CHAPTER 2

Pseudodifferential operators on $\mathbb{R}^n$

Formula (1.85) for the action of a differential operator (with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$) on $\mathcal{S}(\mathbb{R}^n)$ can be written

$$
\begin{aligned}
P(x,D)u &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}P(x,\xi)u(y)dyd\xi \\
&= (2\pi)^{-n}\int e^{ix\cdot\xi}P(x,\xi)\hat{u}(\xi)d\xi
\end{aligned} \quad [3.1] \tag{2.1}
$$

where $\hat{u}(\xi) = \mathcal{F}u(\xi)$ is the Fourier transform of u. We shall generalize this formula by generalizing $P(x,\xi)$ from a polynomial in ξ to a *symbol.* In fact we shall also allow the symbol, or rather the *amplitude*, in the integral (2.1) to depend on the 'incoming' variables, y :

$$
A(x,D)u = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi, u \in \mathcal{S}(\mathbb{R}^n).[3.2] \tag{2.2}
$$

Of course it is not clear that this integral is well-defined. To interpret (2.2) we shall first look into the properties of symbols.

2.1. Symbols

A polynomial, p, in ξ, of degree at most m, satisfies a bound

$$
|p(\xi)| \le C(1+|\xi|)^m \quad \forall\ \xi \in \mathbb{R}^n.[3.3] \tag{2.3}
$$

Since successive derivatives, $D^\alpha_\xi p(\xi)$, are polynomials of degree $m-|\alpha|$, for any multiindex α, we get the family of estimates

$$
|D^\alpha_\xi p(\xi)| \le C_\alpha(1+|\xi|)^{m-|\alpha|} \quad \forall\ \xi \in \mathbb{R}^n, \alpha \in \mathbb{N}^n_0.[3.4] \tag{2.4}
$$

Of course if $|\alpha| > m$ then $D^\alpha_\xi p \equiv 0$, so we can even take the constant C_α to be independent of α. If we consider the characteristic polynomial $P(x,\xi)$ of a differential operator of order m with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ (i.e. all derivatives of the coefficients are bounded) (2.4) is replaced by

$$
\mid D^\alpha_x D^\beta_\xi P(x,\xi) \mid \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \quad \forall\ (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,\ \alpha,\beta \in \mathbb{N}^n_0.[3.5] \tag{2.5}
$$

There is no particular reason to have the same number of x variables as of ξ variables, so in general we define:

DEFINITION 2.1. *[3.6] The space $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ of symbols of order m consists of those functions $a \in \mathcal{C}^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ satisfying all the estimates*

$$
\mid D^\alpha_z D^\beta_\xi a(z,\xi) \mid \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \text{ on } \mathbb{R}^p \times \mathbb{R}^n \quad \forall\ \alpha \in \mathbb{N}^p_0, \beta \in \mathbb{N}^n_0.[3.7] \tag{2.6}
$$

For later reference we even define $S^m_\infty(\Omega;\mathbb{R}^n)$ when $\Omega \subset \mathbb{R}^p$ and $\Omega \subset \operatorname{clos}(\operatorname{int}(\Omega))$ as consisting of those $a \in \mathcal{C}^\infty(\operatorname{int}(\Omega)\times\mathbb{R}^n)$ satisfying (2.6) for $(z,\xi) \in \operatorname{int}(\Omega)\times\mathbb{R}^n$.

The estimates (2.6) can be rewritten

$$\|a\|_{N,m} = \sup_{\substack{z \in \mathrm{int}(\Omega)\\ \xi \in \mathbb{R}^n}} \max_{|\alpha|+|\beta| \le N} (1+|\xi|)^{-m+|\beta|} \times \left| D_z^\alpha D_\xi^\beta a(z,\xi) \right| < \infty.[3.8] \tag{2.7}$$

With these *norms* $S^m_\infty(\Omega;\mathbb{R}^n)$ is a Fréchet space, rather similar in structure to $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Thus the topology is given by the metric

$$d(a,b) = \sum_{N \ge 0} 2^{-N} \frac{\|a-b\|_{N,m}}{1+\|a-b\|_{N,m}},\ a,b \in S^m_\infty(\Omega;\mathbb{R}^n).[3.9] \tag{2.8}$$

The subscript '∞' here is *not* standard notation. It refers to the assumption of uniform boundedness of the derivatives of the 'coefficients'. More standard notation would be just $S^m(\Omega \times \mathbb{R}^n)$, especially for $\Omega = \mathbb{R}^p$, but I think this is too confusing.

A more significant issue is: Why this class precisely? As we shall see below, there are other choices which are not only possible but even profitable to make. The present one has several virtues. It is large enough to cover most of the straightforward things we want to do (at least initially) and small enough to 'work' easily. It leads to what I shall refer to as the 'traditional' algebra of pseudodifferential operators.

Now to some basic properties. First notice that

$$(1+|\xi|)^m \le (1+|\xi|)^{m'} \quad \forall\, \xi \in \mathbb{R}^n \iff m \le m'. \tag{2.9}$$

Thus we have an inclusion

$$S^m_\infty(\Omega;\mathbb{R}^n) \hookrightarrow S^{m'}_\infty(\Omega;\mathbb{R}^n) \quad \forall\, m' \ge m.[3.10] \tag{2.10}$$

Moreover this inclusion is continuous, since from (2.7), $\|a\|_{N,m'} \le \|a\|_{N,m}$ if $a \in S^m(\Omega;\mathbb{R}^n)$ and $m' \ge m$. Since these spaces increase with m we think of them as a *filtration* of the big space

$$S^\infty_\infty(\Omega;\mathbb{R}^n) = \bigcup_m S^m_\infty(\Omega;\mathbb{R}^n).[3.11] \tag{2.11}$$

The *residual* space of this filtration is

$$S^{-\infty}_\infty(\Omega;\mathbb{R}^n) = \bigcap_m S^m_\infty(\Omega;\mathbb{R}^n).[3.12] \tag{2.12}$$

A rather useful technical result is:

Lemma 2.1. *[3.13] For any $m' > m$ the space $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ is dense in the space $S^m_\infty(\Omega;\mathbb{R}^n)$ in the topology of $S^{m'}_\infty(\Omega;\mathbb{R}^n)$.*

One *cannot* take $m' = m$ here, for the same reason that $\mathcal{S}(\mathbb{R}^n)$ is *not* dense in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$; the limiting function must vanish at infinity.

Proof. Choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(\xi) = 1$ if $|\xi| < 1, \phi(\xi) = 0$ if $|\xi| > 2$ and consider the sequence

$$a_k(z,\xi) = \phi(\xi/k)a(z,\xi), \quad a \in S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.13}$$

I claim that $a_k \in S^{-\infty}_\infty(\Omega,\mathbb{R}^n)$ and $a_k \longrightarrow a$ in $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for any $m' > m$. [This gives a slightly stronger result than stated, i.e. the same sequence works for all $m' > m$.] Certainly for each N

$$|a_k(z,\xi)| \le C_{N,k}(1+|\xi|)^{-N} \tag{2.14}$$

and by Leibniz formula we easily conclude that

$$\left| D_z^\alpha D_\xi^\beta a_k(z,\xi) \right| \le C_{N,\alpha,\beta,k}(1+|\xi|)^{-N} \quad \forall\ \alpha,\beta,N,k. \tag{2.15}$$

Thus $a_k \in S_\infty^{-\infty}(\Omega;\mathbb{R}^n)$.

So consider the difference

$$(a-a_k)(z,\xi) = (1-\phi)(\xi/k)\ a(z,\xi).[3.14] \tag{2.16}$$

If $\frac{1}{2} > \epsilon > 0$ is given then

$$\left| (1-\phi)(\frac{\xi}{k})(1+|\xi|)^{-\epsilon} \right| \le C_\epsilon k^{-\epsilon/2},\ k \ge 1.[3.15] \tag{2.17}$$

Indeed, in $|\xi| \ge k^{\frac{1}{2}}$, $(1+|\xi|)^{-\epsilon} \le k^{-\frac{\epsilon}{2}}$ and $(1-\phi)$ is bounded, and in $|\xi| \le k^{\frac{1}{2}}$, by Taylor's formula

$$\begin{gathered}(1-\phi)(\frac{\xi}{k}) \le C|\frac{\xi}{k}| \le Ck^{-\frac{1}{2}}, \\ (1+|\xi|)^{-\epsilon} \le C.\end{gathered}\ [3.16] \tag{2.18}$$

This gives (2.17) and hence, with $\epsilon = m'-m$,

$$(1+|\xi|)^{-m'} \left| (a-a_k)(z,\xi) \right| \le C(1+|\xi|)^{-\epsilon}|(1-\phi)(\frac{\xi}{k})| \le Ck^{-\frac{\epsilon}{2}} \longrightarrow 0. \tag{2.19}$$

This is the first symbol estimate. Next consider the ξ derivatives of (2.16). Using Leibniz' formula

$$\begin{aligned} D_\xi^\beta(a-a_k) &= \sum_{\gamma\le\beta}\binom{\beta}{\gamma} D_\xi^{\beta-\gamma}(1-\phi)(\frac{\xi}{k}) \cdot D_\xi^\gamma a(z,\xi) \\ &= (1-\phi)(\frac{\xi}{k}) \cdot D_\xi^\beta a(z,\xi) - \sum_{\gamma<\beta}\binom{\beta}{\gamma}\left(D^{\beta-\gamma}\phi\right)(\frac{\xi}{k}) \cdot k^{-|\beta-\gamma|} D_\xi^\alpha a(z,\xi). \end{aligned}$$

In the first term, $D_\xi^\beta a(z,\xi)$ is a symbol of order $m-|\beta|$, so

$$\sup_\xi (1+|\xi|)^{-m'+|\beta|}(1-\phi)(\frac{\xi}{k}) D_\xi^\beta a(x,\xi) \longrightarrow 0 \tag{2.20}$$

as $k \longrightarrow \infty$ if $m' > m$. In all the other terms, $(D^{\beta-\gamma}\phi)(\zeta)$ has compact support, i.e. $1 \le |\zeta| \le 2$. Thus for each term we get a bound

$$\sup_{k\le|\xi|\le 2k} (1+|\xi|)^{-m'+|\beta|} \cdot k^{-|\beta-\gamma|} C \cdot (1+|\xi|)^{m-|\gamma|} \le Ck^{-m'+m}. \tag{2.21}$$

The variables z play the rôle of parameters so we have in fact shown that

$$\sup_{\substack{z\in\Omega \\ \xi\in\mathbb{R}^n}} (1+|\xi|)^{-m'} \left| D_z^\alpha D_\xi^\beta (a-a_k) \right| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{2.22}$$

This means $a_k \longrightarrow a$ in $S_\infty^{m'}(\mathbb{R}^p;\mathbb{R}^n)$ as required. □

In fact this proof suggests a couple of other 'obvious' results. Namely

$$S_\infty^m(\Omega;\mathbb{R}^n) \cdot S_\infty^{m'}(\Omega;\mathbb{R}^n) \subset S_\infty^{m+m'}(\Omega;\mathbb{R}^n).[3.17] \tag{2.23}$$

This can be proved directly using Leibniz' formula:

$$\begin{aligned}&\sup_\xi (1+|\xi|)^{-m-m'+|\beta|} \mid D_z^\alpha D_\xi^\beta \left(a(z,\xi)\cdot b(z,\xi)\right) \mid \\ \le \sum_{\substack{\mu\le\alpha\\ \gamma\le\beta}} \binom{\alpha}{\mu}\binom{\beta}{\gamma} &\sup_\xi (1+|\xi|)^{-m+|\gamma|} \mid D_z^\mu D_\xi^\gamma a(z,\xi) \mid \\ &\times \sup_\xi (1+|\xi|)^{-m'+|\beta-\gamma|} \mid D_z^{\alpha-\mu} D_\xi^{\beta-\gamma} b(z,\xi) \mid < \infty.\end{aligned}$$

We note the obvious action of differentiation:

$$\begin{aligned} D_z^\alpha &: S_\infty^m(\Omega;\mathbb{R}^n) \longrightarrow S_\infty^m(\Omega;\mathbb{R}^n) \\ D_\xi^\beta &: S_\infty^m(\Omega;\mathbb{R}^n) \longrightarrow S_\infty^{m-|\beta|}(\Omega;\mathbb{R}^n). \end{aligned} \tag{2.24}$$ [3.18]

In fact, while we are thinking about these things we might as well show the important consequence of *ellipticity*. A symbol $a \in S_\infty^m(\Omega;\mathbb{R}^n)$ is said to be (globally) elliptic if

$$|a(z,\xi)| \ge \epsilon(1+|\xi|)^m - C(1+|\xi|)^{m-1},\ \epsilon > 0 \tag{2.25}$$ [3.19]

or equivalently

$$|a(z,\xi)| \ge \epsilon(1+|\xi|)^m \text{ in } |\xi| \ge C_\epsilon. \tag{2.26}$$ [3.20]

Lemma 2.2. *[3.21] If $a \in S_\infty^m(\Omega;\mathbb{R}^n)$ is elliptic there exists $b \in S_\infty^{-m}(\Omega;\mathbb{R}^n)$ such that*

$$a\cdot b - 1 \in S_\infty^{-\infty}(\Omega;\mathbb{R}^n). \tag{2.27}$$ [3.22]

Proof. Using (2.26) choose ϕ as in the proof of Lemma 2.1 and set

$$b(z,\xi) = \begin{cases} \frac{1-\phi(2\xi/C)}{a(z,\xi)} & |\xi| \ge C \\ 0 & |\xi| \le C. \end{cases} \tag{2.28}$$ [3.23]

Then b is $\mathcal{C}^\infty$ since $b = 0$ in $C \le |\xi| \le C+\epsilon$ for $\epsilon > 0$. The symbol estimates follow by noting that, in $|\xi| \ge C$,

$$D_z^\alpha D_\xi^\beta b = a^{-1-|\alpha|-|\beta|} \cdot G_{\alpha\beta} \tag{2.29}$$

Add problem with hint.

where $G_{\alpha\beta}$ is a symbol of order $(|\alpha|+|\beta|)m - |\beta|$. This can be proved by induction. □

2.2. Pseudodifferential operators

Now we proceed to discuss the formula (2.2) where we shall assume that, for some $p, m \in \mathbb{R}$

$$\begin{aligned} a(x,y,\xi) &= (1+|x-y|^2)^{w/2}\tilde a(x,y,\xi) \\ \tilde a &\in S_\infty^m(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi). \end{aligned} \tag{2.30}$$ [3.24]

The extra 'weight' factor (which allows polynomial growth in the direction of $x-y$) turns out, somewhat enigmatically, to both make no difference and be very useful! Notice that if $a \in \mathcal{C}^\infty(\mathbb{R}^{2n}\times\mathbb{R}^n)$ then $a \in (1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ if and only if

Make into a problem with reference.

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1+|x-y|)^w(1+|\xi|)^{m-|\gamma|}\ \forall\ \alpha,\beta,\gamma\in\mathbb{N}_0^n. \tag{2.31}$$ [3.32]

If $m < -n$ then, for each $u \in \mathcal{S}(\mathbb{R}^n)$ the integral in (2.2) is absolutely convergent, uniformly in x, since

$$\begin{aligned}|a(x,y,\xi)u(y)| &\le C(1+|x-y|)^w(1+|\xi|)^m(1+|y|)^{-N}\\ &\le C(1+|x|)^w(1+|\xi|)^m(1+|y|)^m,\ m<-n.\end{aligned} \tag{2.32}$$

Here we have used the following simple consequence of the triangle inequality

$$(1+|x-y|) \le (1+|x|)(1+|y|).$$

Thus we conclude that, provided $m < -n$,

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow (1+|x|^2)^{w/2}\mathcal{C}^0_\infty(\mathbb{R}^n).[3.33] \tag{2.33}$$

To show that, for general m, A exists as an operator, we prove that its Schwartz kernel exists.

Proposition 2.1. *[3.25] The map, defined for $m < -n$ as a convergent integral,*

$$\begin{aligned}(1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \longmapsto I(a) =\\ (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi \in \mathcal{S}'(\mathbb{R}^{2n})\end{aligned}[3.26] \tag{2.34}$$

extends by continuity to all m.

Proof. Since we already have the density of $S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ in S^m_∞ we only need to show the continuity on this subspace. What we shall show is that, for each $w, m \in \mathbb{R}$ there are integers $N, k \in \mathbb{N}$ such that, in terms of the norms in (2.7) and (1.6)

$$\big|\ I(a)(\phi)\ \big| \le C\|\tilde{a}\|_{N,m}\|\phi\|_k \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^{2n}).[3.27] \tag{2.35}$$

To see this we just use integration by parts!

Set $\tilde{\phi}(x,y,\xi) = (1+|x-y|^2)^{w/2}\phi(x,y)$. Observe that

$$\begin{aligned}(1+\xi\cdot D_x)e^{i(x-y)\cdot\xi} &= (1+|\xi|^2)e^{i(x-y)\cdot\xi}\\ (1-\xi\cdot D_y)e^{i(x-y)\cdot\xi} &= (1+|\xi|^2)e^{i(x-y)\cdot\xi}.\end{aligned}$$

Thus we can write (for $\tilde{a} \in S^{-\infty}_\infty$, with $a = (1+|x-y|^2)^{w/2}\tilde{a}$)

$$\begin{aligned}I(a)(\phi) = \iint (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-2q}\\ (1-\xi\cdot D_x)^q(1+\xi\cdot D_y)^q\left[\tilde{a}(x,y,\xi)\tilde{\phi}(x,y)\right]d\xi dxdy\\ = \sum_{|\gamma|\le 2q}\iint\Big(\int e^{i(x-y)\cdot\xi}a_\gamma(x,y,\xi)d\xi\Big)D^\gamma_{(x,y)}\tilde{\phi}(x,y)dxdy.\end{aligned}\quad[3.28] \tag{2.36}$$

Here $a_\gamma = (1+|\xi|^2)^{-2q}\sum_{\substack{\mu\le\gamma\\|\gamma|\le 2q}} C_\mu\xi^\gamma D^\mu_{(x,y)}\tilde{a}$. So if we take $-2q+m < -n$, e.g. $q > \max(\frac{n+m}{2}, 0)$ and use the integrability of $(1+|x|+|y|)^{-2n-1}$ on $\mathbb{R}^{2n}$, then

$$\big|I(a)(\phi)\big| \le C\|\tilde{a}\|_{2q,m}\|\tilde{\phi}\|_{2q+2n+1} \le C\|\tilde{a}\|_{2q,m}\|\phi\|_{2q+w+2n+1}. \tag{2.37}$$

This proves the desired continuity. □

In fact we do *not* really need to integrate by parts in both x *and* y, either will do the trick.

LEMMA 2.3. *[3.29] If $a \in (1+|x-y|^2)^{w/2} S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ then the operator A, with Schwartz kernel $I(a)$, is a continuous linear map*

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n).[3.30] \tag{2.38}$$

We shall denote by $\Psi^m_\infty(\mathbb{R}^n)$ the linear spaces of operators (2.38), corresponding to $(1+|x-y|^2)^{-w/2} a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w. They are called *pseudodifferential operators* (and I say 'of traditional type' – or type '1,0', the meaning of which is explained in Problem * below).

Put in the problem for ρ, δ with references.

PROOF. Proceeding as in (2.36) but only integrating by parts in y we deduce that, for q large depending on m,

$$Au(\psi) = \sum_{\gamma \le 2q} (2\pi)^{-n} \iint \int e^{i(x-y)\cdot\xi} a_\gamma(x,y,\xi) D^\gamma_y u(y) d\xi \psi(x) dy dx.$$

This shows that Au is given by the convergent integral

$$Au(x) = \sum_{\gamma \le 2q} (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a_\gamma(x,y,\xi) D^\gamma_y u(y) d\xi dy.$$

Notice that

$$D_{x_j} Au(x) = (2\pi)^{-n} \sum_{|\gamma| \le 2q} \iint e^{i(x-y)\cdot\xi} \big(\xi_j + D_{x_j}\big) a_\gamma \cdot D^\gamma_y u(y) dy d\xi$$

and

$$x_j Au(x) = (2\pi)^n \sum_{|\gamma| \le 2q} \iint e^{i(x-y)\cdot\xi} \big(-D_{\xi_j} + y_j\big) a_\gamma \cdot D^\gamma_y u(y) dy d\xi.$$

Proceeding inductively (2.38) follows from (2.33). □

2.3. Composition

There are two extreme cases of $I(a)$, namely where a is independent of x and of y. Below we shall prove:

THEOREM 2.1 (Reduction). *[3.31] Each $A \in \Psi^m_\infty(\mathbb{R}^n)$ can be written uniquely as $I(a')$ where $a' \in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$.*

This is the main step in proving the fundamental result of this Chapter, which is that two pseudodifferential operators can be composed to give a pseudodifferential operator and that the orders are additive. Thus our aim to to demonstrate the fundamental

THEOREM 2.2. *[4.1] The space $\Psi^\infty_\infty(\mathbb{R}^n)$ is a filtered $*$-algebra on $\mathcal{S}(\mathbb{R}^n)$.*

We have already shown that each $A \in \Psi^\infty_\infty(\mathbb{R}^n)$ defines a continuous linear map (2.38). We now want to show that

$$A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A^* \in \Psi^m_\infty(\mathbb{R}^n)[4.35] \tag{2.39}$$

$$A \in \Psi^m_\infty(\mathbb{R}^n),\ B \in \Psi^{m'}_\infty(\mathbb{R}^n) \Longrightarrow A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n).[4.3] \tag{2.40}$$

In fact we will pick up some more information along the way.

2.4. Reduction

We proceed to prove Theorem 2.1, which we can restate as:

PROPOSITION 2.2. *[4.4] The range of* (2.34) *(for any w) is the same as the range of I restricted to the image of the inclusion map*

$$S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto a(x,\xi) \in S^m_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n).$$

PROOF. Suppose $a \in \big(1+|x-y|^2\big)^{w/2} S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w, then

$$I\big((x_j-y_j)a\big) = I\big(-D_{\xi_j}a\big)\ j=1,\dots,n.[4.5] \tag{2.41}$$

Indeed this is just the result of inserting the identity

$$D_{\xi_j}e^{i(x-y)\cdot\xi} = (x_j-y_j)e^{i(x-y)\cdot\xi}$$

into (2.34), with $N=0$, and integrating by parts. Since both sides of (2.41) are continuous on $\big(1+|x-y|^2\big)^{w/2} S^\infty_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ the identity holds in general. Notice that if a is of order m then $D_{\xi_j}a$ is of order $m-1$, so (2.41) shows that even though the operator with amplitude $(x_j-y_j)a(x,y,\xi)$ *appears* to have order m, it actually has order $m-1$.

To exploit (2.41) consider the Taylor series (with Legendre's remainder) for $a(x,y,\xi)$ around $x=y$:

$$\begin{aligned} a(x,y,\xi) = &\sum_{|\alpha|\le N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \big(D^\alpha_y a\big)(x,x,\xi) \\ &+ \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \cdot R_{N,\alpha}(x,y,\xi). \end{aligned} \quad [4.6] \tag{2.42}$$

Here,

$$R_{N,\alpha}(x,y,\xi) = \int_0^1 (1-t)^{N-1}\big(D^\alpha_y a\big)(x,(1-t)x+ty,\xi)dt.[4.7] \tag{2.43}$$

Now,

$$(x-y)^\alpha\big(D^\alpha_y a\big)(x,y,\xi) \in \big(1+|x-y|^2\big)^{\frac{(w+|\alpha|)}{2}} S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n). \tag{2.44}$$

Applying (2.41) repeatedly we see that if A is the operator with kernel $I(a)$ then

$$A = \sum_{j=0}^{N-1} A_j + R_N, \quad A_j \in \Psi^{m-j}_\infty(\mathbb{R}^n),\ R_N \in \Psi^{m-N}_\infty(\mathbb{R}^n)[4.8] \tag{2.45}$$

where the A_j have kernels

$$I\big(\sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!}\big(D^\alpha_y D^\alpha_\xi a\big)(x,x,\xi)\big).[4.9] \tag{2.46}$$

To proceed further we need somehow to *sum* this series. Of course we cannot really do this, but we can come close! □

2.5. Asymptotic summation

Suppose $a_j \in S^{m-j}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$. The fact that the *orders* are decreasing means that those symbols are getting very small, for $|\xi|$ large. The infinite series

$$\sum_j a_j(z,\xi) \tag{2.47}$$

need not converge. However we shall say that it converges asymptotically, or since it is a series we say it is 'asymptotically summable,' if there exists $a \in S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ such that

$$\text{for every } N,\quad a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n).[4.10] \tag{2.48}$$

We write this relation as

$$a \sim \sum_{j=0}^{\infty} a_j.[4.11] \tag{2.49}$$

Proposition 2.3. *[4.12] Any series $a_j \in S^{m-j}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ is asymptotically summable, in the sense of* (2.48)*, and the asymptotic sum is well defined up to an additive term in $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$.*

Proof. The uniqueness part is easy. Suppose a and a' both satisfy (2.48). Taking the difference

$$a - a' = \big(a - \sum_{j=0}^{N-1} a_j\big) - \big(a' - \sum_{j=0}^{N-1} a_j\big) \in S^{m-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n).[4.13] \tag{2.50}$$

Since $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ is just the intersection of the $S^{-N}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ over N it follows that $a - a' \in S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, proving the uniqueness.

So to the existence of an asymptotic sum. To construct this (by Borel's method[1]) we cut off each term "near infinity in ξ". Thus fix $\phi \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ with $\phi(\xi) = 0$ in $|\xi| \le 1$, $\phi(\xi) = 1$ in $|\xi| \ge 2$, $0 \le \phi(\xi) \le 1$. Consider a decreasing sequence

$$\epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \downarrow 0.[4.14] \tag{2.51}$$

We want to set

$$a(z,\xi) = \sum_{j=0}^{\infty} \phi(\epsilon_j\xi) a_j(z,\xi).[4.15] \tag{2.52}$$

Of course we need to consider the seminorms, in $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ of each term. The first of these is

$$\sup_z \sup_\xi (1+|\xi|)^{-m} \;\big|\, \phi(\epsilon_j\xi) \,\big|\; \big|\, a_j(z,\xi) \,\big| \;.[4.16] \tag{2.53}$$

[1] Émile Borel

Now $\phi(\epsilon_j\xi) = 0$ unless $|\epsilon_j\xi| \le 1$, i.e. $|\xi| \le \frac{1}{\epsilon_j}$. Since a_j is of order $m-j$ this allows us to estimate (2.53) by

$$\sup_z \sup_{|\xi|\le\frac{1}{\epsilon_j}} (1+|\xi|)^{-j} \cdot \left[\left(1+|\xi|\right)^{-m+j} \mid a_j(z,\xi) \mid\right]$$
$$\le \left(1+\frac{1}{\epsilon_j}\right)^{-j} \cdot C_j \le \epsilon_j^j \cdot C_j$$

where the C_j's are *fixed* constants, independent of ϵ_j, Let's look at the higher symbol estimates. As usual we can apply Leibniz' formula:

$$\sup_z \sup_\xi (1+|\xi|)^{-m+|\beta|} \mid D_z^\alpha D_\xi^\beta \phi(\epsilon_j\xi) a_j(z,\xi) \mid$$
$$\le \sum_{\mu\le\beta} \sup_z \sup_\xi (1+|\xi|)^{|\beta|-|\mu|-j} \epsilon_j^{|\beta|-|\mu|} \mid \left(D^{\beta-\mu}\phi\right)(\epsilon_j\xi) \mid$$
$$\times (1+|\xi|)^{-m+j+|\mu|} \mid D_z^\alpha D_\xi^\mu a_j(z,\xi) \mid .$$

The term with $\mu = \beta$ we estimate as before and the others, with $\mu \ne \beta$ are supported in $\frac{1}{\epsilon_j} \le |\xi| \le \frac{2}{\epsilon_j}$. Then we find that for all j

$$\|\phi(\epsilon_j\xi) a_j(z,\xi)\|_{N,m} \le C_{N,j}\epsilon_j^j [4.17] \tag{2.54}$$

where $C_{N,j}$ is independent of ϵ_j.

So we see that for each given N we can arrange that, for instance,

$$\|\phi(\epsilon_j\xi) a_j(z,\xi)\|_{N,m} \le C_N \frac{1}{j^2}$$

by choosing the ϵ_j to satify

$$C_{N,j}\epsilon_j^j \le \frac{1}{j^2} \quad \forall\ j \ge j(N).$$

Notice the crucial point here, we can arrange that for *each* N the sequence of norms in (2.54) is dominated by $C_N j^{-2}$ by fixing $\epsilon_j < \epsilon_{j,N}$ for large j. Thus we can arrange convergence of *all* the sums

$$\sum_j \|\phi(\epsilon_j\xi) a_j(z,\xi)\|_{N,m}$$

by diagonalization, for example setting $\epsilon_j = \frac{1}{2}\epsilon_{j,j}$. Thus by choosing $\epsilon_j \downarrow 0$ rapidly enough we ensure that the series (2.52) converges. In fact the same argument allows us to ensure that for every N

$$\sum_{j\ge N} \phi(\epsilon_j\xi) a_j(z,\xi) \text{ converges in } S_\infty^{m-N}(\mathbb{R}^p;\mathbb{R}^n).[4.18] \tag{2.55}$$

This certainly gives (2.48) with a defined by (2.52). □

2.6. Residual terms

Now we can apply Proposition 2.3 to the series (2.46), i.e. we can find $b \in S_\infty^m(\mathbb{R}_x^n;\mathbb{R}_\xi^n)$ satisfying

$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\left(D_y^\alpha a\right)(x,x,\xi).[4.19] \tag{2.56}$$

Let $B = I(b)$ be the operator defined by this amplitude (which is independent of y). Now (2.45) says that

$$A - B = \sum_{j=0}^{N-1} A_j + R_N - B$$

and from (2.48) applied to (2.56)

$$B = \sum_{j=0}^{N-1} A_j + R'_N, \ R'_N \in \Psi_\infty^{m-N}(\mathbb{R}^n)$$

Thus

$$A - B \in \Psi_\infty^{-\infty}(\mathbb{R}^n) = \bigcap_N \Psi_\infty^N(\mathbb{R}^n).[4.20] \tag{2.57}$$

Notice that, at this stage, we do *not* know that $A - B$ has kernel $I(c)$ with $c \in S_\infty^{-\infty}(\mathbb{R}^{2n}, \mathbb{R}^n)$, just that it has kernel $I(c_N)$ with $c_N \in S_\infty^N(\mathbb{R}^{2n}; \mathbb{R}^n)$ for each N.

However:

PROPOSITION 2.4. *[4.21] An operator $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is an element of the space $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies the estimates*

$$\mid D_x^\alpha D_y^\beta K(x,y) \mid \leq C_{N,\alpha,\beta}(1 + |x-y|)^{-N} \ \forall \ \alpha, \beta, N.[4.22] \tag{2.58}$$

PROOF. Suppose first that $A \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$, which means that $A \in \Psi_\infty^N(\mathbb{R}^n)$ for every N. The Schwartz kernel, K_A, of A is therefore given by (2.34) with the amplitude $a_N \in S_\infty^N(\mathbb{R}^{2n}; \mathbb{R}^n)$. For $N << -n-1-p$ the integral certainly converges absolutely and we can integrate by parts to show that

$$(x-y)^\alpha D_x^\beta D_y^\gamma K_A(x,y)$$
$$= (2\pi)^{-N} \int e^{i(x-y)\cdot\xi}(-D_\xi)^\alpha (D_x + i\xi)^\beta (D_y - i\xi)^\gamma a_N(x,y,\xi) d\zeta$$

which converges absolutely, and uniformly in x, y, provided $|\beta| + |\gamma| + N - |\alpha| < -n$. Thus

$$\sup \mid (x-y)^\alpha D_x^\beta D_y^\gamma K \mid < \infty \qquad \forall \ \alpha, \beta, \gamma$$

which is another way of writing (2.58) i.e.

$$\sup\big(1 + |x-y|^2\big)^N \mid D_x^\beta D_y^\gamma K \mid < \infty \qquad \forall \ \beta, \gamma, N.$$

Conversely suppose that (2.58) holds. Define

$$g(x,z) = K(x, x-z).[4.23] \tag{2.59}$$

The estimates (2.58) become

$$\sup \mid D_x^\alpha z^\gamma D_z^\beta g(x,z) \mid < \infty \qquad \forall \ \alpha, \beta, \gamma.[4.24] \tag{2.60}$$

That is, g is rapidly decreasing with all its derivatives in z. Taking the Fourier transform,

$$b(x,\xi) = \int e^{-iz\cdot\xi} g(x,z) dz[4.25] \tag{2.61}$$

the estimate (2.60) translates to

$$\begin{aligned}\sup_{x,\xi} \left| D_x^\alpha \xi^\beta D_\xi^\gamma b(x,\xi) \right| < \infty \qquad \forall\ \alpha,\beta,\gamma \\ \iff b \in S_\infty^{-\infty}(\mathbb{R}_x^n;\mathbb{R}_\xi^n).\end{aligned} \tag{2.62}$$ [4.26]

Now the inverse Fourier transform in (2.61), combined with (2.59) gives

$$K(x,y) = g(x,x-y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} b(x,\xi) d\xi \tag{2.63}$$ [4.27]

i.e. $K = I(b)$. This certainly proves the proposition and actually gives the stronger result.

$$A \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \iff A = I(c),\ c \in S_\infty^{-\infty}(\mathbb{R}_x^n;\mathbb{R}_\xi^n). \tag{2.64}$$ [4.127]

□

This also finishes the proof of Proposition 2.2 since in (2.56), (2.57) we have shown that

$$A = B + R, \quad B = I(b), \quad R \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \tag{2.65}$$ [4.28]

so in fact

$$A = I(e),\ e \in S_\infty^m(\mathbb{R}_x^n;\mathbb{R}_\xi^n),\ e \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \left(D_y^\alpha D_\xi^\alpha a\right)(x,x,\xi). \tag{2.66}$$ [4.29]

2.7. Proof of Theorem 2.2

First consider the adjoint formula. If

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

the adjoint is the operator

$$A^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

defined by duality:

$$A^* u(\bar\phi) = u(\overline{A\phi}) \qquad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{2.67}$$ [4.30]

Certainly $A^*u \in \mathcal{S}'(\mathbb{R}^n)$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ since

$$\begin{aligned} A^* u(\psi) = u(\overline{A\overline{\psi}}) \text{ and} \\ \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \overline{A\overline{\psi}} \in \mathcal{S}(\mathbb{R}^n)\end{aligned} \tag{2.68}$$ [4.31]

is clearly continuous. In terms of Schwartz kernels,

$$\begin{aligned} A\phi(x) &= \int K_A(x,y)\phi(y)dy \ \ \phi \in \mathcal{S}(\mathbb{R}^n) \\ A^*u(x) &= \int K_{A^*}(x,y)u(y) \ \ u \in \mathcal{S}(\mathbb{R}^n)dy.\end{aligned} \tag{2.69}$$ [4.32]

We then see that

$$\begin{aligned}\int K_{A^*}(x,y)u(y)\overline{\phi(x)}dydx = \int \overline{K_A(x,y)\phi(y)}dyu(x)dx \\ \Longrightarrow K_{A^*}(x,y) = \overline{K_A(y,x)}\end{aligned} \tag{2.70}$$ [4.33]

where we are using the uniqueness of Schwartz kernels. This proves the first part of (2.40) since

$$\overline{K_A(y,x)} = \overline{\left[\frac{1}{(2\pi)^n}\int e^{i(y-x)\cdot\xi}a(y,x,\xi)d\xi\right]} \\ = \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}\bar{a}(y,x,\xi)d\xi \tag{2.71}$$ [4.34]

i.e. $A^* = I(\bar{a}(y,x,\xi))$. Thus one advantage of allowing general operators (2.34) is that closure under the passage to adjoint is immediate.

For the composition formula we need to apply Proposition 2.2 twice. First to $A \in \Psi^m_\infty(\mathbb{R}^n)$, to write it with symbol $a(x,\xi)$

$$A\phi(x) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)\phi(y)dyd\xi \\ = (2\pi)^{-n}\int e^{ix\cdot\xi}a(x,\xi)\hat{\phi}(\xi)d\xi.$$

Then we also apply Proposition 2.2 to B^*,

$$B^*u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\bar{b}(x,\xi)\hat{u}(\xi)d\xi$$

Integrating this against a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ gives

$$\langle B\phi, u\rangle = \langle \phi, B^*u\rangle = (2\pi)^{-n}\int\int e^{-ix\cdot\xi}\phi(x)b(x,\xi)\overline{\hat{u}(\xi)}d\xi dx \\ \Longrightarrow \widehat{B\phi}(\xi) = \int e^{-iy\cdot\xi}b(y,\xi)\phi(y)dy.$$

Inserting this into the formula for $A\phi$ shows that

$$\Longrightarrow AB(u) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)b(y,\xi)u(y)dyd\xi.$$

Since $a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi)$ this shows that $AB \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ as claimed.

2.8. Quantization and symbols

We have already noted that as well as the left quantization map leading to the isomorphism (2.175) there is a right quantization map, similarly derived from (2.173):

$$q_R(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(y,\xi)d\xi, \quad a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.72}$$ [5.4]

In fact using the adjoint operator, $*$, on operators and writing as well $*$ for complex conjugation of symbols shows that

$$q_R = * \cdot q_L \cdot * \tag{2.73}$$ [5.5]

is also an isomorphism, with inverse σ_R

$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_R}{\overset{\sigma_R}{\rightleftarrows}} S^m_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.74}$$ [5.6]

These are the two "extreme" quantization procedures, see Problem 3.1 for another (more centrist). Using the proof of the reduction theorem we find:

LEMMA 2.4. *[5.7] For any* $a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$,

$$\sigma_L\left(q_R(a)\right)(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi a(x,\xi) \sim e^{i<D_x,D_\xi>}a.[5.8] \tag{2.75}$$

For the moment the last asymptotic equality is just to help in remembering the formula, which is the same as given by the formal Taylor series expansion at the origin of the exponential.

PROOF. This follows from the general formula (2.66). □

One important thing to note from (2.75) is that

$$D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-|\alpha|}_\infty(\mathbb{R}^n;\mathbb{R}^n) \tag{2.76}$$

so that for *any* pseudodifferential operator

$$A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow \sigma_L(A) - \sigma_R(A) \in S^{m-1}_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.77}$$

For this reason we consider the general quotient spaces

$$S^{m-[1]}_\infty(\mathbb{R}^p;\mathbb{R}^n) = S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\big/S^{m-1}_\infty(\mathbb{R}^p;\mathbb{R}^n) \tag{2.78}$$

and, for $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, write $[a]$ for its image, i.e. equivalence class, in the quotient space $S^{m-[1]}_\infty(\mathbb{R}^p;\mathbb{R}^n)$. The "principal symbol map"

$$\begin{gathered}\sigma_m : \Psi^m_\infty(\mathbb{R}^n) \longrightarrow S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)\\ \text{is defined by } \sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)].\end{gathered}[5.9] \tag{2.79}$$

As distinct from σ_L or σ_R, σ_m depends on m, i.e. one needs to know that the order is at most m before it is defined.

The isomorphism (2.175) is replaced by the weaker exact sequence.

LEMMA 2.5. *[5.10] For every* $m \in \mathbb{R}$

$$0 \hookrightarrow \Psi^{m-1}_\infty(\mathbb{R}^n) \hookrightarrow \Psi^m_\infty(\mathbb{R}^n) \xrightarrow{\sigma_m} S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow 0$$

is a short exact sequence (the 'principal symbol sequence' or simply the 'symbol sequence').

PROOF. This is just the statement that the range of each map is the null space of the next i.e. that σ_m is surjective, which follows from (2.175), and that the null space of σ_m is just $\Psi^{m-1}_\infty(\mathbb{R}^n)$ and this is again (2.175) and the definition of σ_m. □

The fundamental result proved last time is that

$$\Psi^m_\infty(\mathbb{R}^n)\cdot\Psi^{m'}_\infty(\mathbb{R}^n) \subset \Psi^{m+m'}_\infty(\mathbb{R}^n).[5.11] \tag{2.80}$$

In fact we showed that if $A = q_L(a)$, $a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $B = q_R(b)$, $b \in S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ then the composite operator has Schwartz kernel

$$K_{A\cdot B}(x,y) = I\left(a(x,\xi)b(y,\xi)\right)$$

Using the formula (2.66) again we see that

$$\sigma_L(A\cdot B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi\left[a(x,\xi)D^\alpha_x b(x,\xi)\right].[5.12] \tag{2.81}$$

Of course $b = \sigma_R(B)$ so we really want to rewrite (2.81) in terms of $\sigma_L(B)$.

LEMMA 2.6. *[5.13] If $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ then $A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ and*

$$\sigma_{m+m'}(A \circ B) = \sigma_m(A) \cdot \sigma_{m'}(B), [5.14] \tag{2.82}$$

$$\sigma_L(A \circ B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi \sigma_L(A) \cdot D^\alpha_x \sigma_L(B). [5.15] \tag{2.83}$$

PROOF. The simple formula (2.82) is already immediate from (2.81) since all terms with $|\alpha| \geq 1$ are of order $m + m' - |\alpha| \leq m + m' - 1$. To get the "full" formula (2.83) we can insert into (2.81) the inverse of (2.75), namely

$$\sigma_R(x, \xi) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_x \sigma_L(x, \xi) \sim e^{-i<D_x, D_\xi>} \sigma_L(x, \xi).$$

This gives the double sum (still asymptotically convergent)

$$\sigma_L(A \circ B) \sim \sum_\beta \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D^\alpha_\xi \left[\sigma_L(A) D^\alpha_x \frac{i^{|\beta|}}{\beta!} D^\beta_x D^\beta_\xi \sigma_L(B)\right]$$

Setting $\gamma = \alpha + \beta$ this becomes

$$\sigma_L(A \circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} \sum_{0 \leq \alpha \leq \gamma} \frac{\gamma!(-1)^{|\gamma - \alpha|}}{\alpha!(\gamma - \alpha)!} D^\alpha_\xi \left[\sigma_L(A) \times D^{\gamma - \alpha}_\xi D^\gamma_x \sigma_L(B)\right].$$

Then Leibniz formula shows that this sum over α can be rewritten as

$$\sigma_L(A \circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} D^\gamma_\xi \sigma_L(A) \cdot D^\gamma_x \sigma_L(B)$$
$$\sim e^{i<D_y, D_\xi>} \sigma_L(A)(x, \xi) \sigma_L(B)(y, \eta)\big|_{y = x, \eta = \xi}.$$

This is just (2.83). □

The simplicity of (2.82) over (2.83) is achieved at the expense of enormous loss of information. Still, many problems can be solved using (2.82) which we can think of as saying that the principal symbol maps give a homomorphism, for instance from the filtered algebra $\Psi^0_\infty(\mathbb{R}^n)$ to the *commutative* algebra $S^{0-[1]}_\infty(\mathbb{R}^n; \mathbb{R}^n)$.

2.9. Ellipticity

We say that an element of $\Psi^m_\infty(\mathbb{R}^n)$ is *elliptic* if it is invertible modulo an error in $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ with the approximate inverse of order $-m$ i.e.

$$\begin{aligned} &A \in \Psi^m_\infty(\mathbb{R}^n) \text{ is elliptic} \\ &\Longleftrightarrow \exists \quad B \in \Psi^{-m}_\infty(\mathbb{R}^n) \text{ s.t. } \quad A \circ B - \mathrm{Id} \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \end{aligned} [5.16] \tag{2.84}$$

Thus ellipticity, here by definition, is invertibility in $\Psi^m_\infty(\mathbb{R}^n) / \Psi^{-\infty}_\infty(\mathbb{R}^n)$, with the inverse in $\Psi^{-m}_\infty(\mathbb{R}^n) / \Psi^{-\infty}_\infty(\mathbb{R}^n)$. The point about ellipticity is that it is a phenomenon of the *principal symbol.*

THEOREM 2.3. *[5.17] The following conditions on $A \in \Psi^m_\infty(\mathbb{R}^n)$ are equivalent*

$$A \text{ is elliptic } [5.18] \tag{2.85}$$

$$\exists\ [b] \in S^{-m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)\ \ s.t.\ \sigma_m(A)\cdot[b] \equiv 1 \ in\ S^{0-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)\,[5.19] \tag{2.86}$$

$$\exists\ b \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n)\ \ s.t.\ \sigma_L(A)\cdot b - 1 \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n)\,[5.20] \tag{2.87}$$

$$\exists\ \epsilon > 0\ s.t.\ \ \big|\ \sigma_L(A)(x,\xi)\ \big| \geq \epsilon(1+|\xi|)^m\ \ in\ |\xi| > \frac{1}{\epsilon}.[5.21] \tag{2.88}$$

PROOF. We shall show

$$(2.85) \Longrightarrow (2.86) \Longrightarrow (2.87) \Longleftrightarrow (2.88) \Longrightarrow (2.85).[5.22] \tag{2.89}$$

Since we know that $\sigma_0(\mathrm{Id}) = 1$ applying the identity (2.82) to the definition of ellipticity in (2.84) gives

$$\sigma_m(A)\cdot\sigma_{-m}(B) \equiv 1 \text{ in } S^{0-[1]}_\infty(\mathbb{R}^n,\mathbb{R}^n),[5.44] \tag{2.90}$$

i.e. that (2.85) $\Longrightarrow$ (2.86).

Now assuming (2.90) (i.e. (2.86)), and recalling that $\sigma_m(A) = [\sigma_L(A)]$ we find that a representative b_1 of the class $[b]$ must satisfy

$$\sigma_L(A)\cdot b_1 = 1 + e_1,\quad e_1 \in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n),[5.23] \tag{2.91}$$

this being the meaning of the equality of residue classes. Now, since the remainder $e_1 \in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ the Neumann series

$$f \sim \sum_{j\geq 1}(-1)^j e_1^j \tag{2.92}$$

is asymptotically convergent, so $f \in S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ exists, and

$$(1+f)\cdot(1+e_1) = 1 + e_\infty,\quad e_\infty \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n).[5.24] \tag{2.93}$$

Then multiplying (2.93) by $(1+f)$ gives

$$\sigma_L(A)\cdot\{b_1(1+f)\} = 1 + e_\infty[5.25] \tag{2.94}$$

which proves (2.87), since $b = b_1(1+f) \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Of course

$$\sup(1+|\xi|)^N|e_\infty| < \infty \quad \forall\ N \tag{2.95}$$

so

$$\exists\ C \text{ s.t. } \ |e_\infty(x,\xi)| < \frac{1}{2} \text{ in } |\xi| > C. \tag{2.96}$$

From (2.94) this means

$$\big|\ \sigma_L(A)(x,\xi)\ \big|\cdot\big|\ b(x,\xi)\ \big| \geq \frac{1}{2},\ |\xi| > C.[5.26] \tag{2.97}$$

Since $|b(x,\xi)| \leq C(1+|\xi|)^{-m}$ (being a symbol of order $-m$), (2.97) implies

$$\inf_{|\xi|\geq C}\ \big|\ \sigma_L(A)(x,\xi)\ \big|\ (1+|\xi|)^{-m} \geq C > 0.[5.27] \tag{2.98}$$

which shows that (2.87) implies (2.88).

Conversely, suppose (2.98) holds, i.e. (2.88) holds with $C = 1/\epsilon$. Then if $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, has $\phi(\xi) = 1$ when $|\xi| \leq 1$, and $\phi(\xi) = 0$ when $|\xi| \geq 2$,

$$\frac{(1-\phi(\xi/C))}{\sigma_L(A)(x,\xi)} = b(x,\xi) \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n)[5.28] \tag{2.99}$$

as follows from (2.40). This gives (2.87), so (2.88) implies (2.87).

Now suppose (2.87) holds. Set $B_1 = q_L(b)$ then from (2.82) again

$$\sigma_0(A \circ B_1) = [q_m(A)] \cdot [b] \equiv 1. \tag{2.100}$$

That is,

$$A \circ B_1 - \mathrm{Id} = E_1 \in \Psi_\infty^{-1}(\mathbb{R}^n).[5.29] \tag{2.101}$$

Consider the Neumann series of operators

$$\sum_{j\geq 0} (-1)^j E_1^j. \tag{2.102}$$

The corresponding series of symbols is asymptotically summable so we can choose $F \in \Psi_\infty^{-1}(\mathbb{R}^n)$ with

$$\sigma_L(F) \sim \sum_{j\geq 1} (-1)^j \sigma_L(E_1^j).[5.45] \tag{2.103}$$

Then

$$(\mathrm{Id} + E_1)(\mathrm{Id} + F) = \mathrm{Id} + E_\infty,\ E_\infty \in \Psi_\infty^{-\infty}(\mathbb{R}^n).[5.30] \tag{2.104}$$

Then $B = B_1(\mathrm{Id} + F) \in \Psi_\infty^{-m}(\mathbb{R}^n)$ satisfies (2.84), i.e. A is elliptic. $\square$

In the definition of ellipticity in (2.84) we have taken B to be a 'right parametrix', i.e. a right inverse modulo $\Psi_\infty^{-\infty}(\mathbb{R}^n)$. We can just as well take it to be a *left* parametrix.

LEMMA 2.7. *[5.31] $A \in \Psi_\infty^m(\mathbb{R}^n)$ is elliptic if and only if there exists $B' \in \Psi_\infty^{-m}(\mathbb{R}^n)$ such that*

$$B' \circ A = \mathrm{Id} + E',\ E' \in \Psi_\infty^{-\infty}(\mathbb{R}^n)[5.32] \tag{2.105}$$

and then if B satisfies (2.84), $B - B' \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$.

PROOF. Certainly (2.105) implies $\sigma_{-m}(B') \cdot \sigma_m(A) \equiv 1$, and the multiplication here is commutative so (2.86) holds and A is elliptic. Conversely if A is elliptic we get in place of (2.101)

$$B_1 \circ A - \mathrm{Id} = E_1' \in \Psi_\infty^{-1}(\mathbb{R}^n).$$

Then defining F' as in (2.103) with E_1' in place of E_1 we get $(\mathrm{Id} + F')(\mathrm{Id} + E_1') = \mathrm{Id} + E_\infty'$ and then $B' = (\mathrm{Id} + F') \circ B_1$ satisfies (2.105). Thus 'left' ellipticity as in (2.105) is equivalent to right ellipticity. Applying B to (2.105) gives

$$B' \circ (\mathrm{Id} + E) = B' \circ (A \circ B) = (\mathrm{Id} + E') \circ B[5.33] \tag{2.106}$$

which shows that $B - B' \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$. $\square$

Thus a left parametrix of an elliptic element of $\Psi_\infty^m(\mathbb{R}^n)$ is always a right, hence two-sided, parametrix.

2.10. The Laplacian

Suppose that $g_{ij}(x)$ are the components of an "∞-metric" on $\mathbb{R}^n$, i.e.

$$g_{ij}(x) \in \mathcal{C}^\infty_\infty(\mathbb{R}^n), i,j = 1,\dots,n$$

$$| \sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j | \geq \epsilon|\xi|^2 \quad \forall\ x \in \mathbb{R}^n, \xi \in \mathbb{R}^n. \tag{2.107}$$ [5.34]

The *Laplacian* of the metric is the second order differential operator

$$\Delta_g = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} D_{x_i} g^{ij}\sqrt{g} D_{x_j} \tag{2.108}$$ [5.35]

where

$$g(x) = \det g^{ij}(x),\ g^{ij}(x) = (g_{ij}(x))^{-1}.$$

The Laplacian is determined by the integration by parts formula

$$\int_{\mathbb{R}^n} \sum_{i,j} g^{ij}(x) D_{x_i}\phi \cdot \overline{D_{x_i}\psi} dg = \int \Delta_g \phi \cdot \overline{\psi} dg\ \forall\ \phi,\psi \in \mathcal{S}(\mathbb{R}^n) \tag{2.109}$$ [5.37]

where

$$dg = \sqrt{g}dx. \tag{2.110}$$ [5.38]

Our assumption in (2.107) shows that $\Delta = \Delta_g \in \mathrm{Diff}^2_\infty(\mathbb{R}^n) \subset \Psi^2_\infty(\mathbb{R}^n)$ is in fact elliptic, since

$$\sigma_2(\Delta) = \sum_{i,j=1} g^{ij}\xi_i\xi_j. \tag{2.111}$$ [5.39]

Thus Δ has a two-sided parametrix $B \in \Psi^{-2}_\infty(\mathbb{R}^n)$

$$\Delta \circ B \equiv B \circ \Delta \equiv \mathrm{Id} \qquad \mathrm{mod} \quad \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.112}$$ [5.40]

Corollary 2.1. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\Delta u = 0$ then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.*

Proof. $B \circ \Delta = \mathrm{Id} + E$, $E \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ means, given $\Delta u = 0$,

$$u = -Eu. \tag{2.113}$$ [5.41]

So it suffices to prove:

Lemma 2.8. *[5.42] If $E \in \Psi^{-\infty}_\infty$ then*

$$E : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n). \tag{2.114}$$ [5.43]

Proof. This follows from Proposition 2.4 since we can regard the kernel as a $\mathcal{C}^\infty$ function of x taking values in $\mathcal{S}(\mathbb{R}^n_y)$. □

□

2.11. L^2 boundedness

So far we have thought of pseudodifferential operators, the elements of $\Psi^m_\infty(\mathbb{R}^n)$ for some m, as defining continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$ and, by duality, on $\mathcal{S}'(\mathbb{R}^n)$. Now that we have proved the composition formula we can use it to prove other 'finite order' regularity results. The basic one of these is L^2 boundedness:

PROPOSITION 2.5. *[6.1] If $A \in \Psi^0_\infty(\mathbb{R}^n)$ then, by continuity from $\mathcal{S}(\mathbb{R}^n)$, A defines a bounded linear operator*

$$A : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).[6.2] \tag{2.115}$$

Our proof will be in two stages, the first part is by direct estimation. Namely, *Schur's lemma* gives a useful criterion for an integral operator to be bounded on L^2.

LEMMA 2.9 (Schur). *[5.50] If $K(x,y)$ is locally integrable on $\mathbb{R}^{2n}$ and is such that*

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|dy,\ \sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|dx < \infty[5.51] \tag{2.116}$$

then the operator $K : \phi \longmapsto \int_{\mathbb{R}^n} K(x,y)\phi(y)dy$ is bounded on $L^2(\mathbb{R}^n)$.

PROOF. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (see Problem) we only need to show the existence of a constant, C, such that

$$\int \mid K\phi(x) \mid^2 dx \le C\int |\phi|^2 \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).[6.4] \tag{2.117}$$

Writing out the integral on the left

$$\begin{aligned}&\int \mid \int K(x,y)\phi(y)dy \mid^2 dx\\ &= \iiint K(x,y)\overline{K(x,z)}\phi(y)\overline{\phi(z)}\ dydzdx\end{aligned} \quad [6.5] \tag{2.118}$$

is certainly absolutely convergent and

$$\begin{aligned}&\int \mid K\phi(x) \mid^2 dx\\ &\le \left(\iiint \mid K(x,y)K(x,z) \mid\mid \phi(y) \mid^2 dydxdz\right)^{\frac12}\\ &\times \left(\iiint \mid K(x,y)K(x,z) \mid\mid \phi(z) \mid^2 \ dzdxdy\right)^{\frac12}.\end{aligned}$$

These two factors are the same. Changing variables to $y, t = x - y$ and $x - z = s$ gives

$$\int \mid K\phi \mid^2 dx \le \sup_y \iint \mid K(t+y,y) \mid\mid K(t+y,t+y-s) \mid dtds \cdot \int |\phi|^2 dy. \tag{2.119}$$

Thus (2.116) gives (2.117). ∎

This standard lemma immediately implies the L^2 boundedness of the 'residual terms.' Thus, if $K \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ then its kernel satisfies

$$K \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \Longleftrightarrow \Big| \sup_{x,y} |(x-y)^\alpha D_x^\beta D_y^\gamma K(x,y) \Big| < \infty \quad \forall\, \alpha,\beta,\gamma. \text{[6.3]} \tag{2.120}$$

This in particular implies

$$\Big| K(t+y,y) \Big| \le C\,(1+|t|)^{-n-1}\,,$$
$$\Big| K\,(t+y,t+y-s) \Big| \le C\,(1+|s|)^{-n-1}$$

and hence that K satisfies (2.116). Thus

$$\text{each } K \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \text{ is bounded on } L^2(\mathbb{R}^n). \text{[5.52]} \tag{2.121}$$

2.12. Square root

To prove the general result, (2.115), we shall use the clever idea, due to Hörmander, of using the (approximate) square root of an operator. We shall say that an element $[a] \in S_\infty^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)$ is positive if there is some $0 < a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ in the equivalence class.

PROPOSITION 2.6. *[6.6] Suppose $A \in \Psi_\infty^m(\mathbb{R}^n)$ is self-adjoint, $A = A^*$, and elliptic with a positive principal symbol, then there exists $B \in \Psi_\infty^{m/2}(\mathbb{R}^n)$, $B = B^*$, such that*

$$A = B^2 + G, \qquad G \in \Psi_\infty^{-\infty}(\mathbb{R}^n). \text{[6.7]} \tag{2.122}$$

PROOF. This is a good exercise in the use of the symbol calculus. Let $a \in S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)$, $a > 0$, be a positive representative of the principal symbol of A. Now

$$b_0 = a^{\frac12} \in S_\infty^{m/2}(\mathbb{R}^n;\mathbb{R}^n). \text{[6.8]} \tag{2.123}$$

Let $B_0 \in \Psi_\infty^{m/2}(\mathbb{R}^n)$ have principal symbol b_0. We can assume that $B_0 = B_0^*$, since if not we just replace it by $\frac12(B_0 + B_0^*)$ which has the same principal symbol.

The symbol calculus shows that $B_0^2 \in \Psi_\infty^m(\mathbb{R}^n)$ and

$$\sigma_m(B_0^2) = \big(\sigma_{m/2}(B_0)\big)^2 = b_0^2 = a_0 \mod S_\infty^{m-1}.$$

Thus

$$A - B_0^2 = E_1 \in \Psi_\infty^{m-1}(\mathbb{R}^n). \text{[6.9]} \tag{2.124}$$

Then we proceed inductively. Suppose we have chosen $B_j \in \Psi_\infty^{m/2-j}(\mathbb{R}^n)$, with $B_j^* = B_j$, for $j \le N$ such that

$$A - \left(\sum_{j=0}^N B_j\right)^2 = E_{N+1} \in \Psi_\infty^{m-N-1}(\mathbb{R}^n). \text{[6.10]} \tag{2.125}$$

Of course we *have* done this for $N = 0$. Then see the effect of adding $B_{N+1} \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$:

$$A - \left(\sum_{j=0}^{N+1} B_j\right)^2 = E_{N+1} - \left(\sum_{j=0}^{N} B_j\right) B_{N+1} \tag{2.126}$$
$$-B_{N+1}\left(\sum_{j=0}^{N} B_j\right) - B_{N+1}^2. \quad [6.11]$$

On the right side all terms are of order $m - N - 2$, except for

$$E_{N+1} - B_0 B_{N+1} - B_{N+1} B_0 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n).[6.12] \tag{2.127}$$

wThe principal symbol, of order $m - N - 1$, of this is just

$$\sigma_{m-N-1}(E_{N+1}) - 2\ b_0 \cdot \sigma_{\frac{m}{2}-N-1}(B_{N+1}).[6.13] \tag{2.128}$$

Thus if we choose

$$\sigma_{m/2-N-1}(B_{N+1}) = \frac{1}{2}\frac{1}{b_0} \cdot \sigma_{m-N-1}(E_{N+1})$$

and replace B_{N+1} by $\frac{1}{2}(B_{N+1} + B_{N+1}^*)$, we get the inductive hypothesis for $N+1$. Thus we have arranged (2.125) for every N. Now define $B = \frac{1}{2}(B' + (B')^*)$ where

$$\sigma_L(B') \sim \sum_{j=0}^{\infty} \sigma_L(B_j).[6.14] \tag{2.129}$$

Since all the B_j are self-adjoint B also satisfies (2.129) and from (2.125)

$$A - B^2 = A - \left(\sum_{j=0}^{N} B_j + B_{(N+1)}\right)^2 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n) \tag{2.130}$$

for every N, since $B_{(N+1)} = B - \sum_{j=0}^{N} B_j \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$. Thus $A - B^2 \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ and we have proved (2.122), and so Proposition 2.6. $\square$

2.13. Proof of Proposition 2.5

Here is Hörmander's clever argument. We want to show that

$$\|A\phi\| \le C\|\phi\| \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n)[6.15] \tag{2.131}$$

where $A \in \Psi_\infty^0(\mathbb{R}^n)$. The square of the left side can be written

$$\int A\phi \cdot \overline{A\phi} dx = \int \phi \cdot (A^* A\phi)\, dx.$$

So it suffices to show that

$$\langle \phi, A^* A\phi \rangle \le C\|\phi\|^2.[6.16] \tag{2.132}$$

Now $A^*A \in \Psi_\infty^0(\mathbb{R}^n)$ with $\sigma_0(A^*A) = \overline{\sigma_0(A)}\sigma_0(A) \in \mathbb{R}$. If $C > 0$ is a large constant,

$$C > \sup_{x,\xi} \left| \ \sigma_L(A^*A)(x,\xi)\ \right|$$

then $C-A^*A$ has a positive representative of its principal symbol. We can therefore apply Proposition 2.6 to it:

$$C - A^*A = B^*B + G, \quad G \in \Psi_\infty^{-\infty}(\mathbb{R}^n).\text{[6.17]} \tag{2.133}$$

This gives

$$\begin{aligned}\langle \phi, A^*A\phi\rangle =& C\langle \phi,\phi\rangle - \langle \phi, B^*B\phi\rangle - \langle\phi, G\phi\rangle \\ =& C\|\phi\|^2 - \|B\phi\|^2 - \langle \phi, G\phi\rangle.\end{aligned}\text{[6.18]} \tag{2.134}$$

The second term on the right is negative and, since $G \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$, we can use the residual case discussed above to conclude that

$$\mid \langle \phi, G\phi\rangle \mid \le C'\|\phi\|^2 \Longrightarrow \|A\phi\|^2 \le C\|\phi\|^2 + C'\|\phi\|^2,$$

so (2.115) holds and Proposition 2.5 is proved.

2.14. Sobolev boundedness

Using the basic boundedness result, Proposition 2.5, and the calculus of pseudodifferential operators we can prove more general results on the action of pseudodifferential operators on Sobolev spaces.

Recall that for any positive integer, k,

$$H^k(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n);\ D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \le k\right\}.\text{[6.19]} \tag{2.135}$$

Using the Fourier transform we find

$$u \in H^k(\mathbb{R}^n) \Longrightarrow \xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n) \quad \forall\ |\alpha| \le k.\text{[6.20]} \tag{2.136}$$

Now these finitely many conditions can be written just

$$\left(1+|\xi|^2\right)^{k/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n).\text{[6.21]} \tag{2.137}$$

Notice that $a(\xi) = (1+|\xi|^2)^{k/2} = \langle\xi\rangle^k \in S_\infty^k(\mathbb{R}^n)$. Here we use the notation

$$\langle\xi\rangle = \left(1+|\xi|^2\right)^{\frac{1}{2}}\text{[6.22]} \tag{2.138}$$

for a smooth (symbol) of the size of $1+|\xi|$, thus (2.137) just says

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n).\text{[6.23]} \tag{2.139}$$

For *negative* integers

$$H^k(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{|\beta|\le -k} D^\beta u_\beta,\ u_\beta \in L^2(\mathbb{R}^n)\right\},\ -k \in \mathbb{N}.\text{[6.24]} \tag{2.140}$$

The same sort of discussion applies, showing that

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n).\text{[6.25]} \tag{2.141}$$

In view of this we define the Sobolev space $H^m(\mathbb{R}^n)$, for any real order, by

$$u \in H^k,(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^m u \in L^2(\mathbb{R}^n).\text{[6.26]} \tag{2.142}$$

It is a Hilbert space with

$$\|u\|_m^2 = \|\langle D\rangle^m u\|_{L^2}^2 = \int (1+|\xi|^2)^m|\hat{u}(\xi)|^2 d\xi.\text{[6.27]} \tag{2.143}$$

Clearly we have

$$H^m(\mathbb{R}^n) \supseteq H^{m'}(\mathbb{R}^n) \text{ if } m' \ge m.\text{[6.28]} \tag{2.144}$$

Notice that it is rather unfortunate that these spaces get *smaller* as m gets bigger, as opposed to the spaces $\Psi^m_\infty(\mathbb{R}^n)$ which get *bigger* with m. Anyway that's life and we have to think of

$$\begin{cases} H^\infty(\mathbb{R}^n) = \bigcap_m H^m(\mathbb{R}^n) & \text{as the residual space} \\ H^{-\infty}(\mathbb{R}^n) = \bigcup_m H^m(\mathbb{R}^n) & \text{as the big space.} \end{cases} \quad [6.29] \tag{2.145}$$

It is important to note that

$$\mathcal{S}(\mathbb{R}^n) \subsetneq H^\infty(\mathbb{R}^n) \subsetneq H^{-\infty}(\mathbb{R}^n) \subsetneq \mathcal{S}'(\mathbb{R}^n).[6.30] \tag{2.146}$$

In particular we do *not* capture all the tempered distributions in $H^{-\infty}(\mathbb{R}^n)$. We therefore consider *weighted* versions of these Sobolev spaces:

$$\langle x\rangle^q H^m(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle x\rangle^{-q} u \in H^m(\mathbb{R}^n)\right\}.[6.31] \tag{2.147}$$

THEOREM 2.4. *[6.32] For each $q, m, M \in \mathbb{R}$ each $A \in \Psi^M_\infty(\mathbb{R}^n)$ defines a continuous linear map*

$$A : \langle x\rangle^q H^m(\mathbb{R}^n) \longrightarrow \langle x\rangle^q H^{m-M}(\mathbb{R}^n).[6.33] \tag{2.148}$$

PROOF. Let us start off with $q = 0$, so we want to show that

$$A : H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n),\ A \in \Psi^M_\infty(\mathbb{R}^n)[6.34] \tag{2.149}$$

Now from (2.142) we see that

$$\begin{aligned} &u \in H^m(\mathbb{R}^n) \iff \langle D\rangle^m u \in L^2(\mathbb{R}^n) \\ &\iff \langle D\rangle^{m-M}\langle D\rangle^M u \in L^2(\mathbb{R}^n) \iff \langle D\rangle^M u \in H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M.[6.35] \end{aligned} \tag{2.150}$$

That is,

$$\langle D\rangle^M : H^m(\mathbb{R}^n) \longleftrightarrow H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M.[6.36] \tag{2.151}$$

To prove (2.149) it suffices to show that

$$B = \langle D\rangle^{-M+m} \cdot A \cdot \langle D\rangle^{-m} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)[6.37] \tag{2.152}$$

since then $A = \langle D\rangle^{-m+M} \cdot B \cdot \langle D\rangle^m$ maps $H^m(\mathbb{R}^n)$ to $H^{m-M}(\mathbb{R}^n)$:

$$\begin{array}{ccc} H^m(\mathbb{R}^n) & \xrightarrow{A} & H^{m-M}(\mathbb{R}^n) \quad [6.38] \\ {\scriptstyle \langle D\rangle^m}\big\downarrow & & \big\downarrow{\scriptstyle \langle D\rangle^{m-M}} \\ L^2(\mathbb{R}^n) & \xrightarrow[B]{} & L^2(\mathbb{R}^n). \end{array} \tag{2.153}$$

Since $B \in \Psi^0_\infty(\mathbb{R}^n)$, by the composition theorem, we already know (2.152).

Thus we have proved (2.149). To prove the general case, (2.148), we proceed in the same spirit. Thus $\langle x\rangle^q$ is an isomorphism from $H^m(\mathbb{R}^n)$ to $\langle x\rangle^q H^m(\mathbb{R}^n)$, by definition. So to get (2.148) we need to show that

$$Q = \langle x\rangle^{-q} \cdot A \cdot \langle x\rangle^q :\ H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n), [6.39] \tag{2.154}$$

i.e. satisfies (2.149). Consider the Schwartz kernel of Q. Writing A in left-reduced form, with symbol a,

$$K_Q(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \langle x\rangle^{-q} a(x, \xi) d\xi \cdot \langle y\rangle^q.[6.40] \tag{2.155}$$

Now if we check that

$$\langle x\rangle^{-q}\langle y\rangle^{q}a(x,\xi)\in\left(1+|x-y|^2\right)^{\frac{|q|}{2}}S^M_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right)[6.41] \tag{2.156}$$

then we know that $Q\in\Psi^M_\infty(\mathbb{R}^n)$ and we get (2.154) from (2.149). Thus we want to show that

$$\langle x-y\rangle^{-|q|}\frac{\langle y\rangle^q}{\langle x\rangle^q}a(x,\xi)\in S^M_\infty(\mathbb{R}^{2n};\mathbb{R}^n)[6.42] \tag{2.157}$$

assuming of course that $a(x,\xi)\in S^M_\infty(\mathbb{R}^n;\mathbb{R}^n)$. By interchanging the variables x and y if necessary we can assume that $q<0$. Consider separately the two regions

$$\begin{aligned}\left\{(x,y);|x-y|<\frac{1}{4}(|x|+|y|)\right\}&=\Omega_1\\ \left\{(x,y);|x-y|>\frac{1}{8}(|x|+|y|)\right\}&=\Omega_2.\end{aligned}[6.43] \tag{2.158}$$

In Ω_1, x is "close" to y, in the sense that

$$|x|\le|x-y|+|y|\le\frac{1}{4}(|x|+|y|)+|y|\Longrightarrow|x|\le\frac{4}{3}\cdot\frac{5}{4}|y|\le 2|y|. \tag{2.159}$$

Thus

$$\langle x-y\rangle^{-q}\cdot\frac{\langle x\rangle^{-q}}{\langle y\rangle^{-q}}\le C\ \text{ in }\Omega_1.[6.44] \tag{2.160}$$

On the other hand in Ω_2,

$$|x|+|y|<8|x-y|\Longrightarrow|x|<8|x-y| \tag{2.161}$$

so again

$$\langle x-y\rangle^{-q}\frac{\langle x\rangle^{-q}}{\langle y\rangle^{-q}}\le C.[6.45] \tag{2.162}$$

In fact we easily conclude that

$$\langle x-y\rangle^{-q}\frac{\langle y\rangle^q}{\langle x\rangle^q}\in\mathcal{C}^\infty_\infty(\mathbb{R}^n)\quad\forall\ q,[6.46] \tag{2.163}$$

since differentiation by x or y makes all terms "smaller". This proves (2.157), hence (2.156) and (2.154) and therefore (2.148), i.e. the theorem is proved. □

2.15. Consequences

We can capture any tempered distribution in a weighted Sobolev space; this is really Schwartz' representation theorem which says that any $u\in\mathcal{S}'(\mathbb{R}^n)$ is of the form

$$u=\sum_{\text{finite}}x^\alpha D^\beta_x u_{\alpha\beta},\ u_{\alpha\beta}\text{ bounded and continuous.} \tag{2.164}$$

Clearly $\mathcal{C}^0_\infty(\mathbb{R}^n)\subset\langle x\rangle^{1+n}L^2(\mathbb{R}^n)$. Thus as a special case of Theorem 2.4,

$$D^\beta_x:\langle x\rangle^{1+n}L^2(\mathbb{R}^n)\longrightarrow\langle x\rangle^{1+n}H^{-|\beta|}(\mathbb{R}^n)$$

so

Lemma 2.10. *[6.47]*

$$\mathcal{S}'(\mathbb{R}^n)=\bigcup_M\langle x\rangle^M H^{-M}(\mathbb{R}^n). \tag{2.165}$$

The elliptic regularity result we found before can now be refined:

PROPOSITION 2.7. *[6.48] If $A \in \Psi_\infty^m(\mathbb{R}^n)$ is elliptic then*

$$\begin{aligned} & Au \in \langle x\rangle^p H^q(\mathbb{R}^n),\ u \in \langle x\rangle^{p'} H^{q'}(\mathbb{R}^n) \\ & \Longrightarrow u \in \langle x\rangle^{p''} H^{q''}(\mathbb{R}^n),\ p'' = \max(p,p'),\ q'' = \max(q+m,q'). \end{aligned} \tag{2.166}$$ [6.49]

PROOF. The existence of a left parametrix for A, $B \in \Psi_\infty^{-m}(\mathbb{R}^n)$,

$$B \cdot A = \mathrm{Id} + G,\ G \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$$

means that

(2.167)

$$u = B(Au) + Gu \in \langle x\rangle^p H^{q+m}(\mathbb{R}^n) + \langle x\rangle^{p'} H^\infty(\mathbb{R}^n) \subset \langle x\rangle^{p''} H^{q+m}(\mathbb{R}^n).$$ [6.50]

□

2.16. Isotropic calculus

As noted above, there are other sensible choices of the class of amplitudes which should be admitted in the definition of a space of pseudodifferential operators. One of the smallest such choices is the class which is completely symmetric in the variables x and ξ and consists of the symbols on $\mathbb{R}^{2n}$. Thus, $a \in S^m(\mathbb{R}^{2n}_{x,\xi})$ satisfies the estimates

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^{m-|\alpha|-|\beta|}$$

for all multiindices α and β. If $m \le 0$ this is in the space $S^m(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$; if $m > 0$ it is not, however,

$$S^m(\mathbb{R}^{2n}) \subset \begin{cases} \bigcap\limits_{0\ge r\ge m} (1+|x|^2)^r S^{m-r}(\mathbb{R}^n_x;\mathbb{R}^n_\xi) & m \le 0 \\ (1+|x|^2)^{m/2} S^m(\mathbb{R}^n_x;\mathbb{R}^n_\xi), & m > 0. \end{cases} \tag{2.168}$$

These inclusions, plus a little reinterpretation of earlier results, leads us to

THEOREM 2.5. *The spaces $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ of isotropic pseudodifferential operators on $\mathbb{R}^n$, with kernels given by (2.2) with amplitudes $a \in S^m(\mathbb{R}^{2n})$ form an order-filtered $*$-algebra with residual space $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n})$ as spaces of kernels.*

PROOF. In fact the main novelty is in the residual terms. The condition that a continuous linear operator A on $\mathcal{S}(\mathbb{R}^n)$ be am element of $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ is that it be an element of $(1+|x|^2)^{m/2}\Psi^m_\infty(\mathbb{R}^n)$ if $m \ge 0$ or $\Psi^m_\infty(\mathbb{R}^n)$ if $m < 0$ with left-reduced symbol an element of $S^m(\mathbb{R}^{2n}_{x,\xi})$:

$$q_l : S^m(\mathbb{R}^{2n}) \longleftrightarrow \Psi^m_{\mathrm{iso}}(\mathbb{R}^n). \tag{2.169}$$

Consider the right-reduced symbol of this operator. Notice that if $m > 0$ then, by the conjugation invariance shown above, $A = A'(1|x|^2)^{m/2}$ with $A' \in \Psi^m_\infty(\mathbb{R}^n)$. In either case it follows that the right-reduced symbol has an asymptotic expansion, as $\xi \to \infty$ given by the usual formula, namely the inverse of (2.75). Each of the terms in this expansion

$$a_R(A) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D_x^\alpha D_\xi^\alpha a(x,\xi)$$

is in the space $S^{m-2|\alpha|}(\mathbb{R}^{2n})$. Thus we can actually choose an asymptotic sum in the stronger sense that

$$b \in S^m(\mathbb{R}^{2n}),\ b_N = b - \sum |\alpha| < N\frac{(-i)^|\alpha|}{\alpha!} D_x^\alpha D_\xi^\alpha a(x,\xi) \in S^{m-2N}(\mathbb{R}^{2n})\ \forall\ N.$$

In view of (2.168) this means that b and $\sigma_R(A)$ differ by a term in $(1+|x|^2)^{m+/2}S^m(\mathbb{R}^n;\mathbb{R}^n)$. However, consider the difference between A and the operator B with right-reduced symbol b. □

2.17. Problems

PROBLEM 2.1. [P3.1] Show, in detail, that for each $m \in \mathbb{R}$

$$(1+|\xi|^2)^{-\frac{1}{2}m} \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \tag{2.170}$$

for any p. Use this to show that

$$S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \cdot S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) = S^{m+m'}_\infty(\mathbb{R}^p;\mathbb{R}^n).$$

PROBLEM 2.2. [P3.2] Consider $p = 0$ and $n = 2$ in the definition of symbols and show that if $a \in S^1_\infty(\mathbb{R}^2)$ is elliptic then for $r > 0$ sufficiently large the integral

$$\int_0^{2\pi} \frac{1}{2\pi}\frac{1}{a(re^{i\theta})}\frac{d}{d\theta}a(re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{d}{d\theta}\log a(re^{i\theta})d\theta,$$

exists and is an integer independent of r, where $z = \xi_1 + i\xi_2$ is the complex variable in $\mathbb{R}^2 = \mathbb{C}$. Conclude that there is an elliptic symbol, a on $\mathbb{R}^2$, such that there does not exist b, a symbol with

$$b \neq 0 \text{ on } \mathbb{R}^2 \text{ and } a(\xi) = b(\xi) \text{ for } |\xi| > r \tag{2.171}$$

for any r.

PROBLEM 2.3. [P3.3] Show that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies all the estimates

$$|D_x^\alpha D_y^\beta a(x,y)| < C_{\alpha,\beta,N}(1+|x-y|)^{-N} \tag{2.172}$$

for multiindices $\alpha, \beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$.

Possible Changes

Should add some more problems. Charlie says A is $\mathcal{C}^\infty$ off diagonal.

So, what was accomplished in the previous chapter? We showed that there is an 'oscillatory integral' interpretation of

$$K(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi = I(a)[5.1] \tag{2.173}$$

which defines, for any p, a continuous linear map

$$I\ (1+|x-y|^2)^{\frac{p}{2}}S^\infty_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

the range of which is the space of pseudodifferential operators on $\mathbb{R}^n$;

$$\begin{gathered} A \in \Psi^m_\infty(\mathbb{R}^n) \Longleftrightarrow A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ and} \\ \exists\ p \text{ s.t. } K_A(x,y) = I(a),\ a \in \left(1+|x-y|^2\right)^{\frac{p}{2}} S^m_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right). \end{gathered} [5.2] \tag{2.174}$$

We then showed in Proposition 2.2 that the special case, $p = 0$ and $\partial_y a \equiv 0$, gives an isomorphism

$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_L}{\overset{\sigma_L}{\rightleftarrows}} S^m_\infty(\mathbb{R}^n;\mathbb{R}^n).[5.3] \tag{2.175}$$

The maps here, $q_L = I$, is the *left quantization map* and its inverse σ_L is the *left full symbol map.* We also showed that $\Psi^\infty_\infty(\mathbb{R}^n)$ forms a filtered $*$-algebra. Next shall consider invertibility, modulo $\Psi^{-\infty}_\infty(\mathbb{R}^n)$. First we consider some more consequences of the reduction theorem.