FIBRATIONS, COMPACTIFICATIONS AND ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS

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Respectfully dedicated to Lars Hörmander on the occasion of his sixty-fifth unbirthday

Abstract. Some recent, and some new, results on the structure of algebras of pseudodifferential operators on compact manifolds with boundary are discussed. In particular their relationship to compactifications of non-compact spaces is emphasized. It is shown how these relationships allow the methods of pseudodifferential operators ("microlocalization") to be applied to problems in scattering and spectral theory.

Euclidean space

The first algebra of pseudodifferential operators on a general compact manifold with boundary, that I will discuss here, comes directly from the very familiar case of Euclidean space. Namely consider the standard calculus of pseudodifferential operators on $\mathbb{R}^n$. Since there are several variants of this 'standard' calculus, let me be precise. An element of the space $\Psi^m(\mathbb{R}^n)$ is an operator on $\mathcal{S}(\mathbb{R}^n)$ which can be written in the form

$$(1)\qquad Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi),\ \ \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} u(y)dy$$

with the (left-reduced) symbol satisfying the estimates 'of type $1,0$'

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}\ \forall\ \alpha,\beta\in\mathbb{N}_0^n.$$

These spaces form a filtered algebra, but I wish to consider the subalgebra of elements with symbols with complete, 1-step, asymptotic expansions in both variables. Rather than do this in the usual way, let me consider the compactification of Euclidean space to a ball.

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Radial compactification. Let z denote the variable in $\mathbb{R}^n$. Consider the map of 'stereographic projection' type

$$\mathrm{SP} : \mathbb{R}^n \ni z \longmapsto Z = (Z_0, zZ_0) \in \mathbb{R}^{n+1},\ Z_0 = (1 + |z|^2)^{-\frac{1}{2}}. \tag{2}$$

The range of this map is precisely the open upper half sphere

$$(\mathbb{S}^n_+)^\circ = \mathbb{S}^n_+ \setminus \partial\mathbb{S}^n_+,\ \mathbb{S}^n_+ = \{(Z_0, Z_1, \dots, Z_n) \in \mathbb{R}^{n+1}; Z_0 \geq 0,\\ Z_0^2 + Z_1^2 + \dots + Z_n^2 = 1\},\ \partial\mathbb{S}^n_+ = \mathbb{S}^{n-1}.$$

Thus, $\mathbb{S}^n_+$ is a compact manifold with boundary and $\mathbb{R}^n$ is identified with its interior. The function a can therefore be considered as defined on the interior of the product, $\mathbb{S}^n_+ \times \mathbb{S}^n_+$, which is a compact manifold with corner. Let $\rho_\partial = (1 + |x|^2)^{-\frac{1}{2}}$ be the function 'Z_0' on the first factor and $\rho_\sigma = (1 + |\xi|^2)^{-\frac{1}{2}}$ be the same function on the second factor. Then by 'a has a complete 1-step asymptotic expansion' I simply mean that

$$a = (\mathrm{SP} \times \mathrm{SP})^* a',\ \rho_\sigma^m a' \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+). \tag{3}$$

The space of smooth functions $\mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)$ is just the restriction of $\mathcal{C}^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ to $\mathbb{S}^n_+ \times \mathbb{S}^n_+$.

I will come back to this 'radial' compactification of Euclidean space at several points later, for the moment it allows me to define the space, which I shall denote $\Psi^m_{\mathrm{cl}}(\mathbb{R}^n)$, although I shall fairly soon discard this (objectionable) notation. The significance of the map SP is that it identifies the space of symbols of order 0 on $\mathbb{R}^n$ with complete, 1-step, asymptotic expansions with $\mathcal{C}^\infty(\mathbb{S}^n_+)$. Thus $A \in \Psi^m_{\mathrm{cl}}(\mathbb{R}^n)$ if it can be written in the form (1) with left-reduced symbol satisfying (3). I should point out that, even at the level of estimates, the assumption (3) improves the 'type $1, 0$' hypothesis to

$$|D^\alpha_x D^\beta_\xi a(x, \xi)| \leq C_{\alpha,\beta}(1 + |x|)^{-|\alpha|}(1 + |\xi|)^{m-|\beta|}\ \forall\ \alpha, \beta \in \mathbb{N}^n_0.$$

Operators defined by symbols satisfying these estimates were considered by Shubin in [13].

Fredholm properties. Amongst the elements of $\Psi^m_{\mathrm{cl}}(\mathbb{R}^n)$ are the translation invariant differential operators. In particular the flat Laplacian, $\Delta = D_1^2 + D_2^2 + \dots + D_n^2 \in \Psi^2_{\mathrm{cl}}(\mathbb{R}^n)$. Multiplication by a bounded symbol on $\mathbb{R}^n$, V, with complete, 1-step, asymptotic expansion is also in the algebra, so it is quite natural to expect these pseudodifferential operators to be helpful in discussing an operator such as $\Delta + V$, which is one of the central objects treated in scattering theory. Notice that such a V may be (very) long range in the sense that it does not decay at infinity at all. Let me give a simple example of the relationship between this algebra and the spectral theory of $\Delta + V$. The elements of $\Psi^m_{\mathrm{cl}}(\mathbb{R}^n)$ are bounded from the standard Sobolev space $H^m(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. The Fredholm elements have a simple characterization.

Proposition 1. *An element $A \in \Psi^m_{\mathrm{cl}}(\mathbb{R}^n)$ is Fredholm as a map from $H^m(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ if and only if it is 'fully elliptic' in the sense that the function $\rho_\sigma^m a'$ in (3) is non-zero at every point of the boundary of $\mathbb{S}^n_+ \times \mathbb{S}^n_+$.*

The boundary of $\mathbb{S}^n_+ \times \mathbb{S}^n_+$ consists of the two manifolds with boundary (boundary hypersurfaces) at $\rho_\sigma = 0$ and $\rho_\partial = 0$; both are diffeormorphic to $\mathbb{S}^n_+ \times \mathbb{S}^{n-1}$. The first condition, that $\rho_\sigma^m a'$, be non-zero on $\{\rho_\sigma = 0\}$ is just the usual notion of ellipticity (uniformly in x). The second condition, that the same function be non-vanishing on $\{\rho_\partial = 0\}$, is not a condition on the principal symbol at all, but a condition on the 'full symbol at infinity'. Using the composition properties of the calculus it is straightforward to translate Proposition 1 to a (well-known) statement about the spectral theory of $\Delta + V$.

Proposition 2. *If $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ is real-valued and of the form $\mathrm{SP}^*(V')$ for $V' \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ then $\Delta + V$ is self-adjoint with domain $H^2(\mathbb{R}^n)$, has continuous spectrum in the interval $[b, \infty)$, where $b = \inf(V' \restriction \mathbb{S}^{n-1})$ and only discrete spectrum of finite multiplicity in the complement.*

In fact the resolvent $(\Delta + V - \lambda)^{-1}$ is a meromorphic function on $\mathbb{C} \setminus [b, \infty)$ with values in $\Psi^{-2}_{\mathrm{cl}}(\mathbb{R}^n)$.

Let me 'interpret' the properties of $\Psi^m_{\mathrm{cl}}(\mathbb{R}^n)$ in the direction I wish to go. First, the constant coefficient vector fields on $\mathbb{R}^n$ can be transferred to $\mathbb{S}^n_+$ by SP. They become smooth vector fields on $\mathbb{S}^n_+$ and $\Psi^m_{\mathrm{cl}}(\mathbb{R}^n)$ can be thought of as the 'microlocalization' of the Lie algebra, $\mathcal{V}_{\mathrm{sc}}(\mathbb{S}^n_+)$, of vector fields on $\mathbb{S}^n_+$ which is the span, with coefficients from $\mathcal{C}^\infty(\mathbb{S}^n_+)$, of the images of the constant coefficient vector fields from $\mathbb{R}^n$. The subscript 'sc' stands for 'scattering' for the reasons outlined above. The next idea is to replace $\mathbb{S}^n_+$ by an arbitrary compact manifold with boundary.

Scattering calculus

First one can work out exactly what $\mathcal{V}_{\mathrm{sc}}(\mathbb{S}^n_+)$ is, in terms of the $\mathcal{C}^\infty$ structure of $\mathbb{S}^n_+$ as a compact manifold with boundary. Let X be a general compact manifold with boundary. The space $\mathcal{V}(X)$ of *all* smooth vector fields on X is then defined. Here (and below), by 'smooth' I always mean smooth up to the boundary. Now, $\mathcal{V}(X)$ is a Lie algebra. It has a natural subalgebra consisting of the elements, $V \in \mathcal{V}(X)$, which are tangent to the boundary, meaning that if $f \in \mathcal{C}^\infty(X)$ vanishes at ∂X then so does Vf. I shall denote this subalgebra as $\mathcal{V}_{\mathrm{b}}(X)$ (the 'b' just stands for boundary). Let $\rho \in \mathcal{C}^\infty(X)$ be a boundary defining function, i.e. $\rho > 0$ on $X \setminus \partial X$, $\rho = 0$ on ∂X and $d\rho \neq 0$ on ∂X. Then we *define*

$$\mathcal{V}_{\mathrm{sc}}(X) = \rho \mathcal{V}_{\mathrm{b}}(X) = \{V \in \mathcal{V}(X); V = \rho W \text{ with } W \in \mathcal{V}_{\mathrm{b}}(X)\}. \tag{4}$$

Of course this definition would be improper were it not for the fact that it is indeed consistent with the notation above for $\mathcal{V}_{\mathrm{sc}}(\mathbb{S}^n_+)$, defined by compactification.

Localization. Having generalized $\mathcal{V}_{\mathrm{sc}}(X)$ in this way it is perhaps natural to look for a generalization of the algebra $\Psi^*_{\mathrm{cl}}(\mathbb{R}^n)$. Observe that the map SP identifies the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with $\dot{\mathcal{C}}^\infty(\mathbb{S}^n_+) \subset \mathcal{C}^\infty(\mathbb{S}^n_+)$, precisely the space of smooth functions vanishing to infinite order at the boundary, i.e. for any compact manifold with boundary

$$\dot{\mathcal{C}}^\infty(X) = \bigcap_{k\in\mathbb{N}} \rho^k \mathcal{C}^\infty(X).$$

Thus if $A \in \Psi^*_{\mathrm{cl}}(\mathbb{R}^n)$ then $A(\mathrm{SP}^* u) = \mathrm{SP}^*(A'u)$ defines an operator, A' on $\dot{\mathcal{C}}^\infty(\mathbb{S}^n_+)$. I shall denote this image algebra by $\Psi^*_{\mathrm{sc}}(\mathbb{S}^n_+)$ (and drop the previous notation on $\mathbb{R}^n$ as anticipated).

This generalization is quite straightforward; it is carried out in [6]. First it can be seen that $\Psi^*_{\mathrm{sc}}(\mathbb{S}^n_+)$ is invariant under conjugation by F^* for any diffeomorphism F of $\mathbb{S}^n_+$. Secondly the Schwartz kernel of an element of $\Psi^{-\infty}_{\mathrm{sc}}(\mathbb{S}^n_+)$ can be seen to be smooth on $\mathbb{S}^n_+ \times \mathbb{S}^n_+$ and to vanish to infinite order at the boundary. Indeed this is a characterization

$$\Psi^{-\infty}_{\mathrm{sc}}(\mathbb{S}^n_+) = \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+; \Omega_R) \tag{5}$$

(where Ω_R is the density bundle pulled up from the right factor). We *define* $\Psi^{-\infty}_{\mathrm{sc}}(X)$ by replacing $\mathbb{S}^n_+$ by X everywhere in (5). Any point in X has a neighbourhood which can be smoothly identified with an open set in $\mathbb{S}^n_+$. If $F : O \longrightarrow O'$ is such a smooth diffeomorphism and $\phi \in \mathcal{C}^\infty(X)$ has support in O, so $\phi = F^*\phi'$ where ϕ' has compact support in O' then any $A \in \Psi^m_{\mathrm{sc}}(\mathbb{S}^n_+)$ defines an operator on $\dot{\mathcal{C}}^\infty(X)$ by $u \longmapsto F^*(\phi' A(G^*(\phi u)))$, $G = F^{-1}$. Then $\Psi^m_{\mathrm{sc}}(X)$ may be defined as the span of $\Psi^{-\infty}_{\mathrm{sc}}(X)$ and these localized operators. By working over sets where a given pair of vector bundles, E, F, is trivial the space of operators, $\Psi^m_{\mathrm{sc}}(X; E, F)$ ($=\Psi^m_{\mathrm{sc}}(X; E)$ if $E = F$) from sections of E to sections of F can be defined in a similar way. There is a more elegant definition, which I shall not discuss in detail here. It arises by generalizing the direct definition of pseudodifferential operators on a compact manifold without boundary; namely that they are precisely those operators which have Schwartz kernels which are densities conormal to the diagonal. In the corresponding definition of the scattering calculus the product $X \times X$ must first be blown up in an appropriate manner. This is discussed in the Appendix to [6].

The basic properties of $\Psi^*_{\mathrm{sc}}(X)$ can be obtained from those of the special case $\Psi^*_{\mathrm{sc}}(\mathbb{S}^n_+)$. In particular the spaces form a filtered algebra

$$\Psi^m_{\mathrm{sc}}(X; F, G) \circ \Psi^{m'}_{\mathrm{sc}}(X; E, F) \subset \Psi^{m+m'}_{\mathrm{sc}}(X; E, G).$$

To describe the analogue of Proposition 1 in this more general setting it is convenient first to describe the symbol map in an invariant way. The compactification, (2), of $\mathbb{R}^n$ is invariant under linear transformations. That is, if L is an invertible linear transformation of $\mathbb{R}^n$ then there is a smooth diffeomorphism, L', of $\mathbb{S}^n_+$ such that $L' \circ \mathrm{SP} = \mathrm{SP} \circ L$. This means that the compactification makes sense for any real vector space, U, of finite dimension. As a set this 'radial compactification' is

$$\bar{U} = U \sqcup \big((U \setminus \{0\})/\mathbb{R}^+\big) \cong \mathbb{S}^n_+, \ \dim U = n.$$

The smooth lifting of linear isomorphisms shows that the identification with $\mathbb{S}^n_+$ given by any basis of U, identifying U with $\mathbb{R}^n$, gives $\bar{U}$ a $\mathcal{C}^\infty$ structure. In fact the smoothness of this operation in parameters means that the fibre-by-fibre compactification of a vector bundle over a manifold X defines a smooth ball bundle over X.

Structure algebra. Consider again the basic 'structure algebra' $\mathcal{V}_{\mathrm{sc}}(X)$, for a compact manifold with boundary X. If $p \in \partial X$ then there are local coordinates $x, y_1, \dots, y_{n-1}$, based at p, where x is a local boundary defining function, and in terms of any such coordinates an element $V \in \mathcal{V}_{\mathrm{sc}}(X)$ is locally of the form

$$V = a(x,y)x^2\frac{\partial}{\partial x} + \sum_{j=1}^{n-1} b_j(x,y)x\frac{\partial}{\partial y_j}$$

with the coefficients $\mathcal{C}^\infty$ functions in the local coordinates. Conversely if $V \in \mathcal{V}(X)$ is of this form in a covering of the boundary by coordinate systems then it is an element of $\mathcal{V}_{\mathrm{sc}}(X)$. This implies that there is a smooth vector bundle, ${}^{\mathrm{sc}}TX$ over X, with coordinate basis $x^2\partial/\partial x$, $x\partial/\partial y_j$, $j = 1, \dots, n-1$, such that

$$\mathcal{V}_{\mathrm{sc}}(X) = \mathcal{C}^\infty(X; {}^{\mathrm{sc}}TX)$$

just as is the case for $\mathcal{V}(X)$ itself. The fibre of ${}^{\mathrm{sc}}TX$ at any point $p \in X$ can be defined as an abstract vector space by

$${}^{\mathrm{sc}}T_pX = \mathcal{V}_{\mathrm{sc}}(X)/\mathcal{I}_p(X) \cdot \mathcal{V}_{\mathrm{sc}}(X)$$

where $\mathcal{I}_p(X) \subset \mathcal{C}^\infty(X)$ is the ideal of functions vanishing at p and $\mathcal{I}_p(X) \cdot \mathcal{V}_{\mathrm{sc}}(X)$ is the span of the products. If $p \in \partial X$ then $\mathcal{I}_p(X) \cdot \mathcal{V}_{\mathrm{sc}}(X)$ is a Lie ideal in $\mathcal{V}_{\mathrm{sc}}(X)$ with commutative quotient, i.e. ${}^{\mathrm{sc}}T_pX$ in that case is naturally an Abelian Lie algebra.

Symbol map. Combining these two constructions, let ${}^{\mathrm{sc}}\bar{T}X$ be the fibre-by-fibre compactification of ${}^{\mathrm{sc}}TX$. Since X is a compact manifold with boundary, ${}^{\mathrm{sc}}\bar{T}X$ is a compact manifold with corners of codimension two. In case $X = \mathbb{S}^n_+$, ${}^{\mathrm{sc}}\bar{T}X = \mathbb{S}^n_+ \times \mathbb{S}^n_+$ is precisely the space on which the symbols were defined. As in that case ${}^{\mathrm{sc}}\bar{T}X$ has two types of boundary hypersurfaces, that coming from the compactification of the fibres and the part of the bundle over the boundary. Let ρ_σ be a defining function for the first part and ρ_∂ for the second (which may not be connected). The symbol space for operators of order m is then a quotient

$$S^{[m]}_{\mathrm{sc}}(X;E) = \frac{\rho_\sigma^{-m}\mathcal{C}^\infty({}^{\mathrm{sc}}\bar{T}X;\pi^*\hom E)}{\rho_\sigma^{-m+1}\mathcal{C}^\infty({}^{\mathrm{sc}}\bar{T}X;\pi^*\hom E) + \rho_\sigma^{-m}\rho_\partial\mathcal{C}^\infty({}^{\mathrm{sc}}\bar{T}X;\pi^*\hom E)},$$
$$\pi : {}^{\mathrm{sc}}\bar{T}X \longrightarrow X.$$

Thus, if $m = 0$, an element of $S^{[0]}_{\mathrm{sc}}(X;E)$ is just the restriction to the boundary of ${}^{\mathrm{sc}}\bar{T}X$ of a smooth section of $\pi^*\hom E$. The symbol map, $\sigma_{m,\mathrm{sc}}$, is multiplicative, and gives a short exact sequence

$$0 \longrightarrow \rho\Psi^{m-1}_{\mathrm{sc}}(X;E) \hookrightarrow \Psi^m_{\mathrm{sc}}(X;E) \xrightarrow{\sigma_{m,\mathrm{sc}}} S^{[m]}_{\mathrm{sc}}(X;E) \longrightarrow 0. \tag{6}$$

The Sobolev spaces $H^m_{\mathrm{sc}}(X;E)$ can be defined by local identification with $\mathbb{S}^n_+$ (and hence ultimately by local identification with $H^m(\mathbb{R}^n)$ in conic neighbourhoods of infinity) and the elements $A \in \Psi^m_{\mathrm{sc}}(X;E)$ define bounded maps

$$A \in \Psi^m_{\mathrm{sc}}(X;E) \Longrightarrow A : H^m_{\mathrm{sc}}(X;E) \longrightarrow L^2_{\mathrm{sc}}(X;E). \tag{7}$$

Here $L^2_{\mathrm{sc}}(X)$ is the metric space of square-integrable functions, which reduces exactly to $L^2(\mathbb{R}^n)$ (rather than $L^2(\mathbb{S}^n_+)$) in the case of Euclidean space. Furthermore the first space in (6), the null space of $\sigma_{m,\mathrm{sc}}$, is precisely the subset of $\Psi^m_{\mathrm{sc}}(X;E)$ defining compact maps in (7).

Routine symbolic constructions then allow us to obtain a direct generalization of Proposition 1.

Proposition 3. *An element $A \in \Psi^m_{\mathrm{sc}}(X;E)$ is Fredholm as a map from $H^m_{\mathrm{sc}}(X;E)$ to $L^2_{\mathrm{sc}}(X;E)$ if and only if it is 'fully elliptic' in the sense that the symbol $\sigma_{m,\mathrm{sc}}(A)$ has an inverse in $S^{[-m]}_{\mathrm{sc}}(X;E)$.*

To get an analogue of Proposition 2 we first need a generalization of the flat Laplacian to an arbitrary compact manifold with boundary. This is provided by the choice of a 'scattering metric' which makes the space look asymptotically Euclidean at the boundary. Near the boundary of $\mathbb{S}^n_+$, viewed as the compactification of $\mathbb{R}^n$, the function $x = 1/r$ is a boundary defining function so that $\mathbb{S}^n_+ \cong [0,1)_x \times \mathbb{S}^{n-1}$ near $\partial \mathbb{S}^n_+$. In terms of this decomposition the usual formula for the Euclidean metric in polar coordinates becomes

$$g_{\mathrm{Eu}} = \frac{dx^2}{x^4} + \frac{|d\omega|^2}{x^2}$$

where $|d\omega|^2$ is the standard metric on the sphere. This generalizes to an arbitrary compact manifold with boundary by taking a boundary defining function $\rho \in \mathcal{C}^\infty(X)$ and considering any Riemann metric on the interior which is of an analogous form near the boundary

$$g_{\mathrm{sc}} = \frac{d\rho^2}{\rho^4} + \frac{h'}{\rho^2} \text{ near } \partial X. \tag{8}$$

Here h' is a symmetric 2-cotensor which is smooth up to the boundary and such that $h = h' \restriction \partial X$ is positive definite, and so defines a metric on ∂X. Clearly such a metric exists on any compact manifold with boundary.

Proposition 4. *The Laplacian, Δ of a scattering metric, in the sense of* (8), *(acting on any form bundle) has spectrum* $[0,\infty)$ *consisting of continuous spectrum of infinite multiplicity.*

Generalized eigenfunctions. In fact a little more work is required to show that the spectrum is continuous of infinite multiplicity but its location can be deduced directly from Proposition 3 since the resolvent $(\Delta - \sigma)^{-1}$ can be seen to be a holomorphic function on $\mathbb{C} \setminus [0,\infty)$ with values in $\Psi^{-2}_{\mathrm{sc}}(X)$ and having no bounded analytic continuation across any point of $[0,\infty)$. To get stronger results on the structure of the spectrum it is natural to use ideas from scattering theory combined with microlocal analysis. Similar approaches have been made to the analysis of long-range potentials, see [1]. The first step is to discuss the weak limit as $\operatorname{Im}\lambda \uparrow 0$ near $\lambda' > 0$ of the resolvent written in the from $R(\lambda) = (\Delta - \lambda^2)^{-1}$, with the resolvent domain taken to be $\operatorname{Im}\lambda < 0$.

Lemma 1 ([6])**.** *For a scattering Laplacian on a compact manifold with boundary and $f \in \dot{\mathcal{C}}^\infty(X)$, as $\lambda \to \lambda' > 0$, from $\operatorname{Im}\lambda < 0$, $R(\lambda)f$ converges in $\rho^{-\frac{1}{2}-\epsilon}L^2_{\mathrm{sc}}(X)$, for any $\epsilon > 0$, to a function of the form*

$$R(\lambda')f = e^{-i\lambda/\rho}\rho^{\frac{1}{2}(n-1)}g,\ g \in \mathcal{C}^\infty(X).$$

The existence of this limit also depends on the use of the form of the unique continuation theorem of Aronszajn-Cordes type given by Hörmander in [4] since this shows that there can be no point spectrum in $(0,\infty)$.

From this lemma it can be seen by direct construction that for each $0 \neq \lambda \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(\partial X)$ there is a unique eigenfunction u, satisfying

$(\Delta - \lambda^2)u = 0$, of the form

$$u = e^{-i\lambda/x} x^{\frac{1}{2}(n-1)} f' + e^{i\lambda/x} x^{\frac{1}{2}(n-1)} f'' \qquad \text{with } f', f'' \in \mathcal{C}^\infty(X) \text{ and } f' \restriction \partial X = f.$$

The resulting map $\mathcal{C}^\infty(\partial X) \ni f \longmapsto u \in \mathcal{C}^{-\infty}(X)$ is the *Poisson operator* for the problem and its leading part at the boundary,

$$A(\lambda) : \mathcal{C}^\infty(\partial X) \ni f \longmapsto f'' \restriction \partial X \in \mathcal{C}^\infty(\partial X)$$

is the absolute scattering operator for the metric. In the case of Euclidean space, as a scattering metric on $\mathbb{S}^n_+$, it can be seen by direct computation that $A(\lambda)f(\omega) = i^{\frac{1}{2}(n-1)} f(-\omega)$. In recent work with M. Zworski the analogous result in the general case has been obtained.

Theorem ([12]). *The scattering matrix for a scattering metric, at any non-zero real frequency λ, is a Fourier integral operator of order 0 associated to geodesic flow, for the metric h, on ∂X at distance π.*

Microlocalization. Both Lemma 1 and this theorem are proved by the techniques of microlocal analysis. The quotient algebra $\Psi^*_{\mathrm{sc}}(X)/\rho^\infty \Psi^{-\infty}_{\mathrm{sc}}(X)$ is localized at the boundary of ${}^{\mathrm{sc}}\bar{T}^*X$, and hence a notion of wavefront set $\mathrm{WF}_{\mathrm{sc}}(u) \subset \partial^{\mathrm{sc}}\bar{T}^*X$, for extendible distributions $u \in \mathcal{C}^{-\infty}(X)$, can be defined essentially as in the case of a compact manifold without boundary. The precise limiting absorption principle, Lemma 1, is then seen to be a uniform version, in this context, of Hörmander's theorem on the propagation of singularities. The more precise result in the theorem is obtained by constructing a parametrix for the Poisson operator as a Legendre distribution, which is the analogue in this setting of a Lagrangian distribution. Indeed, except for the fact that the Legendre manifold involved is singular, the notion of Legendre distribution in [12] is reduced to the usual notion of a Lagrangian distribution by Fourier transformation.

Fibred cusp calculus

The Lie algebra $\mathcal{V}_{\mathrm{sc}}(X)$, defined in (4) can also be described in the following way

$$(9) \qquad \mathcal{V}_{\mathrm{sc}}(X) = \left\{ V \in \mathcal{V}(X); Vx \in x^2\mathcal{C}^\infty(X) \text{ and } V \text{ vanishes at } \partial X \right\}.$$

In this form the definition can be generalized by replacing the vanishing condition at the boundary by tangency to the fibres of a fibration; vanishing then corresponds to the 'trivial' fibration with each fibre a point. Thus let

$\phi : \partial X \longrightarrow Y$ be a fibration of the boundary. Choose a boundary defining function $x \in \mathcal{C}^\infty(X)$ and set

$$\mathcal{V}_\Phi(X) = \big\{V \in \mathcal{V}(X); Vx \in x^2\mathcal{C}^\infty(X) \text{ and } \\ V \text{ is tangent to the fibres of } \phi \text{ at } \partial X\big\}. \tag{10}$$

Notice that the first condition certainly implies that $V \in \mathcal{V}_{\mathrm{b}}(X)$. The other extreme case, of $\phi : \partial X \longrightarrow \{\mathrm{pt}\}$ the one-fibre fibration, is called the 'cusp' algebra.

Model case. As opposed to the case of the scattering Lie algebra the definition (10) does in fact depend on the choice of defining function; the subscript Φ is supposed to indicate that it depends on more than ϕ. The model case analogous to $\mathbb{R}^n$ for the scattering calculus is $\mathbb{R}^p \times F$ where the compact manifold without boundary F of dimension k, $n = p + k$, is a typical fibre of ϕ. A calculus of operators on this model space can be obtained by localizing on F and considering the operators

$$\begin{aligned} Au(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^k} e^{ix\cdot\xi + (y-y')\cdot\eta} a(x,y,\xi,\eta)\hat{u}(\xi,y')dy'd\eta d\xi, \\ \hat{u}(\xi,y') &= \int_{\mathbb{R}^p} e^{-ix'\cdot\xi} u(x',y')dx', \ \ n = p+k \end{aligned} \tag{11}$$

where the amplitude is of the form

$$a = (\mathrm{SP}_p \times \mathrm{SP}_n)^* a', \ \ \rho_\sigma^m a' \in \mathcal{C}^\infty(\mathbb{S}^p_+ \times \mathbb{S}^n_+ \times O). \tag{12}$$

Here, SP_p is radial compactification in $\mathbb{R}^p$ and SP_n is radial compactification in the product $\mathbb{R}^n = \mathbb{R}^p_\xi \times \mathbb{R}^k_\eta$ to $\mathbb{S}^n_+$, ρ_σ is a boundary defining function in this factor and O is a coordinate patch in F. It is again relatively easy to show that these local kernels near the diagonal, together with the smooth kernels vanishing to infinite order on both boundary faces, sum to give an algebra of operators on $\dot{\mathcal{C}}^\infty(\mathbb{S}^p_+ \times F)$ after radial compactification of $\mathbb{R}^p$ to $\mathbb{S}^p_+$.

Although, as already noted in the case of the scattering calculus, there is a more elegant formulation of this definition directly in terms of the Schwartz kernels I shall not describe this approach here; the detailed discussion can be found in [6].

Suspended algebras. The main formal difference between the scattering and cusp calculi, with intermediate behaviour for a general fibration of the boundary, is that in the latter case the operators 'remain global' at the boundary. The definition in (11) can be modified to define pseudodifferential operators on F which depend symbolically on variables $\xi \in \mathbb{R}^p$ and smoothly on parameters $z \in S \subset \mathbb{R}^k$, S open, by dropping the ξ integrals and considering

$$Au(z,\xi,y) = (2\pi)^{-k} \int_{\mathbb{R}^{2k}} e^{(y-y')\cdot\eta} a(x,y,\xi,\eta)\hat{u}(\xi,y')dy'd\eta,$$

where a is an amplitude as in (12), with the x variables replaced by $z \in S$. These local kernels near the diagonal in $F \times F$ can be extended to global kernels by summing over a partition of unity and admitting global smooth kernels in z, ξ, y, y' which are rapidly decreasing, with all derivatives in ξ. Again this gives an algebra of operators, parametrized by $(z,\xi) \in S \times \mathbb{R}^p$. These algebras are invariant under general diffeomorphisms of F, diffeomorphic change of the parameters in S and linear transformations in the variables ξ. This allows the definition to be globalized to the following geometric arrangement. Let Y, replacing S, be a compact manifold without boundary. Let V be a vector bundle over Y, the fibre of which replaces $\mathbb{R}^p$ and let $\phi : X' \longrightarrow Y$ be a fibration with typical fibre F. The fibre product $G = V \times_Y X'$ then reduces locally to the product $S \times \mathbb{R}^p \times F$ considered above and the local definition of the algebra extends to define a global algebra which we shall denote $\Psi^m_{\mathrm{sus}(V)-\phi}(G)$. Notice that its elements act on each fibre of ϕ and only depend parametrically on the other variables.

Normal operator. Each of the Lie algebras $\mathcal{V}_\Phi(X)$ corresponds to a vector bundle over X, denoted ${}^\Phi TX$, with the property that $\mathcal{V}_\Phi(X) = \mathcal{C}^\infty(X; {}^\Phi TX)$. This bundle comes equipped with a natural vector bundle map $\iota_\Phi : {}^\Phi TX \longrightarrow TX$ into the standard tangent bundle; it is an isomorphism over the interior and over the boundary has range of rank k, the fibre dimension, in $T\partial X$. Let ${}^\Phi N\partial X$ be the bundle over ∂X consisting of the null spaces of ι_Φ. Now, this bundle is naturally trivial over each of the fibres of ϕ, i.e. it is actually the lift under ϕ of a bundle ${}^\phi NY$ over Y. Let $G = {}^\phi N^*Y \times_Y \partial X$ be the fibre product, so the discussion above applies. The fibred cusp algebra then has two symbol maps

$$\begin{gathered} \sigma_{m,\Phi} : \Psi^m_\Phi(X;E) \longrightarrow S^{[m]}_\sigma({}^\Phi\bar{T}^*X;\pi^*E) \text{ and} \\ N : \Psi^m_\Phi(X;E) \longrightarrow \Psi^m_{\mathrm{sus}({}^\phi N^*Y)-\phi}(G;\pi^*E) \text{ where} \\ S^{[m]}_\sigma({}^\Phi\bar{T}^*X;\pi^*E) = \rho_\sigma^{-m}\mathcal{C}^\infty({}^\Phi\bar{T}^*X;\pi^*E)/\rho_\sigma^{-m+1}\mathcal{C}^\infty({}^\Phi\bar{T}^*X;\pi^*E). \end{gathered} \tag{13}$$

In case the fibres of ϕ are not points, i.e. other than for the scattering calculus, the image space of the second map is a non-commutative algebra. Both maps are surjective homomorphisms, but their direct sum is not surjective. Rather the compatibility condition between the two images is that the symbol of the 'normal operator' $N(A)$ is the restriction to the boundary of the symbol $\sigma_{m,\Phi}(A)$. The subspace so defined of the direct sum is an algebra and the null space of the joint symbol map is exactly the space $x\Psi_\Phi^{m-1}(X;E)$.

In this wider setting we can again obtain a complete characterization of the Fredholm properties of these operators acting on the corresponding Sobolev spaces. The latter are defined as before from the basic L^2 space. Notice that the algebra $\mathcal{V}_\Phi(X)$ fixes a positive measure, up to a smooth positive multiple, on X by the condition that it pair smoothly and positively with the element $|v_1 \wedge \cdots \wedge v_n|$ for any local smooth basis. Let $L^2_\Phi(X)$ be defined with respect to this measure and set

$$H^m_\Phi(X;E) = \Psi_\Phi^{-m}(X;E) \cdot L^2(X;E).$$

Lemma 2. *There is an element $A_m \in \Psi^m_\Phi(X;E)$ for each m and any vector bundle E over X such that*

$$H^m_\Phi(X;E) = A_{-m}L^2_\Phi(X;E).$$

From this and the properties of the algebra it is easy to deduce that elements of the calculus define bounded operators

$$A : H^m_\Phi(X;E) \longrightarrow L^2_\Phi(X;E),\ A \in \Psi^m_\Phi(X;E). \tag{14}$$

Proposition 5. *An element $A \in \Psi^m_\Phi(X;E)$ is Fredholm as a map* (14) *if and only if it is totally elliptic in the sense that its symbol $\sigma_{m,\Phi}(A)$ and normal operator $N(A)$ have inverses, respectively, in $S^{[-m]}_\sigma({}^\Phi T^*X;\pi^*E)$ and $\Psi^{-m}_{\mathsf{sus}({}^\phi N^*Y)-\phi}(G;E)$.*

Fibred cusp metrics. As before this leads to an effective description of the spectrum of the Laplacian associated to a metric from the appropriate class. Given the defining function x and the fibration of the boundary, ϕ, it is always possible to find a Riemann metric on the interior of X which takes the following form near the boundary

$$g \sim \frac{dx^2}{x^4} + \frac{\tilde\phi^* g_Y}{x^2} + h'. \tag{15}$$

Here $\tilde\phi : U \longrightarrow Y$ is an extension of the fibration of ∂X to a neighbourhood of ∂X given by the choice of a product decomposition near the boundary $U \sim [0,\epsilon)_x \times \partial X$; g_Y is a metric on Y (possibly depending on x) and h' is a smooth 2-cotensor on X near the boundary which restricts to define a metric on each fibre over the boundary. Using the fact that such a 'fibred cusp metric' defines a non-degenerate fibre metric on the bundle ${}^\Phi TX$ it is straightforward to show that the associated Laplacian is an elliptic element of $\Psi^2_\Phi(X)$.

From Proposition 5 it follows directly that the spectrum of the Laplacian for such a metric is $[0,\infty)$. There are 'thresholds' at which the multiplicity of the spectrum changes; these are given by the eigenvalues of the Laplacian on the fibres of ϕ. Since these may vary with the point in Y, which labels the fibres, the general structure of the spectrum is somewhat complicated. Since the null space of the fibre Laplacians consists always of the constants alone, the spectrum *is* relatively simple near 0. It has a parametrization much as in the extreme case of the scattering metric discussed above.

Proposition 6. *For the Laplacian, acting on functions, of a fibred cusp metric* (15)*, for each $0 \neq \lambda \in \mathbb{R}$ with λ^2 smaller than the infimum over Y of the smallest positive eigenvalue of the fibre Laplacians, there is for each $f \in \mathcal{C}^\infty(Y)$ a unique generalized eigenfunction of the form*

$$u = e^{i\lambda/x} x^{\frac{1}{2}(k-1)} f' + e^{-i\lambda/x} x^{\frac{1}{2}(k-1)} f'',\ \ f', f'' \in \mathcal{C}^\infty(X),\ \ f' \restriction \partial X = \phi^* f. \tag{16}$$

The closure of this set in $\mathcal{C}^{-\infty}(X)$ consists of all the eigenfunctions in this space.

As in the scattering case this can be proved by the use of a variant of Hörmander's theorem on the propagation of singularities for the wavefront set corresponding to the fibred cusp calculus. This parametrization of the generalized eigenspaces, for small positive eigenvalues, also leads to an absolute scattering matrix, $A(\lambda) : f \longmapsto f'' \restriction \partial X$. In fact $f'' \restriction \partial X = \phi^* g$ for $g \in \mathcal{C}^\infty(Y)$ so the scattering operator acts on $\mathcal{C}^\infty(Y)$.

Conjecture. *For small λ (such that λ^2 is less than the infimum of the positive fibre eigenvalues) the absolute scattering matrix for a fibred cusp metric is a Fourier integral operator on Y associated to geodesic flow, at time π, for the metric g_Y on Y.*

Other calculi

I will finish this brief overview of some of the properties of 'cusp-type' algebras of pseudodifferential operators on a compact manifold with boundary with some comments on the relationship to other calculi and further likely developments. For simplicity I shall only consider operators on functions; however operators on sections of vector bundles behave similarly.

Cusp and b-pseudodifferential operators. First let me mention that the cusp calculus, which in terms of the fibred cusp structure discussed above is the opposite extreme to the scattering calculus, is closely related to the b-calculus introduced in [7] and discussed in detail in [4; Chapter XVII] and [8]. This relationship can be described, succinctly, as follows.

The algebra $\Psi^*_{\mathrm{b}}(X)$ is defined for any compact manifold with boundary X. One can change the $\mathcal{C}^\infty$ structure near the boundary of X by introducing as a new boundary defining function

$$\rho = \frac{1}{\log \frac{1}{x}} \tag{17}$$

where $x \in \mathcal{C}^\infty(X)$ is a boundary defining function such that $0 \le x < 1$ on X. The space $\mathcal{C}^\infty(X_{\log})$ of $\mathcal{C}^\infty$ functions for the new $\mathcal{C}^\infty$ structure (on the same set X) consists of function of the form $f(\rho, f_1, \dots, f_N)$ where f is $\mathcal{C}^\infty$ on $\mathbb{R}^{N+1}$ and the $f_i \in \mathcal{C}^\infty(X)$ are real-valued. Since this includes $\mathcal{C}^\infty(X)$ itself, the identity map is smooth $\iota_{\log} : X_{\log} \longrightarrow X$. Over the interior this is a diffeomorphism, so any operator on X of the form $A : \mathcal{C}^\infty_c(X^\circ) \longrightarrow \mathcal{C}^\infty(X)$, $X^\circ = X \setminus \partial X$, lifts to an operator on $X_{\log}$. In this sense the space $\Psi^m_{\mathrm{b}}(X)$ lifts into the cusp algebra $\Psi^m_{\mathrm{cusp}}(X_{\log})$ defined with respect to ρ, which is independent of the choice of x modulo $\rho^2\mathcal{C}^\infty(X_{\log})$, i.e. up to a term vanishing to second order at the boundary of $X_{\log}$. The difference between the two calculi is then only that the coefficients are more general in $\Psi^*_{\mathrm{cusp}}(X_{\log})$ than in $\Psi^*_{\mathrm{b}}(X)$, namely $\mathcal{C}^\infty$ functions on $X_{\log}$ in the first case rather than $\mathcal{C}^\infty$ functions on X in the second.

With this identification, the structure of the scattering matrix and trace formula for the Laplacian of a b-metric carried out by T. Christiansen [2], using results from [8], can be extended to the more general case of cusp metrics.

Index problem. Various problems for cusp and b-pseudodifferential operators are therefore essentially the same. For example, the open problem of finding an explicit index formula for Fredholm pseudodifferential operators (characterized in Proposition 5) is the same in the two calculi. This index problem for pseudodifferential operators is, as yet, unsolved for all the fibred cusp algebras except the scattering algebra (see [10] for that case). The eta homomorphism discussed in [11] can be expected to play a rôle. It is defined on the boundary algebra in (13) for the cusp algebra. The extension of the construction of [11] to the boundary algebras for the other fibred cusp algebras (except the commutative case of the scattering algebra) would therefore be of interest.

Quadratic scattering calculus. The relationship between the scattering and the cusp calculi is rather more complicated than that between the cusp and b-calculi. However, it can be seen directly from (4) that there is such a relationship. Namely, the scattering algebra is obtained by quantization of a subalgebra of the algebra from which the b-pseudodifferential operators are obtained. One can consider smaller subalgebras of a similar nature, for example

$$\mathcal{V}_{\mathrm{qsc}}(X) = x^2\mathcal{V}_{\mathrm{b}} \subset \mathcal{V}_{\mathrm{sc}}(X) = x\mathcal{V}_{\mathrm{b}}(X) \subset \mathcal{V}_{\mathrm{b}}(X). \tag{18}$$

This 'quadratic scattering algebra' is therefore also well defined on any compact manifold with boundary. It has been analyzed in some detail by J. Wunsch, [16], who has used it to give a theorem on the propagation of singularities (in the sense of growth) for the Schrödinger operator. Again the quadratic scattering algebra is only slightly more general than the scattering algebra (as discussed above). Namely if X is any compact manifold with boundary then instead of ρ in (17) one can get a new $\mathcal{C}^\infty$ structure by introducing

$$\rho_{\frac{1}{2}} = x^{\frac{1}{2}} \tag{19}$$

as a new boundary defining function. The resulting $\mathcal{C}^\infty$ structure turns the same set X into a new manifold with boundary $X_{\frac{1}{2}}$ (diffeomorphic to the original one) with $\iota_{\frac{1}{2}} : X_{\frac{1}{2}} \longrightarrow X$ a $\mathcal{C}^\infty$ map which is a diffeomorphism over the interior. Apart from a small question of the smoothness of coefficients, the scattering calculus on X then lifts to $X_{\frac{1}{2}}$ to be the 'quadratic scattering calculus' on $X_{\frac{1}{2}}$. That this can be reasonably associated to the (time dependent) Schrödinger operator $D_t + \Delta$ can be seen by introduction of the appropriate singular coordinates. As Wunsch shows, the propagation statements of [3] (see also [5]) can be interpreted in this way as a theorem on the propagation of wavefront sets.

N-body problem. Another analytic problem to which a microlocal approach similar to that described above is applicable is the quantum N-body problem. Consider N particles with positions $x_j \in \mathbb{R}^n$ for $j = 1, \dots, N$. For simplicity suppose that the potentials describing the relative interactions are real-valued functions $V_{ij} \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. Then the energy states of the quantum system are described by the spectrum of the operator

$$\sum_{j=1}^{N} \Delta_j + \sum_{i>j=1}^{N} V_{ij}(x_i - x_j)$$

as an operator on $\mathbb{R}^{nN}$, where each Δ_j is the Laplacian in the variable $x_j \in \mathbb{R}^n$. The span of the variables on which the potentials depend is a hyperplane of dimension $n(N-1)$. The operator is translation invariant in the extra 'centre of mass' coordinate which therefore, after Fourier transformation in those variables, can be absorbed into the spectral variable. Thus it is traditional to consider the 'reduced operator' which is of the form

$$B = \Delta_x + \sum_p V_p(L_p x),\ x \in \mathbb{R}^M,\ M = n(N-1),$$

with $L_j : \mathbb{R}^M \longrightarrow \mathbb{R}^n$ surjective linear maps which can be assumed to have no joint null space. The spectrum of such operators has been much discussed and the 'asymptotic completeness' has been shown by Sigal and Soffer; see [14] and references therein.

The methods used in this and many related papers do not easily yield explicit information on the structure of the resolvent of B. It is natural to associate to the operator B a calculus of pseudodifferential operators. This calculus has not yet been discussed in detail, but it arises from the following compactification of $\mathbb{R}^M$. First consider the radial compactification to $\mathbb{S}^M_+$ of $\mathbb{R}^M$. The Laplacian lifts to an element of the scattering calculus, as discussed above. However the interaction potentials do not lift to be smooth on $\mathbb{S}^M_+$. Each of the linear maps L_p has as null space a linear subspace of $\mathbb{R}^M$ which defines a spherical submanifold of the boundary of $\mathbb{S}^M_+$. The appropriate configuration space for the N-body problem is $\mathbb{S}^M_+$ with all these subspaces, and all their intersections, blown up in increasing order of dimension. One can expect that the generalized eigenfunctions of B are relatively simple functions on this space. The extra boundary faces introduced at infinity are the 'scattering channels'.

The three-body problem is particularly simple since the (three) channels at infinity are disjoint. A. Vasy, see [15], has recently shown how the propagation properties of the scattering wavefront set can be used to construct a precise parametrix for the resolvent, and the Poisson operator. This is done using a finite iteration of unreduced two-body resolvents and Poisson operators. Since all these operators can be analyzed in detail, this allows rather precise information about the generalized eigenfunctions to be obtained. For example it follows that the absolute scattering matrix in this case is a Fourier integral operator associated to a singular canonical relation which consists of the broken geodesic flow on the boundary $\mathbb{S}^{M-1}$ at distance π. The 'breaks' occur at the channels at infinity. A. Hassell had previously shown that the scattering matrix in the unreduced two-body problem has a similar structure.

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