Workshop: Analysis, Geometry and Topology of Singular PDE (hybrid meeting)

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Abstracts

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HODGE THEORY FOR THE WEIL-PETERSSON METRIC

Using the, rather protracted, project with Jesse Gell-Redman on the Hodge theory for the Weil-Peterrson metric on the Riemann moduli spaces $\mathcal{M}_{g,n}$ as a guide I will outline a notion of 'iterated fibration' structure on a manifold with corners. The discussion here is restricted to codimension two and set in the general context of the resolution/quantization of a Lie algebroid.

Consider first the case of a manifold with boundary, the case of codimension one. Many examples have been extensively discussed in the literature – unfortunately too many to list here. The example I concentrate on comes from a 'real Weil-Petersson' metric on a compact manifold with boundary, X. This is arbitrary in the interior and near the boundary takes the form

$$g = dx^2 + h(y, dy) + x^6\alpha^2 + xe,\ x \geq 0. \tag{1}$$

Here x is a defining function for the boundary, α is a connection form on a circle bundle over (and extended off) the boundary, h is a metric on the base of the circle bundle and e is an 'error term' which is smooth and bounded by the leading part.

Appropriately scaled this corresponds to a Lie algebroid, a Lie algebra of smooth b-vector fields on X spanned locally near the boundary by

$$x^3 \times \left(x^{-3}\partial_\theta,\ x^{-1}(x\partial_x),\ \partial_{y_j}\right) \tag{2}$$

where I have multiplied the vector fields of bounded length by x^3 to make them smooth; ∂_θ is a generator of the circle action.

This is a 'geometric' Lie algebroid; in particular a $\mathcal{C}^\infty(X)$-module of smooth b-vector fields, $\mathcal{V}$, on X (that is the geometric part), and as in this case, I will assume in general that it is unrestricted in the interior (although this should be replaced by tangency to a b-fibration). By assumption (as a Lie algebroid) it has a local smooth basis near each point. The notion of a 'boundary-fibration structure' involves the boundary filtration

$$\mathcal{W}_k = \left(\mathcal{V} \cap \rho^k \mathcal{V}_{\mathrm{b}}(X)\right)/\rho^k,\ W_k = \mathcal{W}_k\big|_{\partial X} \subset \mathcal{C}^\infty(\partial X; {}^{\mathrm{b}}T_{\partial X}X). \tag{3}$$

I will demand that the W_k are subbundles. The b-tangent bundle to X has a canonical line subbundle over ∂X, spanned by $x\partial_x$, with the quotient being $T\partial X$. For each k I require that either W_k meets this b-normal bundle only at the 0 section or else contains it. It follows that for some minimal l – the boundary depth – there is a b-normal vector field (inducing the section $x\partial_x$ at the boundary) $N \in \mathcal{W}_l$.

The space $\mathcal{C}^\infty(\partial X; {}^{\mathrm{b}}T_{\partial X}X)$ is a Lie algebra. I will require four further conditions on $\mathcal{V}$ for it to be an iterated boundary fibration structure:-

$$\begin{gathered}\text{The } W_k \text{ are Lie algebras,}\\ \text{The quotients } W_k/{}^{\mathrm{b}}N \text{ define fibrations of } \partial X,\\ \text{The } W_k \text{ are exact}\\ [N, \mathcal{W}_k] \subset \mathcal{W}_k \ \forall\ k.\end{gathered} \tag{4}$$

The $W_k/{}^{\mathrm{b}}N \subset \mathcal{C}^\infty(\partial X; T\partial X)$, for $k < l$, then have local coordinate bases ∂_{y_j} and these lift to elements $\partial_{y_j} + a_j x\partial_x$ of $\mathcal{W}_k$; the exactness condition requires the closed forms $\sum_j a_j dy_j$ to be exact on the fibres. In fact the first three conditions can be combined by requiring the action of W_k on the normal bundle to the boundary to induce a fibration.

For such a Lie algebroid there is a 'Frobenius' basis analogous to (2). Most significantly such a Lie algebroid can always be resolved by the construction of a of generalized product, and in particular can be quantized to a calculus of pseudo-differential operators. As noted above many cases included here are quite familiar:

$l = 0$: b-calculus, (fibred) edge calculus
$l = 1$: scattering calculus, fibred boundary calculus, Weil-Peterrson
$l = 2$: a-calculus of Grieser and Hunsicker.

Note that, for brevity, I have excluded the 'adiabatic calculi' (where N is not in the Lie algebroid). Ideally the definition should also be broadened further to include the Θ-calculus. Such a generalization is even more relevant in higher codimension to capture the compactifications of reductive Lie groups.

The main aim of this talk is to examine appropriate conditions for an iterated boundary fibration in codimension two (and higher). So now let X be a compact manifold with corners up to codimension 2 and let $\mathcal{V} \subset \mathcal{V}_{\mathrm{b}}(X)$ be a 'geometric' Lie algebroid. I will demand conditions as in (4) at the interior of the boundary hypersurfaces. In fact, by generalizing the initial definition to allow non-trivial interior fibrations and the extra normal direction one can proceed iteratively and simply require that each of the spaces in (3), at each boundary hypersurface, define an iterated boundary fibration structure.

Still we need further restrictions at each corner, F, of codimension two; for simplicity I shall assume there is only one (connected) corner. There the boundary filtration is parameterized by a multiorider κ :

$$\mathcal{U}_\kappa = \left(\mathcal{V} \cap \rho^\kappa \mathcal{V}_{\mathrm{b}}(M)\right)/\rho^\kappa,\ U_\kappa = \mathcal{U}_\kappa\big|_F \subset \mathcal{C}^\infty(F; {}^{\mathrm{b}}T_F X),\ \rho = (\rho_1, \rho_2). \tag{5}$$

Here the ρ_i are defining functions for the two local boundary hypersurfaces. These space are automatically decreasing under the standard partial order $\kappa' \geq \kappa$. Again we assume that

$$U_\kappa = \mathcal{C}^\infty(F; {}^{\mathrm{b}}T_F M) \text{ for some } \kappa. \tag{6}$$

The space $\mathcal{C}^\infty(F; {}^{\mathrm{b}}T_F X)$ is again a Lie algebra (and Lie algebroid over F with two abelian 'fibre' variables) and we demand that the U_κ be Lie subalgebroids.

The additional requirement I wish to emphasize – it is automatic in codimension 1 – is

'Strong iteration': There exists a sequence of distinct multiindices

$$(0,0) = \kappa(0) < \kappa(1) < \cdots < \kappa(N) \tag{7}$$

forming a *chain* and such that for any $\kappa \in \mathbb{N}_0^2$

$$U_\kappa = U_{\kappa(j)},\ j(\kappa) = \max\{k; \kappa(k) \le \kappa\}. \tag{8}$$

Of course the $U_j = U_{\kappa(j)}$ then determine all the U_κ.

Beyond this a generalization of 'boundary depth' above is required. The b-normal bundle to F is a canonically trivial subbundle of ${}^{\mathrm{b}}T_FX$ with fixed basis $x_1\partial_{x_1},\ x_2\partial_{x_2}$ corresponding to (but independent of) any local choice of defining functions.

'b-normality': For each k the intersection

$$^{\mathrm{b}}U_j = U_j \cap {}^{\mathrm{b}}NF \text{ is a subbundle with basis } p_1x_1\partial_{x_1} - p_2x_2\partial_{x_2},\ p_i \in \mathbb{N}_0. \tag{9}$$

By assumption ${}^{\mathrm{b}}U_N = {}^{\mathrm{b}}NF$ since we are assuming that $U_N = {}^{\mathrm{b}}T_FX$. So the ${}^{\mathrm{b}}U_j$ are decreasing starting from full rank two (so containing both generators). If the rank drops from two to one (it could drop from two to zero) then the remaining element is required to be some $p_1x_1\partial_{x_1} - p_2x_2\partial_{x_2}$. The sign condition on the integers corresponds to the fact that this should generate a b-fibration of the inward-pointing normal bundle to F.

Finally we require the fibration condition itself, that the

'Fibration': $U_j(F)/{}^{\mathrm{b}}NF$ define fibrations of F

and that the induced 1-forms on F are exact on fibres. I also require *homogeneity* with respect to the two normal vector fields.

Under these conditions an iterated fibration structure has a resolution by a generalized product and hence quantization to a calculus of pseudodifferential operators.

The Weil-Petersson case shows that the chain condition need not be trivial to arrange. Namely in codimension two the metric assumes the 'product' form

$$g = dx_1^2 + dx_2^2 + h(y, dy) + x_1^6\alpha_1^2 + x_2^6\alpha_2^2 + x_1e_1 + x_2e_2. \tag{10}$$

The conditions above, without the chain condition, are achieved on the single space defined by blow-up of the corner. The chain condition holds on the space defined by parabolic blow-up of the resulting two corners.

Is there a simpler way?

The Hodge theorem asserts that the L^2 null space of the Laplacian for a Weil-Petersson metric is isomorphic to the cohomology of the manifold without boundary obtained by collapsing the circle bundles.

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