Calderón projector for manifolds with corners

Michael Taylor's 65th birthday meeting

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17 July, 2012

Outline

On a compact manifold with boundary the Calderón projector is a very convenient way to capture the boundary behaviour of an elliptic differential operator. In this talk I will describe (to some degree conjecturally) how this can be extended to the case of an elliptic differential operator on a compact manifold with corners. Much of this material arose in discussions with András Vasy.

- Suppose given a linear, elliptic differential operator with smooth coefficients on a compact manifold with corners, $D : \mathcal{C}^\infty(M; V_+) \longrightarrow \mathcal{C}^\infty(M; V_-)$.
- I have in mind examples such as $\partial + \delta$, the Hodge operator, $\overline{\partial} + \overline{\partial}^*$ on complex manifolds and more generally Dirac operators.
- I will assume that all bundles carry inner products and that the manifold has a given smooth positive density.
- In particular D has a formal adjoint $D^* : \mathcal{C}^\infty(M; V_-) \longrightarrow \mathcal{C}^\infty(M : V_+)$.

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- If M is a compact manifold without boundary, then such an operator is Fredholm and for the distributional action there is a short exact sequence

$$\partial M = \emptyset \Longrightarrow$$

$$\mathrm{Nul}(D; \mathcal{C}^{\infty}) \longrightarrow \mathcal{C}^{-\infty}(M; V_+) \xrightarrow{D} \mathcal{C}^{-\infty}(M; V_-) \longrightarrow \mathrm{Nul}(D^*; \mathcal{C}^{\infty})$$

is exact where

$$\mathrm{Nul}(D; \mathcal{C}^{\infty}) = \{u \in \mathcal{C}^{\infty}(M; V_+); Du = 0\}.$$

- Of course, by ellipticity $\mathrm{Nul}(D; \mathcal{C}^{\infty})$ is equal to $\mathrm{Nul}(D; \mathcal{C}^{-\infty})$, the null space on distributions (and is finite dimensional)

- The extension to the case of a compact manifold with boundary follows the idea of Calderón.

$$M \text{ a compact manifold with boundary } \Longrightarrow$$

$$\begin{array}{ccccc}
\mathrm{Nul}(D;\dot{\mathcal{C}}^{\infty}) & & & & \\
\downarrow & & & & \\
\mathrm{Nul}(D;\mathcal{C}^{-\infty}(M)) & \longrightarrow & \mathcal{C}^{-\infty}(M;V_+) & \xrightarrow{D} & \mathcal{C}^{-\infty}(M;V_-) \\
\downarrow B & & & & \downarrow \\
\mathrm{Ran}(\Pi_C;\mathcal{C}^{-\infty}) & & & & \mathrm{Nul}(D^*;\dot{\mathcal{C}}^{\infty})
\end{array}$$

$$\mathrm{Nul}(D;\dot{\mathcal{C}}^{\infty}) = \{u \in \dot{\mathcal{C}}^{\infty}(M;V_+); Du = 0\} \text{ (finite dimensional).}$$

- Here $\dot{\mathcal{C}}^{\infty}(M;V)$ is the space of smooth sections of the vector bundle V which vanish to infinite order at the boundary of M, Π_C is the Calderón projector and B is restriction to the boundary.

- If we assume a unique continuation property for D so $\mathrm{Nul}(D;\dot{\mathcal{C}}^\infty)=\{0\}$ this can be written more succinctly in the form

$$M \text{ with boundary plus unique continuation } \Longrightarrow$$

$$\begin{array}{ccccc}\mathrm{Ran}(\Pi_C;\mathcal{C}^{-\infty}) & \xrightarrow{P} & \mathcal{C}^{-\infty}(M;V_+) & \xrightarrow{D} & \mathcal{C}^{-\infty}(M;V_-) \\ & & & & \downarrow \\ & & & & \mathrm{Nul}(D^*;\dot{\mathcal{C}}^\infty).\end{array}$$

- Where now P is the Poission operator – the inverse of restriction from the null space. I will remind you of the properties of the Calderón projector later.
- In particular D is semi-Fredholm (with or without unique continuation).

- What I want to try to convince you of (the likelihood of, since it is partly conjectural) is the existence of such a picture in the case of a compact manifold with corners. The right half is the same, but what we want is a (generalized) boundary map with a generalized Calderón projector:

- $$M \text{ a compact manifold with corners} \implies \text{(maybe)}$$

$$\begin{array}{ccccc}
\mathrm{Nul}(D;\dot{\mathcal{C}}^{\infty}) & & & & \\
\downarrow & & & & \\
\mathrm{Nul}(D;\mathcal{C}^{-\infty}(M)) & \longrightarrow & \mathcal{C}^{-\infty}(M;V_+) & \xrightarrow{D} & \mathcal{C}^{-\infty}(M;V_-) \\
\downarrow B & & & & \downarrow \\
\mathrm{Ran}(\Pi_C;\mathcal{C}^{-\infty}_D) & & & & \mathrm{Nul}(D^*;\dot{\mathcal{C}}^{\infty})
\end{array}$$

- So the whole issue is to define B and Π_C but also the meaning of the little subscript D.

The case where I can do all this is somewhat special – but problably not significantly so.

- Namely work with Dirac operators throughout – but only because of the unique continuation property.
- Also I shall assume that there is a compact manifold without boundary X in which M is embedded as the closure of an open set.
- Furthermore assume that D has an extension to an elliptic differential operator on X which is globally invertible:

$$M \hookrightarrow X,\ D : \mathcal{C}^{-\infty}(X; V_+) \longrightarrow \mathcal{C}^{-\infty}(X; V_-),$$
$$D^{-1} \in \Psi^{-1}(X; V_-, V_+).$$

- The inverse over X is then an elliptic pseudodifferential operator of order -1.

Under this same extension assumption, the theory in the case of a manifold with boundary is straightforward.

- Consider the null space on extendible distributions on M

$$\begin{aligned} \mathrm{Nul}(D;\mathcal{C}^{-\infty}) &= \{u \in \mathcal{C}^{-\infty}(M;V_+); Du = 0\}, \\ \mathcal{C}^{-\infty}(M;V_+) &= (\dot{\mathcal{C}}^{\infty}(M;V_+))'. \end{aligned} \tag{1}$$

- Partial hypoellipticity up to the boundary implies that the restriction to the boundary is well-defined (as are higher normal derivatives),

$$N(D;\mathcal{C}^{-\infty}) \ni u \longmapsto Bu = u\big|_{\partial M} \in \mathcal{C}^{-\infty}(\partial M;V_+). \tag{2}$$

- The 'jumps formula' is also a consequence of this:- There is a unique $v \in \mathcal{C}^{-\infty}(X;V_+)$ such that

$$\begin{aligned} v = 0 \text{ in } X \setminus M,\ & v = u \text{ on } M \setminus \partial M \\ Dv = w\delta(\rho) &\text{ and } w = L(Bu). \end{aligned} \tag{3}$$

Here L is a bundle map over ∂M – essentially the symbol of D along the conormal direction and $\delta(\rho)$ is the delta function in the normal direction.

- Now we can get the (or a) Calderón projector,

$$\Pi_C \in \Psi^0(\partial M; V_+),$$
$$\Pi_C^2 = \Pi_C,\ \operatorname{Ran}(\Pi_C) = B(\operatorname{Nul}(D; \mathcal{C}^{-\infty})).$$

- Using the various maps discussed above gives

$$\begin{array}{ccccc}
\mathcal{C}^{-\infty}(\partial M; V_+) \ni w & & \xrightarrow{\quad \Pi_C \quad} & & \mathcal{C}^{-\infty}(\partial M; V_+) \\
\downarrow & & & & \uparrow B \\
\mathcal{C}^{-\infty}(X; V_-) \ni (Lw)\delta(\rho) & \xrightarrow{D^{-1}} & \mathcal{C}^{-\infty}(M; V_+) & \xrightarrow{\big|M\backslash\partial M} & \operatorname{Nul}(D; \mathcal{C}^{-\infty})
\end{array}$$

- In the case of corners, the first problem is that partial hypoellipticity does not extend to higher codimension so B is not defined directly.
- Nevertheless, the main idea I want to convey is that it really is possible to define the boundary values of a solution u to $Du = 0$.
- One thing that stays the same, is that on a manifold with corners there are two natural spaces of distributions – the 'supported' and the 'extendible' distributions (recall I am trivializing densities):

$$\begin{aligned} \text{Extendible } \mathcal{C}^{-\infty}(M;V) &= (\dot{\mathcal{C}}^{\infty}(M;V))' \\ \text{Supported } \dot{\mathcal{C}}^{-\infty}(M;V) &= (\mathcal{C}^{\infty}(M;V))'. \end{aligned} \tag{1}$$

- The first of these is the subspace of distributions on the interior which are 'of polynomial growth' at the boundary.
- The second is the subspace of the distributions on X – any extension – which vanish outside M.
- There is a surjective restriction map

$$\dot{\mathcal{C}}^{-\infty}(M;V) \longrightarrow \mathcal{C}^{-\infty}(M;V) \tag{2}$$

with null space the distributions supported by the boundary.

- We consider the same space of solutions on the interior of M
$$\mathrm{Nul}(D;\mathcal{C}^{-\infty}) = \{u \in \mathcal{C}^{-\infty}(M;V_+); Du = 0\}.$$
- So any element has a 'zero extension' $v \in \dot{\mathcal{C}}^{-\infty}(M;V_+)$ but there are many.
- To see that there is a 'preferred' class of extensions, observe that there are now 'delta' distributions along each boundary hypersurface.
- Each hypersurface has a defining function ρ_j and whenever any collection of them vanishes (so the hypersurfaces meet) they have independent differentials. Then each $H_j \subset M$ is itself a manifold with corners and the span is a well-defined subspace
$$\mathcal{B} = \sum_j w_j \delta(\rho_j) \in \dot{\mathcal{C}}^{-\infty}(X;V_+),\ w_j \in \dot{\mathcal{C}}^{-\infty}(H_j;V_+). \tag{3}$$
- However, there is a problem in that w_j might have a term $h\delta(\rho_k)$ at $H_j \cap H_k$ and this can be 'shifted' over into $w_j\delta(\rho_k)$, so the *presentation* in (3) is not unique.

Lemma

For an elliptic first order operator $D \in \mathrm{Diff}^1(M; V_+, v_-)$ *on a compact manifold with corners set*

$$\mathrm{Nul}_\delta(D; \mathcal{C}^{-\infty}) = \left\{ v \in \dot{\mathcal{C}}^{-\infty}(M; V); Dv \in \mathcal{B} \right\}$$
$$\mathrm{Nul}_{\partial M}(D) = \{ v \in N_\delta(D; \mathcal{C}^{-\infty}); \mathrm{supp}(v) \subset \partial M \}.$$

then there is a short exact sequence

$$\mathrm{Nul}_{\partial M}(D) \hookrightarrow \mathrm{Nul}_\delta(D; \mathcal{C}^{-\infty}) \xrightarrow{\ \big|_M\ } N(D; \mathcal{C}^{-\infty})$$

and if $\partial_{(2)} M \subset \partial M$ *is the part of the boundary of codimension two or higher.*

$$N_{\partial M}(D) \subset \{ v \in \dot{\mathcal{C}}^{-\infty}(M; V); \mathrm{supp}(v) \subset \partial_{(2)} M \}.$$

- Restating this, each element of $\mathrm{Nul}(D; \mathcal{C}^{-\infty})$ does have an extension v which is zero outside M and is such that

$$Dv = \sum_j v_j \delta(\rho_j). \tag{4}$$

- So, we wish to define the 'normalized' (because there are bundle coefficients) boundary values from the collection of $v_j \in \dot{\mathcal{C}}^{-\infty}(H_j; V_-)$, but these are not unique; naturally we are obliged to ask

Just how non-unique are the $v_j \in \dot{\mathcal{C}}^{-\infty}(H_j; V_-)$*?*

- To answer this we now switch to the 'formal smooth theory'.

- Now think of ∂M as an articulated manifold – the union of the boundary hypersurfaces with only their boundaries identified in the obvious way. I call these 'articulated' since the angles of approach of the boundary hypersurfaces have been lost. We define a Fréchet space which is just the space of boundary values of smooth sections

$$\mathcal{C}^\infty(\partial M; V) = \{u_i \in \mathcal{C}^\infty(H_i; V); u_i\big|_{H_i\cap H_j} = u_j\big|_{H_i\cap H_j}\} = \mathcal{C}^\infty(M; V)\big|_{\partial M}.$$

- Note that there are no compatibility conditions for the *normal derivatives* at intersections of boundary faces.
- However, a first order elliptic differential operator, gives rise to a subspace of 'compatible' sections

$$\mathcal{C}^\infty_D(\partial M; V_+) = \{u \in \mathcal{C}^\infty(M; V_+); Du \in \dot{\mathcal{C}}^\infty(M; V_+)\}\big|_{\partial M} \subset \mathcal{C}^\infty(\partial M; V_+).$$

Lemma

For an elliptic differential operator on a compact manifold with corners $D \in \mathrm{Diff}^1(M; V_+, V_-)$ restriction to any one of the of the boundary hypersurfaces defines a surjective map

$$\mathcal{C}^\infty_D(\partial M; V_+) \xrightarrow{\big|_H} \mathcal{C}^\infty(H; V_+),\ H \in \mathcal{M}_1(M),$$

and there is a natural extension giving an injective map

$$\bigoplus_{H \in \mathcal{M}_1(M)} \dot{\mathcal{C}}^\infty(H; V_+) \hookrightarrow \mathcal{C}^\infty_D(\partial M; V_+).$$

- This result means that the space $\mathcal{C}^\infty_D(\partial M; V_+)$ 'looks' like $\mathcal{C}^\infty(\tilde{H}; V)$ for a smooth gluing of all the boundary hypersurfaces of ∂M to a compact manifold without boundary $\tilde{H}$, in the sense that once you know the Taylor series at any point (where two or more boundary hypersurfaces meet) coming from one boundary hypersurface, you know it on all the others – effectively there is only one Taylor series at each point.
- However, this new space is not an algebra, since the definition is not multiplicative, nor is it even a module of $\sum_j \mathcal{C}^\infty(H_j)$. On the other hand, it does have a topology very similar to that of $\mathcal{C}^\infty(\tilde{H}; V)$ such that the maps in (2) are continuous, even though (2) does not have a linear right inverse. So, the dual space, which we consider next, is similar to the space of distributional sections of V' over $\tilde{H}$.

Lemma

The topological dual $(\mathcal{C}^\infty_D(\partial M; V_+))'$ *comes equipped with a natural surjection to extendible distributions on the boundary hypersurfaces*

$$(\mathcal{C}^\infty_D(\partial M; V_+))' \longrightarrow \bigoplus_{H\in\mathcal{M}_1(M)} \mathcal{C}^{-\infty}(H; V_+) \tag{1}$$

and injections on supported distributions for each $H \in \mathcal{M}_1(M)$

$$\dot{\mathcal{C}}^{-\infty}(H; V_+) \hookrightarrow (\mathcal{C}^\infty_D(\partial M; V_+))' \tag{2}$$

such that the collective map is surjective

$$\bigoplus_{H\in\mathcal{M}_1(M)} \dot{\mathcal{C}}^{-\infty}(H; V_+) \twoheadrightarrow (\mathcal{C}^\infty_D(\partial M; V_+))'. \tag{3}$$

This answers the question of just how well-defined the boundary data is – they fit together naturally to form a ‘distribution’ on the articulated manifold ∂M now we will set, in terms of the boundary pairing

$$\mathcal{C}_D^{-\infty}(\partial M; V_+) = (\mathcal{C}_{D^*}^{\infty}(\partial M; V_-))'.$$

Theorem

With the global hypotheses above on the first order elliptic differential operator D, there is a well-defined injective boundary map B giving a commutative diagram and an injective boundary map

$$\begin{array}{ccc} \mathrm{Nul}_\delta(D; \mathcal{C}^{-\infty}) & \longrightarrow & \{v_j\} \in (\mathcal{C}^{\infty}(\partial M; V_-))' \\ \downarrow & & \downarrow \\ \mathrm{Nul}(D; \mathcal{C}^{-\infty}) & \overset{B}{\hookrightarrow} & \mathcal{C}_D^{-\infty}(\partial M; V_+). \end{array} \tag{1}$$

This in turn allows us to define the Calderón projector as a linear map precisely as in the case of a manifold with corners except for the extra algebraic overhead

$$\Pi_C : \mathcal{C}_D^{-\infty}(\partial M; V_+) \longrightarrow \mathcal{C}_D^{-\infty}(\partial M; V_+) \text{ by}$$

$$\Pi_C([w_j]) = B\left(D^{-1}\left(\sum_H L_j v_j \delta(\rho_j)\right)\Big|_M\right).$$

Lemma

The Calderón projector is well-defined as a continuous projection on $\mathcal{C}_D^{-\infty}(\partial M; V_+)$ and has range precisely equal to the range of B which maps $\mathrm{Nul}(D; \mathcal{C}^{-\infty})$ *injectively into* $\mathcal{C}_D^{-\infty}(\partial M; V_+)$.

Theorem

Acting on spaces of extendible distributions a first order elliptic differential operator (with the global properties assumed above) is surjective, $D : \mathcal{C}^{-\infty}(M; V_+) \longrightarrow \mathcal{C}^{-\infty}(M; V_-)$ *and has null space naturally isomorphic to* $\Pi_C \mathcal{C}_D^{-\infty}(\partial M; V_+)$.

- These results *should* extend to the general case where D is not assumed to either have the extension property or the unique continuation property.
- The extension to higher order systems would be a more serious pain!

- Continuing under the global assumptions, observe that for $t \in \mathbb{R}$, $|t| < \frac{1}{2}$, and on any compact manifold with corners, the extendible and supported Sobolev spaces are identified

$$\dot{H}^t(H;V) = (H^{-t}(H;V))' \equiv H^t(H;V),\ -\frac{1}{2} < t < \frac{1}{2}.$$

- That is, each element of these Sobolev spaces has a unique zero extension with the same regularity (with which it can therefore be identified).
- Now, in view of the properties of the spaces discussed above it follows that

$$\bigoplus_{H \in \mathcal{M}_1(M)} H^t(H;V_+) \subset \mathcal{C}_D^{-\infty}(\partial M;V_+),\ -\frac{1}{2} < t < \frac{1}{2} \tag{1}$$

are well-defined subspaces for any elliptic D.

- The regularity properties of D^{-1} show that that

$$\Pi_C : \bigoplus_{H\in\mathcal{M}_1(M)} H^t(H;V+-) \longrightarrow \bigoplus_{H\in\mathcal{M}_1(M)} H^t(H;V_+)$$
$$G_t : H^t(H;V_+) \ni U \longmapsto D^{-1}\sum_H \delta(\rho_H)U_H\big|_M \in \mathrm{Nul}(D) \tag{2}$$

where the second map has range precisely the subspace

$$\mathrm{Nul}_s(D) = \{u \in H^s(M;V_+); Du = 0\},\ s = t + \frac{1}{2}. \tag{3}$$

- Thus, for instance, for $\frac{1}{2} < s < 1$ there is a short exact sequence

$$\{U \in H^{s-\frac{1}{2}}(H;V_+); \Pi_C U = U\} \longrightarrow H^s(M;V_+) \xrightarrow{D} H^{s-1}(H;V_-). \tag{4}$$

What more remains to be done and what can one hope to show along these lines? A lot.

- One important question is whether a first order elliptic operator can be sensibly realized on the collective boundary of a manifold with corners (or more generally an 'articulated manifold' modeled on it.
- If $\eth : \mathcal{C}^\infty(M; V_+) \longrightarrow \mathcal{C}^\infty(M; V_-)$ is a Dirac operator in the even dimensional case, on a manifold with corners, then on each boundary hypersurface there is a natural induced Dirac operator, $\eth_j$ on H_j.
- The discussion above applies to each $\eth_j$ on each H_j.
- Do these operators collectively define a Fredholm operator from $H^t(\partial M; V_+)$ to $H^{t-1}(\partial M; V_+)$ for $\frac{1}{2} < t < 1$, where

$$H^t(\partial M; V_+) = \{\{v_j\} \in \sum_j H^t(\partial M; V_+); v_j\big|_{H_j \cap H_k} = v_k\big|_{H_j \cap H_k}\}$$

is the 'transmission space'?

- The structure of the kernel of Π_C should be described more fully. Is there a corresponding algebra of operators in which it lies which generalizes the pseudodifferential operators in the boundary case?
- More particularly, can one find an analogue of the Atiyah-Patodi-Singer boundary condition, which is always elliptic – essentially a reasonable natural projection determined by geometric data on the boundary which is in some sense close to Π_C.
- The discussion above of Sobolev spaces should also works for b-regular spaces $H^k_b H^s(\partial M; V_+)$ for the same range of s. The image of these spaces correspond to solutions with conormal regularity.
- One should be able to prove directly the Fredholm property for say $d + \delta$ with absolute or relative boundary conditions.

Happy birthday and best wishes for the future Michael!