in X, with coordinate $x_1, \dots, x_n$ in terms of which $\bar{y} = 0$ and

16.1 (8.1) $$Y \cap U = \{x, = \cdots = x_k = 0\}.$$

We want to define

16.2 (8.2) $$I^*(X, Y; \Omega^{\frac{1}{2}}) \subset \mathcal{C}^{-\infty}(X; \Omega^{\frac{1}{2}})$$

to consist of distributions which are singular only at Y and small "along Y."

So if $u \in \mathcal{C}_c^{-\infty}(U)$ then in local coordinates (8.1) we can identify u with $u' \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ so $u' \in H_c^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. To say that u is 'smooth along Y' means we want to have

16.3 (8.3) $$D_{x_{k+1}}^{l_1} \dots D_{x_n}^{l_{n-k}} u' \in H_c^{s'}(\mathbb{R}^n) \quad \forall\ l_1, \dots, l_{n-k}$$

and a fixed s', independent of l (but just possibly different from the initial s); of course we can take $s = s'$. Now conditions like (8.3) do *not* limit the singular support of u' at all! However we can add a requirement that multiplication by a function which *vanishes* on Y makes u' smooth, by one degree, i.e.

16.4 (8.4) $$x_1^{p_1} \dots x_k^{p_k} u' \in H^{s+|p|}(\mathbb{R}^n), |p| = p_1 + \dots + p_k.$$

This last condition implies

16.5 (8.5) $$D_1^{q_1} \dots D_k^{q_k} x_1^{p_1} \dots x_k^{p_k} u' \in H^s(\mathbb{R}^n) \text{ if } |q| \le |p|.$$

Consider what happens if we rearrange the order of differentiation and multiplication in (8.5). Since we demand (8.5) for *all* p, q with $|q| \le |p|$ we can show in tial that

16.105 (8.6) $$\forall\ |q| \le |p| \le L$$

(8.7) $$\Longrightarrow$$

16.6 (8.8) $$\prod_{i=1}^{L} (x_{j_i} D_{\ell_i}) u \in H^s(\mathbb{R}^n) \quad \forall \text{ pairs, } (j_{i,\ell_i}) \in (1, \dots, k)^2.$$

Of course we can combine (8.3) and (8.8) and demand

16.7 (8.9)
$$\prod_{i=1}^{L_2} D_{p_i} \prod_{i=1}^{L_1} (x_{j_i} D_{\ell_i}) u' \in H^s_c(\mathbb{R}^n) (j_j, \ell_i) \in (1, \dots, k)^2$$
$$\forall\ L_1, L_2\ p_i \in (k+1, \dots u).$$

PROBLEM 8.1. Show that (8.9) implies (8.3) and (8.4)

The point about (8.9) is that it is easy to interpret in a coordinate independent way. Notice that putting $\mathcal{C}^\infty$ coefficients in front of all the terms makes no difference.

16.8 LEMMA 8.1. *The space of all $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^n$ tangent to the submanifold $\{x_1 = \cdots = x_k = 0\}$ is spanning over $\mathcal{C}^\infty(\mathbb{R}^n)$ by*

16.9 (8.10)
$$x_i D_j, D_p\ i, j \leq k, p > k.$$

PROOF. A $\mathcal{C}^\infty$ vector field is just a sum

(8.11)
$$V = \sum_{j \leq k} a_j D_j + \sum_{p > k} b_p D_p.$$

Notice that the D_p, for $p > k$, *are* tangent to $\{x_1 = \cdots = x_k = 0\}$, so we can assume $b_p = 0$. Tangency is then given by the condition

(8.12)
$$V x) i = 0 \text{ and } \{x_1 = \cdots = x_k = 0\}, i = 1, \dots, h$$

i.e. $a_j = \sum\limits_{\ell=1} a_{j\ell} x_\ell, 1 \leq j \leq h$. Thus

(8.13)
$$V = \sum_{\ell=1} a_{j\ell} x_\ell D_j$$

which proves (8.10). □

This allows us to write (8.9) in the compact form

16.10 (8.14)
$$\mathcal{V}(\mathbb{R}^n, Y_k)^p u' \subset H^s_c(\mathbb{R}^n)\ \forall\ p$$

where $\mathcal{V}(\mathbb{R}^n, Y_k)$ is just the space of all $\mathcal{C}^\infty$ vector fields tangent to $Y_k = \{x_1 = \cdots = x_k = 0\}$. Of course the local coordinate just reduce vector fields tangent to Y to vector fields tangent to Y_k so the *invariant* version of (8.14) is

16.11 (8.15)
$$\mathcal{V}(X, Y)^p u \subset H^s(X; \Omega^{\frac{1}{2}})\ \forall\ p.$$

To interpret (8.15) we only need recall the (Lie) action of vector fields on half-densities. First for densities: The *formal* transpose of V is $-V$, so set

(8.16)
$$^L V \phi(\psi) = \phi(-V\psi)$$

if $\phi \in \mathcal{C}^\infty(X; \Omega), \psi \in \mathcal{C}^\infty(X)$. On $\mathbb{R}^n$ then becomes

16.12 (8.17)
$$\begin{aligned} \int {}^L V \phi \cdot \psi &= - \int \phi \cdot V \psi \\ &= - \int \phi(x) V \psi \cdot dx \\ &= \int (V \phi(x) + \delta_V \phi)\, \psi\ dx \end{aligned}$$
$$\delta_V = \sum_{i=1}^n D_i a_i \quad \text{ if } V = \Sigma a_i D_i.$$

i.e.

16.13 (8.18) $$L_V\left(\phi|dx|\right) = (V\phi)|dx| + \delta_V \phi.$$

Given the tensorial properties of density, set

16.14 (8.19) $$L_V(\phi|dx|^t) = V\phi|dx|^t + t\delta_V\phi.$$

This corresponds to the *natural* trivialization in local coordinates.

16.15 Definition 8.1. *If $Y \subset X$ is a closed embedded submanifold then*

16.16 (8.20) $$\begin{aligned} IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right) &= \left\{u \in H^s\left(X; \Omega^{\frac{1}{2}}\right) \text{ satisfying (11)}\right\} \\ I^*\left(X, Y; \Omega^{\frac{1}{2}}\right) &= \bigcup_s IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right). \end{aligned}$$

Clearly

16.17 (8.21) $$u \in I^*(X, Y; \Omega^{\frac{1}{2}}) \Longrightarrow u \restriction X\backslash Y \in \mathcal{C}^\infty\left(X\backslash Y; \Omega^{\frac{1}{2}}\right)$$

and

16.18 (8.22) $$\bigcap_s IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right) = \mathcal{C}^\infty\left(X; \Omega^{\frac{1}{2}}\right).$$

Let us try to understand these distributions *in some detail!* To do so we start with a very simple case, namely $Y = \{p\}$ is a point; so we only have one coordinate system. So construct $p = 0 \in \mathbb{R}^n$.

16.19 (8.23) $$\begin{gathered} u \in I^*_c\left(\mathbb{R}^n, \{0\}; \Omega^{\frac{1}{2}}\right) \Longrightarrow u = u'|dx|^{\frac{1}{2}} \text{ when} \\ x^\alpha D^\beta_x u' \in H^s_c(\mathbb{R}^n), \quad s \text{ fixed } \ \forall\ |\alpha| \geq |\beta|. \end{gathered}$$

Again by a simple commutative argument this is equivalent to

16.20 (8.24) $$D^\beta_x x^\alpha u' \in H^s_c(\mathbb{R}^n) \ \ \forall\ |\alpha| \geq |\beta|.$$

We can take the Fourier transform of (8.24) and get

16.21 (8.25) $$\xi^\beta D^\alpha_\xi \hat{u}' \in \langle\xi\rangle^{-s} L^2(\mathbb{R}^n)\ \forall\ |\alpha| \geq |\beta|.$$

In this form we can just replace ξ^β by $\langle\xi\rangle^{|\beta|}$, i.e. (8.25) just says

16.22 (8.26) $$D^\alpha_\xi \hat{u}'(\xi) \in \langle\xi\rangle^{-s-|\beta|} L^2(\mathbb{R}^n)\ \forall\ \alpha.$$

Notice that this is *very* similar to a symbol estimate, which would say

16.23 (8.27) $$D^\alpha_\xi \hat{u}'(\xi) \in \langle\xi\rangle^{m-|\alpha|} L^\infty(\mathbb{R}^n)\ \forall\ \alpha.$$

16.24 Lemma 8.2. *The estimate (8.26) implies (8.27) for any $m > -s - \frac{n}{2}$; conversely (8.27) implies (8.26) for any $s < -m - \frac{n}{2}$.*

Proof. Let's start with the simple derivative, (8.27) implies (8.26). This really reduces to the case $\alpha = 0$. Thus

(8.28) $$\langle\xi\rangle^M L^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \Longrightarrow M < -\frac{n}{2}$$

is the inequality

(8.29) $$\left(\int |u|^2 d\xi\right)^{\frac{1}{2}} \leq \sup\langle\xi\rangle^{-M}|u| \left(\int \langle\xi\rangle^{2M} d\xi\right)^{\frac{1}{2}}$$

and

$$\int \langle\xi\rangle^{2M} d\xi = \int \left(1+|\xi|^2\right)^M d\xi < \infty \text{ iff } M < -\frac{n}{2}. \tag{8.30}$$

To get (8.27) we just show that (8.27) implies

16.25
$$\langle\xi\rangle^{s+|\alpha|} D_\xi^\alpha \hat{u}' \in \langle\xi\rangle^{m+s} L^\infty \subset L^2 \text{ if } m+s < -\frac{n}{2}. \tag{8.31}$$

The converse is a little trickier. To really see what is going on we can reduce (8.26) to a one dimensional version. Of course, near $\xi = 0$, (8.26) just says $\hat{u}'$ is $\mathcal{C}^\infty$, so we can assume that $|\xi| > 1$ on $\operatorname{supp} \hat{u}'$ and introduce polar coordinates:

16.26
$$\xi = tw,\ w \in S^{n-1} t > 1. \tag{8.32}$$

Then

Exercise 2. Show that (8.26) (or maybe better, (8.25)) implies that

16.27
$$D_t^k P \hat{u}'(tw) \in t^{-s-k} L^2(\mathbb{R}^+ \times S^{n-1}; t^{n-1} dt dw)\ \forall\ k \tag{8.33}$$

for *any* $\mathcal{C}^\infty$ differential operator on S^{n-1}. □

In particular we can take P to be elliptic of any order, so (8.33) actually implies

16.28
$$\sup_w D_t^k P \hat{u}(t,w) \in t^{-s-k} L^2(\mathbb{R}^+; t^{n-1} dt) \tag{8.34}$$

or, changing the meaning to dt,

16.29
$$\sup_{w \in S^{n-1}} \left| D_t^k P \hat{u}(t,w) \right| \in t^{-s-k-\frac{n-1}{2}} L^2\left(\mathbb{R}^+, dt\right). \tag{8.35}$$

So we are in the one dimensional case, with s replaced by $s + \frac{n-1}{2}$. Now we can rewrite (8.35) as

16.30
$$D_t t^q D_t^k P \hat{u} \in t^r L^2,\ \forall\ k, r-q = -s-k-\frac{n-1}{2} - 1. \tag{8.36}$$

Now, observe the simple case:

16.31
$$f = 0 t < 1, D_t f \in t^r L^2 \Longrightarrow f \in L^\infty \text{ if } r < -\frac{1}{2} \tag{8.37}$$

since

$$\sup |f| = \int_{-\infty}^{t} t^r g \leq \left(\int |g|^2\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{t} t^{2r}\right)^{\frac{1}{2}}. \tag{8.38}$$

Thus from (8.36) we deduce $\leq (\int |g|^2)^{\frac{1}{2}}$

$$D_t^k P \hat{u} \in t^{-q} L^\infty \text{ if } r < -\frac{1}{2},\ \text{i.e. } -q > -s-k-\frac{n}{2}. \tag{8.39}$$

Finally this gives (8.27) when we go back from polar coordinates, to prove the lemma.

16.32 Definition 8.2. *Set, for* $m \in \mathbb{R}$,

$$I_c^m(\mathbb{R}^n, |[0\}) = \{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); \hat{u} \in S^{m-\frac{n}{4}}(\mathbb{R}^n)\} \tag{8.40}$$

with this definition,

16.33
$$IH^s(\mathbb{R}^n, \{0\}) \subset I_c^m(\mathbb{R}^n, \{0\}) \subset I_c^{s'}(\mathbb{R}^n, \{0\}) \tag{8.41}$$

provided

$$s > -m - \frac{n}{4} > s'. \tag{8.42}$$

Exercise 3. Using Lemma 24, prove (8.41) carefully.

So now what we want to do is to *define* $I_c^m(X, \{p\}; \Omega^{\frac{1}{2}})$ for any $p \in X$ by

16.34
$$\begin{aligned} u \in I_c^m(X, \{p\}; \Omega^{\frac{1}{2}}) &\Longleftrightarrow F^*(\phi u) \in I_c^m(\mathbb{R}^n, \{0\}), \\ u \restriction X\backslash\{p\} &\in \mathcal{C}^\infty(X\backslash\{p\}). \end{aligned} \tag{8.43}$$

Here we have a little problem, namely we have to check that $I^m(\mathbb{R}^n, \{0\})$ is invariant under coordinate changes. Fortunately we can do this using (8.41).

17.7 Lemma 8.3. *If $F : \Omega \longrightarrow \mathbb{R}^n$ is a diffeomorphism of a neighbourhood of* 0 *onto its range, with $F(0) = 0$, then*

17.8
$$F^*\{u \in I_c^m(\mathbb{R}^n, \{0\}; \operatorname{supp}(u) \subset F(\Omega)\} \subset I_c^m(\mathbb{R}^n, \{0\}). \tag{8.44}$$

Proof. Start with a simple case, that F is *linear*. Then

$$u = (2\pi)^{-n} \int e^{ix\xi} a(\xi) d\xi, a \in S^{m-\frac{n}{4}}(\mathbb{R}^n). \tag{8.45}$$

so

17.9
$$\begin{aligned} F^*u &= (2\pi)^{-n} \int e^{iAx\cdot\xi} a(\xi) d\xi \ \ Fx = Ax \\ &= (2\pi)^{-n} \int i^{ix\cdot A^t\xi} a(\xi) d\xi \\ &= (2\pi)^{-n} \int e^{ix\cdot\eta} a((A^t)^{-1}\eta) |\det A|^{-1} d\eta. \end{aligned} \tag{8.46}$$

Since $a((A^t)^{-1}\eta)|\det A|^{-1} \in S^{m-\frac{n}{4}}\mathbb{R}^n)$ we have proved the result for linear transformations. We can always factorize F is

17.10
$$F = G \cdot A, \qquad A = (F_*) \tag{8.47}$$

so that the differential of G at 0 is the identity, i.e.

17.11
$$G(x) = x + O(|x|^2). \tag{8.48}$$

Now (8.48) allows us to use an homotopy method, i.e. set

17.12
$$G_s(x) = x + s(G(x) - x) \qquad s \in [0, 1) \tag{8.49}$$

so that $G_0 = \mathrm{Id}$, $G_s = G$. Such a 1-parameter family is given by integration of a vector field:

17.13 (8.50)
$$\begin{aligned}
G_s^*\phi &= \int_0^s \frac{d}{ds} G_s^*\phi dx \\
&= \int_0 s\frac{d}{ds}\phi(G_x(x))ds \\
&= \sum_1 \int_0^s \frac{d}{G}{}^{\xi}_{s,i} ds\,(\partial x_j \phi)\,(G_\delta(x))\,ds \\
&= \int_0^s G_s^*\,(V_s\phi)\,ds
\end{aligned}$$

when the coefficients of V_s are

17.14 (8.51)
$$G_s^* V_{s,j} = \frac{d}{ds} G_{s,i}.$$

Now by (8.49) $\frac{d}{ds}G_{s,i} = \Sigma x_i x_j a_{ij}^s(x)$, so the same is true of the $V_{s,i}$, again using (8.49).

We can apply (8.50) to compute

17.15 (8.52)
$$G^* u = \int_0^{\prime} G_s^*\,(V_s u)\,ds$$

when $u \in I_c^m(\mathbb{R}^n, \{0\})$ has support near 0. Namely, by (8.41), $u \in IH_c^s(\mathbb{R}^n, \{0\})$, with $s < -m - \frac{n}{4}$, but then

(8.53)
$$V_s u \in IH_c^{s+1}(\mathbb{R}^n, \{0\})$$

since $V = \sum_{i,j=1}^n b_{ij}^s(x) x_i x_j D_j$. Applying (8.41) again gives

17.16 (8.54)
$$G_s^*(V_s u) \in I^{m'}(\mathbb{R}^n, \{0\}),\ \forall\ m' > m - 1.$$

This proves the coordinates invariance. □

Last time we defined the space of conormal distributions associated to a closed embedded submanifold $Y \subset X$:

17.1 (8.55)
$$\begin{aligned}
IH^s(X,Y) &= \{u \in H^s(X); \mathcal{V}(X,Y)^k u \subset H^s(X)\ \forall\ k\} \\
IH^*(X,Y) &= I^*(X,Y) = \bigcup sIH^s(X,Y).
\end{aligned}$$

Here $\mathcal{V}(X,Y)$ is the space of $\mathcal{C}^\infty$ vector fields on X tangent to Y. In the special case of a point in $\mathbb{R}^n$, say 0, we showed that

17.2 (8.56)
$$u \in I_c^*(\mathbb{R}^n), \{0\}) \Longleftrightarrow u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \text{ and } \hat{u} \in S^M(\mathbb{R}^n), M = M(u).$$

In fact we then defined the "standard order filtration" by

17.3 (8.57)
$$u \in I_c^m(\mathbb{R}^n, \{0\}) = \left\{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); \hat{u} \in S^{m-\frac{n}{4}}(\mathbb{R}^n)\right\},$$

and found that

17.4 (8.58)
$$IH_c^s(\mathbb{R}^n, \{0\}) \subset I_c^{-s-\frac{n}{4}}(\mathbb{R}^n, \{0\}) \subset IH_c^{s'}(\mathbb{R}^n, \{0\})\ \forall\ s' < s.$$

Our next important task is to show that $I_c^m(\mathbb{R}^n,\{0\})$ is invariant under coordinate changes. That is, if $F : U_1 \longrightarrow \mathbb{R}^n$ is a diffeomorphism of a neighbourhood of 0 to its range, with $F(0)=0$, then we want to show that

17.5 (8.59) $$F^*u \in I_c^m(\mathbb{R}^n,\{0\})\ \forall\ u \in I_c^m(\mathbb{R}^n,\{0\}), \operatorname{supp}(u)\subset F(U_1).$$

Notice that we already know the coordinate independence of the Sobolev-based space, so using (8.58), we deduce that

17.6 (8.60) $$F^*u \in I_c^{m'}(\mathbb{R}^n,\{0\})\ \forall\ u \in I_c^m(\mathbb{R}^n,\{0\}), n' > m, \operatorname{supp}(u)\subset F(U_1).$$

In fact we get quite a lot more for our efforts:

17.17 LEMMA 8.4. *There is a coordinate-independent symbol map:*

17.18 (8.61) $$I^m(X,\{p\};\Omega^{\frac12})@>\sigma_Y^m>>S^{m+\frac n4-[J]}\left(T_p^*\mathbb{R}^n;\Omega^{\frac12}\right)$$

given by the local prescription

17.19 (8.62) $$\sigma_Y^m(u) = \hat u(\xi)|d\xi|^{\frac12}$$

where $u = v|dx|^{\frac12}$ *is local coordinate based at* 0, *with* ξ *the dual coordinate in* T_p^*X.

PROOF. Our definition of $I^m(X,\{p\};\Omega^{\frac12})$ is just that in any local coordinate based at p

17.20 (8.63) $$u\in I^m(X,\{p\};\Omega^{\frac12}) \Longrightarrow \phi u = v|dx|^{\frac12}, v\in I_c^m(\mathbb{R}^n,\{0\})$$

and $u \in \mathcal{C}^\infty(X\backslash\{p\};\Omega^{\frac12})$. So the symbol map is clearly supposed to be

17.21 (8.64) $$\sigma^m(u)^{(\zeta)} \equiv_\downarrow \hat v(\xi)|d\xi|^{\frac12} \in S^{m+\frac n4-[1]}(\mathbb{R}^n;\Omega^{\frac12})$$

where $\zeta \in T_p^*X$ is the 1-form $\zeta = \xi\cdot dx$ in the local coordinates. Of course we have to show that (8.64) is independent of the choice of coordinates. We already know that a change of coordinates changes $\hat v$ by a term of order $m-\frac n4-1$, which disappears in (8.64) so the residue class is determined by the Jacobian of the change of variables. From (8.46) we see exactly how $\hat v$ transforms under the Jacobian, namely as a density on

$$T_0^*\mathbb{R}^n : A\in GL(n,\mathbb{R}) \Longrightarrow \widehat{A^*v}(\eta)|d\eta|^{\frac12}$$
$$= \hat v((A^t)^{-1}\eta)|\det A|^{-1}|dy|$$

so $\eta = A^t\xi \Longrightarrow$

17.22 (8.65) $$\widehat{A^*v}(\eta)|dy| = \hat v(\xi)|d\xi|.$$

However recall from (8.63) that u is a half-density, so actually in the new coordinates $v' = A^*v\cdot|\det A|^{\frac12}$. This shows that (8.64) is well-defined.

Before going on to consider the general case let us note a few properties of $I^m(X,\{p\},\Omega^{\frac12})$: □

Exercise: Prove that

If $P\in \operatorname{Diff}^m(X;\Omega^{\frac12})$ then

17.23 (8.66) $$P : I^m(X,\{p\};\Omega^{\frac12}) \longrightarrow I^{m+M}(X,\{p\};\Omega^{\frac12})\ \forall\ m$$
$$\sigma^{m+M}(Pu) = \sigma^M(P)\cdot\sigma^m(u).$$

To pass to the general case of $Y\subset X$ we shall proceed in two steps. First let's consider a rather 'linear' case of $X = V$ a vector bundle over Y. Then Y can be identified with the zero section of V. In fact V is locally trivial, i.e. each $p \in y$ has a neighbourhood U s.t.

17.24 (8.67) $$\pi^{-1}(U) \simeq \mathbb{R}^n_x \times U'_{y'} U' \subset \mathbb{R}^p$$

by a fibre-linear diffeomorphism projecting to a coordinate system on this base. So we want to define

(8.68) $$I^m(V, Y; \Omega^{\frac{1}{2}}) = \{u \in I^*(V, Y; \Omega^{\frac{1}{2}});$$

of $\phi \in \mathcal{C}^\infty_c(U)$ then under *any* trivialization (8.67)

17.25 (8.69) $$\phi u(x, y) \equiv (2\pi)^{-n} \int e^{ix\cdot\xi} a(y, \xi) d\xi |dx|^{\frac{1}{2}}, \quad \mod \mathcal{C}^\infty, \\ a \in S^{m - \frac{n}{2} - \frac{p}{4}}(\mathbb{R}^p_y, \mathbb{R}^n_\xi).$$

Here $p = \dim Y, p + n = \dim V$. Of course we have to check that (8.69) is coordinate-independent. We can write the order of the symbol, corresponding to u having order m as

17.26 (8.70) $$m - \frac{\dim V}{4} + \frac{\dim Y}{2} = m + \frac{\dim V}{4} - \frac{\operatorname{codim} Y}{2}.$$

These additional shifts in the order are only put there to confuse you! Well, actually they make life easier later.

Notice that we know that the space is invariant under *any* diffeomorphism of the fibres of V, varying smoothly with the base point, it is also obvious that (8.69) in independent the choice of coordinates is U', since that just transforms these variables. So a general change of variables preserving Y is

(8.71) $$(y, x) \longmapsto (f(y, x), X(y, x)) \quad X(y, 0) = 0.$$

In particular f is a local diffeomorphism, which just changes the base variables in (8.69), so we can assume $f(y) \equiv y$. Then $X(y, x) = A(y) \cdot x + O(x^2)$. Since $x \longmapsto A(y) \cdot x$ is a fibre-by-fibre transformation it leaves the space invariant too, So we are reduced to considering

17.27 (8.72) $$G : (y, x) \longmapsto (y, x + \Sigma a_{ij}(x, y) x_i x_j) y + \Sigma b_i(x, y) x_i.$$

To handle these transformations we can use the same homotopy method as before i.e.

17.28 (8.73) $$G_s(x, y_= (y + s) \sum_i b_i(x, y) x_i, x + s \sum_{i,j} a_{ij}(x, y) x_i x_j)$$

is a 1-parameter family of diffeomorphisms. Moreover

(8.74) $$\frac{d}{ds} G^*_s u = G^*_s V_s k$$

where

(8.75) $$V_s = \sum_{i,\ell} \beta_{i,\ell}(s, x, y) x_i \partial_{y_\ell} + \sum_{i,j,k} \alpha_{i,j,k} + \sum_{i,j,k} \alpha_{ijk}(\alpha, y, s) \ell_i, \ell_j \frac{\partial}{\partial x_k}.$$

So all we really have to show is that

17.29 (8.76) $$V_s : I^M(U' \times \mathbb{R}^n, U' \times \{0\}) \longrightarrow I^{M-1}(U' \times \mathbb{R}^n, U' \times \{0\}) \ \forall\ M.$$

Again the spaces are $\mathcal{C}^\infty$-modules so we only have to check the action of $x_i \partial_{y_\ell}$ and $x_i x + j \partial_{x_k}$. These change the symbol to

(8.77) $$D_{\xi_i} \partial_{y_\ell} a \text{ and } i D_{\xi_i} D_{\xi_j} \cdot \xi_k a$$

respectively, all one order lower.

This shows that the definition (8.69) is actually a reasonable one, i.e. as usual it suffices to check it for any covering by coordinate partition.

Let us go back and see what the symbol showed before.

17.30 Lemma 8.5. *If*

$$u \in I^m(V, Y; \Omega^{\frac{1}{2}}) u = v|dx|^{\frac{1}{2}}|d\xi|^{\frac{1}{2}} \tag{8.78}$$

defines an element

$$\sigma^m(u) \in S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(V^*; \Omega^{\frac{1}{2}}) \tag{8.79}$$

independent of choices.

Last time we discussed the invariant symbol for a conormal distribution associated to the zero section of a vector bundle. It turns out that the general case is not any more complicated thanks to the "tubular neighbourhood" or "normal fibration" theorem. This compares $Y \hookrightarrow X$, a closed embedded submanifold, to the zero section of a vector bundle.

Thus at each point $y \in Y$ consider the normal space:

18.1
$$N_y Y = N_y\{X, Y\} = T_y x / T_y Y. \tag{8.80}$$

That is, a normal vector is just *any* tangent vector to X modulo tangent vectors to Y. These spaces define a vector bundle over Y :

18.2
$$NY = N\{X; Y\} = \bigsqcup_{y \in Y} N_y Y \tag{8.81}$$

where smoothness of a section is inherited from smoothness of a section of $T_y X$, i.e.

18.3
$$NY = T_y X / T_y Y. \tag{8.82}$$

Suppose $Y_i \subset X_i$ are $\mathcal{C}^\infty$ submanifolds for $i = 1, 2$ and that $F : X_1 \longrightarrow X_2$ is a $\mathcal{C}^\infty$ map such that

18.4
$$F(Y_1) \subset Y_2. \tag{8.83}$$

Then $F_* : T_y X_1 \longrightarrow T_{F(y)} X_2$, must have the property

18.5
$$F_* : T_y Y_1 \longrightarrow T_{F(y)} Y_2 \ \forall\ y \in Y_1. \tag{8.84}$$

This means that F_* defines a map of the normal bundles

18.6
$$\begin{array}{ccc} F_* : NY_1 & \longrightarrow & NY_2 \\ \downarrow & & \downarrow \\ Y_1 & \xrightarrow[F]{} & Y_2. \end{array} \tag{8.85}$$

Notice the very special case that $W \longrightarrow Y$ is a vector bundle, and we consider $Y \hookrightarrow W$ as the zero section. Then

18.7
$$N_y\{W; Y\} \longleftrightarrow W_y \quad \forall\ y \in Y \tag{8.86}$$

since

18.8
$$T_y W = T_y Y \oplus T_y(W_y) \quad \forall\ y \in W. \tag{8.87}$$

That is, the normal bundle to the zero section is *naturally* identified with the vector bundle itself.

So, suppose we consider $\mathcal{C}^\infty$ maps

18.9 (8.88) $$f : B \longrightarrow N\{X;Y\} = NY$$

where $B \subset X$ is an open neighbourhood of the submanifold Y. We can demand that

18.10 (8.89) $$f(y) = (y, 0) \in N_y Y \quad \forall\ y \in Y$$

which is to say that f induces the natural identification of Y with the zero section of NY and moreover we can demand

18.11 (8.90) $$f_* : NY \longrightarrow NY \text{ is the identity.}$$

Here f_* is the map (8.85), so maps NY to the normal bundle to the zero section of NY, which we have just observed is naturally just NY again.

18.12 THEOREM 8.1. *For any closed embedded submanifold $Y \subset X$ there exists a normal fibration, i.e. a diffeomorphism (onto its range) (8.88) satisfing (8.89) and (8.90); two such maps f_1, f_2 are such that $g = f_2 \circ f_1^{-1}$ is a diffeomorphism near the zero section of NY, inducing the identity on Y and inducing the identity (8.90).*

PROOF. Not bad, but since it uses a little Riemannian geometry I will *not* prove it, see [ ], [ ]. (For those who know a little Riemannian geometry, f^{-1} can be taken as the exponential map near the zero section of NY, identified as a subbundle of $T_Y X$ using the metric.) Of course the uniqueness part is obvious. □

Actually we do *not* really need the global aspects of this theorem. Locally it is immediate by using local coordinates in which $Y = \{x_1 = \cdots = x_k = 0\}$.

Anyway using such a normal fibration of X near Y (or working locally) we can simply *define*

18.13 (8.91) $$\begin{gathered} I^m(X, Y; \Omega^{\frac12}) = \{u \in \mathcal{C}^{-\infty}(X; \Omega^{\frac12}); u \text{ is } \mathcal{C}^\infty \text{ in } X\backslash Y \text{ and} \\ (f^{-1})^*(\phi u) \in I^m(NY, Y; \Omega^{\frac12}) \text{ if } \phi \in \mathcal{C}^\infty(X), \operatorname{supp}(\phi) \subset B\}. \end{gathered}$$

Naturally we should check that the definition doesn't depend on the choice of f. This means knowing that $I^m(NY, Y; \Omega^{\frac12})$ is invariant under g, as in the theorem, but we have already checked this. In fact notice that g is exactly of the type of (8.72). Thus we actually know that

(8.92) $$\sigma^m(g^* u) = \sigma^m(u) \text{ in } S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y; \Omega^{\frac12}).$$

So we have shown that there is a coordinate invariance symbol map

18.14 (8.93) $$\sigma^m : I^m(X, Y; \Omega^{\frac12}) \longrightarrow S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y; \Omega^{\frac12})$$

giving a short exact sequence

18.15 (8.94)
$$0 \hookrightarrow I^{m-1}(X, Y; \Omega^{\frac12}) \longrightarrow I^m(X, Y; \Omega^{\frac12}) @>\sigma^m>> S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y; \Omega^{\frac12}) \longrightarrow 0$$

(8.95) $$\text{where } n = \dim X - \dim Y, p = \dim Y.$$

Asymptotic completeness carries over immediately. We also need to go back and check the extension of (8.66):

18.16 Proposition 8.1. *If $Y \hookrightarrow X$ is a closed embedded submanifold and $A \in \Psi^m_c(X;\Omega^{\frac12})$ then*

18.17 (8.96) $$A : I^M(X,Y;\Omega^{\frac12}) \longrightarrow I^{M+m}(X,Y;\Omega^{\frac12}) \ \forall\ M$$

and

18.18 (8.97) $$\sigma^{m+M}(Au) = \sigma^m(A)\sigma^m(A) \restriction N^*Y\sigma^M(u).$$

Notice that $\sigma^m(A) \in S^{m-[1]}(T^*X)$ so the product here makes perfectly good sense.

Proof. Since everything in sight is coordinate-independent we can simply work in local coordinates where

18.19 (8.98) $$X \sim \mathbb{R}^p_y \times \mathbb{R}^n_x, Y = \{x = 0\}.$$

Then $u \in I^m_c(X,Y;\Omega^{\frac12})$ means just

18.20 (8.99) $$u = (2\pi)^{-n}\int e^{ix\cdot\xi}a(y,\xi)d\xi\cdot|dx|^{\frac12}, a \in S^{m-\frac{n}{4}+\frac{p}{4}}(\mathbb{R}^p,\mathbb{R}^n).$$

Similarly A can be written in the form

18.21 (8.100) $$A = (2\pi)^{-n-p}\int e^{i(x-x')\cdot\xi+i(y-y')\cdot\eta}b(x,y,\xi,\eta)d\xi d\eta.$$

Using the invariance properties of the Sobolev based space if we write

(8.101) $$A = A_0 + \Sigma x_j B_j, A_0 = q_L(b(0,y,\xi,\eta))$$

we see that $Au \in I^{m+M}(X,Y;\Omega^{\frac12})$ is equivalent to $A_0u \in I^{m+M}(X,Y;\Omega^{\frac12})$. Then

(8.102) $$A_0u = (2\pi)^{-n-p}\int e^{ix\cdot\xi+i(y-y')\cdot\eta}b(0,y',\xi,\eta)b(y',\xi)dy'd\eta d\xi,$$

where we have put A_0 in right-reduced form. This means

(8.103) $$A_0u = (2\pi)^{-n}\int e^{ix\cdot\xi}c(y,\xi)d\xi$$

where

(8.104) $$c(y,\xi) = (2\pi)^{-p}\int e^{i(y-y')\cdot\eta}b(0,y',\xi,\eta)a(y',\xi)dy'd\eta.$$

Regarding ξ as a parameter, this is, before y' integration, the kernel of a pseudo-differential operator is y. It can therefore be written in left-reduced form, i.e.

18.22 (8.105) $$c(y,\xi) = (2\pi)^{-p}\int e^{i(y-y')\eta}e(y,\xi,\eta)d\eta dy' = e(y,\xi,0)$$

where $e(y,\xi,\eta) = b(0,y,\xi,\eta)a(y,\xi)$ plus terms of order at most $m+M-\frac{n}{4}+\frac{p}{4}-1$. This proves the formula (18.18 8.97). □

Notice that if A is elliptic then $Au \in \mathcal{C}^\infty$ implies $u \in \mathcal{C}^\infty$, i.e. there are *no* singular solutions. Suppose that P is say a *differential* operator which is not elliptic and we look for solutions of

18.23 (8.106) $$Pu \in \mathcal{C}^\infty(X\Omega^{\frac12}).$$

How can we find them? Well suppose we try

18.24 (8.107) $$u \in I^M(X,Y;\Omega^{\frac12})$$

for some submanifold Y. To know that u is singular we will want to have

18.25 (8.108) $$\sigma(u) \text{ is elliptic on } N^*Y$$

(which certainly implies that $u \notin \mathcal{C}^\infty$).

The simplest case would be Y a hypersurface. In any case from (8.97) and (8.106) we deduce

18.26 (8.109) $$\sigma^m(P)\cdot\sigma^M(u) \equiv 0.$$

So if we assume (8.108) then we *must* have

18.27 (8.110) $$\sigma^m(P) \restriction N^*Y = 0.$$

18.28 Definition 8.3. *A submanifold is said to be* characteristic *for a given operator* $P \in \mathrm{Diff}^m(X;\Omega^{\frac12})$ *if* (8.110) *holds.*

Of course even if P *is* characteristic for y, and so (8.109) holds we do *not* recover (8.106), just

18.29 (8.111) $$Pu \in I^{m+M-1}(X,Y;\Omega^{\frac12})$$

i.e., one order smoother than it "should be". The task might seem hopeless, but let us note that these are examples, and important ones at that!!

Consider the (flat) wave operator

18.30 (8.112) $$P = P_t^2 - \sum_{i=1}^n D_i^2 = D_t^2 - \Delta \text{ on } \mathbb{R}^{n+1}.$$

A hypersurface in $\mathbb{R}^{n+1}$ looks like

18.31 (8.113) $$H = \{h(t,x) = 0\}, (dh \neq 0 \text{ on } H).$$

The symbol of P is

18.32 (8.114) $$\sigma^2(P) = \tau^2 - |\xi|^2 = \tau^2 - \xi_1^2 - \cdots - \xi_n^2,$$

where τ, ξ are the dual variables to t, x. So consider (8.110),

18.33 (8.115) $$N^*Y = \{(t,x;\lambda dh(t,y)); h(t,x) = 0\}.$$

Inserting this into (8.114) we find:

18.34 (8.116) $$\left(\lambda\frac{\partial h}{\partial t}\right)^2 - \left(\lambda\frac{\partial h}{\partial x_1}\right)^2 - \cdots - \left(\lambda\frac{\partial h}{\partial x_n}\right)^2 = 0 \text{ on } h = 0$$

i.e. simply:

18.35 (8.117) $$\left(\frac{\partial h}{\partial t}\right)^2 = |d_x h|^2 \text{ on } h(t,x) = 0.$$

This is the "eikonal equation" for h (and hence H).

Solutions to (8.117) are easy to find – we shall actually find all of them (locally) next time. Examples are given by taking h to be linear:

18.36 (8.118) $$H = \{h = at + b\cdot x = 0\} \text{ is characteristic for } P \Longleftrightarrow a^2 = |b|^2.$$

Since h/a defines the same surface, all the linear solutions correspond to planes

18.37 (8.119) $$t = \omega\cdot x, \omega \in \mathbb{S}^{n-1}.$$

So, do solutions of $Pu \in \mathcal{C}^\infty$ which are conormal with respect to such hypersurfaces exist? Simply take

18.38 (8.120) $$u = v(t - \omega \cdot x) \quad v \in I^*(\mathbb{R}, \{0\}; \Omega^{\frac{1}{2}}).$$

Then

18.39 (8.121) $$Pu = 0, u \in I^*(\mathbb{R}^{n+1}, H; \Omega^{\frac{1}{2}}).$$

For example $v(s) = \delta(s), u = \delta(t - \omega \cdot x)$ is a "travelling wave".

8.1. Hamilton-Jacobi theory

Let X be a $\mathcal{C}^\infty$ manifold and suppose $p \in \mathcal{C}^\infty(T^*X\backslash 0)$ is homogeneous of degree m. We want to find characteristic hypersurfaces for p, namely hypersurfaces (locally) through $\bar{x} \in X$

19.1 (8.122) $$H = \{f(x) = 0\} \quad h \in \mathcal{C}^\infty(x) h(\bar{x}) = 0, dh(\bar{x}) \neq 0$$

such that

19.2 (8.123) $$p(x, dh(x)) = 0.$$

Here we demand that (8.123) hold near $\bar{x}$, not just on H itself. To solve (8.123) we need to impose some additional conditions, we shall demand

19.3 (8.124) p is real-valued

and

19.4 (8.125) $$d_{\text{fibre}} p \neq 0 \text{ or } \Sigma(p) = \{p = 0\} \subset T^*X\backslash 0.$$

This second condition is actually stronger than really needed (as we shall see) but in any case it implies that

19.5 (8.126) $\Sigma(P) \subset T^*X\backslash 0$ is a $\mathcal{C}^\infty$ conic hypersurface

by the implicit function theorem.

The strategy for solving (8.123) is a geometric one. Notice that

19.6 (8.127) $$\Lambda_h = \{(x, dh(x)) \in T^*X\backslash 0\}$$

actually determines h up to an additive constant. The first question we ask is – precisely which submanifold $\Lambda \subset T^*X\backslash 0$ corresponds to graphs of differentials of $\mathcal{C}^\infty$ functions? The answer to this involves the tautologous *contact* form.

19.7 (8.128) $$\begin{aligned} \alpha :\ T^*X &\longrightarrow T^*(T^*X) \not\subset \tilde{\pi} \circ \alpha = \mathrm{Id} \\ \alpha(x, \xi) &= \tilde{\pi}^*\xi \in T^*_{(x,\xi)}(T^*X). \end{aligned}$$

Here $\tilde{\pi} : T^*(T^*X) \longrightarrow T^*X$ is the projection. Notice that if $x_1, \dots, x_n$ are local coordinates in X then $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ are local coordinates T^*X, where $\xi \in T^*_xX$ is written

19.8 (8.129) $$\xi = \sum_{i=1}^n \xi_i dx_i.$$

Since $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ are local coordinates in T^*X they together with the dual coordinates $\Xi_1, \dots, \Xi_n$, $X_1, \dots, X_n$ are local coordinates in $T^*(T^*X)$ where

19.9 (8.130) $$\zeta \in T^*_{(x,\xi)}(T^*X) \Longrightarrow \zeta = \sum_{j=1}^n \Xi_j dx_j + \sum_{j=1}^n X_j d\xi_j.$$

In these local coordinates

19.10 (8.131)
$$\alpha = \sum_{j=1}^{n} \xi_j dx_j!$$

The first point is that α is independent of the original choice of coordinates, as is evident from ([19.7] 8.128).

19.11 LEMMA 8.6. *A submanifold $\Lambda \subset T^*X\backslash 0$ is, near $(\bar{x}, \bar{\xi}) \in \Lambda$, of the form* ([19.6] 8.127) *for some $h \in \mathcal{C}^\infty(X)$, if*

19.12 (8.132)
$$\pi : \Lambda \longrightarrow X \text{ is a local diffeomorphism}$$

and

19.13 (8.133)
$$\alpha \text{ restricted to } \Lambda \text{ is exact.}$$

PROOF. The first condition, ([19.12] 8.132), means that Λ is locally the image of a section of T^*X :

(8.134)
$$\Lambda = \{(x, \zeta(x)), \zeta \in \mathcal{C}^\infty(X; T^*X)\}.$$

Thus the section ζ gives an inverse Z to π in ([19.12] 8.132). It follows from ([19.7] 8.128) that

(8.135)
$$Z^*\alpha = \zeta.$$

Thus if α is exact on Λ then ζ is exact on X, $\zeta = dh$ as required. □

Of course if we are only working locally near some point $(\bar{x}, \bar{\xi}) \in \Lambda$ then ([19.13] 8.133) can be replaced by the condition

19.14 (8.136)
$$\omega = d\alpha = 0 \text{ on } X.$$

Here $\omega = d\alpha$ is the symplectic form on T^*X :

19.15 (8.137)
$$\omega = \sum_{j=1}^{n} d\xi_j \wedge dx_j.$$

19.16 DEFINITION 8.4. *A submanifold $\Lambda \subset T^*X$ of dimension equal to that of X is said to be* Lagrangian *if the fundamental 2-form, ω, vanishes when pulled back to Λ.*

By definition a symplectic manifold is a $\mathcal{C}^\infty$ manifold S with a $\mathcal{C}^\infty$ 2-form $\omega \in \mathcal{C}^\infty(S; \Lambda^2)$ fixed satisfying two constraints

19.17 (8.138)
$$d\omega = 0$$

19.18 (8.139)
$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}} \neq 0 \qquad \dim S = 2n.$$

A particularly simple example of a symplectic manifold is a real vector space, necessarily of even dimension, with a non-degenerate antisymmetric 2-form:

19.19 (8.140)
$$\begin{cases} \omega : E \times E \longrightarrow \mathbb{R} \\ \tilde{\omega} : E \longleftrightarrow E^*. \end{cases}$$

Here $\tilde{\omega}(v)(w) = \omega(v, w)\ \forall\ w \in E$. Now ([19.17] 8.138) is trivially true if we think of ω as a translation-invariant 2-form on E, thought of as a manifold.

Then a subspace $V \subset E$ is Lagrangian if

19.20 (8.141)
$$\begin{aligned} &\omega(v, w) = 0\ \forall\ v, w \in V \\ &2 \dim V = \dim E. \end{aligned}$$

Of course the point of looking at symplectic vector spaces and Lagrangian subspaces is:

19.21 Lemma 8.7. *If S is a symplectic manifold then T_zS is a symplectic vector space for each $z \in S$. A submanifold $\Lambda \subset S$ is Lagrangian iff $T_z\Lambda \subset T_zS$ is a Lagrangian subspace $\forall\ z \in \Lambda$.*

We can treat ω, the antisymmetric 2-form on E, as though it were a Euclidean inner product, at least in some regards! Thus if $W \subset E$ is any subspace set

19.22 (8.142) $$W^\omega = \{v \in E; \omega(v, w) = 0\ \forall\ w \in W\}.$$

19.23 Lemma 8.8. *If $W \subset E$ is a linear subspace of a symplectic vector space then* $\dim W^\omega + \dim W = \dim E$; *$W$ is Lagrangian if and only if*

19.24 (8.143) $$W^\omega = W.$$

Proof. Let $W^0 \subset E^*$ be the usual annihilator:

(8.144) $$W^0 = \{\alpha \in E^*; \alpha(v) = 0\ \forall\ v \in W\}.$$

Then $\dim W^0 = \dim E - \dim W$. Observe that

19.25 (8.145) $$\tilde{\omega} : W^\omega \longleftrightarrow W^0.$$

Indeed if $\alpha \in W^0$ and $\tilde{\omega}(v) = \alpha$ then

(8.146) $$\alpha(w) = \tilde{\omega}(v)(w) = \omega(v, w) = 0\ \forall\ w \in W$$

implies that $v \in W^\omega$. Conversely if $v \in W^\omega$ then $\alpha = \tilde{\omega}(v) \in W^0$. Thus $\dim W^\omega + \dim W = \dim E$.

Now if W is Lagrangian then $\alpha = \tilde{\omega}(w), w \in W$ implies

(8.147) $$\alpha(v) = \tilde{\omega}(w)(v) = \omega(w, v) = 0\ \forall\ v \in w.$$

Thus $\tilde{\omega}(W) \subset W^0 \Longrightarrow W \subset W^\omega$, by (8.145), and since $\dim W = \dim W^\omega$, (8.143) holds. The converse follows similarly. □

The "lifting" isomorphism $\tilde{\omega} : E \longleftrightarrow E^*$ for a symplectic vector space is like the Euclidean identification of vectors and covectors, but "twisted". It is of fundamental importance, so we give it several names! Suppose that S is a symplectic manifold. Then

19.26 (8.148) $$\tilde{\omega}_z : T_zS \longleftrightarrow T^*_zS\ \forall\ z \in S.$$

This means that we can associate (by the inverse of (8.148)) a vector field with each 1-form. We write this relation as

19.27 (8.149) $$H_\gamma \in \mathcal{C}^\infty(S; TS) \text{ if } \gamma \in \mathcal{C}^\infty(S; T^*S) \text{ and } \tilde{\omega}_z(H_\gamma) = \gamma\ \forall\ z \in S.$$

Of particular importance is the case $\gamma = df, f \in \mathcal{C}^\infty(S)$. Then H_{df} is written H_f and called the Hamilton vector field of f. From (8.149)

19.28 (8.150) $$\omega(H_f, v) = df(v) = vf\ \forall\ v \in T_zS,\ \forall\ z \in S.$$

The identity (8.150) implies one important thing immediately:

19.29 (8.151) $$H_f f \equiv 0\ \forall\ f \in \mathcal{C}^\infty(S)$$

since

(8.152) $$H_f f = df(H_f) = \omega(H_f, H_f) = 0$$

by the antisymmetry of ω. We need a generalization of this:

19.30 Lemma 8.9. *Suppose $L \subset S$ is a Lagrangian submanifold of a symplectic manifold then for each $f \in \mathcal{I}(S) = \{f \in \mathcal{C}^\infty(X); f \restriction \{s = 0\}$, H_f is tangent to Λ.*

Proof. H_f tangent to Λ means $H_f(z) \in T_z\Lambda \ \forall \ z \in \Lambda$. If $f = 0$ on Λ then $df = 0$ on $T_z\Lambda$, i.e. $df(z) \in (T_z\Lambda)^0 \subset (T_zS) \ \forall \ z \in \Lambda$. By (8.143) the assumption that Λ is Lagrangian means $\tilde{\omega}_z(df(z)) \in T_z\Lambda$, i.e. $H_f(z) \in T_\zeta\Lambda$ as desired. □

This lemma gives us a necessary condition for our construction of a Lagrangian submanifold

19.31 (8.153) $$\Lambda \subset \Sigma(P).$$

Namely H_p must be tangent to Λ! We use this to construct Λ as a union of integral curves of H_p. Before thinking about this seriously, let's look for a moment at the conditions we imposed on p, (8.124) and (8.125). If p is real then H_p is real (since ω is real). Notice that

19.32 (8.154) If $S = T^*X$ then each fibre $T^*_xX \subset T^*X$ is Lagrangian .

Remember that on $T^*X, \omega = d\alpha, \alpha = \xi \cdot dx$ the canonical 1-form. Thus T^*_xX is just $x =$ const, so $dx = 0$, so $\alpha = 0$ on T^*_xX and hence in particular $\omega = 0$, proving (8.154). This allows us to interpret (8.125) in terms of H_p as

19.33 (8.155) (8.125) $\longleftrightarrow$ H_p is everywhere transversal to the fibres T^*_xX.

Now we want to construct a little piece of Lagrangian manifold satisfying (8.153). Suppose $z \in \Sigma(P) \subset T^*X\backslash 0$ and we want to construct a piece of Λ through z. Since $\pi_*(H_p(z)) \neq 0$ we can choose a local coordinate, $t \in \mathcal{C}^\infty(X)$, such that

19.34 (8.156) $$\pi_*(H_p(z))t \neq 0, \text{ i.e. } H_p(\pi^*t)(z) \neq 0.$$

Consider the hypersurface through $\pi(z) \in X$,

19.35 (8.157) $$H = \{t = t(z)\} \Longrightarrow \pi(z) \in H.$$

Suppose $f \in \mathcal{C}^\infty(H), df(\pi(z)) = 0$. In fact we can choose f so that

19.36 (8.158) $$f = f' \restriction H, f' \in \mathcal{C}^\infty(X), \ df'(\pi(z)) = z$$

where $z \in \Xi(P)$ was our chosen base point.

19.37 Theorem 8.2. *(Hamilton-Jacobi) Suppose $p \in \mathcal{C}^\infty(T^*X\backslash 0)$ satisfies (8.124) and (8.125) near $z \in T^*X\backslash 0$, H is a hypersurface through $\pi(z)$ as in (8.156), (8.153) and $f \in \mathcal{C}^\infty(H)$ satisfies (8.158), then there exists $\tilde{f} \in \mathcal{C}^\infty(X)$ such that*

19.38 (8.159) $$\begin{gathered}\Lambda = \text{ graph } (d\tilde{f}) \subset \Sigma(P) \text{ near } z\\ \tilde{f} \restriction H = f \text{ near } \pi(z)\\ d\tilde{f}(\pi(z)) = z\end{gathered}$$

and any other such solution, $\tilde{f}'$, is equal to $\tilde{f}$ in a neighbourhood of $\pi(z)$.

Proof. We need to do a bit more work to prove this important theorem, but let us start with the strategy. First notice that $\Lambda \cap \pi^{-1}(H)$ is already determined, near $\pi(z)$.

To see this we have to understand the relationship between $df(h) \in T^*H$ and $d\tilde{f}(h) \in T^*X$, $h \in H, \tilde{f} \restriction H = f$. Observe that $H = \{t = 0\}$ lifts to $T^*_HX \subset T^*X$ a *hypersurface*. By (8.151), H_t is tangent to $T^*_H X$ and non-zero. In local coordinates $t, x, \dots, x_{n-1}$, the x's in H,

$$H_t = -\frac{\partial}{\partial \tau} \tag{8.160}$$

where $\tau, \xi_1, \dots, \xi_n$ are the dual coordinates. Thus we see that

19.39 $$\pi_H : T^*_H X \longrightarrow T^* H \qquad \pi_H(\beta)(v) = \beta(v), v \in T_h H \subset T_h X, \tag{8.161}$$

is projection along ∂_τ. Now starting from $f \in \mathcal{C}^\infty(H)$ we have

$$\Lambda_f \subset T^* H. \tag{8.162}$$

Notice that if $\tilde{f} \in \mathcal{C}^\infty(X)$, $\tilde{f}\big|H = f$ then

19.40 $$\Lambda_{\tilde{f}} \cap T^*_H X \text{ has dimension } n-1 \tag{8.163}$$

and

19.41 $$\pi_H(\Lambda_{\tilde{f}} \cap T^*_H X) = \Lambda_f. \tag{8.164}$$

The first follows from the fact that $\Lambda_{\tilde{f}}$ is a graph over X and the second from the definition, (8.161). So we find $\square$

19.42 Lemma 8.10. *If $z \in \Sigma(P)$ and H is a hypersurface through $\pi(z)$ satisfying (8.156) and (8.157) then $\pi^P_H : (\Sigma(P) \cap T^*_H X) \longrightarrow T^* H$ is a local diffeomorphism in a neighbourhood z; if (8.158) is to hold then*

19.43 $$\Lambda_{\tilde{f}} \cap T^*_H X = (\pi^P_H)^{-1}(\Lambda_f) \text{ near } z. \tag{8.165}$$

Proof. We know that H_p is tangent to $\Sigma(P)$ but, by assumption (8.158) is *not* tangent to $T^*_H X$ at z. Then $\Sigma(P) \cap T^*_H X$ does have dimension $2n-1-1 = 2(n-1)$. Moreover π_H is projection along ∂_τ which cannot be tangent to $\Sigma(P) \cap T^*_H X$ (since it would be tangent to $\Sigma(P)$). Thus π^P_H has injective differential, hence is a local isomorphism.

So this is our strategy:

Start with $f \in \mathcal{C}^\infty(H)$, look at $\Lambda_f \subset T^* H$, lift to $\Lambda \cap T^*_H X \subset \Sigma(P)$ by π^P_H. Now let

$$\Lambda = \bigcup \{H_p - \text{curves through } (\pi^P_H)^{-1}(\Lambda_f)\}. \tag{8.166}$$

This we will show to be Lagrangian and of the form $\Lambda_{\tilde{f}}$, it follows that

19.44 $$p(x, d\tilde{f}) = 0, \tilde{f} \restriction H = f. \tag{8.167}$$

$\square$

8.2. Riemann metrics and quantization

Metrics, geodesic flow, Riemannian normal form, Riemann-Weyl quantization.

8.3. Transport equation

The first thing we need to do is to finish the construction of characteristic hypersurfaces using Hamilton-Jacobi theory, i.e. prove Theorem XIX.37. We have already defined the submanifold Λ as follows:

1) We choose $z \in \Sigma(P)$ and $t \in \mathcal{C}^\infty(X)$ s.t. $H_p\pi^*(t) \neq 0$ at dz, then selected $f \in \mathcal{C}^\infty(H), H = \{t=0\} \cap \Omega, \Omega \ni \pi z$ s.t.

20.1 (8.168) $$z(v) = df(v) \ \forall \ v \in T_{\pi z}H.$$

Then we consider

20.2 (8.169) $$\Lambda_f = \text{ graph}\{df\} = \{(x, df(x)), x \in H\} \subset T^*H$$

as our "initial data" for Λ. To move it into $\Sigma(P)$ we noted that the map

20.3 (8.170) $$\Sigma(P) \cap \underset{\{t=0 \text{ in } T^*X\}}{\underset{\|}{T^*_H X}} \longrightarrow T^*H$$

is a local diffeomorphism near z, $df(\pi(z))$ by (8.168). The inverse image of Λ_f in (8.170) is therefore a submanifold $\tilde{\Lambda}_f \subset \Sigma(p) \cap T^*_H X$ of dimension $\dim X - 1 = \dim H$. We define

20.4 (8.171) $$\Lambda = \bigcup \big\{ H_p - \text{ curves of length } \epsilon \text{ starting on } \tilde{\Lambda}_f \big\}.$$

So we already know:

20.5 (8.172) $$\Lambda \subset \Sigma(P) \text{ is a manifold of dimension } n,$$

and

20.6 (8.173) $$\pi : \Lambda \longrightarrow X \text{ is a local diffeomorphism near } n,$$

What we need to know most of all is that

20.7 (8.174) $$\Lambda \text{ is Lagrangian.}$$

That is, we need to show that the symplectic two form vanishes identically on $T_{z'}\Lambda$, $\forall \ z' \in \Lambda$ (at least near z). First we check this at z itself! Now

20.8 (8.175) $$T_z\Lambda = T_z\tilde{\Lambda}_f + \text{ sp}(H_p).$$

Suppose $v \in T_z\tilde{\Lambda}_f$, then

20.9 (8.176) $$\omega(v, H_p) = -dp(v) = 0 \text{ since } \tilde{\Lambda}_f \subset \Sigma(P).$$

Of course $\omega(H_p, H_p) = 0$ so it is enough to consider

20.10 (8.177) $$\omega | (T_z\tilde{\Lambda}_f \times T_z\tilde{\Lambda}_f).$$

Recall from our discussion of the projection (8.170) that we can write it as projection along ∂_τ. Thus

20.11 (8.178) $$\begin{gathered} \omega_X(v, w) = \omega_H(v', w') \text{ if } v, w \in T_z(T_H X), \\ (c^*_H)_* v = v' (c^*_H)_* w = w' \in T_z(T^*H) \end{gathered}$$

where $z = df(\pi(z))$. Thus the form (8.177) vanishes identically because Λ_f is Lagrangian.

In fact the same argument applies at every point of the initial surface $\tilde{\Lambda}_f \subset \Lambda$:

20.12 (8.179) $$T_{z'}\Lambda \text{ is Lagrangian } \ \forall \ z' \in \tilde{\Lambda}_f.$$

To extend this result out into Λ we need to use a little more differential geometry. Consider the local diffeomorphisms obtained by exponentiating H_p :

20.13 (8.180) $$\exp(\epsilon H_p)(\Lambda \cap \Omega) \subset \Lambda \ \forall \ \epsilon \text{ small}, \ \Omega \ni z \text{ small}.$$

This indeed is really the definition of Λ_j more precisely,

20.14 (8.181) $$\Lambda = \bigcup_{\epsilon \text{ small}} \exp(\epsilon H_p)(\tilde{\Lambda}_f).$$

The main thing to observe is that, on T^*H, the local diffeomorphisms $\exp(\epsilon H_p)$ are *symplectic*:

20.15 (8.182) $$\exp(\epsilon H_p)^* \omega_X = \omega_X.$$

Clearly (8.182), (8.180) and (8.179) prove (8.174). The most elegant wary to prove (8.182) is to use Cartan's identity (valid for H_p any vector field, ω any form)

20.16 (8.183) $$\frac{d}{d\epsilon} \exp(\epsilon H_p)^* \omega = \exp(\epsilon H_p)^* (\mathcal{L}_{H_p} \omega)$$

where the Lie derivative is given explicitly by

20.17 (8.184) $$\mathcal{L}_V = d \circ \iota_V + \iota_V \circ d,$$

c_V being contradiction with V (i.e. $\alpha(\cdot, \cdot, \dots) \longrightarrow \alpha(V, \cdot, \cdot, \dots)$). Thus

20.18 (8.185) $$\mathcal{L}_{H_p} \omega = d(\omega(H_p, \cdot)) + \underset{\substack{\| \\ 0}}{\iota_V (d\omega)} = d(dp) = 0.$$

Thus from (8.172), (8.173) and (8.174) we know that

20.19 (8.186) $$\Lambda = \ \text{graph}(d\tilde{f}), \tilde{f} \in \mathcal{C}^\infty(X), \ \text{near } \pi(z),$$

must satisfy the eikonal equation

20.20 (8.187) $$p(x, d\tilde{f}(x)) = 0 \text{ near } \pi(z), H\tilde{f} \restriction H = f$$

where we may actually have to add a constant to $\tilde{f}$ to get the initial condition – since we only have $d\tilde{f} = df$ on TH.

So now we can return to the construction of travelling waves: We want to find

20.21 (8.188) $$u \in I^*(X, G; \Omega^{\frac{1}{2}}) \qquad G = \{f = 0\}$$

such that u is elliptic at $z \in \Sigma(p)$ and

20.22 (8.189) $$Pu \in \mathcal{C}^\infty(X).$$

So far we have noticed that

20.23 (8.190) $$\sigma_{m+M}(Pu) = \sigma_m(P) \restriction N^*G \cdot \sigma(u)$$

so that

20.24 (8.191) $$N^*G \subset \Sigma(p) \Longleftrightarrow p(x, df) = 0 \text{ on } f = 0$$

implies

20.25 (8.192) $$Pu \in I^{m+M-1}(X, G; \Omega^{\frac{1}{2}}) \text{ near } \pi(z)$$

which is one order smoother than without (8.191).

It is now clear, I hope, that we need to make the "next symbol" vanish as well, i.e. we want

20.26 (8.193) $$\sigma_{m+M-1}(Pu) = 0.$$

Of course to arrange this it helps to know what the symbol is!

20.27 PROPOSITION 8.2. *Suppose $P \in \Psi^m(X;\Omega^{\frac12})$ and $G \subset X$ is a $\mathcal{C}^\infty$ hypersurface characteristic for P (i.e. $N^*G \subset \Sigma(P)$) then $\forall\ u \in I^M(X,G;\Omega^{\frac12})$*

20.28 (8.194)
$$\sigma_{m+M-1}(Pu) = (-iH_p + a)\sigma_m(u)$$

where $a \in S^{m-1}(N^*G)$ and H_p is the Hamilton vector field of $p = \sigma_m(P)$.

PROOF. Observe first that the formula makes sense since $\Lambda = N^*G$ is Lagrangian, $\Lambda \subset \Sigma(p)$ implies H_p is tangent to Λ and if p is homogeneous of degree m (which we are implicitly assuming) then

20.29 (8.195)
$$\mathcal{L}_{H_p} : S^r(\Lambda;\Omega^{\frac12}) \longrightarrow S^{r+m-1}(\Lambda;\Omega^{\frac12})\ \forall\ m$$

where one can ignore the half-density terms. So suppose $G = \{x_1 = 0\}$ locally, which we can always arrange by choice of coordinates. Then

(8.196)
$$X = N^*G = \{(0,x',\xi_1,0) \in T^*X\}.$$

To say $N^*G \subset \Sigma(p)$ means $p = 0$ on Λ, i.e.

20.30 (8.197)
$$p = x_1 q(x,\xi) + \sum_{j>1} \xi_j p_j(x,\xi) \text{ near } z$$

with q homogeneous of degree m and the p_j homogeneous of degree $m-1$. Working microlocally we can choose $Q \in \Psi^m(X,\Omega^{\frac12}), P_j \in \Psi^{m-1}(X,\Omega^{\frac12})$ with

20.31 (8.198)
$$\sigma_m(Q) = q, \sigma_{m-1}(P_j) = p_j \text{ near } z.$$

Then, from (8.197)

20.32 (8.199)
$$P = x_1 Q + D_{x_j} P_j + R + P', \quad R \in \Psi^{m-1}(X;\Omega^{\frac12}) z \notin WF'(P'), P' \in \Psi^m(X,\Omega^{\frac12}).$$

Of course P' does not affect the symbol near z so we only need observe that

20.33 (8.200)
$$\begin{aligned} \sigma_{r-1}(x,u) &= -d_{\xi_1}\sigma_r(u) \\ & \qquad\qquad\qquad \forall\ u \in I^r(X,G;\Omega^{\frac12}) \\ \sigma_r(D_{x_j}u) &= D_{x_j}\sigma_r(u). \end{aligned}$$

This follows from the local expression

20.34 (8.201)
$$u(x) = (2\pi)^{-1} \int e^{ix_1\xi_1} a(x',\xi_1) d\xi_1.$$

Then from (8.199) we get

20.35 (8.202)
$$\begin{aligned} \sigma_{m+M-1}(Pu) &= -D_{\xi_1}(q\sigma_M(u)) + \sum_j D_{x_j}(p_j\sigma_M(u)) + r\cdot\sigma_m(u) \\ &= -i\left(\sum_{j>1} p_j \restriction \Lambda \frac{\partial}{\partial x_j} - q \restriction \Lambda \frac{\partial}{\partial \xi_i}\right)\sigma_M(u) + a'\sigma_M(u). \end{aligned}$$

Observe from (8.197) that the Hamilton vector field of p, at $x_1 = \xi' = 0$ is just the expression in parenthesis. This proves (8.194). □

So, now we can solve (8.193). We just set

20.36 (8.203) $$\sigma_M(u)(\exp(\epsilon H_p)z') = e^{i\epsilon A}\exp(\epsilon H_p)^*[b] \ \forall\ z' \in \tilde{\Lambda}_f = \Lambda \cap \{t = 0\}.$$

where A is the solution of

20.37 (8.204) $$H_pA = a,\ A \restriction t = 0 = 0 \quad \text{on } \Lambda_0$$

and $b \in S^r(\Lambda_0)$ is a symbol defined on $\Lambda_0 = \Lambda \cap \{t = 0\}$ near z.

20.38 Proposition 8.3. *Suppose $P \in \Psi^m(X; \Omega^{\frac{1}{2}})$ has homogeneous principal symbol of degree m satisfying*

20.39 (8.205) $$p = \sigma_m(P) \text{ is real}$$

20.40 (8.206) $$d_{\text{fibre}}\, p \neq 0 \text{ on } p = 0$$

and $z \in \Sigma(p)$ is fixed. Then if $H \ni \pi(z)$ is a hypersurface such that $\pi_*(H_p) \cap H$ and $G \subset H$ is an hypersurface in H s.t.

(8.207) $$\bar{z} = c_H^*(z) \in H_{\pi z}^* G$$

there exist a characteristic hypersurface $\tilde{G} \subset X$ for P such that $\tilde{G} \cap H = G$ near $\pi(z)$, $z \in N_{\pi z}^*\tilde{G}$. For each

20.41 (8.208) $$u_0 \in I^{m+\frac{1}{4}}(H, G; \Omega^{\frac{1}{2}}) \text{ with } WF(u_0) \subset \gamma,$$

γ a fixed small conic neighbourhood of $\bar{z}$ n T^*H there exists

20.42 (8.209) $$u \in I(X, \tilde{G}; \Omega^{\frac{1}{2}}) \text{ satisfying}$$

20.43 (8.210) $$u \restriction G = u_0 \text{ near } \pi z \in H$$

20.44 (8.211) $$Pu \in \mathcal{C}^\infty \text{ near } \pi z \in X.$$

Proof. All the stuff about G and $\tilde{G}$ is just Hamilton-Jacobi theory. We can take the symbol of u_0 to be the b in (8.203), once we think a little about half-densities, and thereby expect (8.210) and (8.211) to hold, modulo certain singularities. Indeed, we would get

20.45 (8.212) $$u_1 \restriction G - u_0 \in I^{r+\frac{1}{4}-1}(H, G; \Omega^{\frac{1}{2}}) \text{ near } \pi z \in H$$

20.46 (8.213) $$Pu \in I^{r+m-2}(X, \tilde{G}; \Omega^{\frac{1}{2}}) \text{ near } \pi z \in X.$$

So we have to work a little to remove lower order terms. Let me do this informally, without worrying too much about (8.210) for a moment. In fact I will put (8.212) into the exercises!

All we really have to observe to improve (8.213) to (8.211) is that

20.47 (8.214) $$\begin{gathered} g \in I^r(X, \tilde{G}; \Omega^{\frac{1}{2}}) \Longrightarrow \exists \quad u \in I^{r+m-1}(X; \tilde{G}; \Omega^{\frac{1}{2}}) \\ \text{s.t.} \quad Pu - g \in I^{r-1}(X, \tilde{G}; \Omega^{\frac{1}{2}}) \end{gathered}$$

which we can then iterate and asymptotically sum. In fact we can choose the solution so $u \restriction H \in \mathcal{C}^\infty$, near $\pi\bar{z}$. To solve (8.214) we just have to be able to solve

20.48 (8.215) $$-i(H_p + a)\sigma(u) = \sigma(g)$$

which we can do by integration (duHamel's principle). □

The equation (8.215) for the symbol of the solution is the transport equation. We shall use this construction next time to produce a microlocal parametrix for P!

8.4. Problems

20.49 PROBLEM 8.2. Let X be a $\mathcal{C}^\infty$ manifold, $G \subset X$ on $\mathcal{C}^\infty$ hypersurface and $t \in \mathcal{C}^\infty(X)$ a real-valued function such that

T (8.216) $$dt \neq 0 \text{ on } T_pG \ \forall \ p \in L = G \cap \{t = 0\}.$$

Show that the transversality condition (8.216) ensures that $H = \{t = 0\}$ and $L = H \cap G$ are both $\mathcal{C}^\infty$ submanifolds.

20.50 PROBLEM 8.3. Assuming (8.216) show that dt gives an isomorphism of line bundles

(8.217) $$\Omega^{\frac{1}{2}}(H) \equiv \Omega^{\frac{1}{2}}_H(X) \sim \Omega^{\frac{1}{2}}_H(X)/|dt|^{\frac{1}{2}}$$

and hence one can define a restriction map,

(8.218) $$\mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(H; \Omega^{\frac{1}{2}}).$$

20.51 PROBLEM 8.4. Assuming 1 and 2, make sense of the restriction formula

(8.219) $$\restriction H : I^m\left(X, G; \Omega^{\frac{1}{2}}\right) \longrightarrow I^{m+\frac{1}{4}}\left(H, L; \Omega^{\frac{1}{2}}\right)$$

and prove it, and the corresponding symbolic formula

(8.220) $$\sigma_{m+\frac{1}{4}}(u \restriction H) = (\iota^*_H)^*\left(\sigma_m(u) \restriction N^*_L G\right)/|d\tau|^{\frac{1}{2}}.$$

NB. Start from local coordinates and try to understand restriction at that level before going after the symbol formula!

8.5. The wave equation

We shall use the construction of travelling wave solutions to produce a parametrix, and then a fundamental solution, for the wave equation. Suppose X is a Riemannian manifold, e.g. $\mathbb{R}^n$ with a 'scattering' metrice:

21.1 (8.221) $$g = \sum_{i,j=1}^n g_{ij}(x)dx^i dx^j, g_{ij} = \delta_{ij}|x|R.$$

Then the associates Laplacian, on functions, i.e.

21.2 (8.222) $$\Delta u = -\sum_{i,j=1}^n \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}(\delta g g^{ij}(x))\frac{\partial}{\partial x_i}u$$

where $g^{ij}(x) = (g_{ij}(x))^{-1}$ and $g = \det g_{ij}$. We are interested in the wave equation

21.3 (8.223) $$Pu = (D_t^2 - \Delta)u = f \qquad \text{on } \mathbb{R} \times X$$

For simplicity we assume X is either compact, or $X = \mathbb{R}^n$ with a metric of the form (8.221).

The cotangent bundle of $\mathbb{R} \times X$ is

(8.224) $$T^*(\mathbb{R} \times X) \simeq T^*\mathbb{R} \times T^*X$$

with canonical coordinates (t, x, τ, ξ). In terms of this

21.4 (8.225) $$\sigma(P) = \tau^2 - |\xi|^2|\xi| = \sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j.$$

Thus we certainly have an operator satisfying the conditions of (8.223) and (8.225), since

$$d_{\text{fibre}}\, p = \left(\frac{\partial p}{\partial \tau}, \frac{\partial p}{\partial \xi}\right) = 0 \Longrightarrow \tau = 0 \text{ and } g^{ij}(x)\xi_i = 0 \Longrightarrow \xi = 0. \tag{8.226}$$

As initial surface we consider the obvious hypersurface $\{t = 0\}$ (although it will be convenient to consider others). We are after the two theorems, one local and global, the other microlocal, although made to look global.

21.5 Theorem 8.3. *If X is a Riemannian manifold, as above, then for every $f \in \mathcal{C}_c^{-\infty}(\mathbb{R} \times X)$ $\exists !\ u \in \mathcal{C}^{-\infty}(\mathbb{R} \times X)$ satisfying*

21.6
$$Pu = f, u = 0 \text{ in } t < \inf\{\bar{t};\quad \exists(\bar{t}, x) \in \operatorname{supp}(f)\}. \tag{8.227}$$

21.7 Theorem 8.4. *If X is a Riemannian manifold, as above, then for every $u \in \mathcal{C}^{-\infty}(\mathbb{R} \times X)$,*

21.8
$$WF(u)\backslash WF(Pu) \subset \Sigma(P)\backslash WF(Pu) \tag{8.228}$$

is a union of maximally extended H_o-curves in the open subset $\Sigma(P)\backslash WF(Pu)$ of $\Sigma(P)$.

Let us think about Theorem 8.3 first. Suppose $\bar{x}X$ is fixed on $\delta_{\bar{x}} \in \mathcal{C}^{-\infty}(X;\Omega)$ is the Dirac delta (g measure) at $\bar{x}$. Ignoring, for a moment, the fact that this is not quite a generalized function we can look for the "forward fundamental solution" of P with pole at $(0, \bar{x})$:

21.9
$$\begin{gathered} PE_{\bar{x}}(t,x) = \delta(t)\delta_{\bar{x}}(x) \\ E_{\bar{x}} = 0 \text{ in } t < 0. \end{gathered} \tag{8.229}$$

Theorem 8.3 asserts its existence and uniqueness. Conversely if we can construct $E_{\bar{x}}$ for each $\bar{x}$, and get reasonable dependence on $\bar{x}$ (continuity is almost certain once we prove uniqueness) then

21.10
$$K(t, x; \bar{t}, \bar{x}) = E_{\bar{x}}(t - \bar{t}, x) \tag{8.230}$$

is the kernel of the operator $f \longmapsto u$ solving (8.227).

So, we want to solve (8.229). First we convert it (without worrying about rigour) to an initial value problem. Namely, suppose we can solve instead

21.11
$$\begin{gathered} PG_{\bar{x}}(t,x) = 0 \text{ in } \mathbb{R} \times X \\ G_{\bar{x}}(0,x) = 0,\ D_t G_{\bar{x}}(0,x) = \delta_{\bar{x}}(x) \text{ in } X. \end{gathered} \tag{8.231}$$

Note that

21.12
$$(g(t,x,\tau,0) \notin \Sigma(P) \Longrightarrow (t,x;\tau,0) \notin WF(G). \tag{8.232}$$

This means the restriction maps, to $t = 0$, in (8.231) are well-defined. In fact so is the product map:

21.13
$$E_{\bar{x}}(t,x) = H(t)G_{\bar{x}}(t,x). \tag{8.233}$$

Then if G satisfied (8.231), a simple computation shows that $E_{\bar{x}}$ satisfies (8.229). Thus we want to solve (8.231).

Now (8.231) seems very promising. The initial data, $\delta_{\bar{x}}$, is certainly conormal to the point $\{\bar{x}\}$, so we might try to use our construction of travelling wave solutions. However there is a serious problem. We already noted that, for the wave equation, there cannot be any smooth characteristic surface other than a hypersurface. The point is that if H has codimension k then

$$N^*_{\bar{x}}H \subset T^*_{\bar{x}}(\mathbb{R}\times X) \text{ has dimension } k. \tag{8.234}$$

To be characteristic we must have

21.14 $$N^*_{\bar{x}}H \subset \Sigma(P) \Longrightarrow k = 1 \tag{8.235}$$

Since the *only* linear space contained in a (proper) cone is a line.

However we can easily 'guess' what the characteristic surface corresponding to the point $(x, \bar{x})$ is – it is the *cone* through that point:

This certainly takes us *beyond* our conormal theory. Fortunately there is a way around the problem, namely the possibility of superposition of conormal solutions.

To see where this comes from consider the representation in terms of the Fourier transform:

21.15 $$\delta(x) = (2\pi)^{-n}\int e^{ix\xi}d\xi. \tag{8.236}$$

The integral of course is not quite a proper one! However introduce polar coordinates $\xi = r\omega$ to get, at least formally

21.16 $$\delta(x) = (2\pi)^{-n}\int_0^\infty\int_{\mathbb{S}^{n-1}} e^{irx\cdot\omega}r^{n-1}dr\ d\omega. \tag{8.237}$$

In odd dimensions r^{n-1} is even so we can write

21.17 $$\delta(x) = \frac{1}{2(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty} e^{irx\cdot\omega}r^{n-1}dr\ d\omega, n \text{ odd }. \tag{8.238}$$

Now we can interpret the r integral as a 1-dimensional inverse Fourier transform so that, always formally,

21.18 $$\begin{aligned} \delta(x) &= \frac{1}{2(2\pi)^{n-1}}\int_{\mathbb{S}^{n-1}} f_n(x\cdot\omega)d\omega \\ & \qquad\qquad\qquad\qquad n \text{ odd} \\ f_n(s) &= \frac{1}{(2\pi)}\int e^{irs}\gamma^{n-1}dr. \end{aligned} \tag{8.239}$$

In even dimensions we get the same formula with

21.19 $$f_n(s) = \frac{1}{2\pi}\int e^{irs}|r|^{n-1}dr. \tag{8.240}$$

These formulas show that

21.20 $$f_n(s) = |D_s|^{n-1}\delta(s). \tag{8.241}$$

Here $|S_s|^{n-1}$ is a pseudodifferential operator for n even or differential operator $(= D_s^{n-1})$ if n is odd. In any case

21.21 $$f_n \in I^{n-1+\frac{1}{4}}(\mathbb{R}, \{0\})! \tag{8.242}$$

Now consider the map

21.22 $$\mathbb{R}^n \times \mathbb{S}^{n-1} \ni (x, \omega) \mapsto x\cdot\omega \in \mathbb{R}. \tag{8.243}$$

Thus $\mathcal{C}^\infty$ has different

$$\omega \cdot dx + x \cdot d\omega \neq 0 \text{ or } x \cdot \omega = 0 \tag{8.244}$$

So the inverse image of $\{0\}$ is a smooth hypersurface R.

21.23 LEMMA 8.11. *For each $n \geq 2$*

21.24
$$f_n(x, \omega) = \frac{1}{2\pi} \int e^{i(x\cdot\omega)r} |r|^{n-1} dr \in I^{\frac{n}{4}-\frac{1}{4}}\left(\mathbb{R} \times \mathbb{S}^{n-1}, R\right). \tag{8.245}$$

PROOF. Replacing $|r|^{n-1}$ by $\rho(r)|r|^{n-1} + (1-\rho(r))|r|^{n-1}$, where $\rho(r) = 0$ n $r < \frac{1}{2}, \rho(r) = 1$ in $r > 1$, expresses f_n as a sum of a $\mathcal{C}^\infty$ term and a conormal distribution. Check the order yourself! □

21.25 PROPOSITION 8.4. *(Radon inversion formula). Under pushforward corresponding to $\mathbb{R}^n \times \mathbb{S}^{n-1} @>\pi_1>> \mathbb{R}^n$*

21.26
$$\begin{aligned} (\pi_1)_* f_n' &= 2(2\pi)^{n-1}\delta(x), \\ f_n' &= f_n |d\omega||dx|. \end{aligned} \tag{8.246}$$

PROOF. Pair with a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$:

$$(\pi_1)_* f_n' = \iint f_n(x\cdot\omega)\phi(x) dx \; d\omega \tag{8.247}$$

by the Fourier inversion formula. □

So now we have a superposition formula expressing $\delta(x)$ as an integral:

21.27
$$\delta(x) = \frac{1}{2(2\pi)^{n-1}} \int\limits_{\mathbb{S}^{n-1}} f_n(x\cdot\omega) d\omega \tag{8.248}$$

where for each fixed ω $f_n(x\cdot\omega)$ is conormal with respect to $x \cdot \omega = 0$. This gives us a strategy to solve (21.11 8.231).

21.28 PROPOSITION 8.5. *Each $\bar{x} \in X$ has a neighbourhood, $U_{\bar{x}}$, such that for $\bar{t} > 0$ (independent of $\bar{x}$) there are two characteristic hypersurfaces for each $\omega \in \mathbb{S}^{n-1}$*

$$H^{\pm}_{\bar{x},\omega)} \subset (-\bar{t}, \bar{t}) \times U_{\bar{x}} \tag{8.249}$$

depending on $\bar{x}, \omega$, and there exists

$$u^{\pm}(t, x; \bar{x}, \omega) \in I^*((-\bar{t}|\bar{t}| \times U_{\bar{x}}, H^{\pm}_{(\bar{x},\omega)}) \tag{8.250}$$

such that

21.29
$$Pu^{\pm} \in \mathcal{C}^\infty \tag{8.251}$$

21.30
$$\begin{cases} u^+ + \bar{u} \restriction t = 0 = \delta_{\bar{x}}(x\cdot\omega) & \text{in } U_{\bar{x}} \\ D_t(u^+ + u^-) \restriction \{t = 0\} = 0 & \text{in } U_{\bar{x}}. \end{cases} \tag{8.252}$$

PROOF. The characteristic surfaces are constructed through Hamilton-Jacobi theory:

$$\begin{aligned} N^* H^{\pm} &\subset \Sigma(P), \\ H_0 = H^{\pm} \cap \{t = 0\} &= \{x \cdot \omega = 0\}. \end{aligned} \tag{8.253}$$

There are two or three because the conormal direction to H_0 at 0; ωdx, has two $\Sigma(P)$:

21.31 (8.254)
$$\tau = \pm 1,\ (\tau, \omega) \in T_0^*(\mathbb{R} \times X).$$

With *each* of these two surfaces we can associate a microlocally unique conormal solution

21.32 (8.255)
$$\begin{gathered} Pu^\pm = 0, \quad u^\pm \restriction \{t = 0\} = u_0^\pm \\ u_0^\pm \in I^*(\mathbb{R}^n, \{x \cdot \omega = 0\}) \end{gathered}$$

Now, it is easy to see that there are unique choices

21.33 (8.256)
$$\begin{gathered} u_\delta^+ + u_0^- = \delta(x \cdot \omega) \\ D_t u^+ + D_t u^- \restriction \{t = 0\} = 0. \end{gathered}$$

(See exercise 2.) This solves ([21.30]8.252) and proves the proposition (modulo a fair bit of hard work!).

□

So now we can use the superposition principle. Actually it is better to add the variables ω to the problem and see that

21.34 (8.257)
$$\begin{gathered} u^\pm(t, x; \omega, \bar{x}) \in I^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n; H^\pm) \\ H^\pm \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n \end{gathered}$$

being fixed by the condition that

(8.258)
$$H^\pm \cap \mathbb{R} \times \mathbb{R}^n \times \{\omega\} \times \{\bar{x}\} = H^\pm_{\bar{x},\omega}.$$

Then we set

21.35 (8.259)
$$G'_{\bar{x}}(t, x) = \int\limits_{\mathbb{S}^{n-1}} (u^+ + u^-)(x, x; \omega, \bar{x}).$$

This satisfies ([21.11]8.231) locally near $\bar{x}$ and modulo $\mathcal{C}^\infty$. i.e.

21.36 (8.260)
$$\begin{cases} PG'_{\bar{x}} \in \mathcal{C}^\infty((-\bar{t}(\bar{t})) \times U_{\bar{x}}) \\ G'_{\bar{x}} \restriction \{t = 0\} = xv, & \\ & v_i \in \mathcal{C}^\infty \\ D_t G'_{\bar{x}} = \delta_{\bar{x}}(x) + v_2 \end{cases}$$

Let us finish off by doing a calculation. We have (more or less) shown that $u^\pm$ are conormal with respect to the hypersurfaces $H^\pm$. A serious question then is, what is (a bound one) the wavefront set of $G'_{\bar{x}}$? This is fairly easy provided we understand the geometry. First, since $u^\pm$ are conormal,

(8.261)
$$WF(u^\pm) \subset N^* H^\pm.$$

Then the push-forward theorem says

21.37 (8.262)
$$\begin{gathered} WF(G^\pm) \subset \{(t, x, \tau, \xi);\ \exists \quad (t, x, \tau, \xi, \omega, w) \in WF(u^\pm)\} \\ G^\pm = (\pi_1)_* u^\pm = \int\limits_{\mathbb{S}^{n-1}} u^\pm(t, s; \omega, \bar{x}) d\omega \end{gathered}$$

so here

(8.263)
$$(t, x, \tau, \xi, \omega, w) \in T^*(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1}) = T^*(\mathbb{R} \times \mathbb{R}^n) \times T^*\mathbb{S}^{n-1}.$$

We claim that the singularities of $G'_{\bar{x}}$ lie on a cone:

$$WF(G'_{\bar{x}}) \subset \Lambda_{\bar{x}} \subset T^*(\mathbb{R} \times \mathbb{R}^n) \tag{8.264}$$

where $\Lambda_{\bar{x}}$ is the conormal bundle to a cone:

21.38
$$\begin{gathered}\Lambda_{\bar{x}} = \mathrm{cl}\{(t,x;\tau,\xi);\ t \neq 0, D(x,\bar{x}) = \pm t,\\ (\tau,\xi) = \tau(1, \mp d_x D(x,\bar{x}))\end{gathered} \tag{8.265}$$

where $D(x,\bar{x})$ is the Riemannian distance from x to $\bar{x}$.

8.6. Forward fundamental solution

Last time we constructed a local parametrix for the Cauchy problem:

22.1
$$\begin{cases} PG'_{\bar{x}} = f \in \mathcal{C}^\infty(\Omega) & (0,\bar{x}) \in \Omega \subset \mathbb{R} \times X \\ G'_{\bar{x}} \restriction t = 0 = u' & \\ D_t G'_{\bar{x}} \restriction \{t = 0\} = \delta_{\bar{x}}(x) + u'' & u', u'' \in \mathcal{C}^\infty(\Omega_0) \end{cases} \tag{8.266}$$

where $P = D_t^2 - \Delta$ is the wave operator for a Riemann metric on X. We also computed the wavefront set, and hence singular support of $G_{\bar{x}}$ and deduced that

22.2
$$\operatorname{sing}\cdot\operatorname{supp}.(G_{\bar{x}}) \subset \{(t,x); d(x,\bar{x}) = |t|\} \tag{8.267}$$

in terms of the Riemannian distance.

22.3 (8.268)

This allows us to improve (8.266) in a very significant way. First we can chop $G_{\bar{x}}$ off by replacing it by

22.4
$$\phi\left(\frac{t^2 - d^2(x,\bar{x})}{\epsilon^2}\right). \tag{8.269}$$

where $\phi \in \mathcal{C}^\infty(\mathbb{R})$ has support near 0 and is identically equal to 1 in some neighbourhood of 0. This gives (8.266) again, with $G'_{\bar{x}}$ now supported in say $d^2 < t^2 + \epsilon^2$.

22.5 (8.270)

Next we can improve (8.266) a little bit by arranging that

22.6
$$u' = u'' = 0,\ D_t^k f \big|_{t=0} = 0\ \forall\ k. \tag{8.271}$$

This just requires adding to G' a $\mathcal{C}^\infty, v,$ function, so that

22.7
$$v\big|_{t=0} = u',\ D_t v\big|_{t=0} = -u'',\quad D_t^k(Pu)\big|_{t=0} = -D_t^k f\big|_{t=0}\quad k > 0. \tag{8.272}$$

Once we have done this we consider

$$E'_{\bar{x}} = iH(t)G'_{\bar{x}} \tag{8.273}$$

which now satisfies

22.8
$$\begin{gathered} PE'_{\bar{x}} = \delta(t)\delta_{\bar{t}}(x) + F_{\bar{x}},\ F_{\bar{x}} \in \mathcal{C}^\infty(\Omega_{\bar{x}}) \\ \operatorname{supp}(E'_x) \subset \{d^2(x,\bar{x}) \le t^2 + \epsilon^2\} \cap \{t \ge 0\}. \end{gathered} \tag{8.274}$$

Here F vanishes in $t < 0$, so vanishes to infinite order at $t = 0$.

Next we remark that we can actually do all this with smooth dependence of $\bar{x}$. This should really be examined properly, but I will not do so to save time. Thus we actually have

22.9 (8.275)
$$\begin{cases} E'(t,x,\bar{x}) \in \mathcal{C}^{-\infty}(P(-\infty,\epsilon)\times X\times X) \\ PE' = \delta(t)\sigma_{\bar{x}}(x) + F \\ \operatorname{supp} E' \subset \{d^2(x,\bar{x}) \ge t^2+\epsilon^2\} \cap \{t \ge 0\}. \end{cases}$$

We can, and later shall, estimate the wavefront set of E. In case $X = \mathbb{R}^n$ we can take E to be the *exact* forward fundamental solution where $|x|$ or $\bar{x} \ge R$, so

22.10 (8.276)
$$\begin{gathered} \operatorname{supp}(F) \subset \{t \ge 0\} \cap \{|x|, |\bar{x}| \le R\} \cap \{d^2 \le t^2+\epsilon^2\} \\ F \in \mathcal{C}^{\infty}((-\infty,\epsilon)\times X\times X). \end{gathered}$$

Of course we want to remove F, the error term. We can do this because it is a *Valterra operator*, very similar to an upper triangular metric. Observe first that the operators of the form (22.10 8.276) form an algebra under t-convolution:

22.11 (8.277)
$$F = F_1 \circ F_1,\ F(t,x,\bar{x}) = \int\limits_0^t \int F_1(t,-t',x,x')F_2(t^1,x^1,\bar{x})dx'dt'.$$

In fact if one takes the iterates of a fixed operator

(8.278)
$$F^{(k)} = F^{(k-1)} \circ F$$

One finds exponential convergence:

22.12 (8.279)
$$\left|D_x^\alpha D_t^p F^{(k)}(t,x,\bar{x})\right| \le \frac{C^{k+1}N,\delta}{k!}|t|^N \quad \text{in } t < \epsilon-\delta\ \forall\ N.$$

Thus if F is as in (22.10 8.276) then $Id + F$ has inverse $Id + \tilde{F}$,

22.13 (8.280)
$$\tilde{F} = \sum_{j\ge 1}(-1)^j F^{(j)}$$

again of this form.

Next note that the composition of E' with $\tilde{F}$ is again of the form (22.10 8.276), with R increased. Thus

22.14 (8.281)
$$E = E' + E' \circ F$$

is a forward fundamental solution, satisfying (22.9 8.275) with $F \equiv 0$.

In fact E is also a left parametrix, in an appropriate sense:

22.15 PROPOSITION 8.6. *Suppose* $u \in \mathcal{C}^{-\infty}((-\infty,\epsilon)\times X)$ *is such that*

22.16 (8.282)
$$\operatorname{supp}(u) \cap [-T,\tau]\times X \textit{ is compact } \ \forall\ T \textit{ and for } \tau < \epsilon$$

then $Pu = 0 \Longrightarrow u = 0$.

PROOF. The trick is to make sense of the formula

22.17 (8.283)
$$0 = E \cdot Pu = u.$$

In fact the operators G with kernel $G(t,x,\bar{x})$, defined in $t < \epsilon$ and such that $G * \phi \subset \mathcal{C}^\infty\ \forall\ \phi \in \mathcal{C}^\infty$ and

22.18 (8.284)
$$\{t \ge 0\} \cap \{d(x,\bar{x}) \le R\} \supset \operatorname{supp}(G)$$

act on the space (22.16 8.282) as t-convolution operators. For this algebra $E * P = \mathrm{Id}$ so (22.17 8.283) holds! □

We can use this proposition to prove that E itself is unique. Actually we want to do more.

22.19 THEOREM 8.5. *If X is either a compact Riemann manifold or $\mathbb{R}^n$ with a scattering metric then P has a unique forward fundamental solution, ω.*

22.20
$$\operatorname{supp}(E) \subset \{t \geq 0\}, P^E = \operatorname{Id} \tag{8.285}$$

and

22.21
$$\operatorname{supp}(E) \subset \{(t, x, \bar{x}) \in \mathbb{R} \times X \times X; d(x, \bar{x}) \leq t\} \tag{8.286}$$

and further

22.22
$$WF'(E) \subset \operatorname{Id} \cup \mathcal{F}_+ \tag{8.287}$$

where $\mathcal{F}_+$ is the forward bicharacteristic relation on $T^*(\mathbb{R} \times X)$

22.23
$$\begin{gathered} \zeta = (t, x, \tau, \xi) \notin \Sigma(P) \Longrightarrow \mathcal{F}_+(\zeta) = \emptyset \\ \zeta = (t, x, \tau, \xi) \in \Sigma(P) \Longrightarrow \mathcal{F}_+(\zeta) = \{\zeta' = (t', x', \tau', \xi') \\ t' \geq t \times \zeta' = \exp(TH_p)\zeta \text{ for some } T\}. \end{gathered} \tag{8.288}$$

PROOF. (1) Use E_1 defined in $(-\infty, \epsilon \times X$ to continue E globally.
(2) Use the freedom of choice of $\{t = 0\}$ and uniqueness of E to show that (22.21 8.286)can be arranged for small, and hence all,
(3) Then get (22.23 8.288) by checking the wavefront set of G.

□

As corollary we get proofs of (22.5 8.270) and (22.6 8.271).

PROOF OF THEOREM XXI.5.

$$u(t, x) = \int E(t - t', x, x') f(t', x') dx' dt'. \tag{8.289}$$

□

PROOF OF THEOREM XXI.6. We have to show that if both $\operatorname{WF}(Pu) \not\ni z$ and $\operatorname{WF}(u) \not\ni z$ then $\exp(\delta H_p)z \notin WF(u)$ for small δ. The general case that follows from the (assumed) connectedness of H_p curves. This involves microlocal uniqueness of solutions of $Pu = f$. Thus if $\phi \in \mathcal{C}^\infty(\mathbb{R})$ has support in $t > -\delta$, for $\delta > 0$ small enough, $\pi^* t(z) = \bar{t}$

$$P(\phi(t - \bar{t})u) = g \text{ has } z \notin WF(g), \tag{8.290}$$

and vanishes in $t < \delta$. Then

$$\begin{gathered} \phi(t - \bar{t})u = E \times g \\ \Longrightarrow \exp(\tau H_p)(z) \notin WF(u) \text{ for small } \tau. \end{gathered} \tag{8.291}$$

□

8.7. Operations on conormal distributions

I want to review and refine the push-forward theorem, in the general case, to give rather precise results in the conormal setting. Thus, suppose we have a projection

23.1 (8.292) $$X \times Y @ > x >> X$$

where we can view $X \times Y$ as compact manifolds or Euclidean spaces as desired, since we actually work locally. Suppose

23.2 (8.293) $$Q \subset X \times Y \text{ is an embeded submanifold.}$$

Then we know how to define and examine the *conormal* distribution associated to Q. If

23.3 (8.294) $$u \in I^m(X \times Y, Q; \Omega)$$

when is $\pi_*(u) \in \mathcal{C}^{-\infty}(X;\Omega)$ conormal? The obvious thing we ned is a submanifold with respect to what it should be conormal! From our earlier theorem we know that

23.4 (8.295) $$WF(\pi_*(u)) \subset \{(x,\xi);\quad \exists \quad (x,\xi,y,0) \in WF(u) \subset N^*Q\}.$$

So suppose $Q = \{q_j(x,y) = 0, j = 1, \dots, k\}$, $k = \operatorname{codim} Q$. Then we see that

23.5 (8.296) $$(\bar{x}, \bar{\xi}, \bar{y}, 0) \in N^*Q \Longleftrightarrow (\bar{x}, \bar{y}) \in Q, \bar{\xi} = \sum_{j=1}^{k} \tau_j d_x q_j, \sum_{j=1}^{k} \tau_j dy q_j = 0.$$

Suppose for a moment that Q has a hypersurface, i.e. $k = 1$, and that

23.6 (8.297) $$Q \longrightarrow \pi(Q) \text{ is a fibration}$$

then we expect

23.7 Theorem 8.6. $\pi_* : I^m(X \times Y, Q, \Omega) \longrightarrow I^{m'}(X, \pi(Q))$.

Proof. Choose local coordinates so that

(8.298) $$Q = \{x_1 = 0\}$$

(8.299) $$u = \frac{1}{2\pi} \int e^{ix_1\xi_1} a(x', y, \xi_1) d\xi_1$$

(8.300) $$\pi^* u = \frac{1}{2\pi} \int e^{ix_1\xi_1} b(x', \xi_1) d\xi_1$$

(8.301) $$b = \int a(x', y, \xi) dy.$$

□

Next consider the case of restriction to a submanifold. Again let us suppose $Q \subset X$ is a hypersurface and $Y \subset X$ is an embedded submanifold *transversal* to Q:

23.8 (8.302)
$$\begin{aligned} &Q \pitchfork Y = QY \\ &\text{i.e. } T_qQ + T_qY = T_qX \quad \forall\ q \in Qy \\ &\Longrightarrow Q_y \qquad \text{is a hypersurface in } X. \end{aligned}$$

Indeed locally we can take coordinates in which

23.9 (8.303) $$Q = \{x_1 = 0\}, Y = \{x'' = 0\}, \quad x = (x_1, x', x'').$$

23.10 THEOREM 8.7.

$$C_Y^*: I^m(X,Q) \longrightarrow I^{m+\frac{k}{4}}(Y,Q_Y) k = \operatorname{codim} Y \textit{ in } X. \tag{8.304}$$

PROOF. In local coordinates as in (23.9 8.303)

$$\begin{aligned} u &= \frac{1}{2\pi}\int e^{ix_1\xi_1} a(x(x',x'',\xi_1))d\xi, \\ c^*u &= \frac{1}{2\pi}\int e^{ix_1\xi_1} a(x',0,\xi_1)d\xi_1. \end{aligned} \tag{8.305}$$

Now let's apply this to the fundamental solution of the wave equation. Well rather consider the solution of the initial value problem

23.11 $$\begin{cases} PG(t,x,\bar{x}) = 0 \\ G(0,x,\bar{x}) = 0 \\ D_tG(0,x,\bar{x}) = \delta_{\bar{x}}(x). \end{cases} \tag{8.306}$$

We know that G exists for *all* time and that for short time it is

23.12 $$G - \int_{\mathbb{S}^{n-1}} (u_+(t,x,\bar{x};\omega) + u_-(t,x,\bar{x};\omega))d\omega + \mathcal{C}^\infty \tag{8.307}$$

where $u_\pm$ are conormal for the term characteristic hypersurfaces H_p satisfying

23.13 $$\begin{aligned} N^*H_\pm &\subset \Sigma(P) \\ H_\pm \cap \{t=0\} &= \{(x-\bar{x})\cdot\omega = 0\} \end{aligned} \tag{8.308}$$

Consider the 2×2 matrix of distribution

23.14 $$U(t) = \begin{pmatrix} D_tG & G \\ D_t^2G & D_tG \end{pmatrix}. \tag{8.309}$$

Since $WFU \subset \Sigma(P)$, in polar $\tau \neq 0$ we can consider this as a smooth function of t, with values in distribution on $X \times X$. □

23.15 THEOREM 8.8. *For each $t \in \mathbb{R}$ $U(t)$ is a boundary operator on $L^2(X) \oplus H'(X)$ such that*

23.16 $$U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(t) \\ D_tu(t) \end{pmatrix} \tag{8.310}$$

where $u(t,x)$ is the unique solution of

23.17 $$\begin{aligned} (D_t^2 - \Delta)u(t) &= 0 \\ u(0) &= u_0 \\ D_t + u(0) &= u_1. \end{aligned} \tag{8.311}$$

PROOF. Just check it! □

Consider again the formula (23.12 8.307). First notice that at $x=\bar{x}$, $t=0$, $dH^\pm = dt \pm d(x-\bar{x})\omega)$ (by construction). so

23.18 $$H_\pm \pitchfork \{x=\bar{x}\} = \{t=0\} \subset \mathbb{R}\times X \hookrightarrow \mathbb{R}\times X \times Y \times \mathbb{S}^{n-1}. \tag{8.312}$$

Moreover the projection

23.19 $$\mathbb{R}\times X \times \mathbb{S}^{n-1} \longrightarrow \mathbb{R} \tag{8.313}$$

clearly fibres $\{t = 0\}$ over $\{t = 0\} \in= \{0\} \subset \mathbb{R}$. Then we can apply the two theorems, on push-forward and pull-back, above to conclude that

23.20 (8.314) $$T(t) = \int_X G(t, x, \bar{x}) \restriction x = \bar{x} dx \in \mathcal{C}^{-\infty}(\mathbb{R})$$

is conormal near $t = 0$ i.e. $\mathcal{C}^\infty$ in $(-\epsilon, \epsilon)\backslash\{0\}$ for some $\epsilon > 0$ and conormal at 0. Moreover, we can, at least in principle, work at the *symbol* of $T(t)$ at $t = 0$. We return to this point next time.

For the moment let us think of a more 'fundamental analytic' interpretation of (23.20 8.314). By this I mean

23.21 (8.315) $$T(t) = trU(t).$$

REMARK 8.1. Trace class operators $\Delta\lambda$; Smoother operators are trace order, $tr = \int K(x, x)$

(8.316) $$\int U(t)\phi(t) \text{ is smoothing}$$

(8.317) $$\langle T(t), \phi(t)\rangle = tr\langle U(t), \phi(t)\rangle.$$

8.8. Weyl asymptotics

Let us summarize what we showed last time, and a little more, concerning the trace of the wave group

24.1 PROPOSITION 8.7. *Let X be a compact Riemann manifold and $U(t)$ the wave group, so*

(8.318) $$U(t) : \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X) \ni (u_0, u_1) \mapsto (u, (t), D + tu(t)) \in \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X)$$

where u is the solution to

24.2 (8.319) $$(D_t^2 - \Delta)u(t) = 0 \quad u(0) = u_0 \quad D_t u(0) = u_1.$$

The trace of the wave group, $T \in \mathcal{S}'(\mathbb{R})$, is well-defined by

24.3 (8.320) $$T(\phi) = \operatorname{Tr} U(\phi), U(\phi) = \int U(t)\phi(t)dt \ \forall \ \phi \in \mathcal{S}(\mathbb{R})$$

and satisfies

24.4 (8.321) $$T = Y(\left(1 + \sum_{j=1}^\infty 2\cos(t\lambda_j)\right)$$

(8.322) $$\textit{where } 0 = \lambda_0 < \lambda_1^2 \le \lambda_2^2 \dots \qquad \lambda_j \ge 0$$

is the spectrum of the Laplacian repeated with multiplicity

24.5 (8.323) $$\operatorname{sing.supp}(T) \subset \mathcal{L} \cup \{0\} \cup -\mathcal{L}$$

where $\mathcal{L}$ is the set of lengthes of closed geodesics of X and

24.6 (8.324) $$\textit{if } \psi \in \mathcal{C}_c^\infty(\mathbb{R}), \psi(t) = 0 \textit{ if } |t| \ge \textit{ inf } \mathcal{L} - \epsilon, \epsilon > 0, \quad \psi T \in I(\mathbb{R}, \{0\}) \quad \sigma(\psi T) =$$

Proof. We have already discussed (8.321) and the first part of (8.324) (given (8.323)). Thus we need to show (8.323), the *Poisson relation*, and compute the symbol of T as a cononormal distribution at 0 .

Let us recall that if G is the solution to

24.7 (8.325)
$$\begin{aligned}(D_t^2 = \Delta)G(t,x,\bar{x}) &= 0\\ G(0,x,\bar{x}) &= 0\\ D_tG(0,x,\bar{x}) &= \delta_{\bar{x}}(x)\end{aligned}$$

then

24.8 (8.326)
$$T = \pi_*(\iota_\Delta^* 2D_tG),$$

where

24.9 (8.327)
$$\iota_\Delta : \mathbb{R}\times X \hookrightarrow \mathbb{R}\times X\times X$$

is the embedding of the diagonal and

24.10 (8.328)
$$\pi : \mathbb{R}\times X \longrightarrow \mathbb{R}$$

is projective. We also know about the wavefront set of G. That is,

24.11 (8.329)
$$\begin{aligned}WF(G) \subset \{(t,x,\bar{x},\tau,\xi,\bar{\xi}); \tau^2 = |\xi|^2 = |\bar{\xi}|^2,\\ \exp(sH_p)(0,\bar{x},\tau,\bar{\xi}) = (t,x,\tau,\xi),\ \text{some } s\}.\end{aligned}$$

Let us see what (8.329) says about the wavefront set of T. First under the restriction map to $\mathbb{R}\times\Delta$

24.12 (8.330)
$$\begin{aligned}WF(\iota_\Delta^* D_tG) \subset \{(t,y,\tau,\eta);\quad \exists\\ (t,x,y,\tau,\xi,\bar{\xi});\ \eta = \xi - \bar{\xi}\}.\end{aligned}$$

Then integration gives

24.13 (8.331)
$$WF(T) \subset \{(t,\tau);\quad \exists\quad (t,y,\tau,0) \in WF(D_tG)\}.$$

Combining (8.330) and (8.331) we see

24.14 (8.332)
$$\begin{aligned}t \in \operatorname{sing.supp}(T) &\Longrightarrow \quad \exists\quad (t,\tau) \in WF(T)\\ &\Longrightarrow \quad \exists\quad (t,x,x,\tau,\xi,\xi) \in WF(D_tG)\\ &\Longrightarrow \quad \exists\quad s \quad \text{s.t.}\quad \exp(sH_p)(0,x,\tau,\xi) = (t,x,\tau,\xi).\end{aligned}$$

Now

(8.333)
$$p = \tau^2 - |\xi|^2,\ \text{so } H_p = 2\tau\partial_t - H_g,\ g = |\xi|^2,$$

H_g being a vector field on T^*X. Since WF is *conic* we can take $|\xi| = 1$ in the last condition in (8.332). Then it says

24.15 (8.334)
$$s = 2\tau t, \qquad \exp(tH_{\frac{1}{2}g})(x,\xi) = (x,\xi),$$

since $\tau^2 = 1$.

The curves in X with the property that their tangent vectors have unit length and the lift to T^*X is an integral curve of $H_{\frac{1}{2}g}$ are *by definition* geodesic, parameterized by arclength. Thus (8.334) is the statement that $|t|$ is the length of a closed geodesic. This proves (8.323).

So now we have to compute the symbol of T at 0. We use, of course, our local representation of G in terms of conormal distributions. Namely

24.16
$$G = \sum_j \phi_j G_j, \quad \phi_j \in \mathcal{C}^\infty(X), \tag{8.335}$$

where the ϕ_j has support in coordinate particles in which

24.17
$$\begin{gathered} G_j(t, x, \bar{x}) = \int\limits_{\mathbb{S}^{n-1}} (u_+(t, x, \bar{x}; \omega) + u_-(t, x, \bar{x}; \omega))\, d\omega, \\ u_p m = \frac{1}{2\pi} \int\limits_\xi e^{ih_\pm(t,x,\bar{x},\omega)\xi} a_\pm(x, \bar{x}, \xi, \omega) d\xi. \end{gathered} \tag{8.336}$$

Here $h_\pm$ are solutions of the eikonal equation (i.e. are characteristic for P)

24.18
$$\begin{gathered} |\partial_t h_\pm|^2 = |h_\pm|^2 \\ h_\pm \big|_{t=0} = (x - \bar{x}) \cdot \omega \\ \pm \partial_t h_\pm > 0, \end{gathered} \tag{8.337}$$

which fixes them locally uniquely. The $a_\pm$ are chosen so that

24.19
$$(u_+ + u_\pm \big|_{t=0} = 0, (D_t u_+ D_t u_-) \big|_{t=0} \; \delta((x - \bar{x}) \cdot \omega) P u_\pm \in \mathcal{C}^\infty. \tag{8.338}$$

Now, from (24.17 8.336)

$$u_+ + u_- \big|_{t=0} = \frac{1}{2\pi} \int e^{((x - x\bar{x}) \cdot \omega)\xi} (a_+ + a_-)(x, \bar{x}, \xi, \omega) d\xi = 0 \tag{8.339}$$

so $a_+ - a_-$. Similarly

24.20
$$\begin{aligned} D_t u_+ + D_t u_- \big|_{t=0} &= \frac{1}{2\pi} \int e^{i((x - \bar{x}) \cdot \omega)\xi} \big[(D_t h_+) a_+ + (D_t h_-) a_-\big] d\xi \\ &= \frac{1}{2(2\pi)^{n-1}} f_n((x - \bar{x}) \cdot \omega) \end{aligned} \tag{8.340}$$

From (24.18 8.337) we know that $D_t h_\pm = \mp i |d_x (x - \bar{x}) \cdot \omega| = \mp i |\omega|$ where the length is with respect to the Riemann measure. We can compute the symbols or both sides in (24.20 8.340) and consider that

24.21
$$-2i|\omega| a_+ \equiv \frac{1}{2(2\pi)^{n-1}} |\xi|^{n-1} = D_t h_+ a_+ + D_t h_- a_- \big|_{t=0} \tag{8.341}$$

is necessary to get (24.19 8.338). Then

24.22
$$\begin{aligned} T(t) &= 2\pi_* (\iota_\Delta^* D_t G) \\ &= \frac{1}{2\pi} \sum_{j,\pm} 2 \int\limits_X \int\limits_{\mathbb{S}^{n-1}} e^{ih_\pm(t,x,x,\omega)\xi} (D_t h_\pm a_\pm)(x, \bar{x}, \omega, \xi) d\xi d\omega dx. \end{aligned} \tag{8.342}$$

Here dx is really the Riemann measure on X. From (24.21 8.341) the leading part of this is

$$\frac{2}{2\pi} \sum_{j\pm} \int\limits_X \int\limits_{\mathbb{S}^{n-1}} e^{ih_\pm(t,x,x,\omega)\xi} \frac{1}{4(2\pi)^{n-1}} |\xi|^{n-1} d\xi d\omega dx \tag{8.343}$$

since any term vanishes at t contributes a weaker singularity. Now

$$h_\pm = \pm |\omega| t + (x - \bar{x}) \cdot \omega + 0(t^2). \tag{8.344}$$

From which we deduce that

24.23 (8.345)
$$\psi(t)T(t) = \frac{1}{2\pi}\int e^{it\tau}a(\tau)d\tau$$
$$a(\tau) \sim C_n \operatorname{Vol}(X)|\tau|^{n-1}C_n =$$

where C_n is a universal constant depending only on dimension. Notice that if n is odd this is a "little" function.

The final thing I want to do is to show how this can be used to describe the asymptotic behaviour of the eigenvalue of Δ : □

24.24 PROPOSITION 8.8. *("Weyl estimates with optimal remainder".) If $N(\lambda)$ is the number of eigenvalues at Δ satisfying $\lambda_1^2 \le \lambda$, counted with multiplicity, the*

24.25 (8.346)
$$N(\lambda) = C_n \operatorname{Vol}(X)\lambda^n + o(\lambda^{n-1})$$

The estimate of the remainder terms is the here – weaker estimates are easier to get.

PROOF. (Tauberian theorem). Note that

(8.347)
$$T = \mathcal{F}(\mu) \text{ where } N(\lambda) = \int_0^\lambda \mu(\lambda),$$

$\mu(\lambda)$ being the measure

(8.348)
$$\mu(\lambda) = \sum_{\lambda_j^2 \in \operatorname{spec}(\Delta)} \delta(\lambda - \lambda_j).$$

Now suppose $\rho \in \mathcal{S}(\mathbb{R})$ is even and $\int \rho = 1$, $\rho \ge 0$. Then $N_\rho(\lambda) = \int(\lambda')\rho(\lambda - \lambda')$ is a $\mathcal{C}^\infty$ function. Moreover

24.26 (8.349)
$$\widehat{\frac{d}{d\lambda}N_\rho(\lambda)} = \hat{\mu}\cdot\hat{\rho}.$$

Suppose we can choose ρ so that

24.27 (8.350)
$$\rho \ge 0, \int \rho = 1, \rho \in \mathcal{S},\ \hat{\rho}(t) = 0,\ |t| > \epsilon$$

for a given $\epsilon > 0$. Then we *know* $\hat{\mu}\hat{\rho}$ is conormal and indeed

24.28 (8.351)
$$\frac{d}{d\lambda}N\rho(\lambda) \sim C \operatorname{Vol}(X)\lambda^{n-1} + \dots$$
$$\Longrightarrow N_\rho(\lambda) \sim C' \operatorname{Vol}(X)\lambda^n + \text{ lots.}$$

So what we need to do is look at the *difference*

24.29 (8.352)
$$N_\rho(\lambda) - N(\lambda) = \int N(\lambda - \lambda')\rho(\lambda') - N(\lambda)\rho(\lambda').$$

It follows that a bound for N

24.30 (8.353)
$$|N(\lambda + \mu) - N(\lambda)| \le ((1 + |\lambda| + |\mu|)^{n-1}(1 + |\lambda|)$$

gives

(8.354)
$$N(\lambda) - N_\rho(\lambda) \le C\lambda^{n-1}$$

which is what we want. Now (24.31 8.355) follows if we have

24.31 (8.355)
$$N(\lambda + 1) - N(\lambda) \le C(1 + |\lambda|) \qquad t/\lambda.$$

This in turn follows from the *positivity* of ρ, since

24.32 (8.356)
$$\int \rho(\lambda-\lambda')\mu(\lambda') \le C(1+|\lambda|)^{n-1}.$$

Finally then we need to check the existence of ρ as in (8.350). If ϕ is real and even so is $\hat{\phi}$. Take ϕ with support in $(-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ and construct $\phi * \phi$, real and even with ϕ. □

8.9. Problems

19.45 Problem 8.5. Show that if E is a symplectic vector space, with non-degenerate bilinear form ω, then there is a basis $v_1, \dots, v_n, w_1, \dots, w_n$ of E such that in terms of the dual basis of E^*

D (8.357)
$$\omega = \sum_j v_j^* \wedge w_j^*.$$

Hint: Construct the w_j, v_j successive. Choose $v_1 \neq 0$. Then choose w_1 so that $\omega(v_1, w_1) = 1$. Then choose v_2 so $\omega(v_1, v_2) = \omega(w_1, v_2) = 0$ (why is this possible?) and w_2 so $\omega(v_2, w_2) = 1$, $\omega(v_1, w_2) = \psi(w_1, w_2) = 0$. Then proceed and conclude that (8.357) must hold.

Deduce that there is a linear transformation $T : E \longrightarrow \mathbb{R}^{2n}$ so that $\omega = T^* \omega_D$, with ω_D given by (8.137).

19.46 Problem 8.6. Extend problem 8.5 to show that T can be chosen to map a given Lagrangian plane $V \subset E$ to

(8.358)
$$\{x = 0\} \subset \mathbb{R}^{2n}$$

Hint: Construct the basis choosing $v_j \in V \ \forall\ j$!

19.47 Problem 8.7. Suppose S is a symplectic manifold. Show that the *Poisson bracket*

(8.359)
$$\{f, g\} = H_f g$$

makes $\mathcal{C}^\infty(S)$ into a Lie algebra.

CHAPTER 9

K-theory

9.1. Vector bundles

9.2. The ring $K(X)$

9.3. Chern-Weil theory and the Chern character

9.4. $K_1(X)$ and the odd Chern character

9.5. $\mathcal{C}^*$ algebras

9.6. K-theory of an algeba

9.7. The norm closure of $\Psi^0(X)$

9.8. The index map

9.9. Problems

CHAPTER 10

Hochschild homology

10.1. Formal Hochschild homology

The Hochschild homology is defined, formally, for any associative algebra. Thus if $\mathcal{A}$ is the algebra then the space of *formal* k-chains, for $k \in \mathbb{N}_0$ is the $(k+1)$-fold tensor product

$$\mathcal{A}^{\otimes(k+1)} = \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}. \tag{10.1}$$

The 'formal' here refers to the fact that for the 'large' topological algebras we shall consider it is wise to replace this tensor product by an appropriate completion, usually the 'projective' tensor product. At the formal level the differential defining the cohomolgy is given in terms of the product, $\star$, by

HHdifferential (10.2)

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = b'(a_0 \otimes a_1 \otimes \cdots \otimes a_k) + (-1)^k (a_0 \star a_k) \otimes a_1 \otimes \cdots \otimes a_{k-1},$$
$$b'(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = \sum_{j=0}^{k-1} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_k.$$

Lemma 10.1. *Both the partial map, b', and the full map, b, are differentials, that is*

$$(b')^2 = 0 \text{ and } b^2 = 0. \tag{10.3}$$

Proof. This is just a direct computation. From (HHdifferential 10.2) it follows that

$$\begin{aligned}(10.4)\quad & (b')^2(a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_m) \\ &= \sum_{j=2}^{m-1} \sum_{p=0}^{j-2} (-1)^j (-1)^p (\cdots \otimes a_{p+1} \star a_p \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_m) \\ &\quad - \sum_{j=1}^{m-1} (\cdots \otimes a_{j+1} \star a_j \star a_{j-1} \otimes \cdots) - \sum_{j=0}^{m-2} (\cdots \otimes a_{j+21} \star a_{j+1} \star a_j \star \otimes \cdots) \\ &+ \sum_{j=0}^{m-3} \sum_{p=j+2}^{m-1} (-1)^j (-1)^{p-1} (a_0 \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_{p+1} \star a_p \otimes \cdots) = 0.\end{aligned}$$

Similarly, direct computation shows that

$$(b-b')b'(a_0\otimes\cdots\otimes a_m) = (-1)^{m-1}(a_1\star a_0\star a_m\otimes\cdots a_{m-1})$$
$$+\sum_{i=1}^{m-2}(-1)^{i+m-1}(a_0\star a_m\otimes\cdots\otimes a_{i+1}\star a_i\otimes\cdots)+(a_0\star a_m\star a_{m-1}\otimes\cdots),$$
$$b'(b-b')(a_0\otimes\cdots\otimes a_m) = (-1)^{m}(a_1\star a_0\star a_m\otimes\cdots a_{m-1})$$
$$+\sum_{i=1}^{m-2}(-1)^{i+m}(a_0\star a_m\otimes\cdots\otimes a_{i+1}\star a_i\otimes\cdots)\text{ and}$$
$$(b-b')^2(a_0\otimes\cdots\otimes a_m) = -(a_0\star a_m\star a_{m-1}\otimes\cdots)$$

so

$$(b-b')b'+b'(b-b') = -(b-b')^2. \tag{10.5}$$

□

The difference between these two differentials is fundamental, roughly speaking b' is 'trivial'.

24.89 LEMMA 10.2. *For any algebra with identity the differential b' is acyclic, since it satifies*

$$b's+sb' = \operatorname{Id} \textit{ where} \tag{10.6}$$

$$s(a_0\otimes\cdots\otimes a_m) = \operatorname{Id}\otimes a_0\otimes\cdots\otimes a_m. \tag{10.7}$$

PROOF. This follows from the observation that

$$b'(\operatorname{Id}\otimes a_0\otimes\cdots\otimes a_m) = a_0\otimes\cdots\otimes a_m+\sum_{i=1}^{m}(-1)^i(\operatorname{Id}\otimes\cdots a_i\star a_{i-2}\otimes\cdots). \tag{10.8}$$

□

DEFINITION 10.1. *An associative algebra is said to be* H-unital *if its b' complex is acyclic.*

Thus the preceeding lemma just says that every unital algebra is H-unital.

10.2. Hochschild homology of polynomial algebras

Consider the algebra $\mathbb{C}[x]$ of polynomials in n variables[1], $x\in\mathbb{R}^n$ (or $x\in\mathbb{C}^n$ it makes little difference). This is not a finite dimensional algebra but it is filtered by the finite dimensional subspaces, $P_m[x]$, of polynomials of degree at most m;

$$\mathbb{C}[x]=\bigcup_{m=0}^{\infty}P_m[x],\ \ P_m[x]\subset P_{m+1}[x].$$

Furthermore, the Hochschild differential does not increase the total degree so it is enough to consider the formal Hochschild homology.

The chain spaces, given by the tensor product, just consist of polynomials in $n(k+1)$ variables

$$(\mathbb{C}[x])^{\hat{\otimes}(k+1)}=\mathbb{C}[x_0,x_1,\dots,x_k],\ \ x_j\in\mathbb{R}^n.$$

[1]The method used here to compute the homology of a polynomial algebra is due to Sergiu Moroianu; thanks Sergiu.

Furthermore composition acts on the tensor product by

$$p(x_0)q(x_1) = p \otimes q \longmapsto p(x_0)q(x_0)$$

which is just restriction to $x_0 = x_1$. Thus the Hochschild differential can be written

$$b : \mathbb{C}[x_0, \dots, x_k] \longrightarrow \mathbb{C}[x_0, \dots, x_{k-1}],$$

$$(bq)(x_0, x_1, \dots, x_{k-1}) = \sum_{j=0}^{k-1} (-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\ + (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0).$$

One of the fundamental results on Hochschild homology is

Theorem 10.1. *The Hochschild homology of the polynomail algebra in n variables is*

24.91 (10.9)
$$\mathrm{HH}_k(\mathbb{C}[x]) = \mathbb{C}[x] \otimes \Lambda^k(\mathbb{C}^n),$$

with the identification given by the map from the chain spaces

$$\mathbb{C}[x_0, \dots, x_k] \ni q \longrightarrow \sum_{1 \le j_i \le n} \frac{\partial}{\partial x_1^{j_1}} \cdots \frac{\partial}{\partial x_k^{j_k}} p\big|_{x = x_0 = x_1 = \dots = x_k} dx_1^{j_1} \wedge \dots \wedge dx_k^{j_k}.$$

Note that the appearance of the original algebra $\mathbb{C}[x]$ on the left in (24.91 10.9) is not surprising, since the differential commutes with multilplication by polynomails in the first variable, x_0

$$b(r(x_0)q(x_0, \dots, x_k)) = r(x_0)(bq(x_0, \dots, x_k)).$$

Thus the Hochschild homology is certainly a module over $\mathbb{C}[x]$.

Proof. Consider first the cases of small k. If $k = 0$ then b is identically 0. If $k = 1$ then again

$$(bq)(x_0) = q(x_0, x_0) - q(x_0, x_0) = 0$$

vanishes identically. Thus the homology in dimension 0 is indeed $\mathbb{C}[x]$.

Suppose that $k > 1$ and consider the subspace of $\mathbb{C}[x_0, x_1, \dots, x_k]$ consisting of the elements which are independent of x_1. Then the first two terms in the definition of b cancel and

$$(bq)(x_0, x_1, \dots, x_{k-1}) = \sum_{j=2}^{k-1} (-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\ + (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0),\ \partial_{x_1} q \equiv 0.$$

It follows that bq is also independent of x_1. Thus there is a well-defined subcomplex on polynomails independend of x_1 given by

$$\mathbb{C}[x_0, x_2, \dots, x_k] \ni q \longmapsto (\tilde{b}q)(x_0, x_2, \dots, x_{k-1}) \\ = \sum_{j=2}^{k-1} (-1)^j p(x_0, x_2, x_2, x_3 \dots, x_{k-1}) + \sum_{j=3}^{k-1} (-1)^j \\ p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) + (-1)^k q(x_0, x_2, \dots, x_{k-1}, x_0)$$

The reordering of variables $(x_0, x_2, x_3, \dots, x_k) \longrightarrow (x_2, x_3, \dots, x_k, x_0)$ for each k, transforms $\tilde{b}$ to the reduced Hochschild differential b' acting in k variables. Thus $\tilde{b}$ is acyclic.

Similarly consider the subspace of $\mathbb{C}[x_0, x_1, \dots, x_k]$ consisting of the polynomials which vanish at $x_1 = x_0$. Then the first term in the definition of b vanishes and the action of the differential becomes

24.92 (10.10)
$$\begin{aligned}(bq)(x_0, x_1, \dots, x_{k-1}) = p(x_0, x_1, x_1, x_2, \dots, x_{k-1}) + \\ \sum_{j=2}^{k-1}(-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\ + (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0), \text{ if } b(x_0, x_0, x_2, \dots) \equiv 0.\end{aligned}$$

It follows that bq also vanishes at $x_1 = x_0$.

By Taylor's theorem any polynomial can be written uniquely as a sum

$$q(x_0, x_1, x_2, \dots, x_k) = q_1'(x_0, x_1, x_2, \dots, x_k) + q''(x_0, x_2, \dots, x_k)$$

of a polynomial which vanishes at $x_1 = x_0$ and a polynomial which is independent of x_1. From the discussion above, this splits the complex into a sum of two subcomplexes, the second one of which is acyclic. Thus the Hochschild homology is the same as the homology of b, which is then given by (10.10), acting on the spaces

24.93 (10.11)
$$\{q \in \mathbb{C}[x_0, x_1, \dots, x_k]; q(x_0, x_1, \dots) = 0\}.$$

This argument can be extended iteratively. Thus, if $k > 2$ then b maps the subspace of (10.11) of functions independent of x_2 to functions independent of x_2 and on these subspaces acts as b' in $k-2$ variables; it is therefore acyclic. Similar it acts on the complementary spaces given by the functions which vanish on $x_2 = x_1$. Repeating this argument shows that the Hochschild homology is the same as the homology of b acting on the smaller subspaces

24.94 (10.12)
$$\begin{gathered}\{q \in \mathbb{C}[x_0, x_1, \dots, x_k]; q(\dots, x_{j-1}, x_j, \dots) = 0,\ j = 1, \dots, k\}, \\ (bq)(x_0, x_1, \dots, x_{k-1}) = (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0).\end{gathered}$$

Note that one cannot proceed further directly, in the sense that one cannot reduce to the subspace of functions vanishing on $x_k = x_0$ as well, since this subspace is not linearly independent of the previous ones[2]

$$x_k - x_0 = \sum_{j=0}^{k-1}(x_{j_1} - x_j).$$

It is precisely this 'non-transversality' of the remaining restriction map in (10.12) which remains to be analysed.

Now, let us we make the following change of variable in each of these reduced chain spaces setting

$$y_0 = x_0,\ y_1 = x_j - x_{j-1},\ \text{for } j = 1, \dots, k.$$

Then the differential can be written in terms of the pull-back operation

$$E_P : \mathbb{R}^{nk} \hookrightarrow \mathbb{R}^{n(k+1)},\ E_P(y_0, y_1, \dots, y_{k-1}) = (y_0, y_1, \dots, y_{k-1}, -\sum_{j=1}^{k-1} y_j),$$
$$bq = (-1)^k E_P^* q,$$

[2]Hence Taylor's theorem cannot be applied.

The variable $x_0 = y_0$ is a pure parameter, so can be dropped from the notation (and restored at the end as the factor $\mathbb{C}[x]$ in (10.9)). Also, as already noted, the degree of a polynomial (in all variables) does not increase under any of these pull-back operations, in fact they all preserve the total degree of homogeneity so it suffices to consider the differential b acting on the spaces of homogeneous polynomials which vanish at the origin in each factor

$$Q_k^m = \{q \in \mathbb{C}^m[y_1, \dots, y_k]; q(sy) = s^m q(y),\ q(y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_k) = 0\}$$
$$b : Q_k^m \longrightarrow Q_{k-1}^m,\ bq = (-1)^* E_P^* q.$$

To analyse this non-transversality further, let $J_i \subset \mathbb{C}[y_1, \dots, y_k]$ be the ideal generated by the n monomials y_i^l, $l = 1, \dots, n$. Thus, by Taylor's theorem,

$$J_i = \{q \in \mathbb{C}[y_1, \dots, y_k]; q(y_1, y_2, \dots, y_{j-1}, 0, y_j, y_k) = 0.$$

Similary set

$$J_P = \{q \in \mathbb{C}[y_1, \dots, y_k]; q(y_1, \dots, -\sum_{j=1}^{k-1} y_j) = 0)$$

For any two ideals I and J, let $I \cdot J$ be the span of the products. Thus for these particular ideals an element of the product is a sum of terms each of which has a factor vanishing on the corresponding linear subspace. For each k there are $k+1$ ideals and, by Taylor's theorem, the intersection of any k of them is equal to the span of the product of those k ideals. For the k coordinate ideals this is Taylor's theorem as used in the reduction above. The general case of any k of the ideals can be reduced to this case by linear change of coordinates. The question then, is structure of the intersection of all $k+1$ ideals. The proof of the theorem is therefore completed by the following result. □

Lemma 10.3. *The intersection $Q_k^m \cap J_P = Q^m \cdot J_P$ for every $m \neq k$ and*

24.95 (10.13) $$Q_k^k \cap J_P = \Lambda^k(\mathbb{C}^n).$$

Proof. When $m < k$ the ideal Q_k^m vanishes, so the result is trivial.

Consider the case in (10.13), when $m = k$. A homogeneous polynomial of degree k in k variables (each in $\mathbb{R}^n$) which vanishes at the origin in each variable is necessarily linear in each variable, i.e. is just a k-multilinear function. Given such a multilinear function $q(y_1, \dots, y_k)$ the condition that $bq = 0$ is just that

24.96 (10.14) $$q(y_1, \dots, y_{k-1}, -y_1 - y_2 - \dots - y_{k-1}) \equiv 0.$$

Using the linearity in the last variable the left side can be expanded as a sum of $k-1$ functions each quadratic in one variables y_j and linear in the rest. Thus the vanishing of the sum implies the vanishing of each, so

$$q(y_1, \dots, y_{k-1}, y_j) \equiv 0\ \forall\ j = 1, \dots, k-1.$$

This is the statement that the multlinear function q is antisymmetric between the jth and kth variables for each $j < k$. Since these exchange maps generate the permutation group, q is necessarily totally antisymmetric. This proves the isomorphism (10.13) since $\Lambda^k(\mathbb{C}^n)$ is the space of complex-valued totally antisymmetric k-linear forms.[3]

Thus it remains to consider the case $m \geq k+1$. Consider a general element $q \in Q_k^m \cap J_P$. To show that it is in $Q_k^m \cdot J_P$ we manipulate it, working modulo $Q_k^m \cap J_P$,

[3] Really on the dual but that does not matter at this stage.

and use induction over k. Decompose q as a sum of terms q_l, each homogeneous in the first variable, y_1, of degree l. Since q vanishes at $y_1 = 0$ the first term is q_1, linear in y_1. The condition $bq = 0$, i.e. $q \in J_P$, is again just (10.14). Expanding in the last variable shows that the only term in bq which is linear in y_1 is

$$q_1(y_1, \ldots, y_{k-1}, -y_2 - \cdots - y_{k-1}).$$

Thus the coefficient of $y_{1,i}$, the ith component of y_1 in q_1, is an element of Q^{m-1}_{k-1} which is in the ideal $J_P(\mathbb{R}^{k-1})$, i.e. for $k-1$ variables. This ideal is generated by the components of $y_2 + \cdots + y_k$. So we can proceed by induction and suppose that the result is true for less than k variables for all degrees of homogeneity. Writing $y_2 + \cdots + y_k = (y_1 + y_2 + \cdots + y_k) - y_1$ It follows that, modulo $Q^m_k \cdot J_P$, q_1 can be replaced by a term of one higher homogeneity in y_1. Thus we can assume that $q_i = 0$ for $i < 2$. The same argument now applies to q_2; expanded as a polynomial in y_1 the coefficients must be elements of $Q^{m-2}_{k-1} \cap J_P$. Thus, unless $m - 2 = k - 1$, i.e. $m = k + 1$, they are, by the inductive hypothesis, in $Q^{m-2}_{k-1} \cdot J_P(\mathbb{R}^{k-1})$ and hence, modulo $Q^m_k \cdot J_P$, q_2 can be absorbed in q_3. This argument can be continued to arrange that $q_i \equiv 0$ for $i < m - k + 1$. In fact $q_i \equiv 0$ for $i > m - k + 1$ by the assumption that $q \in Q^m_k$.

Thus we are reduced to the assumption that $q = q_{m-k+1} \in Q^m_k \cap J_P$ is homogeneous of degree $m-k+1$ in the first variable. It follows that it is multilinear in the last $k-1$ variables. The vanishing of bq shows that it is indeed totally antisymmetric in these last $k-1$ variables. Now for each non-zero monomial consider the map $J : \{1, 2, \ldots, n\} \longrightarrow \mathbb{N}_0$ such that $J(i)$ is the number of times a variable $y_{l,i}$ occurs for some $1 \le l \le k$. The decomposition into the sum of terms for each fixed J is preserved by b. It follows that we can assume that q has only terms corresponding to a fixed map J. If $J(i) > 1$ for any i then a factor $y_{1,i}$ must be present in q, since it is antisymmetric in the other $k-1$ variables. In this case it can be written $y_{1,i}q'$ where $bq' = 0$. Since q' is necessarily in the product of the indeals $J_2 \cdot \ldots J_k \cdot J_P$ it follows that $q' \in Q^m \cdot J_P$. Thus we may assume that $J(i) = 0$ or 1 for all i. Since the extra variables now play no rôle we may assume that $n = m$ is the degree of homogeneity and each index i occurs exactly once.

For convenience let us rotate the last $k-1$ variables so the last is moved to the first position. Polarizing q in the first variable, it can be represented uniquely as an n-multilinear function on $\mathbb{R}^n$ which is symmetric in the first $n-k+1$ variables, totally antisymmetric in the last $k-1$ and has no monomial with repeated index. Let $M_{k-1}(n)$ be the set of such multilinear funtions. The vanishing of bq now corresponds to the vanishing of the symmetrization of q in the first $n-k+2$ variables. By the antisymmetry in the second group of variables this gives a complex

$$M_n(n) \xrightarrow{b_n} M_{n-1}(n) \xrightarrow{b_{n-1}} \cdots \qquad \xrightarrow{b_2} M_1(n) \xrightarrow{b_1} M_0 \xrightarrow{b_0} 0.$$

The remaining step is to show that this is exact.

Observe that $\dim(M_k(n)) = \binom{n}{k}$ since there is a basis of $M_k(n)$ with elements labelled by the subsets $I \subset \{1, \ldots, n\}$ with k elements. Indeed let ω be a non-trivial k-multilinear function of k variables and let ω_I be this function on $\mathbb{R}^k \subset \mathbb{R}^n$ identified as the set of variables indexed by I. Then if $a \in M_0(n-k)$ is a basis of this 1-dimensional space and a_I is this function on the complementary $\mathbb{R}^{n-k}$ the tensor products $a_I\omega_I$ give a basis. Thus there is an isomorphism

$$M_k \ni q = \sum_{I\subset\{1,\dots,n\},|I|=k} c_I a_I \otimes \omega_I \longmapsto \sum_{I\subset\{1,\dots,n\},|I|=k} c_I \otimes \omega_I \in \Lambda^k(\mathbb{R}^n).$$

Transfered to the exterior algebra by this isomorphism the differential b is just contraction with the vector $e_1 + e_2 + \cdots + e_n$ (in the first slot). A linear transformation reducing this vector to e_1 shows immediately that this (Koszul) complex is exact, with the null space of b_k on $\Lambda^k(\mathbb{R}^n)$ being spanned by those ω_I with $1 \in I$ and the range of b_{k+1} spanned by those with $1 \notin I$. The exactness of this complex completes the proof of the lemma. $\square$

10.3. Hochschild homology of $\mathcal{C}^\infty(X)$

The first example of Hochschild homology that we shall examine is for the commutative algebra $\mathcal{C}^\infty(X)$ where X is any $\mathcal{C}^\infty$ manifold (compact or not). As noted above we need to replace the tensor product by some completion. In the present case observe that for any two manifolds X and Y

$$\mathcal{C}^\infty(X)\otimes\mathcal{C}^\infty(Y) \subset \mathcal{C}^\infty(X\times Y) \tag{10.15}$$

is dense in the $\mathcal{C}^\infty$ topology. Thus we simply declare the space of k-chains for Hochschild homology to be $\mathcal{C}^\infty(X^{k+1})$, which can be viewed as a natural completion[4] of $\mathcal{C}^\infty(X)^{\otimes(k+1)}$. Notice that the product of two functions can be written in terms of the tensor product as

$$a\cdot b = D^*(a\otimes b),\ a, b \in \mathcal{C}^\infty(X),\ D : X \ni z \longmapsto (z, z) \in X^2. \tag{10.16}$$

The variables in X^{k+1} will generally be denoted $z_0,\ z_1, \dots, z_k$. Consider the 'diagonal' submanifolds

$$D_{i,j} = \{(z_0, z_1, \dots, z_k); z_i = z_j\},\ i, j = 0, \dots, m,\ i \neq j. \tag{10.17}$$

We shall use the same notation for the natural embedding of X^k as each of these submanifolds, at least for $j = i + 1$ and $i = 0$, $j = m$,

$$D_{i,i+1}(x_0, \dots, z_{m-1}) = (z_0, \dots, z_i, z_i, z_{i+1}, \dots, z_{m-1}) \in D_{i,i+1},\ i = 0, \dots, m-1$$
$$D_{m,0}(z_0, \dots, z_{m-1}) = (z_0, \dots, z_{m-1}, z_0).$$

Then the action of b' and b on the tensor products, and hence on all chains, can be written

$$b'\alpha = \sum_{i=0}^{m-1}(-1)^i D^*_{i,i+1}\alpha,\ b\alpha = b'\alpha + (-1)^m D^*_{m,0}\alpha. \tag{10.18}$$

[4] One way to justify this is to use results on smoothing operators. For finite dimensional linear spaces V and W the tensor product can be realized as

$$V\otimes W = \hom(W', V)$$

the space of linear maps from the dual of W to V. Identifying the topological dual of $\mathcal{C}^\infty(X)$ with $\mathcal{C}_c^{-\infty}(X;\Omega)$, the space of distributions of compact support, with the weak topology, we can identify the *projective* tensor product $\mathcal{C}^\infty(X)\hat{\otimes}\mathcal{C}^\infty(X)$ as the space of continuous linear maps from $\mathcal{C}_c^{-\infty}(X;\Omega)$ to $\mathcal{C}^\infty(X)$. These are precisely the smoothing operators, corresponding to kernels in $\mathcal{C}^\infty(X\times X)$.

HH.ciX

THEOREM 10.2. *The differential b' is acyclic and the homology*[5] *of the complex*

$$\dots \xrightarrow{b} \mathcal{C}^\infty(X^{k+1}) \xrightarrow{b} \mathcal{C}^\infty(X^k) \xrightarrow{b} \dots \tag{10.19}$$

is naturally isomorphic to $\mathcal{C}^\infty(X;\Lambda^*)$.

Before proceeding to the proof proper we note two simple lemmas.

HL.ciX

LEMMA 10.4. [6]*For any $j = 0,\dots,m-1$, each function $\alpha \in \mathcal{C}^\infty(X^{k+1})$ which vanishes on $D_{i,i+1}$ for each $i \le j$ can be written uniquely in the form*

$$\alpha = \alpha' + \alpha'',\ \alpha',\ \alpha'' \in \mathcal{C}^\infty(X^{k+1})$$

where α'' vanishes on $D_{i,i+1}$ for all $i \le j+1$ and α' is independent of z_{j+1}.

PROOF. Set $\alpha' = \pi_{j+1}^*(D_{j,j+1}^*\alpha)$ where $\pi_j : X^{k+1} \longrightarrow X^k$ is projection off the jth factor. Thus, essentially by definition, α' is independent of z_{j+1}. Moreover, $\pi_{j+1}D_{j,j+1} = \mathrm{Id}$ so $D_{j,j+1}^*\alpha' = D_{j,j+1}^*\alpha$ and hence $D_{j,j+1}^*\alpha'' = 0$. The decomposition is clearly unique, and for $i < j$,

$$D_{j,j+1} \circ \pi_{j+1} \circ D_{i,i+1} = D_{i,i+1} \circ F_{i,j} \tag{10.20}$$

for a smooth map $F_{i,j}$, so α' vanishes on $D_{i,i+1}$ if α vanishes there. □

24.33

LEMMA 10.5. *For any finite dimensional vector space, V, the k-fold exterior power of the dual, $\Lambda^k V^*$, can be naturally identified with the space of functions*

(10.21)
$$\big\{u \in \mathcal{C}^\infty(V^k); u(sv) = s^k v,\ s \ge 0,\ u \restriction (V^i \times \{0\} \times V^{k-i-1}) = 0 \text{ for } i = 0,\dots,k-1 \\ \text{and } u \restriction G = 0,\ G = \{(v_1,\dots,v_k) \in V^k; v_1 + \dots + v_k = 0\}\big\}.$$

PROOF. The homogeneity of the smooth function, u, on V^k implies that it is a homogeneous polynomial of degree k. The fact that it vanishes at 0 in each variable then implies that it is multlinear, i.e. is linear in each variable. The vanishing on G implies that for any j and any $v_i \in V$, $i \ne j$,

$$\sum_{i \ne j} u(v_1,\dots,v_{j-1},v_i,v_{j+1},\dots,v_k) = 0. \tag{10.22}$$

Since each of these terms is quadratic (and homogeneous) in the corresponding variable v_i, they must each vanish identically. Thus, u vanishes on $v_i = v_j$ for each $i \ne j$; it is therefore totally antisymmetric as a multlinear form, i.e. is an element of $\Lambda^k V^*$. The converse is immediate, so the lemma is proved. □

PROOF OF THEOREM HH.ciX 10.2. The H-unitality[7] of $\mathcal{C}^\infty(X)$ follows from the proof of Lemma 24.69 10.61 which carries over *verbatim* to the larger chain spaces.

By definition the Hochschild homology in degree k is the quotient

$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = \big\{u \in \mathcal{C}^\infty(X^{k+1}); bu = 0\big\}/b\mathcal{C}^\infty(X^{k+2}). \tag{10.23}$$

The first stage in identifying this quotient is to apply Lemma HL.ciX 10.4 repeatedly. Let us carry through the first step separately, and then do the general case.

[5]This homology is properly referred to as the *continuous* Hochschild homology of the topological algebra $\mathcal{C}^\infty(X)$.

[6]As pointed out to me by Maciej Zworski, this is a form of Hadamard's lemma.

[7]Meaning here the *continuous* H-unitality, that is the acyclicity of b' on the chain spaces $\mathcal{C}^\infty(X^{k+1})$.

For $j = 0$, consider the decomposition of $u \in \mathcal{C}^\infty(X^{k+1})$ given by Lemma 10.4, thus

24.37 (10.24)
$$u = u_0 + u_{(1)},\ u_0 \in \pi_1^*\mathcal{C}^\infty(X^k),\ u_{(1)} \in J_1^{(k)} = \big\{u \in \mathcal{C}^\infty(X^{k+1}); u \restriction D_{0,1} = 0\big\}.$$

Now each of these subspaces of $\mathcal{C}^\infty(X^{k+1})$ is mapped into the corresponding subspace of $\mathcal{C}^\infty(X^k)$ by b; i.e. they define subcomplexes. Indeed,

$$u \in \pi_1^*\mathcal{C}^\infty(X^k) \Longrightarrow D_{0,1}^* u = D_{1,2}^* u \text{ so}$$
$$u = \pi_1^* v \Longrightarrow bu = \pi_1^* Bv,\ B^*v = -\sum_{i=1}^{k-1}(-1)^i D_{i,i+1}^* u + (-1)^k D_{k-1,0}^* v.$$

For the other term

24.36 (10.25)
$$bu_{(1)} = \sum_{i=1}^{k-1}(-1)^i D_{i,i+1}^* u_{(1)} + (-1)^k D_{k,0}^* u_{(1)} \Longrightarrow bu_{(1)} \in J_1^{(k-1)}.$$

Thus, $bu = 0$ is equivalent to $bu_0 = 0$ and $bu_{(1)} = 0$. From (10.3), defining an isomorphism by

(10.26)
$$E_{(k-1)} : \mathcal{C}^\infty(X^k) \longrightarrow \mathcal{C}^\infty(X^k),\ E_{(k-1)}v(z_1,\dots,z_k) = v(z_2,\dots,z_k,z_1),$$

it follows that

24.41 (10.27)
$$B = -E_{(k-1)}^{-1} b' E_{(k-1)}$$

is conjugate to b'. Thus B is acyclic so in terms of (10.24)

24.39 (10.28)
$$bu = 0 \Longrightarrow u - u_{(1)} = bw,\ w = \pi_1^* v'.$$

As already noted this is the first step in an inductive procedure, the induction being over $1 \le j \le k$ in Lemma 10.4. Thus we show inductively that

24.38 (10.29)
$$bu = 0 \Longrightarrow u - u_{(j)} = bw,$$
$$u_{(j)} \in J_j^{(k)} = \big\{u \in \mathcal{C}^\infty(X^{k+1}); u \restriction D_{i,i+1} = 0,\ 0 \le i \le j-1\big\}.$$

For $j = 1$ this is (10.28). Proceeding inductively we may suppose that $u = u_{(j)}$ and take the decomposition of Lemma 10.4, so

24.40 (10.30)
$$u_{(j)} = u' + u_{(j+1)},\ u_{(j+1)} \in J_{j+1}^{(k)},\ u' = \pi_{j+1}^* v \in J_j^{(k)}.$$

Then, as before, $bu_{(j)} = 0$ implies that $bu' = 0$. Furthermore, acting on the space $\pi_{j+1}^*\mathcal{C}^\infty(X^k) \cap J_{(j)}^k$, b is conjugate to b' acting in $k + 1 - j$ variables. Thus, it is again acyclic, so $u_{(j)}$ and $u_{(j+1)}$ are homologous as Hochschild k-cycles.

The end point of this inductive procedure is that each k-cycle is homologous to an element of

24.42 (10.31)
$$J^{(k)} = J_k^{(k)} = \big\{u \in \mathcal{C}^\infty(X^{k+1}); D_{i,i+1}^* u = 0,\ i \le i \le k-1\big\}.$$

Acting on this space $bu = (-1)^k D_{k,0}^* u$, so we have shown that

24.43 (10.32)
$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = M^{(k)}/\big(M^{(k)} \cap b\mathcal{C}^\infty(X^{k+1})\big),\ M^{(k)} = \big\{u \in J^{(k)}; D_{k,o}^* u = 0\big\}.$$

Now consider the subspace

24.44 (10.33)
$$\tilde{M}^{(k)} = \big\{u \in \mathcal{C}^\infty(X^{k+1});\\ u = \sum_{\text{finite},\ 0\le j\le k-1} (f(z_j) - f(z_{j+1}))\, u_{f,j},\ u_{f,j} \in M^{(k)},\ f \in \mathcal{C}^\infty(X).\big\}.$$

If $u = (f(z_j) - f(z_{j+1}))v$, with $v \in M^{(k)}$ set

24.45 (10.34)
$$w(z_0, z_1, \dots, z_j, z_{j+1}, z_{j+2}, \dots, z_{k+1}) \\ = (-1)^j (f(z_j) - f(z_{j+1}))v(z_0, \dots, z_j, z_{j+2}, z_{j+3}, \dots, z_k).$$

Then, using the assumed vanishing of v, $bw = u$.[8] Thus all the elements of $\tilde{M}^{(k)}$ are exact.

Let us next compute the quotient $M^{(k)}/\tilde{M}^{(k)}$. Linearizing in each factor of X around the submanifold $z_0 = z_1 = \dots = z_k$ in V^k defines a map

24.47 (10.35)
$$\mu : M^{(k)} \ni u \longrightarrow u' \in \mathcal{C}^\infty(X; TX \otimes \dots \otimes T^*X).$$

The map is defined by taking the term of homogeneity k in a normal expansion around the submanifold. The range space is therefore precisely the space of sections of the k-fold tensor product bundle which vanish on the subbundle defined in each fibre by $v_1 + \dots + v_k = 0$. Thus, by Lemma 10.5, μ actually defines a sequence

24.48 (10.36)
$$0 \longrightarrow \tilde{M}^{(k)} \hookrightarrow M^{(k)} \xrightarrow{\mu} \mathcal{C}^\infty(X; \Lambda^k X) \longrightarrow 0.$$

24.90 Lemma 10.6. *For any k, (10.36) is a short exact sequence.*

Proof. So far I have a rather nasty proof by induction of this result, there should be a reasonably elementary argument. Any offers? □

From this the desired identification, induced by μ,

24.49 (10.37)
$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = \mathcal{C}^\infty(X; \Lambda^k X)$$

follows, once is is shown that no element $u \in M^{(k)}$ with $\mu(u) \neq 0$ can be exact. This follows by a similar argument. Namely if $u \in M^{(k)}$ is exact then write $u = bv$, $v \in \mathcal{C}^\infty(X^k)$ and apply the decomposition of Lemma 10.4 to get $v = v_0 + v_{(1)}$. Since $u = 0$ on $D_{1,0}$ it follows that $bv_0 = 0$ and hence $u = bv_{1)}$. Proceeding inductively we conculde that $u = bv$ with $v \in M^{(k+1)}$. Now, $\mu(bv) = 0$ by inspection. □

10.4. Commutative formal symbol algebra

As a first step towards the computation of the Hochschild homology of the algebra $\mathcal{A} = \Psi^{\mathbb{Z}}(X)/\Psi^{-\infty}(X)$ we consider the formal algebra of symbols with commutative product. Thus,

24.50 (10.38)
$$\mathcal{A} = \big\{(a_j)_{j=-\infty}^{\infty}; a_j \in \mathcal{C}^\infty(S^*X; P^{(j)}),\ a_j = 0 \text{ for } j >> 0\big\}.$$

Here $P^{(k)}$ is the line bundle over S^*X with sections consisting of the homogeneous functions of degree k on $T^*X \setminus 0$. The multiplication is as functions on $T^*X \setminus 0$, so

$$(a_j) \cdot (b_j) = (c_j),\ c_j = \sum_{k=-\infty}^{\infty} a_{j-k} b_k$$

[8] Notice that $v(z_0, \dots, z_j, z_{j+2}, \dots, z_{k+1})$ vanishes on $z_{i+1} = z_i$ for $i < j$ and $i > j+1$ and also on $z_0 + z_1 + \dots + z_{k+1} = 0$ (since it is independent of z_{j+1} and $bv = 0$.

using the fact that $P^{(l)} \otimes P^{(k)} \equiv P^{(l+k)}$. We take the completion of the tensor product to be

Hochchains (10.39)
$$\mathcal{B}^{(k)} = \Big\{u \in \mathcal{C}^\infty((T^*X \setminus 0)^{k+1}); u = \sum_{\text{finite}} u_I,\\ u_I \in \mathcal{C}^\infty(S^*X; P^{(I_0)} \otimes P^{(I_1)} \otimes \cdots \otimes P^{(I_k)}),\ |I| = k\Big\}.$$

That is, an element of $\mathcal{B}^{(k)}$ is a finite sum of functions on the $(k+1)$-fold product of $T^*X \setminus 0$ which are homogeneous of degree I_j on the jth factor, with the sum of the homogeneities being k. Then the Hochschild homology is the cohomology of the subcomplex of the complex for $\mathcal{C}^\infty(T^*X)$

24.54 (10.40)
$$\cdots \xrightarrow{b} \mathcal{B}^{(k)} \xrightarrow{b} \mathcal{B}^{(k-1)} \xrightarrow{b} \cdots$$

24.53 THEOREM 10.3. *The cohomology of the complex (10.40) for the commutative product on $\mathcal{A}$ is*

24.55 (10.41)
$$\mathrm{HH}_k(\mathcal{A}) \equiv \big\{\alpha \in \mathcal{C}^\infty(T^*X \setminus 0; \Lambda^k(T^*X); \alpha \text{ is homogeneous of degree } k\big\}.$$

10.5. Hochschild chains

The completion of the tensor product that we take to define the Hochschild homology of the 'full symbol algebra' is the same space as in (10.39) but with the non-commutative product derived from the quantization map for some Riemann metric on X. Since the product is given as a formal sum of bilinear differential operators it can be take to act on an pair of factors.

HH.psi (10.42)
$$\cdots \xrightarrow{b^{(\star)}} \mathcal{B}^{(k)} \xrightarrow{b^{(\star)}} \mathcal{B}^{(k-1)} \xrightarrow{b^{(\star)}} \cdots$$

The next, and major, task of this chapter is to describe the cohomology of this complex.

24.56 THEOREM 10.4. *The Hochschild homolgy of the algebra, $\Psi^{\mathbb{Z}}_{\mathrm{phg}}(X)/\Psi^{-\infty}_{\mathrm{phg}}(X)$, of formal symbols of pseudodifferential operators of integral order, identified as the cohomology of the complex (10.42), is naturally identified with two copies of the cohomology of S^*X*[9]

24.58 (10.43)
$$\mathrm{HH}_k(\mathcal{A}; \circ) \equiv H^{2n-k}(S^*X) \oplus H^{2n-1-k}(S^*X).$$

10.6. Semi-classical limit and spectral sequence

The 'classical limit' in physics, especially quantuum mechanics, is the limit in which physical variables become commutative, i.e. the non-commutative coupling between position and momentum variables vanishes in the limit. Formally this typically involves the replacement of Planck's constant by a parameter $h \to 0$. A phenomenon is 'semi-classical' if it can be understood at least in Taylor series in this parameter. In this sense the Hochschild homology of the full symbol algebra is semi-classical and (following [**2**]) this is how we shall compute it.

The parameter h is introduced directly as an isomorphism of the space $\mathcal{A}$

$$L_h : \mathcal{A} \longrightarrow \mathcal{A},\ L_h(a_j)^*_{j=-\infty} = (h^j a_j)^*_{j=-\infty},\ h > 0.$$

[9]In particular the Hochschild homology vanishes for $k > 2\dim X$. For a precise form of the identification in (10.43) see (11.1).

Clearly $L_h \circ L_{h'} = L_{hh'}$. For $h \neq 1$, L_h is *not* an algebra morphism, so induces a 1-parameter family of products

24.59 (10.44) $$\alpha \star_h \beta = (L_h^{-1})(L_h\alpha \star L_h\beta).$$

In terms of the differential operators, associated to quantization by a particular choice of Riemann metric on X this product can be written

24.60 (10.45) $$\alpha \star_h \beta = (c_j)_{j=-\infty}^{*},\ c_j = \sum_{k=0}^{*}\sum_{l=-*}^{*} h^k P_k(a_{j-l-k}, b_l).$$

It is important to note here that the P_k, as differential operators on functions on T^*X, do only depend on k, which is the difference of homogeneity between the product $a_{j-l+k}b_l$, which has degree $j+k$ and c_j, which has degree j.

Since $\mathcal{A}$ with product $\star_h$ is a 1-parameter family of algebras, i.e. a deformation of the algebra $\mathcal{A}$ with product $\star = \star_1$, the Hochschild homology is 'constant' in h. More precisely the map L_h induces a canonical isomorphism

$$L_h^* : \mathrm{HH}_k(\mathcal{A}; \star_h) \equiv \mathrm{HH}_k(\mathcal{A}; \star).$$

The dependence of the product on h is smooth, so it is reasonable to expect the cycles to have smooth representatives as $h \to 0$. To investigate the consider Taylor series in h and define

(10.46) $$F_{p,k} = \big\{\alpha \in \mathcal{B}^{(k)};\ \exists\ \alpha(h) \in \mathcal{C}^\infty([0,1)_h; \mathcal{B}^{(k)}) \text{ with } \alpha(0) = \alpha \text{ and } b_h\alpha \in h^p\mathcal{C}^\infty([0,1)_h; \mathcal{B}^{k-1)}\big\},$$

24.61 (10.47) $$G_{p,k} = \big\{\alpha \in \mathcal{B}^{(k)};\ \exists\ \beta(h) \in \mathcal{C}^\infty([0,1)_h; \mathcal{B}^{(k+1)}) \text{ with } b_h\beta(h) \in h^{p-1}\mathcal{C}^\infty([0,1)_h; \mathcal{B}^{(k)} \text{ and } (t^{-p+1}b_h\beta)(0) = \alpha\big\}.$$

Here b_h is the differential defined by the product $\star_h$.

Notice that the $F_{p,k}$ decrease with increasing p, since the condition becomes stronger, while $G_{p,k}$ increases with p, the condition becoming weaker.[10] We define the 'spectral sequence' corresponding to this filtration by

$$E_{p,k} = F_{p,k}/G_{p,k}.$$

These can also be defined successively, in the sense that if

$$F'_{p,k} = \big\{u \in E_{p-1,k}; u = [u'],\ u' \in F_{p,k}\big\}$$
$$G'_{p,k} = \big\{e \in E_{p-1,k}; u = [u'],]u' \in G_{p,k}\big\}$$
$$\text{then } E_{p,k} \equiv F'_{p,k}/G'_{p,k}.$$

The basic idea[11] of a spectral sequence is that each $E_p = \bigoplus_k E_{p,k}$, has defined on it a differential such that the next spaces, forming E_{p+1}, are the cohomology space for the complex. This is easily seen from the definitions of $F_{p,k}$ as follows. If $\alpha \in F_{p,k}$ let $\beta(t)$ be a 1-parameter family of chains as in the defintion. Then consider

24.62 (10.48) $$\gamma(t^{-p}b_h\beta)(0) \in \mathcal{B}^{(k-1)}.$$

[10] If $\alpha \in G_{p,k}$ and $\beta(h)$ is the 1-parameter family of chains whose existence is required for the definition then $\beta'(h) = h\beta(h)$ satisfies the same condition with p increased to show that $\alpha \in G_{p+1,k}$.

[11] Of Leray I suppose, but I am not really sure.

This depends on the choice of β, but only up to a term in $G_{p,k-1}$. Indeed, let $\beta'(t)$ is another choice of extension of α satisfying the condition that $b_h\beta' \in h^p\mathcal{C}^\infty([0,1);\mathcal{B}^{(k-1)}$ and let γ' be defined by (10.48) with β replaced by β'. Then $\delta(t) = t^{-1}(\beta(h) - \beta'(h))$ satisfies the requirements in the definition of $G_{p,k-1}$, i.e. the difference $\gamma' - \gamma \in G_{p,k-1}$. Similarly, if $\alpha \in G_{p,k}$ then $\gamma \in G_{p,k}$.[12] The map so defined is a differential

$$b_{(p)} : E_{p,k} \longrightarrow E_{p,k-1},\ b^2_{(p)} = 0.$$

This follows from the fact that if $\mu = b_{(p)}\alpha$ then, by definition, $\mu = (t^{-p}b_h\beta)(0)$, where $\alpha = \beta(0)$. Taking $\lambda(t) = t^{-p}b_h\beta(t)$ as the extension of μ it follows that $b_h\lambda = 0$, so $b_{(p)}\mu = 0$.

Now, it follows directly from the definition that $F_{0,k} = E_{0,k} = \mathcal{B}^{(k)}$ since $G_{0,k} = \{0\}$. Furthermore, the differential $b_{(0)}$ induced on E_0 is just the Hochschild differential for the limiting product, $\star_0$, which is the commutative product on the algebra. Thus, Theorem 10.3 just states that

$$E_{1,k} = \bigoplus_{k=-\infty}^{*} \big\{u \in \mathcal{C}^\infty(T^*X \setminus 0; \Lambda^k); u \text{ is homogeneous of degree } k\big\}.$$

To complete the proof of Theorem 10.4 it therefore suffices to show that

24.63 (10.49) $$E_{2,k} \equiv H^{2n-k}(S^*X) \oplus H^{2n-1-k}(S^*X),$$

24.64 (10.50) $$E_{p,k} = E_{2,k},\ \forall\ p \geq 2,\ \text{and}$$

24.65 (10.51) $$\mathrm{HH}_k(\Psi^{\mathbb{Z}}_{\mathrm{phg}}(X)/\Psi^{-\infty}_{\mathrm{phg}}(X)) = \lim_{p\to\infty} E_{p,k}.$$

The second and third of these results are usually described, respectively, as the 'degeneration' of the spectral sequence (in this case at the 'E_2 term') and the 'convergence' of the spectral sequence to the desired cohomology space.

10.7. The E_2 term

As already noted, the $E_{1,k}$ term in the spectral sequence consists of the formal sums of k-forms, on $T^*X \setminus 0$, which are homogeneous under the $\mathbb{R}^+$ action. The E_2 term is the cohomology of the complex formed by these spaces with the differential $b_{(1)}$, which we proceed to compute. For simplicity of notation, consider the formal tensor prodoct rather than its completion. As already noted, for any $\alpha \in \mathcal{B}^{(k)}$ the function $b_h\alpha$ is smooth in h and from the definition of b,

24.66 (10.52) $$\frac{d}{dh}b_h\alpha(0) = \sum_{i=0}^{k-1}(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes P_1(a_{i+1}, a_i) \otimes a_{i+2} \otimes \cdots \otimes a_k \\ + (-1)^k P_1(a_0, a_k) \otimes a_1 \otimes \cdots \otimes a_{k-1},\ \alpha = a_0 \otimes \cdots \otimes a_k.$$

The general case is only more difficult to write, not different.[13] This certainly determines $b_1\alpha$ if α is a superposition of such terms with $b_0\alpha = 0$. Although (10.52) is explicit, it is not given directly in terms of the representation of α, assumed to satisfy $b_0\alpha = 0$ as a form on $T^*X \setminus 0$.

[12] Indeed, α is then the value at $h = 0$ of $\beta(t) = t^{-p+1}b_h\phi(t)$ which is by hypothesis smooth; clearly $b_h\beta \equiv 0$.

[13] If you feel it necessary to do so, resort to an argument by continuity towards the end of this discussion.

To get such an explicit formula we shall use the symplectic analogue of the Hodge isomorphism. Recall that in any local coordinates on X, x_i, $i = 1, \dots, n$, induce local coordinates x_i, ξ_i in the part of T^*X lying above the coordinate patch. In these canonical coordinates the symplectic form (which determines the Poisson bracket) is given by

24.73 (10.53) $$\omega = \sum_{k=1}^{n} d\xi_k \wedge dx_k.$$

This 2-form is non-degenerate, i.e. the n-fold wedge product $\omega^n \neq 0$. In fact this volume form fixes an orientation on T^*X. The symplectic form can be viewed as a non-degenerate antisymmetric bilinear form on $T_q(T^*X)$ at each point $q \in T^*X$, and hence by duality as a bilineear form on $T_q^*(T^*X)$. We denote this form in the same way as the Poisson bracket, since with the convention

$$\{a, b\}(q) = \{da, db\}_q$$

they are indeed the same. As a non-degenerate bilinear form on T^*Y, $Y = T^*X$ this also induces a bilinear form on the tensor algebra, by setting

$$\{e_1 \otimes \cdots \otimes e_k, f_1 \otimes \cdots \otimes f_k,\} = \prod_j \{e_j, f_j\}.$$

These bilinear forms are all antisymmetric and non-degenerate and restrict to be non-degnerate on the antisymmetric part, $\Lambda^k Y$, of the tensor algebra. Thus each of the form bundles has a bilinear form defined on it, so there is a natural isomorhism

24.72 (10.54) $$W_\omega : \Lambda_q^k Y \longrightarrow \Lambda_q^{2n-k} Y,\ \alpha \wedge W_\omega \beta = \{\alpha, \beta\}\omega^n,\ \alpha, \beta \in \mathcal{C}^\infty(Y, \Lambda^k Y),$$

for each k.

24.74 Lemma 10.7. *In canonical coordinates, as in (10.53), consider the basis of k-forms given by all increasing subsequences of length k,*

$$I : \{1, 2, \dots, k\} \longrightarrow \{1, 2, \dots, 2n\},$$

and setting

24.76 (10.55) $$\alpha_I = dz_{I(1)} \wedge dz_{I(2)} \wedge \cdots \wedge dz_{I(k)},\qquad (z_1, z_2, \dots, z_{2n}) = (x_1, \xi_1, x_2, \xi_2, \dots, x_n, \xi_n).$$

In terms of this ordering of the coordinates

24.75 (10.56) $$W_\omega(\alpha_I) = (-1)^{N(I)} \alpha_{W(I)}$$

where $W(I)$ is obtained from I by considering each pair $(2p-1, 2p)$ for $p = 1, \dots, n$, erasing it if it occurs in the image of I, inserting it into I if neither $2p-1$ nor $2p$ occurs in the range of I and if exactly one of $2p-1$ and $2p$ occurs then leaving it unchanged; $N(I)$ is the number of times $2p$ appears in the range of I without $2p-1$.

Proof. The Poisson bracket pairing gives, on 1-forms,

$$-\{dx_j, d\xi_j\} = 1 = \{d\xi_j, dx_j\}$$

with all other pairings zero. Extending this to k-forms gives

$$\begin{aligned} &\{\alpha_I, \alpha_J\} = 0 \text{ unless } (I(j), J(j)) = (2p-1, 2p) \text{ or } (2p, 2p-1)\ \forall\ j \text{ and} \\ &\{\alpha_I, \alpha_J\} = (-1)^N, \text{ if } (I(j), J(j)) = (2p-1, 2p) \text{ for } N \text{ values of } j \\ &\qquad\qquad \text{and } (I(j), J(j)) = (2p-1, 2p) \text{ for } N-k \text{ values of } j. \end{aligned}$$

From this, and (10.54), (10.56) follows. □

From this proof if also follows that $N(W(I)) = N(I)$, so $W_\omega^2 = \mathrm{Id}$. We shall let

24.77 (10.57)
$$\delta_\omega = W_\omega \circ d \circ W_\omega$$

denote the differential operator obtained from d by conjugation,

$$\delta_\omega : \mathcal{C}^\infty(T^*X \setminus 0; \Lambda^k) \longrightarrow \mathcal{C}^\infty(T^*X \setminus 0, \Lambda^{k-1}).$$

By construction $\delta_\omega^2 = 0$. The exterior algebra of a symplectic manifold with this differential is called the Koszul complex.[14] All the α_I are closed so

24.79 (10.58)
$$\begin{aligned}\delta_\omega(a\alpha_I) &= W_\omega\big(\sum_j \frac{\partial a}{\partial z_j} dz_j\big) \wedge (-1)^{N(I)} \alpha_{W(I)} \\ &= \sum_j \frac{\partial a}{\partial z_j} (-1)^{N(I)} W_\omega(dz_j \wedge \alpha_{W(I)}),\end{aligned}$$

Observe that[15]

$$\begin{aligned}W_\omega\big(dz_{2p-1} \wedge \alpha_{W(I)}\big) &= \iota_{\partial/\partial z_{2p}} \alpha_I \\ W_\omega\big(dz_{2p} \wedge \alpha_{W(I)}\big) &= \iota_{\partial/\partial z_{2p-1}} \alpha_I,\end{aligned}$$

where, ι_v denotes contraction with the vector field v. We therefore deduce the following formula for the action of the Koszul differential

24.81 (10.59)
$$\delta_\omega(a\alpha_I) = \sum_{i=1}^{2n} \big(H_{z_i} a\big) \iota_{\partial/\partial z_i} \alpha_I.$$

24.67 LEMMA 10.8. *With E_1 identified with the formal sums of homogeneous forms on $T^*X \setminus 0$, the induced differential is*

24.68 (10.60)
$$b_{(1)} = \frac{1}{i} \delta_\omega.$$

PROOF. We know that the bilinear differential operator $2iP_1$ is the Poisson bracket of functions on T^*X. Thus (10.52) can be written

24.69 (10.61)
$$\begin{aligned}2ib_1\alpha = \sum_{i=0}^{k-1} &(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes \{a_{i+1}, a_i\} \otimes a_{i+2} \otimes \cdots \otimes a_k \\ &+ (-1)^k \{a_0, a_k\} \otimes a_1 \otimes \cdots \otimes a_{k-1},\ \alpha = a_0 \otimes \cdots \otimes a_k.\end{aligned}$$

The form to which this maps under the identification of E_2 is just

24.70 (10.62)
$$\begin{aligned}21b_1\alpha = \sum_{i=0}^{k-1} &(-1)^i a_0 \wedge da_{i-1} \wedge \cdots \wedge d\{a_{i+1}, a_i\} \wedge da_{i+2} \wedge a_k \\ &+ (-1)^k \{a_0, a_k\} \wedge da_1 \wedge \cdots \wedge da_{k-1}\end{aligned}$$

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[14] Up to various sign conventions of course!

[15] Check this case by case, as the range of I meets the pair $\{2p-1, 2p\}$ in $\{2p-1, 2p\}$, $\{2p-1\}$, $\{2p\}$ or $\emptyset$. Both sides of the first equation are zero in the second and fourth case as are both sides of the second equation in the third and fourth cases. In the remaining four individual cases it is a matter of checking signs.

Consider the basis elements α_I for k-forms. These arise as the images of the corresponding functions in local coordinates on X^{k+1}

$$\tilde\alpha_I(z_0,z_1,\dots,z_k)=\sum_\sigma(-1)^{\operatorname{sgn}\sigma}(z_{1,\sigma I(1)}-z_{0,\sigma I(1)})\\ \times z_{2,\sigma I(1)}-z_{1,\sigma I(1)})\dots(z_{1,\sigma I(m)}-z_{0,\sigma I(m-1)}).$$

Since these functions are defined in local coordinates they are not globally defined on $(T^*X\backslash 0)^{k+1}$. Nevertheless they can be localized away from $z_0=\dots=z_m$ and then, with a coefficient $(a_j(z_0))^*_{j=-\infty}$, $a_j\in\mathcal{C}^\infty(T^*X\setminus 0)$ homogeneous of degree j with support in the coordinate patch, unambiguously define elements of E_1 which we can simply denote as $a(z_0)\tilde\alpha_I\in E_1$. These elements, superimposed over a coordinate cover, span E_1. Consider $b_{(1)}\tilde\alpha$ given by ([24.70]10.62). In the sum, the terms with P_1 contracting between indices other than $0,1$ or $m,0$ must give zero because the Poisson bracket is constant in the 'middle' variable. Futhermore, by the antisymmetry of $\tilde\alpha$, the two remaining terms are equal so

$$\begin{aligned} ib_{(1)}\big(a\tilde\alpha_I\big)&=\sum_{\sigma\in\mathcal{P}_k}\big(H_{z_{\sigma I(1)}}a\big)(-1)^{\operatorname{sgn}(\sigma)}dz_{\sigma I(2)}\wedge\dots\wedge dz_{\sigma I(k)}\\ &=\sum_i\big(H_{z_i}a\big)\iota_{\partial/\partial_i}\alpha_I. \end{aligned}$$

Since this is just ([24.81]10.59) the lemma follows. □

With this lemma we have identified the differential on the E_1 term in the spectral sequence with the exterior differential operator. To complete the identification ([24.63]10.49) we need to compute the corresponding deRham groups.

24.82 Proposition 10.1. *The cohomology of the complex*

$$\dots\xrightarrow{d}\sum_{j=-\infty}^{*}\mathcal{C}^\infty_{hom(j)}(T^*X\setminus 0;\Lambda^k)\xrightarrow{d}\sum_{j=-\infty}^{*}\mathcal{C}^\infty_{hom(j)}(T^*X\setminus 0;\Lambda^{k+1})\xrightarrow{d}\dots$$

in dimension k is naturally isomorphic to $H^k(S^*X)\oplus H^{k-1}(S^*X)$.

Proof. Choose a metric on X and let $R=|\xi|$ denote the corresponding length function on $T^*X\setminus 0$. Thus, identifying the quotient $S^*X=(T^*X\setminus 0)/\mathbb{R}^+$ with $\{R=1\}$ gives an isomorphism $T^*X\setminus 0\equiv S^*X\times(0,\infty)$. Under this map the smooth forms on $T^*X\setminus 0$ which are homogeneous of degree j are identified as sums

24.86 (10.63)
$$\mathcal{C}^\infty_{\mathrm{hom}(j)}(T^*X\setminus 0,\Lambda^k)\ni\alpha_j\\ =R^j\big(\alpha'_j+\alpha''_j\wedge\frac{dR}{R}\big),\ \alpha'_j\in\mathcal{C}^\infty(S^*X;\Lambda^k),\ \alpha''_j\in\mathcal{C}^\infty(S^*X;\Lambda^{k-1}).$$

The action of the exterior derivative is then easily computed

$$\begin{gathered} d\alpha_j=\beta_j,\ \beta_j=R^j\big(\beta'_j+\beta_j-''\wedge\frac{dR}{R}\big),\\ \beta'_j=d\alpha'_j,\ \beta''_j=d\alpha''_j+j(-1)^{k-1}\alpha'_j. \end{gathered}$$

Thus a k-form $(\alpha_j)^*_{j=-\infty}$ is closed precisely if it satisfies

24.84 (10.64)
$$j\alpha'_j=(-1)^kd\alpha''_j,\ d\alpha'_j=0\forall\ j.$$

It is exact if there exists a $(k-1)$-form $(\gamma_j)^*_{j=-\infty}$ such that

24.85 (10.65)
$$\alpha'_j=d\gamma'_j,\ \alpha''_j=d\gamma''_j+j(-1)^k\gamma'_j.$$

Since the differential preserves homogeneity it is only necessary to analyze these equations for each integral j. For $j \neq 0$, the second equation in (10.64) follows from the first and (10.65) then holds with $\gamma'_j = \frac{1}{j}(-1)^k\alpha''_j$ and $\gamma''_j = 0$. Thus the cohomology lies only in the subcomplex of homogeneous forms of degree 0. Then (10.64) and (10.65) become

$$d\alpha'_0 = 0,\ d\alpha''_0 = 0 \text{ and } \alpha'_0 = d\gamma'_0,\ \alpha''_0 = d\gamma''_0$$

respectively. This gives exactly the direct sum of $H^k(S^*X)$ and $H^{k-1}(S^*X)$ as the cohomology in degree k. The resulting isomorphism is independent of the choice of the radial function R, since another choice replaces R by Ra, where a is a smooth positive function on S^*X. In the decomposition (10.63), for $j = 0$, α''_0 is unchanged whereas α'_0 is replaced by $\alpha'_0 + \alpha''_0 \wedge d\log a$. Since the extra term is exact whenever α''_0 is closed it has no effect on the identification of the cohomology. □

Combining Proposition 10.1 and Lemma 10.8 completes the proof of (10.49). We make the identification a little more precise by locating the terms in E_2.

24.87 Proposition 10.2. *Under the identification of E_1 with the sums of homogeneous forms on $T^*X \setminus 0$, E_2, identified as the cohomology of δ_ω, has a basis of homogeneous forms with the homogeneity degree j and the form degree k confined to*

24.88 (10.66) $$k - j = \dim X,\ -\dim X \leq j \leq \dim X,\ \dim X \geq 2.$$

Proof. Provided $\dim X \geq 2$, the cohomology of S^*X is isomorphic to two copies of the cohomology of X, one in the same degree and one shifted by $\dim X - 1$.[16] The classes in the first copy can be taken to be the lifts of deRham classes from X, while the second is spanned by the wedge of these same classes with the Todd class of S^*X. This latter, $n-1$, class restricts to each fibre to be non-vanishing. Thus in local representations the first forms involve only the base variable and in the second each terms has the maximum number, $n-1$, of fibre forms. The cohomology of the complex in Proposition 10.1 therefore consists of four copies of $H^*(X)$ consisting of these forms and the same forms wedged with dR/R.

With this decomoposition of the cohomology consider the effect on it of the map W_ω. In each case the image forms are again homogeneous. A deRham class on X in degree l therefore has four images in E_2. One is a form of degree $k_1 = 2n - l$ which is homogeneous of degree $j_1 = n - l$. The second is a form of degree $k_2 = 2n - l - 1$ which is homogeneous of degree $j_2 = n - l - 1$. The third image is of form degree $k_3 = n - l + 1$ and homogeneous of degree $j_3 = -l + 1$ and the final image is of form degree $k_4 = n - l$ and is homogeneous of degree $j_4 = -l$. This gives the relations (10.66). □

10.8. Degeneration and convergence

Now that the E_2 term in the spectral sequence has been explicitly computed, consider the induced differential, $b_{(2)}$ on it. Any homogeneous form representing a class in E_2 can be represented by a Hochshild chain α of the same homogeneity. Thus an element of E_2 in degree k corresponds to a function on $\mathcal{C}^\infty((T^*X)\backslash)^{k+1})$ which is separately homogeneous in each variable and of total homogeneity $k - n$. Furthermore it has an extension $\beta(t)$ as a function of the parameter h, of the same

[16]That is, just as though $S^*X = \mathbb{S}^{n-1} \times X$, where $n = \dim X$.

homogeneity, such that $b_t\beta(t) = t^2\gamma(t)$. Then $b_{(2)}\alpha = [\gamma(0)]$, the class of $\gamma(0)$ in E_2. Noting that the differential operator, P_j, which is the jth term in the Taylor series of the product $\star_h$ reduces homogeneity by j and that b_h depends multilinearly on $\star_h$ it follows tha $b_{(2)}$ must decrease homogeneity by r. Thus if the class $[\gamma(0)]$ must vanish in E_2 by (10.66). We have therefore shown that $b_{(2)} \equiv 0$, so $E_3 = E_2$. The same argument applies to the higher differentials, defining the $E_r \equiv E_2$ for $r \geq 2$, proving the 'degeneration' of the spectral sequence, (10.50).

The 'convergence' of the spectral sequence, (10.51), follows from the same analysis of homogneities. Thus, we shall define a map from E_2 to the Hochschild homology and show that it is an isomorphism.

10.9. Explicit cohomology maps

10.10. Hochschild holomology of $\Psi^{-\infty}(X)$

10.11. Hochschild holomology of $\Psi^{\mathbb{Z}}(X)$

10.12. Morita equivalence

CHAPTER 11

The index formula

HHcoId (11.1)

APPENDIX A

Bounded operators on Hilbert space

Some of the main properties of bounded operators on a complex Hilbert space, H, are recalled here; they are assumed at various points in the text.

(1) Boundedness equals continuity, $\mathcal{B}(H)$.
(2) $\|AB\| \le \|A\| \, \|B\|$
(3) $(A - \lambda)^{-1} \in \mathcal{B}(H)$ if $|\lambda| \ge \|A\|$.
(4) $\|A^*A\| = \|AA^*\| = \|A\|^2$.
(5) Compact operators, defined by requiring the closure of the image of the unit ball to be compact, form the norm closure of the operators of finite rank.
(6) Fredholm operators have parametrices up to compact errors.
(7) Fredholm operators have generalized inverses.
(8) Fredholm operators for an open subalgebra.
(9) Hilbert-Schmidt operators?
(10) Operators of trace class?
(11) General Schatten class?

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