HOMOLOGY OF PSEUDODIFFERENTIAL OPERATORS I. MANIFOLDS WITH BOUNDARY

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Abstract. The Hochschild and cyclic homology groups are computed for the algebra of 'cusp' pseudodifferential operators on any compact manifold with boundary. The index functional for this algebra is interpreted as a Hochschild 1-cocycle and evaluated in terms of extensions of the trace functionals on the two natural ideals, corresponding to the two filtrations by interior order and vanishing degree at the boundary, together with the exterior derivations of the algebra. This leads to an index formula which is a pseudodifferential extension of that of Atiyah, Patodi and Singer for Dirac operators; together with a symbolic term it involves the 'eta' invariant on the suspended algebra over the boundary previously introduced by the first author.

Introduction

The Hochschild homology of the algebra of pseudodifferential operators on a compact manifold (without boundary) was computed by Brylinski and Getzler [4]; Wodzicki had earlier obtained related results. Guillemin and Wodzicki [6, 7, 25, 26] defined a 'residue trace functional' by analytic continuation; this generates the Hochschild cohomology in dimension 0 for the algebra of complete symbols, which is the quotient by the ideal of smoothing operators. These computations are closely related to the Atiyah-Singer index formula since the index can be expressed in terms of a Hochschild pairing for the symbol algebra.

For any compact manifold with boundary the algebra of 'cusp' pseudodifferential operators is a natural generalization of the usual algebra of pseudodifferential operators in the boundaryless case. We describe the Hochschild homology of this algebra and various of its ideals and so deduce a pseudodifferential generalization of the Atiyah-Patodi-Singer index theorem. Such a generalization was previously discussed by Piazza ([23]) but without explicit evaluation of the component functionals, here interpreted as Hochschild cycles.

In case X is a compact manifold without boundary, let $\Psi^{\mathbb{Z}}(X)$ be the algebra of 1-step polyhomogeneous (i.e. classical) pseudodifferential operators of any integral order. The smoothing operators form an ideal, $\mathcal{I} = \Psi^{-\infty}(X)$, and the quotient $\mathcal{A} = \Psi^{\mathbb{Z}}(X)/\mathcal{I}$ is identified, by choice of a quantization map, with the algebra of formal sums of homogeneous functions of integral order on $T^*X \setminus 0$ with order bounded above and with a $\star$-product. A spectral sequence argument, see [4], allows

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the (continuous) Hochschild homology and cohomology to be identified with de Rham cohomology spaces

$$\mathrm{HH}_k(\mathcal{A}) \simeq H^{2n-k}(S^*X \times \mathbb{S}_\sigma),\ \mathrm{HH}^k(\mathcal{A}) \simeq H^k(S^*X \times \mathbb{S}_\sigma) \tag{1}$$

where $S^*X = (T^*X \setminus 0)/\mathbb{R}^+$ is the cosphere bundle; the additional circle factor can be viewed as a quotient (or compactification) of the $\mathbb{R}^+$ action. The pairing uses the symplectic form. In particular $\mathrm{HH}_0(\mathcal{A})$ is one-dimensional; the dual space $\mathrm{HH}^0(\mathcal{A})$ is spanned by the residue trace, $\mathrm{Tr_R}$, of Wodzicki. This can be defined as follows. If $Q \in \Psi^1(X)$ is self-adjoint, elliptic and positive its complex powers, defined by Seeley [24], allow elements of $\Psi^{\mathbb{Z}}(X)$ to be regularized. In particular the function

$$Z(A; z) = \mathrm{Tr}(AQ^{-z}),\ A \in \Psi^m(X),\ m \in \mathbb{Z}, \tag{2}$$

is holomorphic in $\mathrm{Re}\, z > m + \dim X$ and extends to be meromorphic. The residue trace $\mathrm{Tr_R}(A)$ is the residue at $z = 0$ and is independent of the choice of Q; it only depends on the class of A in $\mathcal{A}$.

Although $\log Q$ is not an element of $\Psi^{\mathbb{Z}}(X)$ commutation with it defines a derivation on the algebras, $D_{\log Q} : \mathcal{A} \ni A \longmapsto [\log Q, A]$. This exterior derivation induces a map on Hochschild cohomology which we denote $i_{\log Q}$; it corresponds under (1) with exterior (i.e. cup) product with a generator of $H^1(\mathbb{S}_\sigma)$.

The regularized value, $\widehat{\mathrm{Tr}}(A)$, of $Z(A; z)$ at $z = 0$ is a functional on $\Psi^{\mathbb{Z}}(X)$ which is not a trace functional. Fedosov's formula suggests that the *index functional* be defined in terms of its boundary, as a Hochschild cochain

$$\begin{aligned} \mathrm{IF}(A, B) &= -(b\,\mathrm{Tr})(A, B) = \widehat{\mathrm{Tr}}([A, B]) = (\partial\,\mathrm{Tr})(A, B) \\ &= \mathrm{Tr_R}(A[\log Q, B]) = (i_{\log Q}\,\mathrm{Tr_R})(A, B). \end{aligned} \tag{3}$$

Here ∂ is the boundary map in the long exact sequence

$$0 \longrightarrow \mathrm{HH}^0(\mathcal{A}) \longrightarrow \mathrm{HH}^0(\Psi^{\mathbb{Z}}(X)) \longrightarrow \mathrm{HH}^0(\mathcal{I}) \xrightarrow{\partial} \mathrm{HH}^1(\mathcal{A}) \longrightarrow \cdots \tag{4}$$

which arises from the short exact sequence $0 \longrightarrow \mathcal{I} \longrightarrow \Psi^{\mathbb{Z}}(X) \longrightarrow \mathcal{A} \longrightarrow 0$ since $\mathcal{I}$ is H-unital (see [26]). Thus the index functional is also the image under the boundary map of the class Tr on $\mathcal{I}$. The ideal $\mathcal{I}$ is homologous to a matrix algebra so $\mathrm{HH}^k(\mathcal{A}) \simeq \mathrm{HH}^k(\Psi^{\mathbb{Z}}(X))$ except in dimension 1, where the index functional in (3) is an additional generator of $\mathrm{HH}^1(\mathcal{A})$. The other equalities in (3) show that the index functional can be expressed in terms of Wodzicki's residue trace and the derivation defined by $\log Q$ on the symbol algebra.

The identification of IF as the 'index functional' may be justified as follows. By Morita equivalence the matrix algebras over the algebras under discussion have the same Hochschild homology. If $A = [A_{ij}]$ is an $N \times N$ matrix in $\Psi^{\mathbb{Z}}(X)$ which is elliptic, and hence Fredholm as an operator on $\mathcal{C}^\infty(X)^N$, then its image $[A]$ in the matrix algebra over $\mathcal{A}$ is invertible (and conversely). The index formula of Fedosov then reduces to

$$\mathrm{Ind}(A) = \mathrm{IF}([A], [A]^{-1}). \tag{5}$$

Since IF reduces to the fundamental class in $H^0(S^*X)$, under the isomorphism in (1), the Atiyah-Singer index theorem becomes the identification of the image in $H^{2n-1}(S^*X)$ of the Hochschild cycle $[A] \otimes [A]^{-1}$ for $\mathcal{A}$. Even though one might only be interested in operators of order 1, or even of order 0, it is very convenient to discuss the formula (5) in terms of the Hochschild homology for the algebra of operators of all integral orders, since this is finite dimensional whereas the homology of the operators of order 0 is not.

It is this general discussion which we extend to the case of a compact manifold with boundary, so that (5) becomes a pseudodifferential index formula in that context, extending the Atiyah-Patodi-Singer index theorem for Dirac operators.

The basic algebra considered here consists of the 'cusp' pseudodifferential operators on a compact manifold with boundary. It appears, at least implicitly, in the work of Livingston [13]. The properties of the algebra are discussed more systematically in [17], which is largely based on joint work of Rafe Mazzeo and the first author. This algebra, denoted $\Psi_{\mathrm{c}}^{\mathbb{Z}}(X)$, is determined by a choice of defining function, or more precisely by a trivialization of the normal bundle to the boundary of the compact manifold with boundary X. A change of defining function gives an isomorphic algebra. A general process of microlocalization allows $\Psi_{\mathrm{c}}^{\mathbb{Z}}(X)$ to be considered as the quantization of an algebra of vector fields on X which, with the convention for coordinate systems that at a boundary point the first function x defines the boundary and its trivialization locally, is spanned by $x^2\partial_x$, and the tangential vector fields ∂_y. Thus, at least informally, elements of $\Psi_{\mathrm{c}}^{\mathbb{Z}}(X)$ can be regarded as 'symbolic functions' of these vector fields, with smooth coefficients. If $x \in \mathcal{C}^\infty(X)$ is a global boundary defining function fixing the trivialization of the normal bundle then this Lie algebra of 'cusp' vector fields is

$$\mathcal{V}_{\mathrm{c}}(X) = \left\{V \in \mathcal{C}^\infty(X;TX); Vx \in x^2\mathcal{C}^\infty(X)\right\}. \tag{6}$$

We look at the cusp calculus instead of the more familiar b-calculus to avoid certain problems with completeness. The relationship between the two algebras (which means in particular that the index formula we obtain is valid for elliptic b-pseudodifferential operators) is briefly described in Appendix C. The b-calculus $\Psi_b^0(X)$ and the 'corresponding' cusp calculus $\Psi_{\mathrm{c}}^0(X)$ have isomorphic norm closures in the bounded operators on L^2. In [18], the K-theory of the b-calculus is described. Since the K-theory is given in terms of the norm completion the conclusions are equally valid for the cusp calculus. At a $\mathcal{C}^\infty$ level the 'cusp commutation' $[x^2\partial_x, x] = x^2$ (or rather $[x^{-1}, x^2\partial_x] = 1$) in place of the 'b-commutation' $[x\partial_x, x] = x$ makes quite a lot of difference as regards the computation of both Hochschild and cyclic homology. Nevertheless methods similar to those used here can be applied directly to the b-calculus.

The algebra $\Psi_{\mathrm{c}}^{\mathbb{Z}}(X)$ reduces to $\Psi^{\mathbb{Z}}(X)$ in the boundaryless case and consists in the general case of continuous linear operators on $\mathcal{C}^\infty(X)$. The subspace $\dot{\mathcal{C}}^\infty(X) \subset \mathcal{C}^\infty(X)$ of functions vanishing to all orders at the boundary is preserved by this action. Furthermore conjugation by any complex power x^z of a boundary defining function preserves the algebra. Just as we consider the algebra of all integral order pseudodifferential operators it is very helpful to expand the algebra to

$$x^{-\mathbb{Z}}\Psi_{\mathrm{c}}^{\mathbb{Z}}(X) = \bigcup_{k\in\mathbb{Z}}\bigcup_{m\in\mathbb{Z}} x^k\Psi_{\mathrm{c}}^m(X).$$

Here, and below, the arbitrary integral powers of x are denoted $x^{-\mathbb{Z}}$ to emphasize that this filtration is in the opposite direction to the other, order, filtration. There is an ideal of smoothing operators quite analogous to the boundaryless case, namely

$$\mathcal{I} = x^\infty\Psi_{\mathrm{c}}^{-\infty}(X) = \bigcap_{k\in\mathbb{Z}}\bigcap_{m\in\mathbb{Z}} x^k\Psi_{\mathrm{c}}^m(X).$$

This is again an H-unital algebra, homologous to a matrix algebra, and writing the quotient as $\mathcal{A}$ therefore leads to a long exact sequence in Hochschild cohomology, as in (4). It is again the case that a matrix of elements of $x^{-\mathbb{Z}}\Psi_{\mathrm{c}}^{\mathbb{Z}}(X)$ which is 'fully elliptic', in the sense that its image in the matrix algebra on $\mathcal{A}$ is invertible, is Fredholm as an operator on $\dot{\mathcal{C}}^{\infty}(X)^N$. Its index is given by the same 'Fedosov formula' as the first part of (3) and (5)

$$\operatorname{Ind}(A) = \dim\ker(A) - \dim\ker(A^*) = \mathrm{IF}([A],[A]^{-1}) = (\partial\operatorname{Tr})([A],[A]^{-1}). \tag{7}$$

However the direct analogue of the second part of (3) does not hold. To derive a formula for the new index functional, IF, we need to consider two other ideals.

The smoothing algebra $\mathcal{I}$ is the residual ideal for the joint filtration by order and boundary power. Each of these filtrations gives rise to a separate ideal; we consider the quotient ideals in $\mathcal{A}$

$$\mathcal{I}_\partial = x^{-\mathbb{Z}}\Psi_{\mathrm{c}}^{-\infty}(X)/\mathcal{I},\ x^{-\mathbb{Z}}\Psi_{\mathrm{c}}^{-\infty}(X) = \bigcup_{k\in\mathbb{Z}}\bigcap_{m\in\mathbb{Z}} x^k\Psi_{\mathrm{c}}^m(X), \tag{8}$$

$$\mathcal{I}_\sigma = x^{\infty}\Psi_{\mathrm{c}}^{\mathbb{Z}}(X)/\mathcal{I},\ x^{\infty}\Psi_{\mathrm{c}}^{\mathbb{Z}}(X) = \bigcap_{k\in\mathbb{Z}}\bigcup_{m\in\mathbb{Z}} x^k\Psi_{\mathrm{c}}^m(X), \tag{9}$$

$$\mathcal{A}_\sigma = \mathcal{A}/\mathcal{I}_\partial,\ \mathcal{A}_\partial = \mathcal{A}/\mathcal{I}_\sigma \text{ and } \mathcal{A}_{\partial,\sigma} = \mathcal{A}/(\mathcal{I}_\sigma + \mathcal{I}_\partial). \tag{10}$$

These algebras are related by the following diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[d] & 0 \arrow[d] & \\
& \mathcal{I}_\sigma + \mathcal{I}_\partial \arrow[dr] & \mathcal{I}_\sigma \arrow[d] & \mathcal{I}_\sigma \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_\partial \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] \arrow[dr] & \mathcal{A}_\sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_\partial \arrow[r] & \mathcal{A}_\partial \arrow[r] \arrow[d] & \mathcal{A}_{\partial,\sigma} \arrow[r] \arrow[d] \arrow[dr] & 0 \\
& & 0 & 0 & 0.
\end{tikzcd}$$

The two filtrations are delineated by maps, respectively the symbol map and the indicial map. The first is quite analogous to the symbol map in the boundaryless case, and reduces to it over the interior. The choice of a cusp structure defines a modified tangent bundle such that $\mathcal{V}_{\mathrm{c}}(X) = \mathcal{C}^\infty(X;{}^{c}TX)$. This bundle is isomorphic to the usual tangent bundle. If ${}^{c}T^*X$ is the dual bundle then the symbol map gives a short exact sequence for each m

$$\begin{gathered} 0 \longrightarrow \Psi_{\mathrm{c}}^{m-1}(X) \hookrightarrow \Psi_{\mathrm{c}}^m(X) \xrightarrow{\sigma_m} \mathcal{G}^m({}^{c}T^*X) \longrightarrow 0 \\ \text{where } \mathcal{G}^m({}^{c}T^*X) = \{a\in\mathcal{C}^\infty({}^{c}T^*X\setminus 0);\ \text{homogeneous of degree } m\}. \end{gathered} \tag{11}$$

Similarly, if $\Psi_{\mathrm{sus}}^m(\partial X)$ is the 'suspended algebra' of pseudodifferential operators as discussed in [16], consisting of the pseudodifferential operators on $\mathbb{R}\times\partial X$ which are translation-invariant in $\mathbb{R}$ and have convolution kernels vanishing rapidly with all derivatives at infinity, then the indicial map gives a short exact sequence for each m

$$0 \longrightarrow x\Psi^m_{\mathrm{c}}(X) \hookrightarrow \Psi^m_{\mathrm{c}}(X) \xrightarrow{\mathrm{In}} \Psi^m_{\mathrm{sus}}(\partial X) \longrightarrow 0. \tag{12}$$

The choice of a quantization and an operator $Q \in \Psi^1_{\mathrm{c}}(X)$ which is elliptic, positive and selfadjoint gives isomorphisms

$$\mathcal{I}_\sigma \simeq \dot{\mathcal{C}}^\infty({}^cS^*X)[q]],\ \mathcal{A}_\sigma \simeq x^{-\mathbb{Z}}\mathcal{C}^\infty({}^cS^*X)[q]] \tag{13}$$

where $q = \sigma_1(Q)$ is the symbol of Q and ${}^cS^*X = ({}^cT^*X\setminus 0)/\mathbb{R}^+$ is the sphere bundle associated to ${}^cT^*X$; here $[q]]$ stands for Laurent series in q^{-1}, i.e. for each element there is an upper bound on the powers of q but no lower bound. The product is a $\star$-product induced by the quantization. Similarly the choice of the boundary defining function x and a normal fibration of X at ∂X induces isomorphisms

$$\mathcal{I}_\partial \simeq \Psi^{-\infty}_{\mathrm{sus}}(\partial X)[[x],\ \mathcal{A}_\partial \simeq \Psi^{\mathbb{Z}}_{\mathrm{sus}}(\partial X)[[x]. \tag{14}$$

Here, the commutation between the Laurent series variable x and the suspended algebra of pseudodifferential (or smoothing) operators is through $[x, \partial_t] = x^2$. Thus the variable on the linear factor in $\mathbb{R}_t \times \partial X$ can be identified as $t = 1/x$. The double quotient is then doubly a Laurent series algebra

$$\mathcal{A}_{\partial,\sigma} \simeq \mathcal{C}^\infty({}^cS^*_{\partial X}X)[[x, q^{-1}]. \tag{15}$$

These identifications and filtrations allow us to compute the Hochschild homology groups of all these algebras. The Hochschild cohomology groups can be expressed in terms of appropriate cohomology groups and duality:

$$\begin{gathered}
\mathrm{HH}^*(\mathcal{I}_\sigma) \simeq H^*({}^cS^*X \times \mathbb{S}_\sigma),\\
\mathrm{HH}^*(\mathcal{A}_\sigma) \simeq H^*_{\mathrm{rel}}({}^cS^*X \times \mathbb{S}_\sigma) \oplus H^*({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma),\\
\mathrm{HH}^k(\mathcal{I}_\partial) \simeq \begin{cases} \mathbb{C} & \text{for } k = 0, 1,\\ 0 & \text{otherwise}\end{cases}\\
\mathrm{HH}^k(\mathcal{A}_\partial) \simeq \begin{cases} H^k({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial)/\mathbb{C} & \text{for } k = 1, 2,\\ H^k({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) & \text{otherwise}\end{cases}\\
\mathrm{HH}^*(\mathcal{A}_{\partial,\sigma}) \simeq H^*({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) \text{ and}\\
\mathrm{HH}^k(\mathcal{A}) \simeq \begin{cases} \mathbb{C} \oplus H^*_{\mathrm{rel}}({}^cS^*X \times \mathbb{S}_\sigma) \oplus H^*({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma)/\mathbb{C} & \text{for } k = 1\\ H^*_{\mathrm{rel}}({}^cS^*X \times \mathbb{S}_\sigma) \oplus H^*({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma) & \text{otherwise.}\end{cases}
\end{gathered} \tag{16}$$

In particular we show that in each case HH^0 is one-dimensional, with a generating 'trace functional'. These functionals are, for $\mathcal{I}$ the ordinary trace, for $\mathcal{A}_\sigma$, $\mathcal{A}_\partial$ and $\mathcal{A}_{\partial,\sigma}$ a 'double residue trace' $\mathrm{Tr}_{\partial,\sigma}$ which we define by analytic continuation, for $\mathcal{I}_\sigma$ the residue trace of Wodzicki denoted Tr_σ and for $\mathcal{I}_\partial$ a functional Tr_∂ induced from the trace functional $\overline{\mathrm{Tr}}$ defined in [16] on $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(Y)$ for any boundaryless manifold Y (its definition is recalled below). Analytic continuation arguments give extensions of the last two functionals to $\widehat{\mathrm{Tr}}_\sigma$ on $\mathcal{A}_\sigma$ and $\widehat{\mathrm{Tr}}_\partial$ on $\mathcal{A}_\partial$. These are not trace functionals. Rather, commutation with the operators $\log Q$ and $\log x$ defines derivations and the Hochschild boundaries of these functionals are given by

(17)
$$(\partial\,\mathrm{Tr}_\partial)(A,B) = \widehat{\mathrm{Tr}}_\partial([A,B]) = \mathrm{Tr}_{\partial,\sigma}(A[\log Q, B]) = (i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma})(A,B) \text{ and}$$
$$(\partial\,\mathrm{Tr}_\sigma)(A,B) = \widehat{\mathrm{Tr}}_\sigma([A,B]) = -\,\mathrm{Tr}_{\partial,\sigma}(A[\log x, B]) = -(i_{\log x}\,\mathrm{Tr}_{\partial,\sigma})(A,B) \quad \forall\ A,B\in\mathcal{A}.$$

The ideals $\mathcal{I}_\sigma$ and $\mathcal{I}_\partial$ are H-unital and, in (17), $\partial\,\mathrm{Tr}_\sigma$ and $\partial\,\mathrm{Tr}_\partial$ represent the images under the respective boundary maps of the functionals Tr_σ and Tr_∂ in the long exact sequences

$$0 \longrightarrow \mathrm{HH}^0(\mathcal{A}_\sigma) \longrightarrow \mathrm{HH}^0(\mathcal{A}_{\partial,\sigma}) \longrightarrow \mathrm{HH}^0(\mathcal{I}_\sigma) \xrightarrow{\partial} \mathrm{HH}^1(\mathcal{A}_\sigma) \longrightarrow \cdots \tag{18}$$

$$0 \longrightarrow \mathrm{HH}^0(\mathcal{A}_\partial) \longrightarrow \mathrm{HH}^0(\mathcal{A}_{\partial,\sigma}) \longrightarrow \mathrm{HH}^0(\mathcal{I}_\partial) \xrightarrow{\partial} \mathrm{HH}^1(\mathcal{A}_\partial) \longrightarrow \cdots. \tag{19}$$

These formulæ are analogues of the index formula (3). Indeed the sum of them gives a homotopy invariant of invertible elements of $\mathcal{A}_{\partial,\sigma}$ which we call the 'boundary index'

$$\mathrm{Ind}_\partial(A) = \mathrm{Bf}(A,A^{-1}),\ \mathrm{Bf}(A,B) = (-i_{\log x}\,\mathrm{Tr}_{\partial,\sigma} + i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma})(A,B). \tag{20}$$

This represents an obstruction to the lifting of an invertible element $A\in\mathcal{A}_{\partial,\sigma}$ to an invertible element of $\mathcal{A}$.

The index functional itself, defined as the image in $\mathrm{HH}^1(\mathcal{A})$ of the trace functional on Tr in (4) in the boundary case, can be expressed in terms of these same operations

$$\mathrm{IF}(A,B) = \frac{1}{2}(\widehat{\mathrm{Tr}}_\partial + \widehat{\mathrm{Tr}}_\sigma)(B\widetilde{D}(A) + \widetilde{D}(A)B), \quad A,B\in\mathcal{A},\ \widetilde{D}(A) = [\log Q - \log x, A]. \tag{21}$$

This leads to a pseudodifferential generalization of the Atiyah-Patodi-Singer index theorem in which the index is expressed as the sum of four terms. If $A=[A_{ij}]$ is a matrix in $\Psi^m_{\mathrm{c}}(X)$ which is elliptic and also translation-invariant near the boundary, just as is usually assumed in the case of Dirac operators, then one of these terms can be arranged to vanish. If it is further assumed that the indicial family $A_0 = \mathrm{In}(A)$ is normal (for Dirac operators it is selfadjoint) then a second term can be arranged to vanish and the resulting pseudodifferential index formula takes the form of the Atiyah-Patodi-Singer formula

$$\mathrm{Ind}(A) = \overline{\mathrm{AS}}(A) - \frac{1}{2}\eta(A_0), \tag{22}$$

$$\overline{\mathrm{AS}}(A) = \frac{1}{2}\,\mathrm{Tr}_{\mathrm{R}}(A^{-1}[\log Q, A] + [\log Q, A]A^{-1}), \tag{23}$$

$$\eta(A_0) = 2\,\overline{\mathrm{Tr}}(A_0^{-1}[\log x, A_0]) \tag{24}$$

where $Q=(A^*A)^{\frac{1}{2}}$, the term $\overline{\mathrm{AS}}(A)$ is purely symbolic and $\eta(A_0)$ is the 'eta' functional defined on $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(\partial X)$ in [16].

In Section 1 the Hochschild homology of algebras of functions on a compact manifold with boundary is discussed. This is used in Sections 2, 3 and 4 to analyze the Hochschild homology of the algebras $\mathcal{I}_\sigma$, $\mathcal{I}_\partial$, $\mathcal{A}_\sigma$ and $\mathcal{A}_{\partial,\sigma}$. The various trace functionals are introduced explicitly in Section 5 by analytic continuation and the exterior derivations on the algebras are considered in Section 6; their relation to the boundary maps is discussed in Section 7. This leads to a description of the Hochschild homology of $\mathcal{A}_\partial$ and $\mathcal{A}$ in Section 8. Morita invariance is then discussed and formula (20) for the boundary index is obtained in Section 10. The representation (21) of the index functional is derived in Section 11 and the index formula, (24), is discussed in Section 12. The cyclic homology of the main algebras is deduced in Section 13, the extension to the $\mathbb{Z}_2$-graded case is described in Section 14 and the homology of some other related algebras is examined in Section 15. The appendices are devoted to a description of the relevant properties of the algebra of cusp pseudodifferential operators.

1. Functions and de Rham cohomology

In this section we obtain some preliminary results including the homology of various weighted de Rham complexes.

Consider a compact manifold with boundary M and let $x \in \mathcal{C}^\infty(M)$ be a boundary defining function. We shall consider the algebra of smooth functions on the interior of M with Laurent series expansions at the boundary

$$x^{-\mathbb{Z}}\mathcal{C}^\infty(M) = \bigcup_{j\in\mathbb{Z}} x^j\mathcal{C}^\infty(M).$$

Let $\dot{\mathcal{C}}^\infty(M) \subset \mathcal{C}^\infty(M)$ be the ideal of smooth functions vanishing to infinite order at the boundary. Then, with $\mathcal{C}^\infty(\partial M)[[x] = \{\sum_{k=-n}^\infty f_k x^k\}$ denoting the Laurent series decomposed in any local product decomposition, Taylor's theorem gives a short exact sequence

$$0 \longrightarrow \dot{\mathcal{C}}^\infty(M) \longrightarrow x^{-\mathbb{Z}}\mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(\partial M)[[x] \longrightarrow 0. \tag{25}$$

Recall, [26], that an algebra $\mathcal{I}$ is H-unital if its b' complex is acyclic; the differential b' is defined in (36) below. The following result is a refinement of the same result for $\mathcal{C}^\infty(M)$, which is well known.

Lemma 1. *The algebra $\dot{\mathcal{C}}^\infty(M)$ is H-unital.*

Proof. For any two manifolds M_1 and M_2 the topological tensor product is

$$\mathcal{C}^\infty(M_1) \otimes_{\mathrm{top}} \mathcal{C}^\infty(M_2) = \mathcal{C}^\infty(M_1 \times M_2).$$

Since $\dot{\mathcal{C}}^\infty(M) \subset \mathcal{C}^\infty(M)$ is a closed subspace it follows that the space of k-chains for Hochschild homology is

$$\dot{\mathcal{C}}^\infty(M) \otimes_{\mathrm{top}} \dot{\mathcal{C}}^\infty(M) \otimes_{\mathrm{top}} \cdots \otimes_{\mathrm{top}} \dot{\mathcal{C}}^\infty(M) = \dot{\mathcal{C}}^\infty(M^{k+1}),$$

the space of smooth functions, on the manifold with corners M^{k+1}, vanishing to infinite order at all boundary faces. Let $z^{(j)}$ denote the variable in the jth factor, for $j = 0, 1, \ldots, k$.

Choose $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\phi(0) = 1$. With $x^{(j)}$ denoting a fixed defining function for the boundary of M but in the jth factor consider

$$Ef(z^{(0)}, z^{(1)}, \ldots, z^{(k+1)}) = \phi\Big(\frac{x^{(0)} - x^{(1)}}{x^{(0)} + x^{(1)}}\Big) f(z^{(1)}, \ldots, z^{(k+1)}),\ f \in \dot{\mathcal{C}}^\infty(M^{k+1}). \tag{26}$$

The rapid vanishing of f at all boundaries, together with the fact that the singular coefficient is supported near the diagonal as $x^{(0)}$ or $x^{(1)}$ approaches 0, shows that

$$E : \dot{\mathcal{C}}^\infty(M^{k+1}) \longrightarrow \dot{\mathcal{C}}^\infty(M^{k+2}).$$

By inspection,

$$b'E + Eb' = \mathrm{Id}\,.$$

$\square$

Denote by

$$\chi(f_0 \otimes f_1 \otimes \ldots \otimes f_n) = (n!)^{-1} f_0 df_1 \ldots df_n$$

the Hochschild-Kostant-Rosenberg map, where $f_0, \ldots, f_n \in A$, for a commutative algebra A. The map χ satisfies $\chi \circ b = 0$, where b is the Hochschild differential. For the algebra of smooth functions on a compact manifold χ induces a morphism $\mathrm{HH}_n(\mathcal{C}^\infty(M)) \to \mathcal{C}^\infty(M, \Lambda^n)$, see [10, 14], where $\mathcal{C}^\infty(M; \Lambda^k)$ denotes the space of smooth k-forms on M. For a compact manifold with boundary the same multilinear differential operator gives rise to maps

$$\begin{gathered}
\chi : \dot{\mathcal{C}}^\infty(M^{k+1}) \longrightarrow \dot{\mathcal{C}}^\infty(M; \Lambda^k), \\
\chi : x_1^{-\mathbb{Z}} \cdots x_{k+1}^{-\mathbb{Z}} \mathcal{C}^\infty(M^{k+1}) \longrightarrow x^{-\mathbb{Z}} \mathcal{C}^\infty(M; \Lambda^k) \text{ and} \\
\chi : \mathcal{C}^\infty((\partial M)^{k+1})[[x_1, \ldots, x_{k+1}] \longrightarrow \mathcal{C}^\infty(M; \Lambda^k)[[x] \oplus \mathcal{C}^\infty(M; \Lambda^{k-1})[[x]\frac{dx}{x^2}
\end{gathered} \tag{27}$$

where x_i is a fixed defining function for the boundary of M but on the ith factor.

Also recall from [26] that, whenever the kernel is H-unital a short exact sequence of algebras induces a long exact sequence in Hochschild homology. From this it follows directly that the maps in (27) all induce isomorphism in Hochschild homology.

Lemma 2. *The short exact sequence* (25) *induces a long exact sequence in Hochschild homology which decomposes into the short exact sequences*

$$\begin{multline}
0 \longrightarrow \dot{\mathcal{C}}^\infty(M; \Lambda^k) \longrightarrow x^{-\mathbb{Z}} \mathcal{C}^\infty(M; \Lambda^k) \longrightarrow \\
\mathcal{C}^\infty(\partial M; \Lambda^k)[[x] \oplus \mathcal{C}^\infty(\partial M; \Lambda^{k-1})[[x]\frac{dx}{x^2} \longrightarrow 0.
\end{multline} \tag{28}$$

Consequently the Hochschild-Kostant-Rosenberg maps in (27) *give isomorphisms of the Hochschild homologies of the three algebras in* (25).

Notice that we could just as well use the basis element dx/x (or indeed dx) in the third space in (28). We use dx/x^2 because we think in terms of c- (cusp) geometry rather than b-geometry.

Proposition 1. *The de Rham differential commutes with the maps in* (28) *so induces a long exact sequence of de Rham cohomology spaces*

$$\begin{multline}
\cdots \longrightarrow H^k_{\mathrm{rel}}(M) \longrightarrow H^k_{\mathrm{abs}}(M) \oplus H^{k-1}(\partial M) \longrightarrow \\
H^k(\partial M) \oplus H^{k-1}(\partial M) \longrightarrow \cdots
\end{multline} \tag{29}$$

which is just the direct sum of the Meyer-Vietoris sequence for M relative to its boundary with the identity on $H^{k-1}(\partial M)$.

Proof. The de Rham cohomology of the complex $\dot{\mathcal{C}}^\infty(M; \Lambda^*)$ is isomorphic to the cohomology of M relative to its boundary by the de Rham theorem, since it contains, and can be smoothly retracted onto, the complex of forms with compact support in $M \setminus \partial M$. Similarly the cohomology of the third complex can be retracted onto the image of

$$\mathcal{C}^\infty(\partial M; \Lambda^k) \oplus \mathcal{C}^\infty(\partial M; \Lambda^{k-1}) \ni (\alpha, \beta) \longmapsto \alpha + \beta \wedge \frac{dx}{x} \in x^{-\mathbb{Z}} \mathcal{C}^\infty(M; \Lambda^k),$$

where the Laurent series is decomposed using a local product decomposition of M near ∂M. It follows that the cohomology is $H^k(\partial M) \oplus H^{k-1}(\partial M)$. For the middle space a similar filtration argument shows that the cohomology retracts onto the image of

$$\mathcal{C}^\infty(M;\Lambda^k) \oplus \mathcal{C}^\infty(\partial M;\Lambda^{k-1}) \ni (\gamma,\beta) \longmapsto \gamma + \beta \wedge \frac{dx}{x} \in z^{-\mathbb{Z}}\mathcal{C}^\infty(M;\Lambda^k).$$

Again by the de Rham theorem the cohomology of the first summand is the absolute cohomology of M. Thus, the cohomology groups are as indicated in (29) and the long exact sequence arises from the short exact sequence of complexes

(30)
$$0 \to \dot{\mathcal{C}}^\infty(M;\Lambda^k) \to \mathcal{C}^\infty(M;\Lambda^k) \oplus \mathcal{C}^\infty(\partial M;\Lambda^{k-1}) \to \mathcal{C}^\infty(\partial M;\Lambda^k \oplus \Lambda^{k-1}) \to 0,$$

where the second map is the sum of pull-back to the boundary and the identity map on $\mathcal{C}^\infty(\partial M;\Lambda^{k-1})$. Since, again by a form of the de Rham theorem on a compact manifold with boundary, (30) induces the Meyer-Vietoris sequence, the lemma is proved. □

The second of the image spaces in (27) consists of the forms on the manifold with boundary M which are smooth in the interior and have Laurent series expansions at the boundary. It is interesting to note that the homology of the de Rham complex for these spaces

$$(31) \qquad 0 \longrightarrow x^{-\mathbb{Z}}\mathcal{C}^\infty(M;\Lambda^0) \xrightarrow{d} \cdots\cdots \xrightarrow{d} x^{-\mathbb{Z}}\mathcal{C}^\infty(M;\Lambda^k) \xrightarrow{d} \cdots\cdots$$

turns out to be the same as that of various natural subcomplexes defined in terms of Lie algebras of vector fields on the manifold. In particular we consider the tangent (b-) vector fields $\mathcal{V}_{\mathrm{b}}(M)$ and the vector fields corresponding to a choice of cusp structure $\mathcal{V}_{\mathrm{c}}(M)$; these are described briefly in Appendix A and Appendix C. Both Lie algebras define vector bundles, ${}^bTM$ and ${}^cTM$, over M which are each isomorphic to the usual tangent bundle over the interior and are such that for the corresponding form bundles

$$\mathcal{C}^\infty(M;\Lambda^k) \subset \mathcal{C}^\infty(M;{}^b\Lambda^k) \subset \mathcal{C}^\infty(M;{}^c\Lambda^k),$$
$$x^{-\mathbb{Z}}\mathcal{C}^\infty(M;\Lambda^k) = x^{-\mathbb{Z}}\mathcal{C}^\infty(M;{}^b\Lambda^k) = x^{-\mathbb{Z}}\mathcal{C}^\infty(M;{}^c\Lambda^k).$$

Furthermore (essentially because of the origins in terms of Lie algebras of vector fields) the de Rham differential restricts to give complexes

$$(32) \qquad \cdots \xrightarrow{d} \mathcal{C}^\infty(M;{}^b\Lambda^k) \xrightarrow{d} \mathcal{C}^\infty(M;{}^b\Lambda^{k+1}) \xrightarrow{d} \ldots \text{ and}$$

$$(33) \qquad \cdots \xrightarrow{d} \mathcal{C}^\infty(M;{}^c\Lambda^k) \xrightarrow{d} \mathcal{C}^\infty(M;{}^c\Lambda^{k+1}) \xrightarrow{d} \cdots.$$

We shall denote by ${}^bH^*(M)$ the 'b-cohomology' of M which is the homology of the first of these complexes. The b-cohomology of a compact manifold with boundary was computed by Nest and Tsygan [19],

$${}^bH^k(M) \equiv H^k_{\mathrm{abs}}(M) \oplus H^{k-1}(\partial M).$$

This is a natural isomorphism at the level of cohomology where the map onto the second factor is the evaluation of the coefficient of dx/x. Cohomologies of this type had been computed in unpublished work of the first author and Rafe Mazzeo; these (easy) computations show that the homology spaces of the c-de Rham complex determined by any cusp structure on M are also isomorphic to the b-cohomology spaces.

Consider a real vector bundle, V, over a compact manifold, X, possibly with boundary or corners. Removal of the zero section gives a non-compact manifold, $V \setminus 0_V$ which has a natural $\mathbb{R}^+$-action on it. We shall denote by S^*V the quotient sphere bundle and consider $S^*V \times \mathbb{S}_\sigma \equiv (V \setminus 0_V)/\mathbb{Z}$ the quotient by the discrete group $\mathbb{Z}$ acting through multiplication by e^k, $k \in \mathbb{Z}$, on the fibres. Here the identification is fixed by the choice of a positive function $r \in \mathcal{C}^\infty(V \setminus 0_V)$ which is homogeneous of degree 1.

If X has no boundary then the cohomology of the complex of formal sums of homogeneous forms on $V \setminus 0_V$ of integral degree bounded above,

$$\begin{gathered} 0 \longrightarrow \mathcal{G}(V) \xrightarrow{d} \mathcal{G}(V;\Lambda^1) \xrightarrow{d} \mathcal{G}(V;\Lambda^2) \longrightarrow \cdots \\ \text{where } \mathcal{G}(V;\Lambda^k) = \bigoplus_{j=-\infty}^{*} \mathcal{G}^j(V;\Lambda^k) \text{ and} \\ \mathcal{G}^j(V;\Lambda^k) = \{u_j \in \mathcal{C}^\infty(V \setminus 0_V;\Lambda^k) \text{ homogeneous of degree } j\} \end{gathered} \tag{34}$$

is easily seen to be concentrated in homogeneity degree 0. That is, if α_i is a basis of de Rham cohomology on S^*X and r is a positive function as above then the cohomology of (34) is spanned by the forms α_i and $\alpha_i \wedge \frac{dr}{r}$. These forms map to a basis of the cohomology of $S^*V \times \mathbb{S}_\sigma$. That is, we have proved the following

Lemma 3. *The cohomology of the de Rham complex* (34) *is naturally identified, as a graded space, with* $H^*(S^*V \times \mathbb{S}_\sigma)$.

In case X is a manifold with boundary consider the Laurent forms on this vector bundle, that is the formal sums of smooth forms on the interior of $V \setminus 0_V$ which are homogeneous, with degree bounded above, and have at most rational singularities at the boundary:

$$\begin{gathered} 0 \longrightarrow x^{-\mathbb{Z}}\mathcal{G}(V) \xrightarrow{d} x^{-\mathbb{Z}}\mathcal{G}(V;\Lambda^1) \xrightarrow{d} x^{-\mathbb{Z}}\mathcal{G}(V;\Lambda^2) \longrightarrow \cdots \\ x^{-\mathbb{Z}}\mathcal{G}(V;\Lambda^k) = \bigcup_{k\in\mathbb{Z}} x^{-k} \bigoplus_{j=-\infty}^{*} \{u_j \in \mathcal{C}^\infty(V \setminus 0_V;\Lambda^k) \text{ homogeneous of degree } j\}. \end{gathered}$$

Again the cohomology is seen to be concentrated in homogeneity zero so, by a simple computation, the cohomology of the de Rham complex for these spaces is naturally identified with

$$^bH^k(S^*V \times \mathbb{S}_\sigma) \tag{35}$$

where $\mathbb{S}^*V$ is the sphere bundle of V, and $\mathbb{S}_\sigma$ is the circle arising from a homogeneous function on the fibres.

2. Spectral sequence

In this we compute the Hochschild homology groups for some of the algebras described in the Introduction, namely those which can be readily deduced by spectral sequence arguments similar to those of Brylinski and Getzler in [4].

Recall that if A is a topological algebra the Hochschild homology groups $\mathrm{HH}_*(A)$ are the homology groups of the complex $(\mathcal{H}_n(A), b)$ where $\mathcal{H}_n(A)$ is the completion of $A^{\otimes n+1}$ and b is the (complete) Hochschild differential

$$
\begin{aligned}
b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\
b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1}.
\end{aligned}
\tag{36}
$$

The first algebra we consider for a compact manifold with boundary is an analogue of the symbol algebra in the boundaryless case.

Proposition 2. *Set $\mathcal{A}_\sigma = x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)/x^{-\mathbb{Z}}\Psi_c^{-\infty}(X)$, the algebra of 'complete symbols with arbitrary polynomial blow up at the boundary'. Then*

$$\mathrm{HH}_*(\mathcal{A}_\sigma) \simeq {}^bH^{2n-*}({}^cS^*X \times \mathbb{S}_\sigma).$$

Here we have used the notation of (35) for the 'homogeneous' cohomology of a vector bundle over a manifold with boundary. Thus, if ${}^cS^*X$ is the (projective) sphere bundle of the cusp cotangent bundle, which is diffeomorphic to the usual one and has boundary ${}^cS^*_{\partial X}X$, then

$$
\begin{aligned}
&\mathrm{HH}_*(\mathcal{A}_\sigma) \simeq {}^bH^{2n-*}({}^cS^*X) \oplus {}^bH^{2n-1-*}({}^cS^*X) \simeq \\
&\big(H^{2n-*}_{\mathrm{abs}}({}^cS^*X) \oplus H^{2n-1-*}({}^cS^*_{\partial X}X)\big) \oplus \big(H^{2n-1-*}_{\mathrm{abs}}({}^cS^*X) \oplus H^{2n-2-*}({}^cS^*_{\partial X}X)\big).
\end{aligned}
$$

Proof. This proceeds to a large extent as in the boundaryless case [4]. Consider the complex $(\mathcal{H}_n(\mathcal{A}_\sigma), b)$ which computes the *continuous* Hochschild homology of $\mathcal{A}_\sigma$. The spaces $\mathcal{H}_n(\mathcal{A}_\sigma)$ are obtained from ${\mathcal{A}_\sigma}^{\otimes n+1}$ by topological completion. As discussed in Appendix A, a choice of quantization map gives an isomorphism

$$\mathcal{A}_\sigma = \operatorname*{ind\,lim}_{k,m\to\infty} \left(x^{-m} \prod_{j=-\infty}^{k} \mathcal{G}^j({}^cT^*X) \right) \tag{37}$$

where $\mathcal{G}^j(V)$ is given by (34). Topological completion therefore gives the chain spaces a similar structure

$$\mathcal{H}_{n-1}(\mathcal{A}_\sigma) = \operatorname*{ind\,lim}_{k,|\alpha|\to\infty} \left(x^{-\alpha} \prod_{j=-\infty}^{k} \mathcal{G}^j_n({}^cT^*X) \right) \tag{38}$$

where x_l is the boundary defining function in the lth factor, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, is a multiindex, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, and $\mathcal{G}^j_n(V) \subset \mathcal{C}^\infty((V \setminus 0_V)^n)$ is the subspace consisting of the functions which are finite sums of terms homogeneous in each factor with total degree j. The filtration of these spaces, given by 'the total order' is independent of the quantization used, so the corresponding graded spaces are naturally defined and can be identified with the terms in (38).

The product on $\mathcal{A}_\sigma$ is filtered with respect to the filtration by the order, k, in (37). It therefore gives rise to a spectral sequence as in [4]. The E^1 term of this spectral sequence computes the Hochschild homology of the graded-commutative algebra $x^{-\mathbb{Z}}\mathcal{G}(\mathcal{A}_\sigma)$ formed by the $x^{-k}\mathcal{G}^j({}^cT^*X)$. Corollary 2 shows that the Hochschild-Kostant-Rosenberg map induces an isomorphism on Hochschild homology

$$\mathrm{HH}_i(\mathcal{G}(\mathcal{A}_\sigma)) = x^{-\mathbb{Z}}\mathcal{G}({}^cT^*X; \Lambda^i),$$

where the image is the same formal sum of homogeneous i-forms (with homogeneity bounded above) on $V \setminus 0_V$, for $V = {}^cTX$.

The differential d_1, can be computed as in [4], Theorem 1 and Proposition 1. It is therefore the extension by continuity of the Poisson differential on the interior; we denote this extension still by δ. The naturality of the cusp exterior algebra, bearing as it does the same relationship to cusp vector fields as the usual exterior algebra does to all vector fields, shows that δ is the Poisson differential with respect to the (degenerate) Poisson structure defined by the singular symplectic form on ${}^cT^*X$. In canonical local coordinates near the boundary this form is just

$$\omega_{\mathrm{c}} = \frac{dx}{x^2} \wedge d\lambda + \sum_{j=1}^{n-1} dy_j \wedge d\eta_j .$$

Since it is homogeneous and the singularity at the boundary is of finite order, symplectic duality as discussed by Brylinski ([3]) gives isomorphisms

$$\begin{gathered} *_\omega : x^{-\mathbb{Z}}\mathcal{G}^j({}^cT^*X;\Lambda^k) \longrightarrow x^{-\mathbb{Z}}\mathcal{G}^{j+n-k}({}^cT^*X;\Lambda^{2n-k}), \\ *(f dx^I d\xi^J) = \pm x^2 f dx^{J^{\complement}} d\xi^{I^{\complement}}, \end{gathered}$$

transforming δ to the de Rham differential. Here $I^{\complement}$ denotes the complement of the index set I. These identifications provide an isomorphism of complexes

$$(E^1_{p,q}, \delta) \simeq \left(x^{-\mathbb{Z}}\mathcal{G}^{n-q}({}^cT^*X;\Lambda^{2n-p-q}), d\right).$$

Thus the E^2 term of the spectral sequence vanishes, except for $q = n$, the dimension of the manifold X. From Lemma 3, as modified in the case of a manifold with boundary to (35)

$$\begin{aligned} E^2_{p-n,n} &= {}^bH^{2n-p}({}^cS^*X \times \mathbb{S}_\sigma) \\ &= {}^bH^{2n-p}({}^cS^*X) \oplus {}^bH^{2n-p-1}({}^cS^*X) d\log r \\ &= H^{2n-p}({}^cS^*X \times \mathbb{S}_\sigma) \oplus H^{2n-p-1}({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma) x^{-1}dx \end{aligned}$$

As in the boundaryless case it follows that all further differentials are zero, so the spectral sequence degenerates at E_2.

The convergence of the spectral sequence to the Hochschild homology also follows directly, as in the boundaryless case. □

Next we consider the corresponding relative object which is an ideal in $\mathcal{A}_\sigma$.

Proposition 3. *The algebra of 'complete symbols vanishing rapidly at the boundary', $\mathcal{I}_\sigma = x^\infty\Psi^{\mathbb{Z}}_c(X)/x^\infty\Psi^{-\infty}_c(X)$, is H-unital and*

$$\mathrm{HH}_*(\mathcal{I}_\sigma) \simeq H^{2n-*}_{\mathrm{rel}}({}^cS^*X \times \mathbb{S}_\sigma) \simeq H^{2n-*}_{\mathrm{rel}}({}^cS^*X) \oplus H^{2n-1-*}_{\mathrm{rel}}({}^cS^*X).$$

Here the 'relative cohomology' groups, $H^*_{\mathrm{rel}}({}^cS^*X)$ are the de Rham cohomology groups computed relative to the boundary.

Proof. The computation of the Hochschild homology proceeds very much as in the previous proposition. One can also deduce it by retracting the algebra to have compact support in the interior and then directly applying the results of [4].

The H-unitality of $\mathcal{I}_\sigma$, i.e. the fact that the corresponding b' complex is acyclic, follows much as in Lemma 1. Indeed, the quantization can be chosen so that Id has full symbol expansion 1. The cut-off factor in (26) then behaves as the identity with respect to the $\star$ product. □

3. Boundary sequence

Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_\sigma \longrightarrow \mathcal{A}_\sigma \longrightarrow \mathcal{A}_{\partial,\sigma} \longrightarrow 0 \tag{39}$$

where $\mathcal{A}_{\partial,\sigma}$ is therefore the algebra of 'Laurent series in x of complete symbols at the boundary'.

Proposition 4. *For the algebra* $\mathcal{A}_{\partial,\sigma} = x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)/\big(x^{\infty}\Psi_c^{\mathbb{Z}}(X) + x^{-\mathbb{Z}}\Psi_c^{-\infty}(X)\big)$

$$\mathrm{HH}_*(\mathcal{A}_{\partial,\sigma}) \simeq H^{2n-*}({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial). \tag{40}$$

Proof. Again the same approach can be used as in the proof of Proposition 2. This symbol algebra consists of formal power series in the boundary variable x as well as the homogeneous variable in the fibres of ${}^cT^*X$. This results in the two circles in (40), with the de Rham cohomology being generated by $1, d\log x, d\log r$ and $d\log x \wedge d\log r$ over $H^*({}^cS^*_{\partial X}X)$. □

Proposition 5. *The long exact sequence in Hochschild homology arising from* (39) *is the long exact sequence for b- de Rham cohomology, of the manifold* $Y_X = {}^cS^*X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial$, *with respect to its boundary; i.e. it is the usual cohomology long exact sequence together with the identity on* $H^*(\partial Y_X)$:

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{HH}_k(\mathcal{A}_\sigma) \arrow[r] \arrow[d, equal] & \mathrm{HH}_k(\mathcal{A}_{\partial,\sigma}) \arrow[r] \arrow[d, equal] & \mathrm{HH}_{k-1}(\mathcal{I}_\sigma) \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & \begin{matrix} H^{2n-k}(Y_X) \\ \oplus \\ H^{2n-k-1}(\partial Y_X) \end{matrix} \arrow[r, "i^*", "\mathrm{Id}"'] & \begin{matrix} H^{2n-k}(\partial Y_X) \\ \oplus \\ H^{2n-k-1}(\partial Y_X) \end{matrix} \arrow[r, "\imath\partial", "0"'] & H^{2n-k+1}_{\mathrm{rel}}(Y_X) \arrow[r] & {}
\end{tikzcd}$$

We have denoted by $i^* = i^*_{\partial Y_X}$ the restriction to the boundary. Also note that the boundary map has an extra factor of $\imath = \sqrt{-1}$.

Proof. Since $\mathcal{I}_\sigma$ is H-unital as a topological algebra it follows from the result of Wodzicki, [26], that (39) induces a long exact sequence in Hochschild homology

$$\cdots \longrightarrow \mathrm{HH}_p(\mathcal{A}_\sigma) \longrightarrow \mathrm{HH}_p(\mathcal{A}_{\partial,\sigma}) \xrightarrow{\partial} \mathrm{HH}_{p-1}(\mathcal{I}_\sigma) \longrightarrow \mathrm{HH}_{p-1}(\mathcal{A}_\sigma) \longrightarrow \cdots .$$

Except for the connecting morphism ∂, the maps here come from algebra morphisms. Since these morphisms preserve the order, they give morphisms of the corresponding spectral sequences. This identifies the composition

$$\begin{aligned} H^{2n-p}_{\mathrm{rel}}({}^cS^*X \times \mathbb{S}_\sigma) \simeq \mathrm{HH}_p(\mathcal{I}_\sigma) \longrightarrow & \\ \mathrm{HH}_p(\mathcal{A}_\sigma) \simeq H^{2n-p}({}^cS^*X \times \mathbb{S}_\sigma) &\oplus H^{2n-1-p}({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma)x^{-1}dx \end{aligned}$$

as $\iota^* \oplus 0$, where ι^* is the natural map from relative to absolute cohomology. Similarly, the composition

$$\begin{aligned} H^{2n-p}({}^cS^*X \times \mathbb{S}_\sigma) \oplus H^{2n-1-p}({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma)x^{-1}dx \simeq \mathrm{HH}_p(\mathcal{A}_\sigma) \longrightarrow & \\ \mathrm{HH}_p(\mathcal{A}_{\partial,\sigma}) \simeq H^{2n-p}({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma) &\oplus H^{2n-1-p}({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma)x^{-1}dx \end{aligned}$$

preserves the direct sum decomposition, is the identity on the second component and the restriction to the boundary on the first component.

It remains to determine the connecting morphism ∂. The results of [26] identify this map in the following way. The inclusion of the complex $\mathcal{H}(\mathcal{I}_\sigma)$ into the complex $\mathcal{L}$, the kernel of the projection $\mathcal{H}(\mathcal{A}_\sigma) \to \mathcal{H}(\mathcal{A}_{\partial,\sigma})$, is shown to induce an isomorphism of homology groups (in our case this can also be seen from the associated spectral sequences). This shows that ∂ can be computed as the connecting morphism of the homology exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{H}(\mathcal{A}_\sigma) \longrightarrow \mathcal{H}(\mathcal{A}_{\partial,\sigma}) \longrightarrow 0. \tag{41}$$

To carry out this computation assume first that $X = Y \times [0,1]$ where Y is a manifold without boundary. Put $Z_i = {}^cS^*_{Y\times\{i\}}X$, $i = 0, 1$. This gives, by restriction, an isomorphism $\mathcal{A}_{\partial,\sigma} \simeq {\mathcal{A}_{\partial,\sigma}}^{(0)} \oplus {\mathcal{A}_{\partial,\sigma}}^{(1)}$. The group $\mathrm{HH}_p(\mathcal{A}_{\partial,\sigma})$ decomposes then as a direct sum of two naturally isomorphic groups

$$\mathrm{HH}_p(\mathcal{A}_{\partial,\sigma}) \simeq H^{2n-p}({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) \simeq H^{2n-p}(Z_0 \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) \oplus H^{2n-p}(Z_1 \times \mathbb{S}_\sigma \times \mathbb{S}_\partial)$$

and accordingly, so does the connecting morphism

$$\partial = \partial_0 \oplus \partial_1 : \mathrm{HH}_p(\mathcal{A}_{\partial,\sigma}) \to \mathrm{HH}_{p-1}(\mathcal{I}_\sigma) \simeq H^{2n-p+1}_{\mathrm{rel}}({}^cS^*X \times \mathbb{S}_\sigma).$$

For reasons of symmetry it follows that $\partial_1 = -\partial_0$, so it is enough to compute ∂_0. Consider the subalgebra $(\mathcal{A}_{\partial,\sigma})_{\mathrm{inv}}$ consisting of those operators that commute with $x^2(1-x)^2\partial_x$. The algebra $\mathcal{A}_{\partial,\sigma}$ is generated as a topological algebra by the subalgebra $(\mathcal{A}_{\partial,\sigma})_{\mathrm{inv}}$ and $x^{-\mathbb{Z}}(1-x)^{-\mathbb{Z}}\mathcal{C}^\infty([0,1])$. The algebra ${\mathcal{A}_{\partial,\sigma}}^{(0)}$ is isomorphic to $x^{-\mathbb{Z}}(\mathcal{A}_{\partial,\sigma})_{\mathrm{inv}}$. This defines a splitting $s_0 : {\mathcal{A}_{\partial,\sigma}}^{(0)} \to \mathcal{A}_\sigma$. We remark that the existence of this splitting is a feature of the product and does not hold for arbitrary X.

Choose a smooth function φ on $[0,1]$, which vanishes in a neighborhood of 1, and is 1 in a neighborhood of 0. and define

$$s : \mathcal{H}({\mathcal{A}_{\partial,\sigma}}^{(0)}) \to \mathcal{H}(\mathcal{A}_{\partial,\sigma})$$
$$s(f_0 \otimes f_1 \otimes \ldots \otimes f_n) = s_0(f_0)\varphi \otimes s_0(f_1) \otimes \ldots \otimes s_0(f_n) \text{ and}$$
$$\sigma : \mathcal{H}({\mathcal{A}_{\partial,\sigma}}^{(0)}) \to \mathcal{L}$$
$$\sigma(f_0 \otimes f_1 \otimes \ldots \otimes f_n) = s_0(f_0)[\varphi, s_0(f_1)] \otimes s_0(f_2) \otimes \ldots \otimes s_0(f_n)$$

where $\mathcal{L}$ is as in (41) above.

The relation $\sigma = [b, s]$, where b is the Hochschild boundary, follows from the fact that s_0 is an algebra morphism and the definitions. An immediate consequence is that σ is a chain map with $[\sigma, b] = 0$ in terms of graded commutators. By definition, the natural morphism σ_*, induced by σ, from the Hochschild homology groups of ${\mathcal{A}_{\partial,\sigma}}^{(0)}$ to the homology of $\mathcal{L}$, coincides with the connecting morphism ∂. The explicit formula for σ shows that it preserves the filtrations of both complexes (shifting orders by -1). Since both spectral sequences degenerate at E^2, it is then enough to compute the morphism $E^2_{p,q}(\sigma) : E^2_{p,q}({\mathcal{A}_{\partial,\sigma}}^{(0)}) \to E^2_{p-1,q}(\mathcal{L})$.

Let η with $d\eta = 0$, be a closed $(2n-p)$-form on $Z_0 \times \mathbb{S}_\sigma \times \mathbb{S}_\partial$. The computations in the proof of Proposition 2 shows that we can represent the class of η as

$$[\eta] \in H^{2n-p}(Z_0 \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) \simeq E^2_{p-n,n} \simeq \mathrm{HH}_p({\mathcal{A}_{\partial,\sigma}}^{(0)}),$$

by a cycle $z \in F_{p-n}\mathcal{H}({\mathcal{A}_{\partial,\sigma}}^{(0)})$, $z \in ({\mathcal{A}_{\partial,\sigma}}^{(0)})^{\otimes p+1}$.

Denote by $\overline{z}$ the image of z in the quotient $\mathcal{G}^{p-n}({\mathcal{A}_{\partial,\sigma}}^{(0)})$. We can assume that $\overline{z}$ is antisymmetric in all variables, and maps, via the Hochschild-Rosenberg-Kostant map χ, to the form $*\eta$, $\chi(\overline{z}) = *\eta$. Let H_φ be the Hamiltonian vector field associated to φ, $H_\varphi(f) = \{\varphi, f\}$. Here, as usual, we have denoted the Poisson bracket by $\{\, ,\, \}$; it is determined by the singular symplectic structure on ${}^cT^*X$. The antisymmetry of $\overline{z}$ gives

$$\chi(\sigma(\overline{z})) = \imath i_{H_\varphi}(\chi(\overline{z})).$$

Taking into account the symplectic duality $*$ and the relation

$$i_{H_f}(*\eta) = *(df \wedge \eta)$$

we obtain the desired description of σ_*

$$\partial([\eta]) = \sigma_*([\eta]) = [d\varphi \wedge \eta] = [d(\varphi\eta)].$$

where the brackets mean 'the cohomology class' of that form. Since $\varphi\eta$ restricted to Z_0 is η it follows that $[d(\varphi\eta)]$ is exactly the image of $[\eta]$ in the relative group, via the connecting morphism in (de Rham) cohomology.

For arbitrary manifolds X, choose an open tubular neighborhood $V \simeq \partial X \times [0, 1)$ of ∂X in X. The operators in $\mathcal{A}_\sigma$ with support in V identify with a subalgebra of $\mathcal{A}_\sigma(\partial X \times [0, 1])$. The computation follows then from the naturality of the exact sequences in de Rham and Hochschild homology, and from the naturality of the isomorphisms in Propositions 2, 3 and 4. $\square$

4. Indicial quotient

Now consider the more interesting, indicial, quotient. That is, set

$$\mathcal{I}_\partial = x^{-\mathbb{Z}}\Psi_{\mathrm{c}}^{-\mathbb{Z}}(X)/x^{\infty}\Psi_{\mathrm{c}}^{-\mathbb{Z}}(X).$$

This is the algebra of 'indicial operators of order $-\infty$ with Laurent series coefficients in x' as described in (14). That is, $\mathcal{I}_\partial$ is filtered by the powers of x, x being given a negative degree,

$$x^{-p}\Psi_{\mathrm{c}}^{-\mathbb{Z}}(X)/x^{-p+1}\Psi_{\mathrm{c}}^{-\mathbb{Z}}(X) \equiv x^{-p}\Psi_{\mathrm{sus}}^{-\infty}(\partial X).$$

This isomorphism to the suspended algebra (which we also call the indicial algebra) on the boundary is discussed in the appendix. This filtration has completion properties similar to the the order filtration of the algebra of pseudodifferential operators.

Proposition 6. *The algebra $\mathcal{I}_\partial$ of Laurent series in the indicial algebra is H-unital and*

$$\mathrm{HH}_*(\mathcal{I}_\partial) \simeq H^{1-*}(\mathbb{S}_\partial).$$

Proof. We use the filtration by powers of x and the arguments similar to those in the preceding propositions.

The indicial algebra of order $-\infty$ is the topological completion

$$\Psi_{\mathrm{sus}}^{-\infty}(Y) \simeq \mathcal{S}(\mathbb{R})\hat{\otimes}\Psi^{-\infty}(Y),$$

where $\Psi^{-\infty}(Y)$ is the algebra of smoothing operators on Y, $\mathcal{S}(\mathbb{R})$ is the algebra of Schwartz functions on the line with the convolution product, and $\hat{\otimes}$ is the projective tensor product.

The algebra $\Psi^{-\infty}(Y)$ is a projective module over its 'double'

$$\Psi^{-\infty}(Y)\hat{\otimes}\Psi^{-\infty}(Y)^{\mathrm{op}}$$

and hence is homologous to a matrix algebra, i.e. its homology is concentrated in dimension 0, where it is $\mathbb{C}$, given by the trace. This shows that the structure of the graded algebra is

$$\Psi^{-\infty}(\partial X)\hat{\otimes}\mathcal{S}(\mathbb{R})\hat{\otimes}\mathbb{C}[x,x^{-1}].$$

where all tensor products are projective tensor products and all three algebras are nuclear. Using the topological Künneth formula for Hochschild homology [12] we see that the algebra of smoothing operators does not change the result and hence the E^1-term of the spectral sequence is

$$E^1_{p,q} \simeq \mathrm{HH}_{p+q}(\mathcal{S}(\mathbb{R})\hat{\otimes}\mathbb{C}[x]) = x^{-p}\mathcal{S}(\mathbb{R};\Lambda^{p+q-1}). \tag{42}$$

The commutation relation $[x^{-1}, x^2\partial_x] = 1$ makes $\mathbb{R}\times\mathbb{S}_\partial$ into a symplectic manifold, and the differential d_1 coincides with the Poison differential as in the proof of Proposition 2. The argument continues in the same way and show that the only non vanishing E^2-terms are $E^2_{p,1} \simeq H^{2-p}_{\mathrm{comp}}(\mathbb{R}\times\mathbb{S}_\partial)$, for $p = 0, 1$.

Since the spectral sequence clearly degenerates at E^2 and convergence follows directly, we find

$$\mathrm{HH}_*(\mathcal{I}_\partial) \simeq H^{2-*}_{\mathrm{comp}}(\mathbb{R}\times\mathbb{S}_\partial) \simeq H^{1-*}(\mathbb{S}_\partial).$$

The spectral sequence associated to the same filtration and Lemma 1 shows that the b'-homology vanishes. This proves H-unitality which can also be seen directly as in Lemma 1. □

For the fully residual algebra $\mathcal{I} = x^{\mathbb{Z}}\Psi^{-\infty}_{\mathrm{c}}(X)$ one concludes as in the boundaryless case that the Hochschild homology is just $\mathbb{C}$ in dimension 0.

5. Residue functionals

In this section traces on various of the algebras are identified. As in the boundaryless case, see [6, 25], we consider functionals which arise as the residue of the analytic continuation of 'zeta-type' functions. Fix a boundary defining function $x \in \mathcal{C}^\infty(X)$ and a positive, elliptic and invertible element $Q \in \Psi^1_{\mathrm{c}}(X)$. For example, Q can be taken to be $(\Delta+1)^{\frac{1}{2}}$ where Δ is the Laplacian of an exact cusp metric, as discussed in Appendix B.

Lemma 4. *Suppose $A = A(z,\tau) \in x^p\Psi^m_c(X)$ is a holomorphic function on $\Omega \subset \mathbb{C}^2$, where the connected open set Ω contains a set of the form $[R,\infty)\times[R,\infty)$, then the function $\mathrm{Tr}(A(z,\tau)x^zQ^{-\tau})$ is defined and holomorphic in the component of $\{\mathrm{Re}\, z > -p+1\}\cap\{\mathrm{Re}\,\tau > m+\dim X\}\cap\Omega$ meeting $[R',\infty)\times[R',\infty)$ for R' large, and extends to a meromorphic function on Ω with at most simple poles on the surfaces $z = -p+1-j$, $j\in\mathbb{N}_0$ and $\tau = m+\dim X - k$, $k\in\mathbb{N}_0$.*

Proof. The general fact underlying this result is described in Lemma 26 in Appendix B. The present result follows by applying this to the kernel of $A(z,\tau)$ as a conormal density on X^2_{c} with respect to the lifted diagonal $\varDelta_{\mathrm{c}}$. The function x lifted from the left factor is not quite a defining function for the boundary hypersurface (defined by blow up) which the lifted diagonal meets, but is a product $\rho\rho'$ where ρ is such a function and ρ' is a product of boundary defining functions for boundary hypersurfaces at which the kernel vanishes to infinite order. Thus the additional factor of $(\rho')^z$ can be absorbed into the kernel to give a conormal distribution which is entire in z. Then Lemma 26 applies, with the poles in z shifted as a consequence of the fact that the resulting density has an additional factor of x^{-2}. □

We are interested in the following generalized (and 'double') zeta function for $A \in x^p\Psi^m_c(X)$

$$Z(A;\tau,z) = Z_{x,Q}(A;\tau,z) = \frac{1}{2}\big(\operatorname{Tr}(Ax^zQ^{-\tau}) + \operatorname{Tr}(AQ^{-\tau}x^z)\big). \tag{43}$$

Lemma 4, applied twice, shows that $\tau z Z_{x,Q}(A;\tau,z)$ is holomorphic in a neighborhood of $0 \in \mathbb{C}^2$. We shall examine the three functionals defined by

$$\tau z Z(A;\tau,z) = \operatorname{Tr}_{\partial,\sigma}(A) + \tau\widehat{\operatorname{Tr}}_\partial(A) + z\widehat{\operatorname{Tr}}_\sigma(A) + \tau^2 W + \tau z W' + z^2 W'' \tag{44}$$

where W, W' and W'' are holomorphic near 0.

It also follows that the function

$$\widetilde{Z}(A;z) = \widetilde{Z}_{x,Q}(A;z) = Z_{x,Q}(A;z,z) = \frac{1}{2}\big(\operatorname{Tr}(Ax^zQ^{-z}) + \operatorname{Tr}(AQ^{-z}x^z)\big)$$

has at most a double pole at $z = 0$. The double residue of $\widetilde{Z}(A;z)$ at $z = 0$, (i.e. the coefficient of z^{-2}) is just $\operatorname{Tr}_{\partial,\sigma}(A)$. Clearly the residue of $\widetilde{Z}(A;z)$ at $z = 0$ is $\widehat{\operatorname{Tr}}_\sigma(A) + \widehat{\operatorname{Tr}}_\partial(A)$.

Although we use the symmetrized functional in (43) the 'left' version

$$Z^L_{x,Q} = \operatorname{Tr}(Ax^zQ^{-\tau})$$

is only slightly different.

Lemma 5. *For any $A \in x^p\Psi^m_c(X)$, $Z_{x,Q}(A;z,\tau) - Z^L_{x,Q}(A;z,\tau)$ is regular at $\tau = 0$ and $z = 0$.*

Proof. The difference is given by the analytic continuation of $\operatorname{Tr}(AC(z,\tau)x^zQ^\tau)$ where $C(z,\tau) = \frac{1}{2}(\operatorname{Id} - Q^{-\tau}x^zQ^\tau x^{-z})$. Since $C(z,\tau)$ is entire of order 0 and vanishes at $\tau = 0$ and at $z = 0$ the result follows. □

Lemma 6. *The double residue $\operatorname{Tr}_{\partial,\sigma}(A)$, of $\widetilde{Z}(A;z)$ at $z = 0$, is a trace functional on the algebra $x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_c(X)$; it is independent of the choice of Q or x, vanishes on $\mathcal{I}_\sigma + \mathcal{I}_\partial$ and therefore defines a trace functional on $\mathcal{A}_{\partial,\sigma}$.*

Proof. We can consider the double residue of the non-symmetrized function

$$\widetilde{Z}^L_{x,Q}(A;z) = Z^L_{x,Q}(A;z,z). \tag{45}$$

If x' is another defining function for the boundary then the analytic family $A(z,\tau) = A - A(x'/x)^z$ also vanishes at $z = 0$ so Lemma 4 now shows that the difference takes the form

$$\widetilde{Z}^L_{x,Q}(A;z) - \widetilde{Z}^L_{x',Q}(A;z) = \operatorname{Tr}(Ax^zQ^{-z} - A(x')^zQ^{-z}) = z\operatorname{Tr}\big(B(z,\tau)x^zQ^{-z}\big)$$

where $B(z,\tau)$ is holomorphic near 0 and takes values in $x^p\Psi^m_c(X)$. Thus the difference has at most a simple pole at $z = 0$ and it follows that the functional $\operatorname{Tr}_{\partial,\sigma}(A)$ does not depend on the choice of the boundary defining function. A similar argument shows independence of the choice of Q.

To prove that $\operatorname{Tr}_{\partial,\sigma}$ is a trace, consider A and B in the 'full' algebra $x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_c(X)$. Expanding the commutator $[Q^{-\tau}A, Bx^z]$ and using the trace property of Tr we deduce the following relation for large real values of z and τ

(46)

$$\begin{aligned} Z^L_{x,Q}([A,B];z,\tau) &= \operatorname{Tr}([A,B]x^zQ^{-\tau}) \\ &= -\operatorname{Tr}\big([Q^{-\tau},B]Ax^z + BQ^{-\tau}[A,x^z] + B[Q^{-\tau},x^z]A\big) \\ &= -\operatorname{Tr}\big((Q^\tau[Q^{-\tau},B]A + Q^\tau BQ^{-\tau}[Ax^{-z},x^z] + B[Q^{-\tau},x^z]AQ^\tau x^{-z})x^zQ^{-\tau}\big). \end{aligned}$$

Since each of the families $Q^\tau[Q^{-\tau},B]A$, $Q^\tau BQ^{-\tau}[Ax^{-z},x^z]$ and $B[Q^{-\tau},x^z]AQ^\tau x^{-z}$ is holomorphic as a family of classical pseudodifferential operators of fixed order, Lemma 4 shows that (46) is valid in the domain of holomorphy. Due to the commutators, each of these families vanishes at $z=\tau=0$ so $\mathrm{Tr}_{\partial,\sigma}([A,B])=0$. $\square$

Before deriving an explicit formula for $\mathrm{Tr}_{\partial,\sigma}(A)$ consider the functional $\widehat{\mathrm{Tr}}_\sigma$. Recall (see Appendix A) that if $2X$ is a compact manifold without boundary obtained by doubling X across its boundary then

$$x^\infty\Psi^{\mathbb{Z}}_{\mathrm{c}}(X)\subset\Psi^{\mathbb{Z}}(2X) \tag{47}$$

is an ideal, consisting precisely of those elements which have Schwartz kernels supported in $X\times X$. The smaller ideal, with kernels supported in the interior, i.e. in $X^\circ\times X^\circ$, is dense. The residue trace of Wodzicki for $2X$ is defined on the latter space; we shall denote it Tr_{R}. We shall denote by Tr_σ the restriction of $\widehat{\mathrm{Tr}}_\sigma$ to $\mathcal{I}_\sigma$.

Lemma 7. *The restriction,* Tr_σ*, of* $\widehat{\mathrm{Tr}}_\sigma$ *to* $\mathcal{I}_\sigma$*, is a trace functional which is the extension by continuity of Wodzicki's residue trace,* Tr_R*, for the double of* X.

Proof. If $A\in\mathcal{I}_\sigma$ then Ax^z is entire with values in $\mathcal{I}_\sigma$, as a family of fixed order. It follows that $Z_{x,Q}(A;\tau,z)$ is entire in z. Thus we only need consider the simpler function

$$Z_Q(A;\tau)=\mathrm{Tr}(AQ^{-\tau}),\ A\in\mathcal{I}_\sigma. \tag{48}$$

From a simplified version of Lemma 4 this is meromorphic in τ with the pole at $\tau=0$ necessarily simple. The residue of this function at 0 is therefore just $\widehat{\mathrm{Tr}}_\sigma(A)=\mathrm{Tr}_\sigma(A)$, by definition from (44). That is,

$$\tau Z_Q(A;\tau)=\mathrm{Tr}_\sigma(A)+\tau U(\tau),\ \text{for } A\in\mathcal{I}_\sigma \tag{49}$$

with $U(\tau)$ holomorphic near 0.

Suppose that $A\in\Psi^{\mathbb{Z}}(2X)$ is in the smaller ideal, i.e. has Schwartz kernel supported in $X^\circ\times X^\circ$. Then (49) is precisely the definition of the residue trace, except that Q should be positive and elliptic in $\Psi^1(2X)$. However, the ellipticity of Q in the cusp calculus implies that for any given B with kernel supported in $X^\circ\times X^\circ$ there exists $Q'\in\Psi^1(2X)$ which is positive and elliptic and such that $Q'B-QB\in\Psi^{-\infty}(2X)$. Since such a smoothing term does not contribute to the residue trace, it follows that

$$\mathrm{Tr}_\sigma(A)=\mathrm{Tr}_{\mathrm{R}}(A),\ A\in\Psi^{\mathbb{Z}}(2X),\ \mathrm{supp}(A)\subset X^\circ\times X^\circ. \tag{50}$$

Since Tr_{R} is itself a trace functional, the continuity of Tr_σ in the topology of $\mathcal{I}_\sigma$ completes the proof. Alternatively the trace property can be seen directly by a simple computation similar to that in (46). The independence of choice of Q follows as before. $\square$

This identification of $\mathrm{Tr}_\sigma(A)$ and $\mathrm{Tr}_{\mathrm{R}}(A)$ allows us to derive an explicit formula for the former from the formula for the latter in [6] or [25], see also [4]. Namely, any element $A\in\Psi^{\mathbb{Z}}(2X)$ can be identified with its Schwartz kernel, a conormal right density with respect to the diagonal in $(2X)^2$. The collar neighborhood theorem allows a neighborhood of the diagonal to be identified with a neighborhood of the zero section of TX. By dropping a smooth term the kernel can be assumed to have support in this neighborhood. Fourier transformation on the fibres converts it to a smooth density on T^*X. The polyhomogeneity of the original operator is just the statement that this density has a complete asymptotic expansion

$$(\sum_{j=-\infty}^{m} a_j)\omega^n,\ a_j \text{ homogeneous of degree } j,$$

(with the nth power of the symplectic form used to remove the density factor). Even though the terms in this expansion depend on the choice of normal fibration of $(2X)^2$ around the diagonal, the residue trace is always given by

$$\mathrm{Tr}_{\mathrm{R}}(A) = (2\pi)^{-n} \int_{S^*(2X)} a_{-n}\nu$$

where ν is the $(2n-1)$-form obtained by contraction of the nth power of the symplectic form on $T^*(2X)$ with the radial vector field, $\zeta \cdot \partial_\zeta$ from the $\mathbb{R}^+$ action on the fibres.

From Lemma 7 and the continuity of Tr_σ we deduce a similar formula for $\mathrm{Tr}_\sigma(A)$ when $A \in \mathcal{I}_\sigma$. This formula can be rewritten in terms of the choice of a full symbol map as described in the Appendix, and already used in the proof of Lemma 4.

Proposition 7. *If $A \in \mathcal{I}_\sigma$ and $\sum_{j=-\infty}^{m} a_j$ is the full symbol expansion of A arising from a choice of normal fibration to the lifted diagonal then*

$$\mathrm{Tr}_\sigma(A) = (2\pi)^{-n} \int_{{}^cS^*X} a_{-n}\,{}^c\nu \tag{51}$$

where ${}^c\nu$ is the (singular) volume form on ${}^cS^*X$ defined by contraction of the nth power of the symplectic form with the radial vector field on the fibres of ${}^cT^*X$.

Notice that ${}^c\nu$ is of the form $x^{-2}\nu'$ where ν' is a positive smooth volume form on ${}^cS^*X$. Thus we can see directly that Tr_σ extends by continuity from $\mathcal{I}_\sigma$ to $x^p\Psi^{\mathbb{Z}}_{\mathrm{c}}(X)$, provided $p > 1$. In particular we can consider the function

$$\mathrm{Tr}_\sigma(x^z A),\ A \in x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{\mathrm{c}}(X) \tag{52}$$

and observe that it is holomorphic for $\operatorname{Re} z$ large with a meromorphic continuation to $z \in \mathbb{C}$.

Now, consider a general element $A \in x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{\mathrm{c}}(X)$, so $A \in x^p\Psi^m_{\mathrm{c}}(X)$ for some p and $m \in \mathbb{Z}$. As already noted this has a Laurent series expansion at the boundary, with respect to any normal fibration and boundary defining function x

$$A \sim \sum_{j=-\infty}^{-p} x^{-j}A_j,\ A_j \in \Psi^m_{\mathrm{sus}}(\partial X). \tag{53}$$

In terms of this expansion we can obtain a formula for $\mathrm{Tr}_{\partial,\sigma}(A)$. Observe that the elements of $\Psi^m_{\mathrm{sus}}(\partial X)$ are pseudodifferential operators on $\mathbb{R} \times \partial X$, which are translation invariant in the first variable. A normal fibration of $(\partial X)^2$ around the diagonal therefore leads to a complete symbol expansion for each A_j of the form $\sum_k a_{k,j}|\omega_\partial^{n-1}dtd\xi|$ where ω_∂ is the symplectic form on $T^*\partial X$ and ξ is the variable in the dual line (dual to t).

Lemma 8. *The functional $\mathrm{Tr}_{\partial,\sigma}(A)$ for $A \in x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{\mathrm{c}}(X)$ is given explicitly by*

$$\mathrm{Tr}_{\partial,\sigma}(A) = (2\pi)^{-n} \int_{{}^cS^*_{\partial X}X} a_{-n,-1}\,{}^c\nu_\partial \tag{54}$$

where $a_{k,l}$ is the term homogeneous of degree k in the asymptotic expansion of the symbol of A_l with respect to a normal fibration of X^2 around the diagonal and ${}^c\nu_\partial$ is the measure obtained by contracting the form $\omega_\partial^{n-1}d\xi$ with the radial vector field on ${}^cT^*_\partial X = T^*\partial X \times \mathbb{R}_\xi$.

Proof. Comparing the various functionals that we have so far defined it follows directly that $\mathrm{Tr}_{\partial,\sigma}(A)$ is the residue at $z=0$ of the meromorphic function in (52). Thus (54) follows directly from (51). □

Similarly we can deduce an explicit formula for $\widehat{\mathrm{Tr}}_\sigma$. Namely it is just the Hadamard regularization of the usual formula for Wodzicki's residue trace,

$$\widehat{\mathrm{Tr}}_\sigma(A) = \lim_{\epsilon\downarrow 0}\left((2\pi)^{-n}\int_{{}^cS^*X\cap\{x>\epsilon\}} a_{-n}\,{}^c\nu + (\log\epsilon)\,\mathrm{Tr}_{\partial,\sigma}(A) + \sum_{l>0}\gamma_l\epsilon^{-l}\right), \tag{55}$$

where the coefficients are chosen so that the limit exists.

Lemma 9. *The functional $\widehat{\mathrm{Tr}}_\sigma$ vanishes on $\mathcal{I}_\partial$ and so defines a continuous functional on $\mathcal{A}_\sigma$, it is independent of the choice of Q. If $x' = ax$, with $0 < a \in \mathcal{C}^\infty(X)$, are two boundary defining functions the difference of the two functionals is given by*

$$\widehat{\mathrm{Tr}}_\sigma(A;x') - \widehat{\mathrm{Tr}}_\sigma(A;x) = \mathrm{Tr}_{\partial,\sigma}(A\log a),\ \ A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X). \tag{56}$$

Proof. Independence of the choice of Q follows from the fact that Tr_{R} does not depend on Q. The relation (56) between the functionals obtained by regularization with respect to different boundary defining functions follows from the fact that it is given by the regular value at $z=0$ of

$$\mathrm{Tr}_\sigma(x^z(a^z-1)A) = \mathrm{Tr}_\sigma(zx^zB(z))$$

where $B(z) = z^{-1}(A^z-1)A$ is entire. Since the residue of $\mathrm{Tr}_\sigma(x^zB(z))$ at $z=0$ is $\mathrm{Tr}_{\partial,\sigma}(B(0))$ the result follows. □

Next we reverse the roles of x and Q and proceed to consider the functional $\widehat{\mathrm{Tr}}_\partial$, defined on $x^{-\mathbb{Z}}\Psi_{\mathrm{c}}^{\mathbb{Z}}(X)$ by (44). Since $Z(A;\tau,z)$ is entire if $A\in\mathcal{I}$, the functional $\widehat{\mathrm{Tr}}_\partial$ is well-defined on $\mathcal{A}_\partial$, we shall write its restriction to $\mathcal{I}_\partial$ as Tr_∂. If $A \in \Psi_{\mathrm{c}}^{-\infty}(X)$ then $Z(A;\tau,z)$ is entire in τ, as a meromorphic function of z. It follows then that $\mathrm{Tr}_\partial(A)$ is the residue at $z=0$ of $\mathrm{Tr}(Ax^z)$. Recall from [16] that on the indicial algebra $\Psi_{\mathrm{sus}}^{-\infty}(\partial X) \simeq \mathcal{S}(\mathbb{R})\widehat{\otimes}\Psi^{-\infty}(\partial X)$ there is a trace functional given by integration of the trace of the indicial family

$$\overline{\mathrm{Tr}}(B) = (2\pi)^{-1}\int_{\mathbb{R}} \mathrm{Tr}\,\widehat{B}(\xi)d\xi. \tag{57}$$

Lemma 10. *The functional Tr_∂ is a trace functional on $\mathcal{I}_\partial$, defined independently of choice of x or Q, and is fixed in terms of the trace (57) on the indicial algebra through*

$$\mathrm{Tr}_\partial(A) = \overline{\mathrm{Tr}}(A_{-1}),\ A\in\mathcal{I}_\partial,\ A \sim \sum_{j=-\infty}^{-p} x^{-j}A_j,\ A_j \in \Psi_{\mathrm{sus}}^{-\infty}(\partial X).$$

Proof. This follows directly from the formula for the trace as the integral of the Schwartz kernel over the diagonal. Lifting to X^2_{c} this becomes

$$\mathrm{Tr}(Ax^z) = \int_{\Delta_{\mathrm{c}}} x^z A_{\Delta_{\mathrm{c}}} = \int_X x^z B\nu$$

where $\nu = x^{-2}\nu'$, with ν' a smooth positive density on X and $B \in x^{-\mathbb{Z}}\mathcal{C}^\infty(X)$ represents the kernel restricted to the diagonal. The residue at $z = 0$ therefore arises from the term B_{-1} in the expansion

$$\mathrm{Tr}_\partial(A) = \int_{\partial X} B_{-1}\nu_\partial,\ \nu' = dx\nu_\partial \text{ at } \partial X,\ B \simeq \sum_{k=-\infty}^{-p} x^{-k}B_k,$$

Since $B_{-1}\nu_\partial$ is the restriction of the kernel of A_{-1} to $\{0\} \times \Delta_{\partial X}$ it follows that the trace is given by (57) with $B = A_{-1}$. Clearly this is independent of the choice of Q, and independence of the choice of x follows as above. $\square$

As shown in [16] the functional $\overline{\mathrm{Tr}}$ extends from $\Psi^{-\infty}_{\mathrm{sus}}(\partial X)$ to $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(\partial X)$ still as a trace functional. This extension is given by regularization of (57). Namely, if $B \in \Psi^m_{\mathrm{sus}}(\partial X)$ then

$$h_p(\xi) = \mathrm{Tr}\left((\frac{\partial}{\partial \xi})^p \widehat{B}(\xi) \right),\ p \geq m + \dim X \tag{58}$$

is defined and the asymptotic expansion of the $(p+1)$-fold integral

$$\int_{-T}^{T} \int_0^{\xi_p} \cdots \int_0^{\xi_1} h_p(\xi) d\xi d\xi_1 \cdots d\xi_p \sim \sum_{j < m + \dim X} c_j T^j + P(T)\log T \tag{59}$$

fixes the coefficient $\overline{\mathrm{Tr}}(B) = c_0/2\pi$ independent of p. Here P is a polynomial and the independence of p follows from the fact that increasing p changes the inner p-fold integral in (59) by a polynomial and hence changes the full integral by a polynomial without constant term, leaving c_0 unchanged.

Let $Q_0 = (D_t^2 + \Delta_{\partial X} + 1)^{\frac{1}{2}}$ be a positive and elliptic element of $\Psi^1_{\mathrm{sus}}(\partial X)$, given by the Laplacian of a metric on ∂X. Then consider the function

$$\overline{\mathrm{Tr}}(BQ_0^{-\tau}). \tag{60}$$

This is holomorphic for $\mathrm{Re}\,\tau$ large and has a meromorphic continuation to the complex plane. Indeed, if $B \in \Psi^m_{\mathrm{sus}}(\partial X)$ then the poles can occur only at $\tau = -\dim X + \mathbb{N}_0$.

Proposition 8. *For $B \in \Psi^m_{\mathrm{sus}}(\partial X)$, $m \in \mathbb{Z}$, the residue at $z = 0$ of the function in* (60) *is the functional $\widetilde{\mathrm{Tr}}(B) = \overline{\mathrm{Tr}}([t, B])$ introduced in* [16] *and the regularized value is $\overline{\mathrm{Tr}}(B)$, i.e.*

$$\tau\,\overline{\mathrm{Tr}}(BQ_0^{-\tau}) = \widetilde{\mathrm{Tr}}(B) + \tau\,\overline{\mathrm{Tr}}(B) + \tau^2 U'(\tau)$$

with $U'(\tau)$ holomorphic near $\tau = 0$.

Proof. For p large, let $h_p(\xi;\tau)$ be the function given by (58) with B replaced by $BQ_0^{-\tau}$. This is entire in τ and for τ near zero the asymptotic expansion as $|\xi| \to \infty$ is also uniform with

$$h_p(\xi;\tau) \sim \sum d_k^\pm(\tau)|\xi|^{m-p+n-1-\tau-k},\ \xi \to \pm\infty,$$

the coefficients being holomorphic, as is any finite order remainder. Inserting this into (59) shows that $c_0(\tau)$ has only a simple pole at $\tau = 0$ with residue coming from $d_k^{\pm}$ for $k = m + n$ and moreover the regular value at $\tau = 0$ is just $\overline{\mathrm{Tr}}(B)$. The identification of the residue with $\widetilde{\mathrm{Tr}}(B)$ follows from the results of [16]. □

One direct consequence of this formula is the identification of the residue trace $\mathrm{Tr}_{\partial,\sigma}$ in terms of the functional $\widetilde{\mathrm{Tr}}$ on the indicial algebra.

Corollary 1. *If $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ then the residue of $\mathrm{Tr}_\partial(AQ^{-\tau})$ at $\tau = 0$ is*

$$\mathrm{Tr}_{\partial,\sigma}(A) = \widetilde{\mathrm{Tr}}(A_{-1}). \tag{61}$$

Corollary 2. *If the regularizer Q is asymptotically translation invariant with respect to a given fibration of X near ∂X, so that the expansion (53) for Q becomes $Q \sim Q_0$, then in terms of the corresponding expansion for a general element $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$,*

$$\widehat{\mathrm{Tr}}_\partial(A) = \overline{\mathrm{Tr}}(A_{-1}).$$

Indeed, $AQ \sim \sum_j x^{-j}A_jQ_0$, so this follows from Proposition 8.

Note that $\widehat{\mathrm{Tr}}_\partial$ is independent of the choice of x but depends on the choice of Q. Using this corollary it is straightforward to deduce the formula for a general regularizer Q. If Q' and Q are both positive and elliptic then the formal difference of logarithms

$$\log(Q'/Q) = \frac{d}{d\tau}\left((Q')^\tau Q^{-\tau}\right)\big|_{\tau=0}$$

is a well-defined element of $\Psi_c^0(X)$ which satisfies the cocycle condition

$$\log(Q''/Q') + \log(Q'/Q) = \log(Q''/Q). \tag{62}$$

Lemma 11. *If $\widehat{\mathrm{Tr}}_\partial(A;Q)$ denotes the value of the functional $\widehat{\mathrm{Tr}}_\partial$ on an element $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ given by a regularizer Q then*

$$\widehat{\mathrm{Tr}}_\partial(A;Q') - \widehat{\mathrm{Tr}}_\partial(A;Q) = -\,\mathrm{Tr}_{\partial,\sigma}\left(A\log(Q'/Q)\right).$$

Proof. In view of the cocycle condition it suffices to compute the difference when Q satisfies the assumption of Corollary 2. Since $\widehat{\mathrm{Tr}}_\partial(A;Q')$ is the regular value at $\tau = 0$ of the analytic continuation of $\mathrm{Tr}_\partial(A(Q')^{-\tau})$ the difference is the value at $\tau = 0$ of $\tau\,\mathrm{Tr}_\partial(AB(\tau)Q^{-\tau})$ where

$$B(\tau) = \tau^{-1}\left((Q')^{-\tau}Q^\tau - \mathrm{Id}\right)$$

is regular at $\tau = 0$. Thus $B(0) = -\log(Q'/Q)$ and the difference is $\mathrm{Tr}_{\partial,\sigma}(AB(0))$. □

6. Exterior derivations

If D is a derivation on an algebra $\mathcal{A}$ then the map

$$i_D(a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_p) = a_0D(a_1) \otimes a_2 \otimes \cdots \otimes a_p$$

extends to a map on the Hochschild chain spaces, $i_D : \mathcal{H}_p(\mathcal{A}) \longrightarrow \mathcal{H}_{p-1}(\mathcal{A})$, which anticommutes with the Hochschild differential, $bi_D + i_Db = 0$. It therefore induces a morphism on Hochschild homology, which we shall also denote as

$$i_D : \mathrm{HH}_p(\mathcal{A}) \longrightarrow \mathrm{HH}_{p-1}(\mathcal{A}).$$

If D is an inner derivation, given by commutation with an element of $\mathcal{A}$, then this map is trivial.

For the algebras considered here there are two natural exterior derivations. These are both given formally by commutation with the logarithm of an operator

$$D_{\log Q}:\mathcal{A}\ni A\longmapsto[\log Q,A]=\frac{d}{d\tau}\left(Q^{\tau}AQ^{-\tau}-A\right)\big|_{\tau=0}\ \text{and}\tag{63}$$

$$D_{\log x}:\mathcal{A}\ni A\longmapsto[\log x,A]=\frac{d}{dz}\left(x^{z}Ax^{-z}-A\right)\big|_{z=0}.\tag{64}$$

Here Q is a positive element as considered in Section 5 and x is a boundary defining function. Since these exterior derivations are well defined modulo interior derivations, the resulting maps on homology do not depend on the choice of Q or x.

7. Explicit cocycles

To obtain explicit representations for Hochschild cocycles we begin with the simple but important case of traces.

Lemma 12. *The functional* $\mathrm{Tr}_{\partial,\sigma}$ *spans the groups* $\mathrm{HH}^0(\mathcal{A}_\sigma)$ *and* $\mathrm{HH}^0(\mathcal{A}_{\partial,\sigma})$, Tr_σ *spans* $\mathrm{HH}^0(\mathcal{I}_\sigma)$ *and* Tr_∂ *spans* $\mathrm{HH}^0(\mathcal{I}_\partial)$; .

Proof. The homology groups in dimension 0 are known to be singly generated from the computations in Section 2 and the results of Section 5. Since the traces $\mathrm{Tr}_{\partial,\sigma}, \mathrm{Tr}_\sigma$ and Tr_∂ are all non zero, the result follows. □

The arguments in Section 2 can also be used to compute the Hochschild cohomology groups of the various algebras, as the homology groups of the corresponding spaces. Using the canonical orientations coming from the symplectic structure and the counterclockwise orientation of the circles, we shall replace homology by cohomology using Lefschetz and Poincaré duality. Recalling that we always use complex coefficients, we therefore obtain isomorphisms

$$\Phi_1:H^*({}^cS^*_{\partial X}X\times\mathbb{S}_\sigma)\oplus H^*_{\mathrm{rel}}({}^cS^*M\times\mathbb{S}_\sigma)\simeq\mathrm{HH}^*(\mathcal{A}_\sigma)\tag{65}$$

$$\Phi_2:H^*({}^cS^*X\times\mathbb{S}_\sigma)\simeq\mathrm{HH}^*(\mathcal{I}_\sigma)\tag{66}$$

$$\Phi_3:H^*({}^cS^*_{\partial X}X\times\mathbb{S}_\sigma\times\mathbb{S}_\partial)\simeq\mathrm{HH}^*(\mathcal{A}_{\partial,\sigma})\ \text{and}\tag{67}$$

$$\Phi_4:H^*(\mathbb{R}\times\mathbb{S}_\partial)\simeq H^*(\mathbb{S}_\partial)\simeq\mathrm{HH}^*(\mathcal{I}_\partial).\tag{68}$$

Denote by $d\theta_r$ the class of $\imath r^{-1}dr$ in $H^*(\mathbb{S}_\sigma)$ and by $d\theta_x$ the class of $\imath x^{-1}dx$ in $H^*(\mathbb{S}_\partial)$.

Proposition 9. *Under the isomorphism* Φ_i, *the action of the morphism* $i_{\log Q}$ *is transformed to exterior product with* $d\theta_r$ *for* $i=1,2,3$ *and the action of the morphism* $i_{\log x}$ *is transformed to exterior product with* $d\theta_x$ *for* $i=3,4$:

$$\Phi_i(d\theta_r\xi)=i_{\log Q}\Phi_i(\xi),\ \ \textit{and}\ \Phi(d\theta_x\xi)=i_{\log x}\Phi(\xi).$$

Proof. The isomorphisms Φ_i are all defined through variants of the Hochschild-Kostant-Rosenberg map on the symbol algebra. Consider for example $i=1$. Denote by A the operator of antisymmetrization in the variables $f_1,\dots,f_p$

$$A(f_0\otimes f_1\otimes\ldots\otimes f_p)=(p!)^{-1}\sum_\sigma\epsilon(\sigma)f_0\otimes f_{\sigma(1)}\otimes\ldots\otimes f_{\sigma(p)}$$

where $\epsilon(\sigma)$ is the sign of an element σ in the symmetric group. Explicitly, if $[\eta]$ is any p-cohomology class then the relation

$$\Phi([\eta])\big(A(f_0 \otimes f_1 \otimes \ldots \otimes f_p)\big) = \int\limits_{S^*M \times \mathbb{S}_\sigma} *(\chi_0(f_0 \otimes f_1 \otimes \ldots \otimes f_p))\eta \tag{69}$$

for $f_0 \otimes f_1 \otimes \ldots \otimes f_p \in F_{n-p}(\mathcal{A}_\sigma)^{\otimes(p+1)}$ completely determines $\Phi([\eta])$. We can assume that the principal symbol of Q is r. The relation between the contraction $i_{\log Q}$ on the Hochschild complex and the Hochschild-Kostant-Rosenberg map χ_0, is expressed by

$$\chi_0 \circ i_{\log Q} \circ A(f_0 \otimes f_1 \otimes \ldots \otimes f_p) = \imath i_{H_{\log r}} \chi_0(f_0 \otimes f_1 \otimes \ldots \otimes f_p)$$

where the on the right hand side $i_{H_{\log r}}$ is the contraction with the Hamiltonian vector field of $\log r$. The equation (69) proves the result in this case. The other relations follows in the same way. □

It follows that $\mathrm{HH}^1(\mathcal{I}_\partial) \simeq \mathbb{C}$ is generated by $\beta = i_{\log x} \operatorname{Tr}_\partial$. If $B \sim \sum_{-\infty}^m x^{-j} B_j$ and $C \sim \sum_{-\infty}^m x^{-j} C_j$ then

$$\beta(B, C) = \sum_{j,k} \int_{-\infty}^{\infty} \int_{\partial X^2} (-1)^k k^{-1} B_{-j}(t, y, y') C_{j-k}(t, y', y) t^k \, dt. \tag{70}$$

We shall now briefly describe the construction of explicit Hochschild cocycles corresponding to other forms. For any element of $(x^{-\mathbb{Z}}\Psi^0_{\mathrm{c}}(X))^{\otimes(p+1)}$, of the form $b = f_0 \otimes f_1 \otimes \ldots \otimes f_p$ consider the multilinear map $\tau_b = \operatorname{Tr}_\sigma \circ i_b$

$$\tau_b(g_0, \ldots, g_n) = \operatorname{Tr}_\sigma(f_0 g_0 [f_1, g_1][f_2, g_2] \ldots [f_p, g_p]).$$

This operation extends to any element $b \in \mathcal{H}_p(x^{-\mathbb{Z}}\Psi^0_{\mathrm{c}}(X))$. Let η be a closed form representing a class $[\eta] \in H^p({}^cS^*X)$. We can choose the element b so that $\chi_0(b) = \eta$, where χ_0 is the Hochschild-Kostant-Rosenberg map applied to the principal symbols, and such that τ_b is a Hochschild cocycle. It follows then from the spectral sequence computation in the proof of Proposition 2 that τ_b represents the class of η, $\tau_b = \Phi_2([\eta])$. (One can recognize here the action of the Gerstenhaber algebra $\mathrm{HH}^*(A, A)$ on the trace Tr_σ.)

The description of the cocycles for $\mathcal{A}_\sigma$ corresponding to classes $[\eta] \in H^*_{\mathrm{rel}}({}^cS^*M \times \mathbb{S}_\sigma)$ is also obtained as multilinear forms τ_b or $\tau_b \circ i_{\log Q}$, defined using the residue trace and tensors b with components vanishing to infinite order at the boundary. In order to obtain explicit formulae for classes in $H^*({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma) \subset \mathrm{HH}_*(\mathcal{A}_\sigma)$, it is enough to do so for $\mathcal{A}_{\partial,\sigma}$, in view of the previous proposition.

Similar results hold for $\mathcal{A}_{\partial,\sigma}$. A cocycle φ represents the class of the form η, then $\varphi \circ i_{\log x}$ represents the class of $d\theta_x \eta$, and $\varphi \circ i_{\log Q}$ represents the class of $d\theta_r \eta$. Also the classes not containing any of $d\theta_x$ or $d\theta_r$ can be obtained as multilinear functionals of the form τ_b, constructed using invariant operators and the boundary residue $\operatorname{Tr}_{\partial,\sigma}$.

8. The homology of $\mathcal{A}_\partial$ and $\mathcal{A}$

In this section we shall complete the computation of the Hochschild homology groups announced in the introduction. The main tool will be to compute the boundary morphisms in the exact sequence in Hochschild homology for an H-unital algebra. These results will be used in the proof of the index formula in the sections

10, 12. Using also results from [20] we shall give a cohomological interpretation of the index formula.

Next we wish to consider the 'Laurent series in finite order indicial operators' i.e.

$$\mathcal{A}_\partial = x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)/x^{\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X).$$

Clearly $\mathcal{I}_\partial \subset \mathcal{A}_\partial$ is an ideal so consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_\partial \longrightarrow \mathcal{A}_\partial \longrightarrow \mathcal{A}_{\partial,\sigma} \longrightarrow 0.$$

Since the structure of this algebra is somewhat more involved we do not attempt to compute its Hochschild homology directly. The H-unitality of $\mathcal{I}_\partial$ means that this gives rise to a long exact sequence in Hochschild homology which, following Proposition 6, breaks up into

$$\begin{gathered}0 \leftarrow \mathrm{HH}_0(\mathcal{A}_{\partial,\sigma}) \longleftarrow \mathrm{HH}_0(\mathcal{A}_\partial) \longleftarrow \mathrm{HH}_0(\mathcal{I}_\partial) \xleftarrow{\partial} \mathrm{HH}_1(\mathcal{A}_{\partial,\sigma}) \longleftarrow \\ \mathrm{HH}_1(\mathcal{A}_\partial) \longleftarrow \mathrm{HH}_1(\mathcal{I}_\partial) \xleftarrow{\partial} \mathrm{HH}_2(\mathcal{A}_{\partial,\sigma}) \longleftarrow \mathrm{HH}_2(\mathcal{A}_\partial) \longleftarrow 0,\end{gathered} \tag{71}$$

$$\mathrm{HH}_k(\mathcal{A}_{\partial,\sigma}) \simeq \mathrm{HH}_k(\mathcal{A}_\partial),\ k \geq 3. \tag{72}$$

We shall need the following lemma.

Lemma 13. *The following commutation relations hold*

$$\widehat{\mathrm{Tr}}_\partial([A,B]) = \mathrm{Tr}_{\partial,\sigma}(A[\log Q, B]) = (i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma})(A,B),$$
$$\widehat{\mathrm{Tr}}_\sigma([A,B]) = -\,\mathrm{Tr}_{\partial,\sigma}(A[\log x, B]) = -(i_{\log r}\,\mathrm{Tr}_{\partial,\sigma})(A,B).$$

Proof. By definition, $\widehat{\mathrm{Tr}}_\partial(C)$ is the regular value at $\tau = 0$ of the analytic continuation of $\mathrm{Tr}_\partial(CQ^{-\tau})$. Now for $C = [A,B]$, using the fact that Tr_∂ is a trace,

$$\mathrm{Tr}_\partial([A,B]Q^{-\tau}) = \mathrm{Tr}_\partial(A[B,Q^{-\tau}]) = \tau\,\mathrm{Tr}_\partial(AM(\tau)Q^{-\tau})$$

where $M(\tau) = \tau^{-1}(B - Q^{-\tau}BQ^{\tau})$ is entire. Since $M(0) = [\log Q, B]$ it follows from Corollary 2 that

$$\widehat{\mathrm{Tr}}_\partial([A,B]) = \mathrm{Tr}_{\partial,\sigma}(A[\log Q,B]) = (i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma})(A,B) \tag{73}$$

since $i_{\log Q}(A \otimes B) = A[\log Q, B]$, by definition.

The second relation is proved in a similar manner. □

Consider the dual exact sequence to (71). The boundary map

$$\partial : \mathrm{HH}^i(\mathcal{I}_\partial) \longrightarrow \mathrm{HH}^{i+1}(\mathcal{A}_{\partial,\sigma}) \tag{74}$$

is defined explicitly, for $i = 0$, as follows. First the functional Tr_∂ on $\mathcal{I}_\partial$ should be extended continuously to $\mathcal{A}_\partial$. This is already done in Section 5, with the result denoted $\widehat{\mathrm{Tr}}_\partial$. Then the image $[f] = \partial[\mathrm{Tr}_\partial]$ is given by evaluation on the boundary, i.e. the commutator

$$f(A,B) = (b\widehat{\mathrm{Tr}}_\partial)(A,B) = \widehat{\mathrm{Tr}}_\partial([A,B]),\ A,B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X).$$

Here the functional is defined on $\mathcal{A}_\partial$ since $\widehat{\mathrm{Tr}}_\partial([A,B]) = 0$ if either element is in $\mathcal{I}_\partial$ (otherwise the H-unitality would need to be used explicitly). By the above lemma $f = i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma}$.

In particular this shows that $\Phi_3^{-1}(\partial[\mathrm{Tr}_\partial])$ is a nonzero multiple of $d\theta_r$. Extending the argument a little gives

Proposition 10. *The boundary maps in* (71) *are both injective; their duals,* (74), *are given by*

$$\partial[\mathrm{Tr}_\partial] = [i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma}]$$
$$\partial[i_{\log x}\,\mathrm{Tr}_\partial] = -[i_{\log x} i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma}].$$

Proof. The preceding computation, on the dual complex, shows this for $i = 0$. Now, apply the morphism $i_{\log x}$ to Tr_∂ . By Proposition 9 we know that the image spans $\mathrm{HH}^1(\mathcal{I}_\partial)$.

In order to compute $\partial[i_{\log x}\,\mathrm{Tr}_\partial]$ we proceed as above. The functional $g = i_{\log x}\widehat{\mathrm{Tr}}_\partial$ is an extension of $g = i_{\log x}\,\mathrm{Tr}_\partial$, hence

$$\partial[i_{\log x}\,\mathrm{Tr}_\partial] = [b i_{\log x}\widehat{\mathrm{Tr}}_\partial] = -[i_{\log x} b\widehat{\mathrm{Tr}}_\partial] = -[i_{\log x} i_{\log Q}\widehat{\mathrm{Tr}}_\partial].$$

It follows that the second boundary map is also injective. □

We can now complete the description of the homology of $\mathcal{A}_\partial$.

Proposition 11. *The Hochschild homology groups of $\mathcal{A}_\partial$ are*

$$\mathrm{HH}_i(\mathcal{A}_\partial) \simeq \begin{cases} \ker\partial : \mathrm{HH}_i(\mathcal{A}_{\partial,\sigma}) \to \mathrm{HH}_{i-1}(\mathcal{I}_\partial) & \textit{for } i = 1, 2, \\ \mathrm{HH}_i(\mathcal{A}_{\partial,\sigma}) & \textit{otherwise.} \end{cases} \tag{75}$$

Proof. This is an immediate consequence of the previous result applied to the exact sequence (71). □

Combining the results above we can also describe the homology of the the 'most important' algebra

$$\mathcal{A} = x^{-\mathbb{Z}}\Psi_{\mathrm{c}}^{\mathbb{Z}}(X)/x^{\mathbb{Z}}\Psi_{\mathrm{c}}^{-\mathbb{Z}}(X). \tag{76}$$

The exact sequence

$$0 \longrightarrow \mathcal{I}_\partial \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}_\sigma \longrightarrow 0 \tag{77}$$

determines a long exact sequences in homology.

Proposition 12. *The connecting morphisms $\partial : \mathrm{HH}^i(\mathcal{I}_\partial) \to \mathrm{HH}^{i+1}(\mathcal{A}_\sigma)$ are given on generators by $\partial[\mathrm{Tr}_\partial] = [i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma}]$ and $\partial[i_{\log x}\,\mathrm{Tr}_\partial] = 0$; it follows that*

$$\mathrm{HH}^k(\mathcal{A}) \simeq \begin{cases} \mathbb{C} \oplus H^*_{\mathrm{rel}}({}^cS^*X \times \mathbb{S}_\sigma) \oplus H^*({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma)/\mathbb{C} & \textit{for } k = 1 \\ H^*_{\mathrm{rel}}({}^cS^*X \times \mathbb{S}_\sigma) \oplus H^*({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma) & \textit{otherwise.} \end{cases}$$

Proof. The naturality of the connecting morphism and the previous proposition together reduce the proof to the computation of the morphism

$$p^* : \mathrm{HH}^*(\mathcal{A}_{\partial,\sigma}) \to \mathrm{HH}^*(\mathcal{A}_\sigma)$$

induced by the natural map $p : \mathcal{A}_\sigma \to \mathcal{A}_{\partial,\sigma}$. Using the isomorphisms in the equations (67) and (65), we deduce from the definition of the Lefschetz isomorphism that the composition

$$\begin{aligned} \Phi_1^{-1} \circ p^* \circ \Phi_3 : H^k({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma) \oplus H^k({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma)d\theta_x = \\ = H^k({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial) \to H^k({}^cS^*_{\partial X}X \times \mathbb{S}_\sigma) \oplus H^k_{\mathrm{rel}}({}^cS^*M \times \mathbb{S}_\sigma) \end{aligned}$$

preserves the direct sum decomposition, $\Phi_1^{-1} \circ p^* \circ \Phi_3 = i^* \oplus (-1)^q\partial$, where i is the inclusion $i : {}^cS^*_{\partial X}X \times \mathbb{S}_\sigma \to {}^cS^*X \times \mathbb{S}_\sigma$ and ∂ is the connection morphism in the cohomology exact sequence. The form $d\theta_\sigma$ has an extension to the whole manifold ${}^cS^*X$ and hence $\partial d\theta_\sigma = 0$. Thus we find that

$$\partial[i_{\log x}\operatorname{Tr}_\partial] = p^*\partial[i_{\log x}\operatorname{Tr}_\partial] = -p^*[i_{\log x}i_{\log Q}\operatorname{Tr}_{\partial,\sigma}] = -(-1)^q\Phi_1(\partial d\theta_\sigma) = 0.$$

This completes the proof. □

9. Morita invariance

In the following sections we will be concerned with index formulæ involving the cocycles studied in the previous sections. In order to obtain operators with nontrivial index we need to consider pseudodifferential operators acting between sections of two bundles E_+ and E_-, with isomorphic pull-backs to the cusp-cosphere bundle ${}^cS^*X$.

In this section we shall explain how the previous results extend to the simple case where $E_+ = E_- = E$ over X. We shall replace all the algebras considered above by the corresponding algebras of operators between sections of E; for instance $x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{\mathrm{c}}(X)$ is to be replaced by $x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{\mathrm{c}}(X;E)$. In general by choosing an embedding of E into a trivial bundle, $E \subset X \times \mathbb{R}^N$, we obtain an idempotent e in $M_n(\mathcal{C}^\infty(X))$, the projection onto the bundle E, and all our algebras A will be replaced by the bi-ideals ('corners') $eM_N(A)e$. This will entail changing the algebras $x^{-\mathbb{Z}}\mathcal{G}({}^cT^*X)$ to $x^{-\mathbb{Z}}\mathcal{G}({}^cT^*X, \hom(E))$ and so on. These new algebras are Morita equivalent to the original ones and hence have the same Hochschild homology.

Proposition 13. *If E is a vector bundle over a compact manifold with boundary then the Hochschild homology of $x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_c(X;E)$ and the analogues of $\mathcal{A}$, $\mathcal{A}_\sigma$, $\mathcal{I}_\sigma$, $\mathcal{A}_{\partial,\sigma}$, $\mathcal{I}_\partial$, $\mathcal{A}_\partial$ and $\mathcal{I}$ are of the same dimensions as in the case that E is trivial, since all these algebras are Morita equivalent to their analogues when E is trivial.*

Proof. An algebra of the form $eM_N(A)e$ is Morita equivalent to A if $e = e^2 \in M_N(A)$ is an idempotent not contained in any proper two-sided ideal. The rest follows from the Morita invariance of Hochschild homology see [5] and the references therein. □

From now on $\mathcal{A}$, $\mathcal{A}_\sigma$, $\mathcal{I}_\sigma$, $\mathcal{A}_{\partial,\sigma}$, $\mathcal{I}_\partial$, $\mathcal{A}_\partial$ and $\mathcal{I}$ will denote the new algebras, acting between sections of E. The results on traces extend trivially to the new algebras as well.

10. Boundary index

As discussed in the boundaryless case in the Introduction, we will define an 'index class' using the regularization discussed in Section 5. In fact $\widetilde{Z}([A,B];z)$ will in general have a pole at $z = 0$, so we discuss this functional first. Let

$$\operatorname{Bf}_{x,Q}(A,B) = \lim_{z\to 0} z\widetilde{Z}([A,B];z),\ A,B \in x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{\mathrm{c}}(X). \tag{78}$$

Proposition 14. *The functional $\operatorname{Bf}_{x,Q}$ vanishes if either component is in $\mathcal{I}_\sigma$ or $\mathcal{I}_\partial$ and so defines a functional on $\mathcal{A}_{\partial,\sigma}$ which is a non-trivial Hochschild cocycle mapping under Φ_3^{-1} to a multiple of $-d\theta_x + d\theta_r$.*

Proof. The existence of Bf depends on the trace property for $\operatorname{Tr}_{\partial,\sigma}$. Consider the identity

$$\widetilde{Z}([A,B];z) = \operatorname{Tr}([A,B]x^zQ^{-z}) + \frac{1}{2}z^2\operatorname{Tr}([A,B]C(z)x^zQ^{-z}) \tag{79}$$

where $C(z) = z^{-2}(Q^{-z}x^zQ^zx^{-z} - \mathrm{Id})$ is an entire family with

$$C(0) = 2[\log Q, \log x]. \tag{80}$$

Thus the second term in (79) cannot affect Bf and $\mathrm{Bf}(A, B)$ is the value at $z = 0$ of

$$\begin{aligned} z\,\mathrm{Tr}([A, B]x^zQ^{-z}) &= z\,\mathrm{Tr}(B[x^z, Q^{-z}A]) \\ &= z^2\,\mathrm{Tr}(BL_1(z)x^zQ^{-z}) + z^2\,\mathrm{Tr}(L_2(z)Bx^zQ^{-z}) \end{aligned} \tag{81}$$

where $zL_1(z) = [x^z, A]x^{-z}$ and $zL_2(z) = Q^z[Q^{-z}, A]$. Thus both L_1 and L_2 are entire. Moreover $L_1(0) = [\log x, A]$ and $L_2(0) = -[\log Q, A]$. Thus Bf can be expressed in term of $\mathrm{Tr}_{\partial,\sigma}$ as

$$\mathrm{Bf}(A, B) = \mathrm{Tr}_{\partial,\sigma}(B[\log x - \log Q, A]) = (-i_{\log x}\,\mathrm{Tr}_{\partial,\sigma} + i_{\log Q}\,\mathrm{Tr}_{\partial,\sigma})(A, B). \tag{82}$$

Here we have used the notation of Section 6. The statements of the proposition now follow from Proposition 9. □

Notice that the formula for this 'boundary index functional' can also be written

$$\mathrm{Bf}(A, B) = (\widehat{\mathrm{Tr}}_\partial + \widehat{\mathrm{Tr}}_\sigma)([A, B]).$$

It is therefore the image in $\mathrm{HH}^1(\mathcal{A}_{\partial,\sigma})$ of the 0-cocycle $\mathrm{Tr}_\partial + \mathrm{Tr}_\sigma$ on the ideal $\mathcal{I}_\sigma + \mathcal{I}_\partial$ under the long exact sequence in Hochschild cohomology arising from the short exact sequence

$$0 \longrightarrow \mathcal{I}_\sigma + \mathcal{I}_\partial \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}_{\partial,\sigma} \longrightarrow 0.$$

Proposition 15. *Suppose $A \in \mathcal{A}_{\partial,\sigma}$ is invertible then the pairing in Hochschild homology*

$$\text{c-ind}(A) = \mathrm{Bf}(A, A^{-1})$$

is homotopy invariant and vanishes if A arises from an invertible element in $\mathcal{A}$.

The first invertibility condition is equivalent to the existence of a representative in $x^p\Psi^m_{\mathrm{c}}(X; E)$ of the form x^pB where $B \in \Psi^m_{\mathrm{c}}(X; E)$ is elliptic near the boundary; the last invertibility condition corresponds to the existence of such a B which is globally elliptic and has invertible indicial family.

Proof. To see the homotopy invariance of $\text{c-ind}(A)$ as a function on the open set of elliptic elements of $\mathcal{A}_{\partial,\sigma}$ consider a smooth family A_t with derivative $\dot{A}_t$. The function $\text{c-ind}(A_t)$ is smooth while A_t remains elliptic and differentiation gives

$$\begin{aligned} \frac{d}{dt}\mathrm{Bf}_{x,Q}(A_t, A_t^{-1}) &= \mathrm{Bf}_{x,Q}(\dot{A}_t, A_t^{-1}) - \mathrm{Bf}_{x,Q}(A_t, A_t^{-1}\dot{A}_tA_t^{-1}) \\ &= \mathrm{Tr}_{\partial,\sigma}\left(A_t^{-1}\widetilde{D}(\dot{A}_t) - A_t^{-1}\dot{A}_tA_t^{-1}\widetilde{D}(A_t)\right). \end{aligned}$$

Here

$$\widetilde{D}(A) = [\log x - \log Q, A] \tag{83}$$

and (82) has been used. Since $B \mapsto \widetilde{D}(B)$ is a derivation and $\mathrm{Tr}_{\partial,\sigma}$ is a trace this reduces to $\mathrm{Tr}_{\partial,\sigma}(\widetilde{D}(A_t^{-1}\dot{A}_t))$. However

$$\mathrm{Tr}_{\partial,\sigma}(\widetilde{D}(B)) = 0 \ \forall\ B \in x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{\mathrm{c}}(X) \tag{84}$$

since $F(z) = z^2\,\mathrm{Tr}(T(z)x^zQ^{-z}) \equiv 0$ where $zT(z) = x^zQ^{-z}BQ^zx^{-z} - B$ is entire so $F(0) = \mathrm{Tr}_{\partial,\sigma}(T(0)) = 0$ with $T(0) = \widetilde{D}(B)$.

Now if $A \in \mathcal{A}_{\partial,\sigma}$ arises from an invertible element $A' \in \mathcal{A}$, i.e. a fully elliptic element of $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$, then it has a parametrix $B' \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ modulo $x^\infty\Psi_c^{-\infty}(X;E)$. In this case c-ind$(A)$ vanishes since it is the residue at $z=0$ of

$$\tilde{Z}_{x,Q}([A',B'];z) = \operatorname{Tr}([A',B']x^zQ^{-z})$$

which is entire. ☐

11. Index 1-cocycle

For any pair of elements $A, B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ we now consider the bilinear functional corresponding to the regularized value of $\widetilde{Z}$, for the commutator, at $z=0$. Thus set

$$\operatorname{IF}(A,B) = \lim_{z\to 0}\left(\widetilde{Z}_{x,Q}([A,B];z) - z^{-1}\operatorname{Bf}_{x,Q}(A,B)\right),\ A,B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X). \tag{85}$$

The existence of this limit follows from the definition of Bf in (78).

Lemma 14. *If $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ is fully elliptic in the sense that it is invertible in $\mathcal{A}$ then it defines a Fredholm operator on $\dot{\mathcal{C}}^\infty(X;E)$ and its index is*

$$\operatorname{Ind}(A) = \operatorname{IF}(A,B),\ B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X),\ [B]=[A]^{-1} \text{ in } \mathcal{A}. \tag{86}$$

In this case it follows from Proposition 15 that $\widetilde{Z}([A,B];z)$ is regular near $z=0$. The fact that $\operatorname{Ind}(A)$ is the value at $z=0$ is Fedosov's formula. We prove this lemma and discuss the index formula which follows from it in the next section. This lemma can also be viewed as expressing the compatibility between the boundary map in Hochschild homology and the boundary (or index) map in K-theory for a particular cocycle, namely the Fredholm trace. This compatibility is proved in general in [20] and will be used in further extensions of the index theorem, to families for example. See also [21].

Proposition 16. *The index functional defined by* (85) *descends from $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ to a cocycle on $\mathcal{A}$ and can be written*

$$\operatorname{IF}(A,B) = \frac{1}{2}(\widehat{\operatorname{Tr}}_\partial + \widehat{\operatorname{Tr}}_\sigma)(B\widetilde{D}(A) + \widetilde{D}(A)B). \tag{87}$$

Proof. By definition $\operatorname{IF}(A,B)$ is the regularized value at $z=0$ of $\widetilde{Z}([A,B];z)$. Thus is just

$$\operatorname{IF}(A,B) = \frac{d}{dz}z\widetilde{Z}([A,B];z)\big|_{z=0}.$$

The symmetrized form of (79) gives

$$\begin{aligned}(88)\quad & 2\widetilde{Z}([A,B];z)\\ &= z\operatorname{Tr}\left((BL_1(z)+L_2(z)B)x^zQ^{-z}\right) + z\operatorname{Tr}\left((BL_3(z)+L_4(z)B)Q^{-z}x^z\right)\\ &\qquad \text{with } zL_1(z) = x^zAx^{-z} - A,\ zL_2(z) = A - Q^zAQ^{-z},\\ &\qquad\qquad zL_3(z) = Q^{-z}AQ^z - A \text{ and } zL_4(z) = A - x^{-z}Ax^z.\end{aligned}$$

The functions $z^2\operatorname{Tr}(Cx^zQ^{-z})$ and $z^2\operatorname{Tr}(CQ^{-z}x^z)$ are both holomorphic near, and are equal to second order at, $z=0$ with common value $\operatorname{Tr}_{\partial,\sigma}(C)$ and derivative

$(\widehat{\operatorname{Tr}}_\partial + \widehat{\operatorname{Tr}}_\sigma)(C)$. It follows that

$$\begin{aligned} 2\,\mathrm{IF}(A,B) = (\widehat{\operatorname{Tr}}_\partial + \widehat{\operatorname{Tr}}_\sigma)\,(BL_1(0) + L_2(0)B + BL_3(0) + L_4(0)B) \\ + \operatorname{Tr}_{\partial,\sigma}(BL_1'(0) + L_2'(0)B + BL_3'(0) + L_4'(0)B)\,, \end{aligned}$$

Now, $L_1(0) = L_4(0) = [\log x, A]$, $L_2(0) = L_3(0) = -[\log Q, A]$, $L_1'(0) + L_4'(0) = 0$ and $L_2'(0)+L_3'(0) = 0$. Since $\operatorname{Tr}_{\partial,\sigma}$ is a trace functional, (87) follows. This functional certainly vanishes if either factor is in $\mathcal{I}$ so it descends to $\mathcal{A}$. □

This computation of the index class can be interpreted in terms of the previous computations of the Hochschild homology groups. Consider the class of the functional IF in $\mathrm{HH}^1(\mathcal{A})$. The restrictions of IF to $\mathcal{I}_\sigma$ and $\mathcal{I}_\partial$ are given by extension of the Atiyah-Singer cocycle and the Toeplitz-index cocyle, respectively. These cocycles correspond in our computations to $i_{\log Q}\operatorname{Tr}_\sigma$ and, respectively, $-i_{\log x}\operatorname{Tr}_\partial$. Their sum extends to the whole algebra if and only if the compatibility condition

$$\partial i_{\log Q}\operatorname{Tr}_\sigma = \partial i_{\log x}\operatorname{Tr}_\partial$$

in $\mathrm{HH}^2(\mathcal{A}_{\partial,\sigma})$ is satisfied. This checks with the computations of lemma 13 and Proposition 10 which give that their common value as the class of the cocycle $i_{\log x} i_{\log Q}\operatorname{Tr}_{\partial,\sigma}$.

Put in an other way, IF gives an extension of $-i_{\log x}\operatorname{Tr}_\partial$ from $\mathcal{I}_\partial$ to $x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{c}(X)$ and this explains the vanishing of $\partial[i_{\log x}\operatorname{Tr}_\partial]$ in 12.

12. Index formula

Proof of Lemma 14. If $A \in x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{c}(X)$ is fully elliptic then it has a generalized inverse B which satisfies

$$AB - \mathrm{Id} = -\Pi_1,\ BA - \mathrm{Id} = -\Pi_0$$

where Π_0 and Π_1 are projections onto the null space and a complement to the range respectively. Thus $AB - \mathrm{Id}$ and $BA - \mathrm{Id}$ are both elements of $\mathcal{I}$ and

$$\begin{aligned} \operatorname{Ind}(A) = \operatorname{Tr}(\Pi_0) - \operatorname{Tr}(\Pi_1) &= -\operatorname{Tr}(BA - \mathrm{Id}) + \operatorname{Tr}(AB - \mathrm{Id}) \\ &= -\widetilde{Z}((BA - \mathrm{Id}); 0) + \widetilde{Z}((AB - \mathrm{Id}); 0) = \mathrm{IF}(A,B). \end{aligned}$$

Now, we know that $\mathrm{IF}(A,B)$ only depends on the classes $[A]$ and $[B] = [A]^{-1}$ in $\mathcal{A}$, so the lemma follows. □

To investigate the index formula (86) further we name various of the component functionals. For any invertible elliptic element $A \in \mathcal{A}$ set

$$\overline{\eta}(A) = -\widehat{\operatorname{Tr}}_\partial(A^{-1}[\log x, A] + [\log x, A]A^{-1}) \tag{89}$$

$$\overline{\mathrm{AS}}(A) = \frac{1}{2}\widehat{\operatorname{Tr}}_\sigma(A^{-1}[\log Q, A] + [\log Q, A]A^{-1})) \tag{90}$$

$$\beta(A) = \frac{1}{2}\widehat{\operatorname{Tr}}_\partial(A^{-1}[\log Q, A] + [\log Q, A]A^{-1}) \text{ and} \tag{91}$$

$$\gamma(A) = \frac{1}{2}\widehat{\operatorname{Tr}}_\sigma(A^{-1}[\log x, A] + [\log x, A]A^{-1}). \tag{92}$$

In fact $\overline{\eta}(A)$ and $\beta(A)$ are defined for all invertible elements of $\mathcal{A}_\partial$ and $\overline{\mathrm{AS}}(A)$ and $\gamma(A)$ are defined for all invertible elements of $\mathcal{A}_\sigma$.

With this change of notation we grace the index formula with the name

Index Theorem. *For any fully elliptic element of* $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$

$$\operatorname{Ind}(A) = \overline{\mathrm{AS}}(A) - \frac{1}{2}\overline{\eta}(A) + \beta(A) + \gamma(A). \tag{93}$$

Here, the notation is supposed to indicate that $\overline{\mathrm{AS}}(A)$ is a generalization of the Atiyah-Singer integrand involving the $\hat{A}$-genus (or, in this case, the Todd genus) and $\overline{\eta}(A)$ is a generalization of the 'eta' invariant. Formally it is certainly the case that $\overline{\mathrm{AS}}(A)$ only depends on the (full) symbol of A whereas $\overline{\eta}(A)$ only depends on the (full) indicial family of A, i.e. the respective images in $\mathcal{A}_\sigma$ and $\mathcal{A}_\partial$. To proceed further we make successive simplifying assumptions on A.

Proposition 17. *If* $A \in \Psi_c^m(X;E)$ *is fully elliptic then*

$$\overline{\eta}(A) = \eta(A_0) \ \textit{where } A_0 = \operatorname{In}(A) \tag{94}$$

is the indicial family of A and the 'eta' invariant on the right is as defined in [16].

The indicial family $A_0 = \operatorname{In}(A)$ is defined as the class of $A \in \Psi_c^m(X;E)$ in the quotient $\Psi_c^m(X;E)/x\Psi_c^m(X;E)$. If $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X)$ is fully elliptic then for some p $x^pA \in \Psi_c^m(X;E)$ is elliptic with A_0 invertible. Furthermore A and x^pA have the same index. Thus the assumption of this Proposition involves no essential restriction as far as the index is concerned.

Proof. If $A \in \Psi_c^m(X;E)$ is fully elliptic then it has a parametrix $B \in \Psi_c^{-m}(X;E)$ and the indicial families are inverses of each other, $\operatorname{In}(B) = B_0$, $B_0 = A_0^{-1}$. For the commutator

$$[\log x, A] \in x\Psi_c^{m-1}(X;E),\ \operatorname{In}(x^{-1}[\log x, A]) = -[t, A_0] \tag{95}$$

where $A_0 \mapsto [t, A_0]$ is the exterior derivation on the indicial algebra (which is the suspended algebra for ∂X) given by commutation with the linear function t. Formula (54) from Lemma 8 then shows that

$$\widehat{\operatorname{Tr}}_\partial(B[\log x, A]) = -\overline{\operatorname{Tr}}(A_0^{-1}[t, A_0]) = -\overline{\operatorname{Tr}}([t, A_0]A_0^{-1}) = \widehat{\operatorname{Tr}}_\partial([\log x, A]B)$$

since $\overline{\operatorname{Tr}}$ is a trace functional on the full suspended algebra. Thus

$$\overline{\eta}(A) = -\overline{\operatorname{Tr}}(A_0^{-1}[t, A_0])$$

which is the definition from [16]. □

Corollary 3. *If* $\eth$ *is an admissible Dirac operator on* X *then*

$$\overline{\eta}(\eth) = \eta(\eth_0) \tag{96}$$

is the Atiyah-Patodi-Singer 'eta' invariant of the boundary Dirac operator.

Proof. In [16] the 'eta' invariant on the right of (96) is identified with the original definition from [1] for admissible Dirac operators. □

A further simplifying assumption, which can always be arranged by deformation, is that the operator A is translation invariant near the boundary.

Lemma 15. *If* $A \in \Psi_c^m(X;E)$ *is fully elliptic and translation invariant with respect to some normal fibration, with normal variable x' near the boundary then the boundary defining function x can be chosen of the form $f(x')$ so that dx is supported in the collar neighborhood and if this is done then*

$$\gamma(A) = 0. \tag{97}$$

Proof. Certainly taking $x = f(x')$ it can be arranged to be constant outside any preassigned neighborhood of the boundary. If A is translation invariant in a neighborhood containing the support of dx then the Taylor series of the commutator takes the form

$$[\log x, A] \simeq \sum_{j=1}^{\infty} \left(\frac{d}{dx'}\right)^j f(x')R_j$$

with R_j translation invariant and of order at most $m-j$. The composites $B[\log x, A]$ and $[\log x, A]B$ therefore have similar expansions with different coefficients R_j. The explicit formula given in Proposition 7 for the functional $\widehat{\mathrm{Tr}}_\sigma$, as the Hadamard regularization of Wodzicki's residue trace, shows that all terms with $j > 1$ integrate to zero (and only terms with $j \leq \dim X$ can be non-trivial). Furthermore the term with $j = 1$ integrates exactly to be logarithmic, so is removed by the regularization. $\square$

Simplified Index Theorem. *If $A \in \Psi_c^0(X; E)$ is fully elliptic and translation invariant with respect to a normal fibration near the boundary then, for a choice of boundary defining function as in Lemma 15,*

$$\mathrm{Ind}(A) = \overline{\mathrm{AS}}(A) - \frac{1}{2}\eta(A_0) + \beta(A) \tag{98}$$

where $\eta(A_0)$ is the 'eta' functional of [16].

As already noted above the assumptions here do not limit the applicability of the index formula since they can be arranged by deformation.

To simplify the remaining two terms in the index formula appreciably we need to make a generally non-trivial assumption on the indicial family.

Lemma 16. *If $A \in \Psi_c^m(X; E)$ has invertible indicial family and is normal to first order at the boundary in the sense that for some inner product*

$$[A^*, A] \in x^2\Psi_c^{2m-1}(X; E)$$

then Q can be so chosen that $\beta(A) = 0$ and, assuming that A is fully elliptic, the integral for $\overline{\mathrm{AS}}(A)$ requires no regularization.

Proof. If A is fully elliptic and of positive order then we can choose

$$Q = (A^*A + 1)^{1/2m}.$$

If it is only elliptic near the boundary, or is of negative order, then an appropriate operator which is positive in the interior and vanishes near the boundary can be used in place of Id so that Q is positive and of order 1. In any case Q can be chosen so that $[A, \log Q] \in x^2\Psi_c^m(X; E)$. Since $\widehat{\mathrm{Tr}}_\partial$ only depends on the coefficient of x in the Laurent series of its argument it vanishes on $A^{-1}[\log Q, A]$ and $[\log Q, A]A^{-1}$ in this case. Similarly the regularization of the integrand in the definition of $\overline{\mathrm{AS}}(A)$ involves only the terms which do not vanish quadratically. $\square$

Combining these various result we arrive at an index formula which takes the same form as that of Atiyah, Patodi and Singer obtain for Fredholm Dirac operators.

Reduced Index Theorem. *If $A \in \Psi_c^m(X; E)$ is fully elliptic, translation invariant near the boundary and has normal indicial family A_0 then*

$$\mathrm{Ind}(A) = \overline{\mathrm{AS}}(A) - \frac{1}{2}\eta(A_0) \tag{99}$$

where the Atiyah-Singer integral is given by a convergent integral in terms of Wodzicki's residue trace

$$\overline{\mathrm{AS}}(A) = \mathrm{Tr}_R(A^{-1}[\log Q, A]) \tag{100}$$

for any $Q \in \Psi^1_c(X;E)$ which is positive, translation invariant near the boundary and commutes there with A and where $\eta(A_0)$ is the eta invariant on the suspended algebra of the boundary as defined in [16]*.*

13. Cyclic homology

For the algebras that we have considered the cyclic homology can be computed directly from the Hochschild homology; we proceed to explain how to do this.

Recall [5, 14] that the cyclic homology of a unital algebra A, denoted $\mathrm{HC}_n(A)$, is the homology of the cyclic complex $(\mathcal{C}(A), b+B)$ where

$$\mathcal{C}(A)_n = \bigoplus_{k\geq 0} A \otimes (A/\mathbb{C}1)^{\otimes n-2k}. \tag{101}$$

Completed (i.e. projective) tensor products are to be used if topological algebras are considered, as it is the case here. The differential B is defined by

$$B(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = s\sum_{k=0}^{n} t^k(a_0 \otimes a_1 \otimes \ldots \otimes a_n). \tag{102}$$

Here we have used the notation of [5], that $s(a_0\otimes a_1\otimes \ldots \otimes a_n) = 1\otimes a_0\otimes a_1\otimes\ldots\otimes a_n$ and $t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$.

Cyclic and Hochschild homology are related by a long exact sequence due to Connes in cohomology [5] and to Loday and Quillen in homology [14].

$$\ldots \to \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \xrightarrow{I} \ldots \tag{103}$$

S being the periodicity operator.

For the algebras $\mathcal{A}_\sigma, \mathcal{I}_\sigma, \mathcal{A}_{\partial,\sigma}$ and $\mathcal{I}_\partial$, the periodic cyclic cohomology can be deduced from the Hochschild homology as follows.

Lemma 17. *The operator B in the above long exact sequence vanishes for the algebras $\mathcal{A}_\sigma$, $\mathcal{I}_\sigma$, $\mathcal{A}_{\partial,\sigma}$ and $\mathcal{I}_\partial$.*

Proof. We proceed by induction on m to show that the morphism

$$B : \mathrm{HC}_{m-1}(\mathcal{A}') \to \mathrm{HH}_m(\mathcal{A}')$$

is zero for any m if $\mathcal{A}'$ is any of the algebras $\mathcal{A}_\sigma$, $\mathcal{I}_\sigma$, $\mathcal{A}_{\partial,\sigma}$ or $\mathcal{I}_\partial$.

The statement is trivially true for $m = 0$. We begin with the following remark. The proofs of Propositions 2, 3, 4 and 5 show that the groups $\mathrm{HH}_q(\mathcal{A}')$ are generated by elements of order $-n+q$ with respect to the order filtration, and that all cycles of order less that $-n+q$ are boundaries. Here n denotes the dimension of the manifold M. This is a direct consequence of the computation of the E^1 terms of the spectral sequences associated to the order filtration.

Assuming the statement to be true for all values less than a given m we find, from the Connes' exact sequence, that the groups $\mathrm{HC}_{m-1}(\mathcal{A}')$ are isomorphic to $\mathrm{HC}_{m-1}(\mathcal{A}') \simeq \oplus_{k\geq 0}\mathrm{HH}_{m-2k-1}(\mathcal{A}')$. This shows that the groups $\mathrm{HC}_{m-1}(\mathcal{A}')$ are generated by elements of order at most $-n+m-1$ in the order filtration. It follows that they map to elements of order less than $-n+m$ in $\mathrm{HH}_m(\mathcal{A}')$, and hence they vanish by the remark above.

The last statement is a direct consequence of the vanishing of B in the Connes' exact sequence. ☐

For any exact sequence of algebras we obtain an exact sequence in cyclic homology [26] completely similar to the exact sequence in Hochschild homology. Moreover the boundary maps of these two exact sequences are compatible. We therefore obtain the following result for the cyclic homology of these algebras.

Proposition 18. *The cyclic homology groups are given by*

$$\mathrm{HC}_m(\mathcal{A}') = \oplus_{k\geq 0}\, \mathrm{HH}_{m-2k}(\mathcal{A}') \tag{104}$$

where $\mathcal{A}'$ is any of the algebras $\mathcal{A}_\sigma, \mathcal{I}_\sigma, \mathcal{A}_{\partial,\sigma}, \mathcal{I}_\partial$ or $\mathcal{A}_\partial$. Explicitly

$$\mathrm{HC}_j(\mathcal{A}_\sigma) \simeq \oplus_{k\geq 0}{}^b H^{2n-j+2k}({}^c S^*X \times \mathbb{S}_\sigma)$$
$$\mathrm{HC}_j(\mathcal{I}_\sigma) \simeq \oplus_{k\geq 0} H^{2n-j+2k}_{\mathrm{rel}}({}^c S^*X \times \mathbb{S}_\sigma).$$
$$\mathrm{HC}_j(\mathcal{A}_{\partial,\sigma}) \simeq \oplus_{k\geq 0} H^{2n-j+2k}({}^c S^*_{\partial X}X \times \mathbb{S}_\sigma \times \mathbb{S}_\partial)$$
$$\mathrm{HC}_*(\mathcal{I}_\partial) \simeq \mathrm{HH}_*(\mathcal{I}_\partial) \simeq H^{1-j}(\mathbb{S}_\partial)$$
$$\mathrm{HC}_i(\mathcal{A}_\partial) \simeq \mathrm{HC}_i(\mathcal{A}_{\partial,\sigma})/\mathbb{C},\ \ \textit{if } i>0\ , \mathrm{HC}_0(\mathcal{A}_\partial) \simeq \mathrm{HC}_0(\mathcal{A}_{\partial,\sigma}).$$

The periodicity operator S is identified with the canonical projection, so the periodic cyclic homology groups are obtained by removing the restriction that k be positive.

Proof. The computations for the first three algebras follow from the computation of Hochschild homology and the previous lemma. The cyclic homology groups of $\mathcal{A}_\partial$ are obtained from the cyclic homology groups of $\mathcal{A}_{\partial,\sigma}$ and $\mathcal{I}_\sigma$ using the exact sequence in cyclic homology. ☐

14. Bundles and grading

The discussion so far has been concerned with elliptic matrices of pseudodifferential operators. These results were extended directly to operators on sections of a particular vector bundle by selecting a complementary bundle, such that the sum is trivial. However, this still does not include the case of Dirac operators, since these in general give operators between non-isomorphic bundles. To overcome this we now briefly show how the results can be extended to the $\mathbb{Z}_2$-graded case.

For a $\mathbb{Z}_2$-graded algebra, let ϵ be the parity operator, so $\epsilon(a) = (-1)^{\deg a}a$ if a is of pure type. Let $E = E_+ \oplus E_-$ be a $\mathbb{Z}_2$-graded vector bundle over a compact manifold with boundary X. Choose a parity-preserving embedding of $E \subset X \times (\mathbb{C}^N \oplus \mathbb{C}^N)$ into a trivial $\mathbb{Z}_2$-graded bundle such that $E = e\big(X \times (\mathbb{C}^N \oplus \mathbb{C}^N)\big)$ for an even degree matrix-valued projecton e. Denote by $\mathrm{str}(a) = \mathrm{tr}(\epsilon a)$ the supertrace functional on $\hom(E) = eM_{2N}(C^\infty(X))e$. The embedding into a trivial bundle gives to rise a canonical connection $\nabla(\xi) = ed\xi$, the Grassman connection on E.

Define the Hochschild-Kostant-Rosenberg map in this case by

$$\chi(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \mathrm{str}\,(a_0 da_1 \wedge \cdots \wedge da_n) = \mathrm{str}\,(a_0 \nabla a_1 \wedge \cdots \wedge \nabla a_n) + \ldots \tag{105}$$

where the dots represent terms involving the curvature $ede \wedge de$ of the Grassman connection.

Reviewing the discussion in Section 1 we find

Lemma 18. *If $E = E_+ \oplus E_-$ is a $\mathbb{Z}_2$-graded vector bundle over a compact manifold with boundary X the Hochschild homology, in the graded sense, of the following algebras of sections of the homomorphism bundle reduce to the corresponding de Rham spaces*

$$\mathrm{HH}_*(x^{-\mathbb{Z}}\mathcal{C}^\infty(X;\hom(E)) \simeq x^{-\mathbb{Z}}\mathcal{C}^\infty(X;\Lambda^*) \tag{106}$$

$$\mathrm{HH}_*(\dot{\mathcal{C}}^\infty(X;\hom(E)) \simeq \dot{\mathcal{C}}^\infty(X;\Lambda^*) \tag{107}$$

with identification given by the Hochschild-Kostant-Rosenberg map χ, as defined above. This isomorphism does depend on the choice of a degree preserving embedding $E \subset X \times (\mathbb{C}^N \oplus \mathbb{C}^N)$.

Proof. This is proved in essentially the same manner as Proposition 13 using Proposition 1 of [22] to first reduce the graded Hochschild homology to the usual Hochschild homology (of a different algebra). □

Proposition 19. *If $E = E_+ \oplus E_-$ is a $\mathbb{Z}_2$ graded vector bundle over a compact manifold with boundary then all the Hochschild homology groups, as $\mathbb{Z}_2$-graded algebras, of $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X;E)$ and the analogues of $\mathcal{A}$, $\mathcal{A}_\sigma$, $\mathcal{I}_\sigma$, $\mathcal{A}_{\partial,\sigma}$, $\mathcal{I}_\partial$, $\mathcal{A}_\partial$ and $\mathcal{I}$ are of the same dimensions as in the case that E is trivial; moreover the isomorphisms do not depend on the choice of the embedding of E into a trivial bundle.*

Proof. These isomorphisms are proved as in the proposition above. The periodic cyclic homology is independent on the choice of the embedding since any two embedding are homotopic by a smooth homotopy (eventually by increasing N) and periodic cyclic homology is homotopy invariant [5]. Since for the algebras listed above Hochschild homology embeds into periodic cyclic homology (Theorem 18) the result follows. □

Suppose that the bundles E_+ and E_- are canonically isomorphic in a neighborhood of the boundary. The embeddings $E_\pm = e_\pm(X \times \mathbb{C}^N)$ $(e = e_+ \oplus e_-)$ can then be chosen such that their images *coincide* in a neighborhood of the boundary, that is $e_+ = e_-$ there.

We now proceed as in section 12. We shall use the extensions of the traces $\mathrm{Tr}_{\partial,\sigma}$, Tr_∂, Tr_σ and the functionals $\widehat{\mathrm{Tr}}_\partial$ and $\widehat{\mathrm{Tr}}_\sigma$ to the algebra $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X;\mathbb{C}^{2N}) = M_{2N}(x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X))$. These extensions vanish on the off-diagonal terms. Using the embeddings introduced above, we obtain by restriction functionals on $x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X;E)$. We shall use the same notation for these restrictions. Define as before

$$\mathrm{IF}(A,B) = \lim_{z\to 0}\big(\widetilde{Z}_{x,Q}([A,B];z) - z^{-1}\,\mathrm{Bf}_{x,Q}(A,B)\big),\ A,B \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X;E). \tag{108}$$

This cocycle descends to a cocycle on $\mathcal{A}$, and Proposition 16 extends without change to this case giving

$$\mathrm{IF}(A,B) = \frac{1}{2}(\widehat{\mathrm{Tr}}_\partial + \widehat{\mathrm{Tr}}_\sigma)(B\widetilde{D}(A) + \widetilde{D}(A)B). \tag{109}$$

We will also need a correction term ν which potentially depends on the embeddings, which we now proceed to define. Denote by $Q_N = Q \otimes I$ the operator Q extended to act diagonally on the trivial graded-bundle $X \times (\mathbb{C}^N \oplus \mathbb{C}^N)$. The functions $Tr(e_\pm Q_N^{-s})$ have no pole at 0 because e_+ and e_- are differential operators [26]. Then define

$$\nu = Tr(e_+Q_N^{-s}) - Tr(e_-Q_N^{-s})\big|_{s=0} = \mathrm{str}(eQ_{2N}^{-s})\big|_{s=0} \tag{110}$$

Theorem 1. *If E is a $\mathbb{Z}_2$-graded vector bundle over a compact manifold with boundary and $A \in x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X;E)$ is fully elliptic and odd with respect to the grading then $[A] \otimes [A]^{-1}$ is a $\mathbb{Z}_2$-graded Hochschild cycle and*

$$\operatorname{Ind}(A) = \dim(A_+) - \dim(A_-) = \mathrm{IF}([A],[A]^{-1}) + \nu. \tag{111}$$

where ν is as defined above and the cocycle IF *satisfies* (109). *If A is self-adjoint and translation-invariant near the boundary then the analogue of* (99) *holds.*

Proof. We consider as before the function $\operatorname{Tr}((e_+ - BA)Q_N^{-s}) - \operatorname{Tr}((e_- - AB)Q_N^{-s})$ which is defined and analytic for large s, and extends to a holomorphic function at 0 whose value at 0 is $\operatorname{Ind}(A)$ the index of A. We obtain

$$\operatorname{Ind}(A) = \operatorname{Tr}((e_+ - BA - e_- + AB)Q_N^{-s})\big|_{s=0} = \mathrm{IF}([A],[A]^{-1}) + \nu \tag{112}$$

All the other statements are proved in exactly the same way as their analogues for $E_+ = E_-$. □

Although this theorem involves an assumption on the 'boundary triviality' of the operator A this is satisfied by admissible Dirac operators and in this sense it is a direct pseudodifferential extension of the Atiyah-Patodi-Singer index theorem. This is reflected in our assumption that $E_+ = E_-$ in a neighborhood of the boundary.

All our previous results on the cohomology classes of the various cocycles appearing in the index formula extend to the above case ($E_+ \neq E_-$) with identical proof. The correction term ν is an interior term, in the sense that it can be computed as the integral of a local term supported in the interior of the manifold X. This shows that the identification of the eta-invariant with one of the terms appearing in our index cocycle extends to this case as well.

15. Other algebras

The methods employed above can also be used to compute the Hochschild homology of the algebras $\Psi_c^{\mathbb{Z}}(X)$, $\Psi_c^0(X)$, algebras of complete symbols and the b-calculus algebras [15]. However these groups can be infinite dimensional, and hence the relation with the homology is not readily seen. As an example consider the cusp operators of order 0,

$$\mathcal{A}_6 = x^{-\mathbb{Z}}\Psi_c^0(X)/x^{-\mathbb{Z}}\Psi_c^{-\infty}(X).$$

Proposition 20. *The $E^1_{p,q}$-term of the spectral sequence associated to the order filtration $\mathcal{A}_6$ is the same as that of $\mathcal{A}_{\partial,\sigma}$ for $p \leq 0$ and vanishes for $p > 0$. The $E^2_{p,q}$-term is unchanged for $p < 0$ and vanishes for $p > 0$. The $E^2_{0,q}$-terms are given by*

$$E^2_{0,q} \simeq \mathcal{C}^\infty({}^cS^*X;\Lambda^{2n-1-q}) \oplus \mathcal{C}^\infty({}^cS^*_{\partial X}X;\Lambda^{2n-q-2})$$

if $q \neq n$ and

$$E^2_{0,n} \simeq Z^{n-1}({}^cS^*X) \oplus Z^n({}^cS^*X) \oplus Z^{n-2}({}^cS^*_{\partial X}X) \oplus Z^{n-1}({}^cS^*_{\partial X}X).$$

The spectral sequence degenerates at E^2.

Here have denoted by $Z^k(M) = \ker d$, the space of closed forms of degree k on the manifold M.

Proof. The statement about the E^1-term follows from the definition. The differential $d_1 : E^1_{p,q} \to E^1_{p-1,q}$ turns out to be the same if $p \leq 0$ and to vanish otherwise.

Using the computations in the proof of proposition 2, we see that we need to compute the homology of the truncated de Rham complexes

$$(113)\quad 0\longleftarrow\mathcal{G}^{q'}({}^cS^*X\times\mathbb{S}_\sigma;\Lambda^{2n})\longleftarrow\mathcal{G}^{q'}({}^cS^*X\times\mathbb{S}_\sigma;\Lambda^{2n-1})\longleftarrow\cdots$$
$$\longleftarrow\mathcal{G}^{q'}({}^cS^*X\times\mathbb{S}_\sigma;\Lambda^{2n-q})\longleftarrow 0$$

where $q'=n-q$ and $\mathcal{G}^k({}^cS^*X\times\mathbb{S}_\sigma;\Lambda^j)$ denotes the space of k-homogeneous forms. The result then follows. □

Appendix A. Cusp calculus

We give a definition of the 'cusp' calculus on a manifold with boundary, by use of blow up techniques. This is also done in somewhat greater generality in [15]. Here we describe the construction and refer to [15] for more details. A more traditional characterization of cusp pseudodifferential operators in terms of standard pseudodifferential operators on Euclidean space is also given and the basic properties of the calculus are described.

An algebra of cusp pseudodifferential operators on a compact manifold with boundary, X, is determined by a choice of trivialization of the normal bundle to the boundary, which need not be connected. This is equivalent to the choice of a defining function ρ up to a positive constant multiple near each boundary hypersurface and the addition of a term vanishing quadratically at the boundary. We call such a choice a *cusp structure* on X.

Lemma 19. *Given any two cusp structures on a manifold with boundary X there is a diffeomorphism of X, connected to the identity through diffeomorphisms, reducing one to the other.*

Proof. Let ρ be a defining function inducing one of two given cusp structures on X. Then, by scaling ρ by a constant as necessary near each boundary hypersurface it can be assumed that there is a product decomposition of a neighborhood of the boundary of X of the form

$$[0,1]_\rho\times Y,\ Y=\partial X.$$

Let V be the normal vector field for this product decomposition, so satisfying $V\rho=1$ in $\{\rho\le 1\}$. A defining function for the other cusp structure is necessarily of the form $a\rho$, with $\mathcal{C}^\infty(X)\ni a>0$. Only the restriction of a to the boundary affects the cusp structure, so it can be assumed that the second defining function is of the form $\rho'=a\rho$ with $a=1$ near $\rho=1$ and $V\rho'>0$ in $\{\rho\le 1\}=\{\rho'\le 1\}$. It follows that $\rho_t=(1-t)\rho+t\rho'$ is a 1-parameter family of defining functions with $V(\rho_t)=a_t>0$ in $\{\rho\le 1\}$. A suitable diffeomorphism can be defined by integration of the vector field $V_t={a'_t}^{-1}V$ from the fixed surface $\{\rho_t=1\}$ towards the boundary. □

The choice of a cusp structure on X defines a Lie algebra of smooth vector fields by

$$(114)\qquad \mathcal{V}_{\mathrm{c}}(X)=\left\{V\in\mathcal{C}^\infty(X;TX);V\rho\in\rho^2\mathcal{C}^\infty(X)\right\}$$

The space of pseudodifferential operators on X described here corresponds to the *microlocalization* of this Lie algebra of vector fields. The local structure of these vector fields follows by elementary computation.

Lemma 20. *In local coordinates, $x, y_1, \dots, y_{n-1}$ near a boundary point, with x being equal to an admissible defining function, an element of $\mathcal{V}_c(X)$ takes the form*

$$V = x^2 \frac{\partial}{\partial x} + \sum_{j=1}^{n-1} b_j \frac{\partial}{\partial y_j}$$

where the coefficients a and b_j are smooth in the coordinate patch. Conversely a vector field on X is in $\mathcal{V}_c(X)$ if it is of this form with respect to a covering of the boundary by such coordinate systems.

Proof. The last part of the lemma follows from the first part and the elementary fact, directly from the definition in (114), that $\mathcal{V}_c(X)$ is a $\mathcal{C}^\infty(X)$ module. □

The local form of the vector fields allows (114) to be reversed in the sense that a defining function ρ' defines the same cusp structure as ρ if

$$\mathcal{V}_c(X)\rho' \in \rho^2 \mathcal{C}^\infty(X)$$

where $\mathcal{V}_c(X)$ is defined from the ρ. Said a different way this is

Corollary 4. *The Lie algebra $\mathcal{V}_c(X)$ determines the cusp structure from which it is defined.*

The first definition we give is of the 'ignorable' operators in the cusp calculus, these are actually the same for all cusp structures on the given manifold.

Definition 1. The space $\rho^\infty \Psi_c^{-\infty}(X)$ consists of the integral operators with kernels in $\dot{\mathcal{C}}^\infty(X^2; \pi_R^* \Omega)$, where $\pi_R : X^2 \longrightarrow X$ is projection onto the second factor and Ω is the density bundle on X, so

$$Af(z) = \int_X A(z, z') f(z'), \ f \in \dot{\mathcal{C}}^\infty(X)$$

where the same letter is used for operator and kernel.

By Fubini's theorem these operators form an algebra.

The first, and intrinsically global, definition we give of the cusp calculus uses the properties of real blow-up of p-submanifolds of a manifold with corners. First consider the space on which kernels of b-pseudodifferential operators become simple,

$$X_b^2 = [X^2; \{H \times H\}_{H \in \mathcal{M}_1(X)}] \tag{115}$$

Here $\mathcal{M}_1(X)$ is the set of boundary hypersurfaces of X. The sets of their products $H \times H$ forms a disjoint collection of p-submanifolds of X^2, so the blow up is well defined independent of order. As a manifold with corners X_b^2 has an additional boundary hypersurface (compared to X^2) for each H. These faces are each naturally isomorphic (by definition) to the inward-pointing spherical normal bundle to the corresponding $H \times H \in \mathcal{M}_2(X^2)$. The choice of cusp structure on X induces an analogous choice of cusp structure on X^2, since $\mathcal{M}_1(X^2) = \mathcal{M}_1(X) \sqcup \mathcal{M}_1(X)$, and this provides a trivialization of the normal bundle to $H \times H$. Thus, using the given cusp structure, the extra faces of X_b^2 are of the form $[-1, 1] \times H^2$. The 'fibre diagonal submanifolds'

$$\Phi^2(H) = \{0\} \times H^2 \subset [-1, 1] \times H^2, \ H \in \mathcal{M}_1(X)$$

are therefore fixed by the choice of cusp structure. These p-submanifolds are disjoint in X^2_{b}, so we may define, without dependence on the order

$$X^2_{\mathrm{c}} = [X^2_{\mathrm{b}}; \{\Phi^2(H)\}_{H\in\mathcal{M}_1(X)}],\ \beta^2_{\mathrm{cb}} : X^2_{\mathrm{c}} \longrightarrow X^2_{\mathrm{b}}, \tag{116}$$

a manifold with corners defined from X and the choice of cusp structure.

The following basic result is proved in [15].

Lemma 21. *The 'stretched projections' $\pi^2_{c,S} : X^2_c \longrightarrow X$ for $S = L, R$, defined as the composites of the blow-down maps and projection onto the left or right factor*

$$\begin{tikzcd} X^2_c \arrow[r, "\beta^2_{cb}"] \arrow[rrr, bend left, "\pi^2_{c,S}"] \arrow[rr, bend right, "\beta^2_c"'] & X^2_b \arrow[r, "\beta^2_b"] & X^2 \arrow[r, "\pi_S"] & X \end{tikzcd}\ ,\ S = L \text{ or } R, \tag{117}$$

are both b-fibrations.

Let $\dot{\mathcal{C}}^\infty_{\mathrm{ff_c}}(X^2_{\mathrm{c}}) \subset \mathcal{C}^\infty(X^2_{\mathrm{c}})$ be the subspace of functions which vanish to infinite order at all boundary hypersurfaces *other than* those, $\mathrm{ff_c}(H)$ for $H \in \mathcal{M}_1(X)$, produced in the second stage of blow up, i.e. in (116).

Lemma 22. *The closure of the inverse image under β^2_c of the diagonal over the interior of X is a p-submanifold, $\Delta_c \subset X^2_c$, meeting the boundary only in the set $\mathit{ff}_c \subset \partial X^2_c$.*

Both the notion of a conormal distribution and the more refined notion of a polyhomogeneous, or classical, conormal distribution with respect to a p-submanifold is defined, for example by extension across the boundary. The space of 1-step polyhomogeneous conormal distributional sections of order m of a vector bundle E where the conormality is with respect to a p-submanifold $M \subset X$ of a manifold with corners X, will be denoted $I^m_{\mathrm{phg}}(X, M; E)$. Those vanishing to all orders at boundary hypersurfaces other than those in the subset $S \subset \mathcal{M}_1(X)$ will be denoted $I^m_{S,\mathrm{phg}}(X, M; E) \subset I^m_{\mathrm{phg}}(X, M; E)$. In fact in terms of the $\mathcal{C}^\infty(X)$ module structure of $I^m_{\mathrm{phg}}(X, M; E)$,

$$I^m_{S,\mathrm{phg}}(X, M; E) = \dot{\mathcal{C}}^\infty_S(X) \cdot I^m_{\mathrm{phg}}(X, M; E).$$

Definition 2. The space of cusp pseudodifferential operators, acting on functions, corresponding to a given cusp structure on a manifold with boundary X is the space of kernels

$$\begin{aligned} \Psi^m_{\mathrm{c}}(X) \equiv & I^m_{\mathrm{ff_c},\mathrm{phg}}(X^2_{\mathrm{c}}, \Delta_{\mathrm{c}}; \mathrm{Kd}) \text{ with} \\ & \mathrm{Kd} = \Omega_{\mathrm{c}}^{\frac12} \otimes (X^2_{\mathrm{c,L}})^*(\rho^{\frac12}\Omega^{-\frac12}) \otimes (X^2_{\mathrm{c,R}})^*(\Omega_{\mathrm{c}}^{\frac12}), \end{aligned} \tag{118}$$

and where X^2_{c} is the blown-up space determined by the cusp structure.

Lemma 23. *Let $\mathcal{G} \subset \mathcal{C}^\infty((X^\circ)^2)$ be such that $\dot{\mathcal{C}}^\infty_{\mathit{ff}_c}(X^2_c)$ is a $\mathcal{G}$-module with respect to point wise multiplication, then the space of kernels in* (118) *is a $\mathcal{G}$-module over $\mathcal{C}^\infty(X^2)$; it is also invariant under exchange of the two factors.*

Since the lift of $\mathcal{C}^\infty(X^2)$ to X^2_{c} has the module structure required of $\mathcal{G}$ in this lemma, $\Psi^m_{\mathrm{c}}(X)$ is a $\mathcal{C}^\infty(X^2)$-module. This allows the space of kernels acting 'from sections of one vector bundle over X to sections of an other vector bundle over X' to be defined as a tensor product.

Definition 3. The space of cusp pseudodifferential operators, acting from sections of any given vector bundle E over X to sections of any vector bundle F over X and corresponding to a given cusp structure on the manifold with corners X is the space of kernels

$$\Psi_{\mathrm{c}}^m(X;E,F) = \Psi_{\mathrm{c}}^m(X) \otimes_{\mathcal{C}^\infty(X^2)} \mathcal{C}^\infty(X^2;\mathrm{Hom}(E,F))$$

where $\mathrm{Hom}(E,F)$ is the bundle over X^2 with fibre $\mathrm{hom}(E_{z'},F_z) \equiv F_z \otimes E'_{z'}$ at $(z,z') \in X^2$.

We shall abbreviate the name to c-pseudodifferential operator and $\Psi_{\mathrm{c}}^m(X;E,E)$ to $\Psi_{\mathrm{c}}^m(X;E)$. For any smooth map of compact manifolds with corners $F : X \longrightarrow Y$, the push-forward operation is well defined on supported distributions. If G is any bundle over Y then

$$F_* : \dot{\mathcal{C}}^{-\infty}(X;F^*G\otimes\Omega) \longrightarrow \dot{\mathcal{C}}^{-\infty}(Y;G\otimes\Omega). \tag{119}$$

Here, $\dot{\mathcal{C}}^{-\infty}(X;L)$, for any vector bundle L, is the dual of $\mathcal{C}^\infty(X;\Omega\otimes L')$. Since the elements of $\Psi_{\mathrm{c}}^*(X;E,F)$ are smooth up to all boundaries (in the normal variables to those boundaries),

$$\Psi_{\mathrm{c}}^*(X) \subset \dot{\mathcal{C}}^{-\infty}(X_{\mathrm{c}}^2;\mathrm{Kd}). \tag{120}$$

Proposition 21. *The kernel density bundle on X_c^2 has a natural identification over $(X_c^2)^\circ$ with $(\pi_{L}{}_c^2)^*\Omega^{-1}\otimes\Omega$ so* (119) *and* (120) *give a push-forward map*

$$\Psi_c^*(X) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X)$$

which in fact takes values in $\mathcal{C}^\infty(X)$.

Lemma 24. *Each c-pseudodifferential operator $A \in \Psi_c^*(X;E,F)$ defines consistent continuous linear maps*

$$A : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;F), \tag{121}$$

$$A : \dot{\mathcal{C}}^\infty(X;E) \longrightarrow \dot{\mathcal{C}}^\infty(X;F), \tag{122}$$

$$A : \dot{\mathcal{C}}^{-\infty}(X;E) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X;F), \tag{123}$$

$$A : \mathcal{C}^{-\infty}(X;E) \longrightarrow \mathcal{C}^{-\infty}(X;F). \tag{124}$$

The consistency here is with respect to the inclusions and projections

$$\begin{tikzcd} & & \dot{\mathcal{C}}^{-\infty}(X;E) \arrow[dd, two heads] \\ \dot{\mathcal{C}}^\infty(X;E) \arrow[r, hook] & \mathcal{C}^\infty(X;E) \arrow[ur, hook] \arrow[dr, hook] & \\ & & \mathcal{C}^{-\infty}(X;E). \end{tikzcd} \tag{125}$$

If ρ_H is a defining function for the boundary hypersurface $H \in \mathcal{M}_1(X)$ then $g = \pi_L^*(\rho_H^{-1})\pi_R^*\rho_H$ is smooth on X^2 except at the boundary face $H \times X$. Moreover it is equal to 1 on the diagonal and lifts to X_{b}^2 to be smooth up to the front face, i.e. there is again only one hypersurface up to which it is not smooth. The same holds for the lift to X_c^2, and since the singularity is of finite order g is a multiplier on $\Psi_c^m(X)$. Thus

$$\Psi_c^m(X) \ni A \longmapsto \rho_H^{-1} \cdot A \cdot \rho_H \in \Psi_c^m(X)$$

is a morphism. Combined with (121) this shows that there is a unique operator A_H completing the commutative diagrame with vertical maps given by restriction

$$\begin{array}{ccc} \mathcal{C}^\infty(X) & \xrightarrow{A} & \mathcal{C}^\infty(X) \\ \downarrow & & \downarrow \\ \mathcal{C}^\infty(H) & \xrightarrow{A_H} & \mathcal{C}^\infty(H). \end{array} \tag{126}$$

This can be generalized by considering the more singular function

$$\mu_H = \frac{1}{\pi_L^* \rho_H} - \frac{1}{\pi_R^* \rho_H}. \tag{127}$$

On X^2 this is singular at both $H \times X$ and $X \times H$. Written in terms of $s_H = \pi_L^* \rho_H / \pi_R^* \rho_H$ and $r_H = \pi_R^* \rho_H + \pi_L^* \rho_H$ it is

$$\mu_H = \frac{1 - s_H^2}{r_H s_H} \tag{128}$$

The surface blown up (in this boundary face) to produce X_c^2 is $s_H = 1, r_H = 0$. It follows that μ_H is smooth up to the new boundary face so produced and hence up to all boundary hypersurfaces of X_c^2 except the three arising from the lifts of $H \times X$, $X \times X$ and the front face of X_b^2 corresponding to H. Since μ_H is real and only has a finite order singularity at these faces, it follows that for any $\zeta \in \mathbb{R}$ the function $\exp(i\zeta\mu_H)$ is a multiplier on $\Psi_c^m(X)$. Lemma 23 then shows that conjugation gives a smooth map

$$\mathbb{R} \ni \zeta \longmapsto \exp(i\zeta/\rho_H) A \exp(-i\zeta/\rho_H) \in \Psi_c^m(X), \text{ for } A \in \Psi_c^m(X). \tag{129}$$

Definition 4. The indicial family of an element $A \in \Psi_c^m(X)$ at a boundary hypersurface $H \in \mathcal{M}_1(X)$, fixed by the choice of a boundary defining function ρ_H, is the family of operators on $\mathcal{C}^\infty(H)$

$$\mathrm{In}_H(A, \zeta) = \big(\exp(\frac{i\zeta}{\rho_H}) A \exp(-\frac{i\zeta}{\rho_H})\big)_H$$

where the operator restriction is defined by (126).

Consider the special case of this construction for the manifold $[-1, 1]$. The interior of this interval can be identified with $\mathbb{R}$ by radial compactification. That is, consider the diffeomorphism

$$\mathrm{SP} : \mathbb{R} \ni z \longmapsto \frac{z}{(1 + |z|^2)^{\frac{1}{2}}} \in (-1, 1) \subset [-1, 1]. \tag{130}$$

This 'fixes a cusp structure on $[-1, 1]$', in fact in dimension one there is a unique cusp structure.

Proposition 22. *The c-pseudodifferential operators on $[-1, 1]$ are identified by (130) with the pseudodifferential operators on $\mathbb{R}$ with 'polyhomogeneous coefficients', that is*

$$Au(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(z-z')\zeta} a(\frac{1}{2}(z + z'), \zeta) u(z') dz' \tag{131}$$

where the amplitude a is determined by the fact that there is an element $a' \in \mathcal{C}^\infty([-1,1]\times\mathbb{R})$ which is a polyhomogeneous symbol in the usual sense and such that

$$a = (\mathrm{SP}\times\mathrm{Id})^* a'. \tag{132}$$

The 'usual sense' in which $a'(x,\zeta)$ is a polyhomogeneous symbol is that $a' \in \mathcal{C}^\infty([-1,1]\times\mathbb{R})$ and there are functions $a'_j \in \mathcal{C}^\infty([-1,1]\times\{-1,+1\})$ such at

$$a' \sim \sum_{j=m}^{-\infty} |\zeta|^j a_j(x, \frac{\zeta}{|\zeta|}) \text{ as } |\zeta|\to\infty,$$

meaning in turn that for each N and all $\alpha,\beta\in\mathbb{N}_0$,

$$\sup_{|\zeta|\ge 1} |\zeta|^{N+1+|\beta|} |D_x^\alpha D_\zeta^\beta\big(a' - \sum_{j=m}^{-N} |\zeta|^j a_j\big)| < \infty.$$

Proof. Since μ_H and r_H, give coordinates near $\varDelta_{\mathrm{c}}$ the problem is reduced to operators of order $-\infty$ where it is a matter of simple estimates arising from the infinite order vanishing. □

This identification in the case of the model manifold $[-1,1]$ suggests a characterization of c-pseudodifferential operators by localization. For an open set $U\subset\mathbb{R}^{n-1}$ consider pseudodifferential operators on $\mathbb{R}\times U$ with kernels of the form

$$A(z,y,z',y') = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(z-z')\zeta+(y-y')\cdot\eta} a(z,z',y,y',\xi,\eta)d\zeta d\eta \tag{133}$$

where $a = (\mathrm{SP}\times\mathrm{SP}\times\mathrm{Id})^* a'$ with $a'\in\mathcal{C}^\infty([-1,1]^2\times U^2\times\mathbb{R}^n)$ a symbol in the same sense,

$$a' \sim \sum_{j=m}^{-\infty} |(\zeta,\eta)|^j a_j(x,x',y,y',\frac{(\zeta,\eta)}{|(\zeta,\eta)|}) \text{ as } |(\zeta,\eta)|\to\infty. \tag{134}$$

Proposition 23. *If (x,y) are coordinates in a coordinate patch $O=[0,1)\times U$. $U\subset\mathbb{R}^{n-1}$, near a boundary point of a compact manifold with boundary, with x an admissible boundary defining functions for a given cusp structure, then a kernel as in* (133) *and* (134) *such that $A' = (T\times T)^*A$, $T(x,y) = (\mathrm{SP}^{-1}(x+1),y)$ which has relatively compact support in O^2 in these local coordinates defines an element of $\Psi_c^m(X)$. Conversely any element of $\Psi_c^m(X)$, for X a compact manifold with boundary is a finite sum of such local operators, plus an element of $\Psi_c^{-\infty}(X)$ and a classical pseudodifferential operator of order m with kernel compactly supported in the interior of X^2.*

Although this proposition reduces cusp pseudodifferential operators to rather familiar objects on $\mathbb{R}\times U$ the global representation discussed above is very convenient. In particular the conormal distributions to a submanifold (in this case transversal to the boundary) always have a global representation, modulo smooth functions, in terms of symbols.

Proposition 24. *For any compact manifold with boundary there is a global quantization map*

$$x^{-\mathbb{Z}} S^{\mathbb{Z}}({}^cT^*X) \longrightarrow x^{-\mathbb{Z}}\Psi_c^{\mathbb{Z}}(X) \tag{135}$$

which is order-filtered and induces the isomorphism (13).

Proof. Choose a Riemann metric on X^2_{c} with respect to which the boundary hypersurfaces are totally geodesic (this is always possible on a compact manifold with corners, see for example [8]). Let $\phi \in \mathcal{C}^\infty(X^2_{\mathrm{c}})$ be identically one in a neighborhood of the lifted diagonal and have support in a collar neighborhood, O, defined by normal geodesic flow with respect to this metric. Writing the Riemannian identification as

$$G^{-1} : {}^cTX \equiv N\varDelta \supset O' \longrightarrow O \subset X^2_{\mathrm{c}}$$

the quantization map is defined by Fourier transformation on the fibres of ${}^cT^*X$

$$A_a = \phi G^*(A'_a)\nu,\ A'_a(p,v) = (2\pi)^{-n}\int_{{}^cT^*X} e^{-iv\cdot\xi}a(p,\xi)d\xi. \tag{136}$$

Here ν is a non-vanishing section of the kernel density bundle and $d\xi$ is the Riemannian volume form on the fibres of ${}^cT^*X$, which is identified with the conormal bundle to the lifted diagonal by the lifting of cusp vector fields from the left factor.

That this map induces an isomorphism (13) follows from the fact that the Fourier transform in the normal bundle gives an identification of conormal distributions at the submanifold, modulo $\mathcal{C}^\infty$ functions, with symbols modulo Schwartz functions. □

Now $x^{-\mathbb{Z}}\Psi^{-\infty}_{\mathrm{c}}(X)$ is an ideal, so (135) induces a product on $x^{-\mathbb{Z}}\mathcal{C}^\infty({}^cS^*X)[q]]$ which we denote by $\star$. It is indeed a star product in the sense of deformation quantization . That is,

$$a \star b = \sum_{j=0}^{\infty} P_j(a,b) \tag{137}$$

where P_j is a bilinear differential operator of total homogeneity $-j$ satisfying

$$P_0(a,b) = ab,\ P_1(a,b) - P_1(b,a) = \frac{1}{i}\{a,b\}. \tag{138}$$

Here, $\{,\}$ is the degenerate Poisson bracket on functions on ${}^cT^*X$ induced by the singular symplectic structure which arises from its identification with T^*X over the interior. The form (137) follows directly from the symbol calculus and then the explicit identifications in (138) follow by continuity from the interior. Notice that in terms of the admissible coordinates x, y near a boundary point and the induced canonical coordinates (x, y, τ, η) in ${}^cT^*X$

$$\begin{gathered} {}^cT^*X \ni \alpha = \tau\frac{dx}{x^2} + \eta\cdot dy, \\ \{a,b\} = \partial_\tau a\cdot x^2\partial_x b - x^2\partial_x a\cdot\partial_\tau b + \partial_\eta a\cdot\partial_y b - \partial_y a\cdot\partial_\tau b. \end{gathered} \tag{139}$$

The algebra $x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{\mathrm{c}}(X)$ is also filtered by the boundary order; the corresponding residual ideal being $x^\infty\Psi^{\mathbb{Z}}_{\mathrm{c}}(X)$. The global representation of the operators allows the quotient to be readily identified. To do so, consider a compact manifold without boundary, Y. The 'suspension' of Y is the product $\mathbb{R}\times X$. Using the stereographic compactification of $\mathbb{R}$ to $[-1,1]$ via (130) $\mathbb{R}\times Y$ can is identified with the interior of the compact manifold with boundary $Y_{\mathrm{sus}} = [-1,1]\times Y$. The action of $\mathbb{R}$ on $\mathbb{R}$ by translation lifts to a smooth action of $\mathbb{R}$ on $[-1,1]$ by projective linear transformations. Consider the subalgebra of $\Psi^{\mathbb{Z}}_{\mathrm{c}}(Y_{\mathrm{sus}})$ consisting of the operators which are invariant under this $\mathbb{R}$-action, we shall denote this algebra by $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(Y)$ since it is precisely the 'suspended algebra of pseudodifferential operators' considered in [16].

The choice of a boundary defining function (admissible for a given cusp structure) trivializes the normal bundle to the boundary and so gives an identification

$$(\partial X)_{\mathrm{sus}} \equiv N\partial X.$$

Proposition 25. *The choice of a boundary defining function admissible for a given cusp structure on a compact manifold with boundary and a normal fibration of the manifold near the boundary induces an isomorphism*

$$x^{-\mathbb{Z}}\Psi^{\mathbb{Z}}_{c}(X)/x^{\infty}\Psi^{\mathbb{Z}}_{c}(X) \equiv \Psi^{\mathbb{Z}}_{\mathrm{sus}}(\partial X)[[x] \tag{140}$$

where the linear variable on the suspension of ∂X can is identified with $1/x$ in the product.

Proof. This is just the Taylor series for the Schwartz kernel at the front face. □

Appendix B. Cusp metrics and holomorphic families

Although we only make we limited use of the analytic properties of cusp metrics we recall briefly from [17] and [15] the basic properties of a cusp metric. Given a cusp structure on a compact manifold with boundary, a(n exact) cusp metric is a metric on the interior of X of the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2}$$

where x is an admissible boundary defining function and h is a smooth symmetric 2-cotensor which restricts to the boundary to be a metric. Clearly such a metric always exists and the Laplacian, Δ, is a cusp differential operator. Furthermore the operator $\Delta + 1$ is invertible on $\dot{\mathcal{C}}^{\infty}(X)$ and the family $(\Delta + 1)^{z/2}$ is an entire family of c-pseudodifferential operators of complex order z.

Lemma 25. *If $S \subset X$ is a closed submanifold of a compact manifold and $F(\tau) \in I(X,S)$ is an analytic family of 1-step polyhomogeneous conomal functions of complex order $-\tau$ for $\tau \in \Omega$ where $\Omega \subset \mathbb{C}$ then for any $\tau' \in \Omega$ with $\operatorname{Re}\tau' > \dim X - \dim S$ and any smooth density ν on X the pairing*

$$\langle F(\tau), \nu\rangle = \int_X F(\tau)\nu \tag{141}$$

is defined and extends to be meromorphic in the connected component of Ω containing τ' with only simple poles at of $\tau = \frac{1}{4}\dim X + \frac{1}{2}\dim S - j$, for $j = 0, 1, 2, \ldots$.

The position of the poles is determined by the convention for orders of conormal distributions introduced by Hörmander in [11].

In fact it is by no means necessary that the measure here be smooth, provided it is smooth in the directions of a normal fibration to S. In particular, in the case of a manifold with corners the same result holds if ν is smooth up to the boundary provided that S is transversal to the boundary. More generally we need to consider the case of a density which is itself entire in another parameter in the sense that

$$\nu = \rho^z \nu_0.$$

Here, ν_0 is a fixed smooth density and ρ is a defining function for some boundary face of X.

Lemma 26. *If X is a compact manifold with corners and $S \subset X$ is a closed submanifold which meets the boundary only in a boundary hypersurface, with defining function ρ, to which it is transversal, then for $F(\tau, z) \in I(X, S)$ an analytic family of 1-step polyhomogeneous conomal functions of complex order $-\tau$ for $(\tau, z) \in \Omega \times \mathbb{C}$ then for any $\tau' \in \Omega$ with $\operatorname{Re} \tau' > \dim X - \dim S$ and any smooth density ν the function*

$$\langle F(\tau), \rho^z \nu \rangle \tag{142}$$

is defined and extends to be meromorphic in the connected component of $\Omega \times \mathbb{C}$ containing $\tau' \times \mathbb{C}$ with simple poles in the two variables at $\tau = \frac{1}{4} \dim X + \frac{1}{2} \dim S - j$, for $j = 0, 1, 2, \ldots$ and $z = -1, -2, \ldots$.

Proof. Choose a function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ which is equal to 1 near 0 and has sufficiently small support. Then the function $\phi(\rho)$ localizes near the boundary hypersurfaces which S meets. In the decomposition of the pairing

$$\langle F(\tau), \rho^z \nu_0 \rangle = \langle F(\tau), \phi(\rho) \rho^z \nu_0 \rangle + \langle F(\tau), (1 - \phi(\rho)) \rho^z \nu_0 \rangle$$

the second term is entire in z so Lemma 26 applies with the minor addition of an entire parameter. Taking ϕ to have small enough support, allows X to be replaced by a product decomposition $H \times [0, 1)$ where H is the boundary hypersurface defined by ρ and, in view of the transversality, S can also be taken to be a product in this set, $S = S' \times [0, 1)$. Then the distribution $F(\tau)$ is a smooth function of the coordinate $x = \rho$ with values in $I(H, S')$. The result then follows by combining Lemma 26 in the tangential variables with the meromorphy of the distribution x^z. □

Appendix C. Cusp and b-calculi

We briefly remark on the relationship between the cusp calculus considered here and the b-calculus considered of which the K-theory is discussed in [18]. This can be understood easily in terms of the two Lie algebras of vector fields of which they are the respective microlocalizations. The b-pseudodifferential operators are, in the usual informal sense, symbolic functions of the smooth vector fields tangent to the boundary. This is a completely natural Lie algebra of vector fields $\mathcal{V}_{\mathrm{b}}(M)$ on any manifold with boundary, M, with a local basis at a boundary point, as a $\mathcal{C}^\infty$ module,

$$t \frac{\partial}{\partial t}, \ \frac{\partial}{\partial y_j}$$

where t is a local boundary defining function and the y_j are tangential coordinates. This should be compared with Lemma 20, in which x must define the cusp structure. Consider the smooth transcendental map

$$[0, \infty) \ni x \longmapsto t = e^{-\frac{1}{x}} \in [0, \infty). \tag{143}$$

The vector field $t\partial/\partial_t$ pulls back under this map to the smooth vector field $x^2 \partial/\partial_x$. Thus it follows that differential operators which are polynomials, with smooth coefficients in the vector fields $t\partial/\partial_t$ and $\partial/\partial y_j$ on $[0, 1)_t \times \mathbb{R}^{n-1}$ pull back to be polynomials in the vector fields $x^2 \partial/\partial_x$ and $\partial/\partial y_j$.

The map in (143) is not a diffeomorphism, so it can be considered instead as the introduction of a singular coordinate, $x = \frac{1}{\log \frac{1}{t}}$, in place of t. That is, as a 'transcendental blow up' of the boundary $t = 0$. This point of view is discussed in [9] where it is shown that the new manifold with boundary (diffeomorphic in fact to the old one) has a natural $\mathcal{C}^\infty$ structure. Observe that a change of t in (143) to another defining function $t' = ta(t,y)$, depending on the tangential variables, changes x to

$$x' = \frac{1}{\log\frac{1}{t'}} = \frac{1}{\log\frac{1}{t} - \log a} = \frac{x}{1 - x\log a(e^{-1/x}, y)} = x + O(x^2).$$

Thus the new manifold, with boundary defining function x has a natural cusp structure arising from the transcendental blow up. It is then straightforward to analyze the behavior to the pseudodifferential operators.

Proposition 26. *Under transcendental blow up of the boundary of a compact manifold with boundary as in* (143) *the b-pseudodifferential operators lift to be cusp pseudodifferential operators and the span of the lifts over $\mathcal{C}^\infty$ of the blown up space is dense in the cusp algebra.*

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