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CONTACT MANIFOLDS AND THE INDEX OF HEISENBERG PSEUDODIFFERENTIAL OPERATORS

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I will spend a considerable part of my lecture today describing two well-known index theorems, the second perhaps better known than the first. One of my aims in doing so is to try to convince you of the usefulness of the algebra of 'Heisenberg' psuedodifferential operators on a compact contact manifold. Rather than concentrate on the technicalities of this algebra I will talk about applications. However I should point out that the Heisenberg algebra was discussed in detail by Beals and Greiner [1] and by Taylor [19] and in the real analytic setting by Geller [12]. These treatments codify ideas of Dynin [6, 7] and are also outgrowths of older ideas of Boutet de Monvel, Grigis, Melin, Sjöstrand, Stein and others. Of coure I prefer the presentation which collaborators Epstein and Mendoza and I have given in [11]. For the moment it suffices to think of the Heisenberg algebra, $\Psi^*_{\mathrm{H}}(X;E)$, as a well-defined order-filtered algebra of operators on $\mathcal{C}^\infty$ sections of any vector bundle E over a compact contact manifold X.

That X is a compact contact manifold means that it is an odd-dimensional compact manifold, say of dimension $2n-1$, with a smooth (real) 1-form α satisfying the non-degeneracy condition that

$$\mu = \alpha \wedge (d\alpha)^{n-1} \neq 0.$$

Since this form μ is a non-vanishing volume form the manifold is necessarily orientable, and oriented by μ. In fact it is better to note that any other positive smooth multiple of α, $\alpha' = \phi\alpha$, $0 < \phi \in \mathcal{C}^\infty(X)$ is also a contact form and so think of the oriented line bundle $L \subset T^*X$ spanned by α as the contact structure.

Now there are two basic model cases of contact manifolds. The most important is the boundary of a strictly pseudoconvex complex manifold. On a complex manifold M the usual deRham operator splits into $d = \partial + \overline{\partial}$ and if f is a smooth real function then $\omega = \frac{1}{i}\overline{\partial}\partial f$ is a smooth real 2-form. If M is a complex manifold with (real) boundary $X = \partial M$ and ρ is a defining function for the boundary then $\alpha = \frac{1}{i}\partial\rho$ pulls back to be a real 1-form on the boundary. Its exterior derivative is ω and the domain is said to be (strictly) pseudoconvex if $\overline{\partial}\partial\rho$ is a positive-definite Hermitian form on the null space of α on the boundary. This implies that $X = \partial M$ is a contact manifold, with α the contact form.

The second basic example of compact contact manifolds is the cosphere bundle of any manifold. Namely if Y is any compact manifold and T^*Y is its cotangent bundle then we may take the quotient $S^*Y = (T^*Y \setminus 0_Y)/\mathbb{R}^+$ – this is the space of radial half-lines in the fibres of T^*Y. It is more usual to take the unit sphere for some Riemann metric on Y but this definition is more intrinsic and shows that there really is not a well defined contact *form* on S^*Y, but the multiples of the usual fundamental 1-form give a contact line bundle over $X = S^*Y$.

Although it is perhaps the more familiar setting, it is significant to note that the second class of examples is actually contained in the first. This is a theorem of Grauert [13], who showed that the cotangent bundle of any manifold could be given a complex structure so that the cosphere bundle – thought of as the boundary of a coball bundle of small radius – is strictly pseudoconvex, The contact structures arrived at by these two methods are the same (or at least can be made to be so, see [14]).

Now, one of the important objects in the first setting, of the boundary of a strictly pseudoconvex complex manifold, is the Szegő projection. The boundary values of the smooth (up to the boundary) holomorphic functions on M form a subspace of $\mathcal{C}^\infty(X)$, $X=\partial M$, and orthogonal projection, with some sensible measure such as μ, onto the closure of this space in L^2 is the Szegő projection; I will denote it S.

Louis Boutet de Monvel's index theorem is about this setting, at least his original theorem is. Namely suppose we look at some $N\times N$ matrix, a, of smooth functions defined over $X=\partial M$. The operator

$$SaS : L^2(X;\mathbb{C}^N) \longrightarrow L^2(X;\mathbb{C}^N) \tag{1}$$

is bounded. It is in fact a bounded operator on $SL^2(X;\mathbb{C}^N)$, the range of S (acting diagonally); such an operator is generally called a Toeplitz operator (well really a generalized Toeplitz operator since the original operators were in one dimension, but we need to generalize them further so they are Toeplitz operators!) As an operator on the space $SL^2(X;\mathbb{C}^N)$, SaS is Fredholm (meaning has finite dimensional null space and closed range of finite codimension) if and only if a is an invertible matrix at each point of X. Boutet de Monvel's index theorem (see [3]) is

$$\operatorname{ind}(SaS) = \int_X \operatorname{Td}(H)\operatorname{Ch}_1(a). \tag{2}$$

The integral here is of a volume form on X, given as the top order term in the product of two characteristic classes – the Todd class of the complex part of the tangent bundle of X and the (odd) Chern character of the isomorphism a. I will talk a little more about the formula later on, but it is an integral of topological-geometric objects over X.

Boutet generalized this theorem and then further generalized it to an arbitrary compact contact manifold in work with Victor Guillemin. To do so, the main thing that is needed on an arbitrary contact manifold is a replacement for S. Such generalized Szegő projections can be characterized and constructed in the Heisenberg calculus; this is done in terms of the Hermite calculus in [4]. In fact there are lots of these projections but they have a reasonably simple characterization and the index of a (generalized) Toeplitz operator given by (1) is independent of the choice of S. The same formula (2) holds in general, even if a is a bundle isomorphism on some possibly non-trivial vector bundle E over X.

Recall that what such an index formula represents is an obstruction to the invertibility of the operator in question. If it is non-zero then no small (either norm small or else large but smoothing) perturbation of the same type can make it invertible, and conversely.

The second index theorem I want to recall at the beginning is the Atiyah-Singer theorem for Dirac operators. In fact let me specialize a little in the interests of brevity. If M is a complex manifold – now compact and without boundary – then the operator $\overline{\partial}$ gives an elliptic complex

$$\mathcal{C}^\infty(M) \xrightarrow{\overline{\partial}} \mathcal{C}^\infty(M;\Lambda^{0,1}) \xrightarrow{\overline{\partial}} \cdots$$

The signature-type operator $\overline{\partial}+\overline{\partial}^*$ is self-adjoint as an operator on sections of $\Lambda^{0,*}$ and maps odd forms to even and vice-versa. The symbol of this operator is a Clifford multiplication on $\Lambda^{0,*}$ which is therefore a Clifford module. Namely at any point $\xi \in T^*_xM$ (the real cotangent bundle) the symbol of $\overline{\partial}+\overline{\partial}^*$ is an element $\operatorname{cl}(\xi) \in \hom(\Lambda^{0,*}_x)$. The properties of $\overline{\partial}$ show that

$$\operatorname{cl}(\xi)\operatorname{cl}(\xi') + \operatorname{cl}(\xi')\operatorname{cl}(\xi) = 2\langle \xi, \xi' \rangle$$

in terms of the metric inner product. A first order differential operator with principal symbol having this property is called a generalized Dirac operator. Here we can assume that the symbol is exactly the one coming from a complex structure on the fibres of TM. Somewhat more generally we get a Clifford module structure on $\Lambda^{0,*} \otimes F$ for any complex bundle F over M. For this to make sense it is not necessary that the manifold be complex, it is enough for it to be almost complex, i.e. that the tangent bundle has a complex structure without requiring it to be integrable (and hence complex by the Newlander-Nirenberg theorem.) In fact it is useful to get an actual Spin-$\mathbb{C}$ Dirac operator, $\eth_F$, but not important for the statment of the index theorem of Atiyah and Singer:

$$\operatorname{ind}(\eth_{F,+}) = \int_M \operatorname{Td}(TM)\operatorname{Ch}_0(F) \tag{3}$$

since the index is determined by the principal symbol. Here $\eth_{F,+}$ is the part of the Dirac operator from even to odd forms.

So, these formulæ are very similar. There is of course a meta-theorem that there is basically only one index theorem so they must *really* be the same. I would like to give you an idea of a direct reduction of the theorem of Boutet de Monvel to the Atyah-Singer theorem; this is joint work with Charlie Epstein and Gerardo Mendoza. Namely we have a direct analytic proof of the following fact.

Theorem 1. *The index of a Toeplitz operator SaS on a compact contact manifold is equal to the index of the Dirac operator on $M = X \times \mathbb{S}^1$ associated with the natural (positive) almost complex structure arising from the contact structure on X and twisted by the bundle $F = E_a$ obtained from E by gluing with a.*

The gluing construction here is the obvious one, where we consider $X \times [0, 2\pi]$ and glue the ends together to get $M = X \times \mathbb{S}^1$, but put a twist in the lift of E from X to get the bundle F. This construction is an explicit form of suspension.

I will describe some applications and extensions of this construction after I give an idea of the proof.

The proof involves manipulation in the Heisenberg and a larger algebra, the 'extended Heisenberg' algebra, which is also discussed in [11]. I would like to describe this without the 'Heisenberg' overhead. I hope to give you a rather precise description of the proof by considering the 1 and 2 dimensional case with which you are likely to be familiar.

So, let me start with the 1 dimensional contact manifold, namely the circle. The contact bundle *is* the cotangent bundle and the Szegő projection is the projection onto the non-negative Fourier components – this is of course thinking of the circle as the boundary of the disk.

Thus our Toeplitz operators are 'genuine' Toeplitz operators ([20, 21]) and the index theorem is due to Szegő ([17, 18]). It says that the index of SaS for a complex map

$$a : \mathbb{S}^1 \longmapsto \mathrm{GL}(N, \mathbb{C}) \tag{4}$$

is the winding number of a (or if you prefer of its determinant as a map $\mathbb{S}^1 \longrightarrow \mathbb{C}\backslash 0$.)

There are rather straightforward proofs of this formula, especially if a little topology is used. Namely it is known that, for a map (4), the winding number is the precise obstruction to the existence of an extension of a to a map $\tilde{a} : \mathbb{D} \longrightarrow \mathrm{GL}(N, \mathbb{C})$. Such an extension allows the operator SaS to be deformed to S; it follows that the index depends only on the winding number. That it depends additively on it is clear by taking products and then it is enough to check one example, for instance $a = e^{-i\theta}$. Nevertheless, I will give a complicated proof (from [11]) which has the virtue of applying in the higher dimensional setting.

We are aiming to connect the Toeplitz operator to the Cauchy-Riemann operator on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Recall that in this case we have an operator of index 0 (though not actually invertible)

$$\overline{\partial} : \mathcal{C}^\infty(\mathbb{T}^2) \longrightarrow \mathcal{C}^\infty(\mathbb{T}^2; \Lambda^{0,1})$$

into the bundle of anti-complex forms on $\mathbb{T}^2$. More generally we can twist $\overline{\partial}$ as above by a (not necessarily holomorphic) complex bundle F and get a well-defined (well, up to various choices) Dirac operator $\eth_{F,+}$ which 'looks like' $\overline{\partial}$ acting on sections of F and giving 1-form valued sections.

Let me try to describe the relationship is a series of steps.

Step 1. Is rather easy but fundamental. SaS is an operator on $SL^2(\mathbb{S}; E)$ (remember there is some bundle around). Since a is invertible we may consider instead the Fredholm operator

$$a^{-1}SaS : SL^2(\mathbb{S}; E) \longrightarrow a^{-1}SL^2(\mathbb{S}; E) = S_a L^2(\mathbb{S}; E)$$

where $S_a = a^{-1}Sa$ is *another* projection. Thus our Toeplitz index is the index of the composite $S_a \circ S$ as a map from the range of S to the range of S_a. This is called the 'relative index' of the two projections

$$\mathrm{ind}(SaS) = \text{R-ind}(S, S_a).$$

At this stage it is a good idea to take the polar decomposition of a (with respect to some Hermitian inner product) and so get $a = bc$ where $c > 0$ is self-adjoint and b is unitary. We may then deform c to Id without changing the index and so assume that a was unitary from the start. This has the effect that S_a is also self-adjoint, as is S.

The relative index of any two generalized Szegő projections is well defined and it defines a map into Z. This map is surjective and is constant in each variable only on the components of the space of generalized Szegő projectors; i.e. this space has components in 1-1 correspondence with $\mathbb{Z}$. The relative index of true Szegő projections plays an importatn rôle in the discussion of the embedding problem for CR manifolds in 3 dimensions by Epstein [9].

Step 2. Consider the linear homotopy from S to S_a, $S_t = (1-t)S + tS_a$. If we think of this as a curve of pseudodifferential operators on the circle, they all have the same principal symbol. Thus they are projections modulo lower order, hence compact, terms. This means that the spectrum of S_t is discrete (it is of course real) outside $0, 1$ (the values of the symbol) and of finite multiplicity. Said another way, $S_t - \lambda$ is Fredholm for any $\lambda \in (0, 1)$. It follows that the *spectral flow* of the family is well-defined around any such point. This is the net number of eigenvalues crossing λ as t runs from 0 to 1, where increase is counts as $+1$ and decrease is counted as -1. There is a simple finite formula for the spectral flow as follows. For each $t \in [0, 1]$ we may choose a $\lambda(t) \in (0, 1)$ such that $S_t - \lambda(t)$ is invertible. Thus it stays invertible nearby in t, so we may find a finite decomposition of $[0, 1]$ into intervals $[t_j, t_{j+1}]$, ($t_0 = 0$, $t_{N+1} = 1$) and a $\lambda_j \in (0, 1)$ so that $S_t - \lambda_j$ is invertible for $t \in [t_j, t_{j+1}]$. We may also suppose that $\lambda_0 = \lambda = \lambda_N$. Then set

$$\mathrm{SF}(S_\bullet) = \sum_{j=1}^{N} \mathrm{mult}(S_{t_j}, [\lambda_{j-1}, \lambda_j]).$$

Here $\mathrm{mult}(S_{t_j}, [\lambda_{j-1}, \lambda_j])$ is the number of eigenvalues (necessarily finite) counted, of course with multiplicity, of S_{t_j} in the interval $[\lambda_{j-1}, \lambda_j]$. By construction the endpoints are not in the spectrum and the number is taken with positive sign if $\lambda_j > \lambda_{j-1}$ and with negative signe if $\lambda_j < \lambda_{j-1}$. It is easy to check that $\mathrm{SF}(S_\bullet, \lambda)$ is well-defined, independent of the decomposition. It make sense for any curve of self-adjoint operators provided $S_t - \lambda$ is Fredholm for all $t \in [0, 1]$ and λ is not in the spectrum at the end points. In fact it depends on λ but is locally constant and changes as λ crosses an eigenvalue (of either end) by the appropriately-signed multiplicity.

With this definition of the spectral flow it is quite straightforward to show that

$$\text{R-ind}(S, S_a) = -\,\mathrm{SF}(S_\bullet, \frac{1}{2}),\ S_t = (1-t)S + tS_a.$$

Step 3. This step is perhaps rather unmotivated in dimension one, but it is important to gain some ellipticity for the family. Instead of S_t we can consider the curve of operators which joins the corresponding involutions $S - (\mathrm{Id} - S)$ and $S_a - (\mathrm{Id} - S_a)$:

$$B_t = (1-t)[2S - \mathrm{Id}] + t[2S_a - \mathrm{Id}].$$

The additional terms are essentially negative, so do not change the result that

$$\text{R-ind}(S, S_a) = -\,\mathrm{SF}(S_\bullet, \frac{1}{2}) = -\,\mathrm{SF}(B_\bullet, \lambda),\ \lambda \in (-1, 1).$$

To see this first check the result for $\lambda = \frac{1}{2}$ and then the right side is independent of $\lambda \in (-1, 1)$; the spectrum at the end points (which are conjugate) is ± 1.

Step 4. Next we modify the family to be self-adjoint pseudodifferential operators of order s, $s \in [0, 1]$. Consider

$$B_t(s) = (1-t)|D_\theta|^{\frac{s}{2}}(2S - \mathrm{Id})|D_\theta|^{\frac{s}{2}} + ta^{-1}|D_\theta|^{\frac{s}{2}}(2S - \mathrm{Id})|D_\theta|^{\frac{s}{2}}a.$$

For each s this is a curve of self-adjoint unbounded Fredholm operators with fixed domain (the Sobolev space $H^s(\mathbb{S}; E)$) and with conjugate endpoints. It follows that the spectral flow around 0 is well-defined for each s and is independent of s. Thus

$$\text{R-ind}(S, S_a) = -\,\mathrm{SF}(S_\bullet(1), 0).$$

The operator

$$|D_\theta|^{\frac{s}{2}}(2S - \mathrm{Id})|D_\theta|^{\frac{s}{2}}\big|_{s=1} = D_\theta.$$

Thus at $s = 1$ the curve of self-adjoint elliptic pseudodifferential operators with fixed symbol joins D_θ to $a^{-1}D_\theta a$. The invariance properties of the spectral flow show that the same is true for the linear curve joining these two operators.

Step 5. Now let t run over $[0, 2\pi]$ instead of $[0, 1]$ and choose the curve connecting A_t from D_θ to $a^{-1}D_\theta a$ to be constant near the end points. The index of the Dirac operator $\eth_{F,+}$ is the same as that of the operator $\partial_t + A_t$ acting on the subspace of the Sobolev space of those $u \in H^1(\mathbb{S}_\theta \times [0, 2\pi]_t; E)$ satisfying the boundary condition $u|_{t=2\pi} = au|_{t=0}$. From this in turn it follows that

$$\operatorname{ind}(\eth_{F,+}) = \operatorname{SF}(A_\bullet, 0) = -\operatorname{ind}(SaS).$$

This is the claimed direct identification of the Toeplitz index with the index of a Dirac operator. The higher dimensional case follows very much the same argument with additional geometric complexity.

Next I shall describe some consequences of this identification of Toeplitz and Dirac index theorems.

The index theorem for Dirac operators is analytically direct. That is it is essentially computational, using the arguments of Atiyah, Singer, Bott, Patodi and Getzler (other related results were shown by Gilkey and Bismut); see [2]. This combines the heat kernel formula for the index with the special scaling properties of the Clifford module to deduce (3) rather directly. In particular combined with the generalization of the identities above to higher dimensions we obtain a 'directly analytic' proof of Boutet's index theorem.

As was observed already by Boutet de Monvel, the index theorem for Toeplitz operators is essentially equivalent to the index theorem for pseudodifferential operators, rather than just Dirac operators. In [3] he deduced the former from the latter, but we can now go in the other direction. This is based on ideas of Guillemin and Boutet de Monvel, arising from the theorem of Grauert discussed above. Namely we may consider, on the cosphere bundle S^*Y of some compact manifold Y, the Szegő projection arising from the identification with the boundary of the coball bundle of covectors of length at most ϵ for some small positive ϵ and some metric. It is convenient to take the Szegő projection acting on maximally anti-complex forms $\Lambda^{0,n-1}$ where Y has dimension n. This is a trivial holomorphic line bundle over the coball bundle with restriction to the spherical fibres of $X = S^*Y$ which is canonically identified with the fibre density bundle. Thus there is a natural push-forward, or integration, map

$$G : \mathcal{C}^\infty(S^*Y; \Lambda^{0,n-1}) \longrightarrow \mathcal{C}^\infty(Y).$$

The restriction of this map to the range of the Szegő projection, i.e. to the boundary values of holomorphic $(0, n)$-forms on the ball bundle, is Fredholm (see [4].) This can be seen using the calculus of Fourier integral operators with complex phase of Melin and Sjöstrand [16] and the results of Boutet de Monvel and Sjöstrand, [5], on the structure of the Bergman and Szegő kernels. Lebeau showed this map to be an isomorphism in the case of the sphere [15]. In as-yet-unfinished work with C. Epstein ([10]) it is shown that, for appropriate choices of ϵ (small) and the metric, this map is an isomorphism for any compact manifold. It is rather instructive to write down the isomorphism in the case of Euclidean space – this is indeed the starting point for the proof in [10]. Here is the result for the circle (which must be well-known).

Proposition 1. *For every smooth 2π-periodic function, f, on the line there is a unique holomorphic function $\tilde{f}$ in the strip $\{z \in \mathbb{C}; |\operatorname{Im} z| < 1\}$ which is smooth up to the boundary, 2π-periodic in $\operatorname{Re} z$ and satisfies*

$$\frac{1}{2}(\tilde{f}(x+i) + \tilde{f}(x-i)) = f(x).$$

Proof. Exercise in case you are bored with the lecture! □

In any case the Fredholm relation of $S\mathcal{C}^\infty(S^*Y; \Lambda^{0,n-1})$ to $\mathcal{C}^\infty(X)$ extends to sections of bundles and relates pseudodifferential operators on X to Toeplitz operators on S^*Y. In particular if A is an elliptic pseudodifferential operator of order 0 on Y then it has the same index as GAG^* which in turn has the same index as the Toeplitz operator SaS, where a is the symbol of A. Thus the index formula (2) may be applied to SaS to give the Atiyah-Singer index formula for pseudodifferential operators as

$$\operatorname{ind}(A) = \int_{S^*X} \operatorname{Td}(H) \operatorname{Ch}_1(a).$$

Here $\operatorname{Td}(H)$ is the Todd class of the annihilator of the contact bundle with its natural positive complex structure.

Thus one application of the construction above is to give an 'analytic' and rather deductive proof of the Atiyah-Singer index theorem, avoiding K-theory (at least formally).

A second application of the approach above, if not the actual results, is to give an index formula for Fourier integral operators; this is work with Epstein [8]. Again suppose that $X = S^*Y$ is the cosphere bundle of a compact manifold Y. Now consider a contact diffeomorphism ψ of X. This is the same thing as a (global) homogeneous symplectic (i.e. canonical) diffeomorphism of $T^*Y \setminus 0_Y$. With such a geometric object Hörmander associated a class of Fourier integral operators. The symbol of such an operator may now be considered as a section of a certain line bundle over S^*Y; the Maslov bundle. In fact in the global case this bundle is naturally trivial; this is incorporated in the idea of Maslov's 'canonical quantization.' In any case the contact diffeomorphism ψ again defines a relative index. Namely if S is a generalized Szegő projection on S^*Y, for example the true Szegő projection arising from Grauert's theorem, then the *contact degree* of ψ

$$\text{C-Deg}(\psi) = \text{R-ind}(S, S_\psi),\ S_\psi = (\psi^{-1})^* S \psi^* \tag{5}$$

is well-defined independently of the choice of S. This is based on two facts. First that the characterization of generalized Szegő projections is invariant under contact diffeomorphisms, so S_ψ is also a generalized Szegő projection and secondly that any two generalized Szegő projections are relatively Fredholm. Both follow from the characterization in the Heisenberg calculus (and its invariance under contact diffeomorphisms). This contact degree is actually defined on any arbitrary compact contact manifold and is a homomorphism from the group of contact diffeomorphism to $\mathbb{Z}$ – since it is homotopy invariant it is really a homomorphism from the contact mapping class group (i.e. the group of components).

A more geometric version of the discussion above is used in [8] to show that on the mapping torus Z, obtained from $X \times [0,1]$ by using ψ to identify the ends, there is a natural Dirac operator $\eth_\psi$ and

$$\text{C-Deg}(\psi) = -\operatorname{ind}(\eth_\psi).$$

It is still unknown whether this invariant is non-trivial; that is it could always be zero. The non-triviality is currently unknown in either the restricted class of cosphere bundles or the general class of contact manifolds. It *is* trivial in the 3 dimensional case. If $A \in I^0(Y,\psi)$ is chosen to have as symbol a natural trivializing section of the Maslov bundle then

$$\operatorname{ind}(A_\psi) = \text{C-Deg}(\psi). \tag{6}$$

A formula equivalent to this one was apparently suggested by Atiyah (see the paper of Weinstein [22] for this and even more general questions) in reply to a question of Weinstein in the '70s; Zelditch [24] has also discussed a similar approach to Weinstein's question. In the general case of an elliptic element $B = I^0(Y,\psi)$ we may write $B = B' \circ A_\psi$ where B' is an elliptic pseudodifferential operator. Then

$$\operatorname{ind}(B) = \text{C-Deg}(\psi) + \operatorname{ind}(B')$$

where the index of B' may be computed from the Atiyah-Singer theorem as discussed above.

There are further extensions of these ideas. In particular the results above only apply to a fixed manifold. Thus if one has a contact diffeomorphism $\psi : S^*Y_1 \longrightarrow S^*Y_2$, between the cosphere bundles of different manifolds, one can again look for the index of an associated elliptic Fourier integral operator. There *are* such operators (although it is less clear that there are interesting examples of such contact diffeomorphisms) and again there is a natural class with 'trivial symbol.' In this case the formula suggested by Atiyah is a little more complicated. The mapping cylinder Z should be replaced by the manifold $Z' = B^*Y_1 \cup_\psi B^*Y_2$ obtained by gluing the two cosphere bundles, which are the boundaries of the two ball bundles, using the contact diffeomorphism. This does not have an associated almost complex structure but it has a 'stable' almost complex structure which again gives a Dirac operator $\eth'_\psi$. Epstein and I believe we can prove the corresponding formula that

$$\text{R-ind}(S_2, (\psi^{-1})^* S_1 \psi^*) = \operatorname{ind}(\eth'_\psi); \tag{7}$$

here S_i is the *true* Szegő projection on S^*Y_i arising from Grauert's theorem.

Let me sketch the proof of (7). First we start with the coball bundles. Thus $M_i = \{\xi \in T^*Y_i; |\xi| < \epsilon\}$ is a complex manifold with strictly pseudoconvex boundary $X_i = S^*Y_i$, $i = 1, 2$. We may give these manifolds Kähler metrics, in which case $\overline{\partial} + \overline{\partial}^*$ is also the Dirac operator acting on the anti-complex exterior bundle $\Lambda^{0,*}M_i$. It is an elliptic operator. We may impose elliptic boundary conditions (of Atiyah-Patodi-Singer type) or subelliptic boundary conditions (of Kohn-Spencer type) but we actually wish to do something rather in between. If we consider the even part of the bundle $\Lambda^{0,\text{even}}M_i$ then over the boundary there is an isomorphism (induced by the metric)

$$\Lambda^{0,\text{even}} M_i\big|_{X_i} \equiv \Lambda^{0,*}X_i. \tag{8}$$

Here, as above, $\Lambda^{0,*}X_i$ is the (full) anti-complex exterior bundle for the complex subbundle of TX_i. There is an induced $\overline{\partial}_b$ complex on $\Lambda^{0,*}X_i$ and we consider the corresponding self-adjoint operator $\overline{\partial}_b + \overline{\partial}_b^*$. As is well known, this is not elliptic. In fact its treatment by pseudodifferential operator methods is one of the main reasons that the Heisenberg calculus was developed.

For the coball bundle (being a Stein space) it is known that the boundary complex $\overline{\partial}_b + \overline{\partial}_b^*$ has no null space outside form dimensions 0 and $n_i - 1$ (where Y_i has dimension n_i), see the work of S.S.T. Yau [23]. In the extreme dimensions the null space is infinite dimensional. Let S_i and T_i be the (Szegő) projections onto these infinite dimensional subspaces of L^2. This takes care of the null space of $\overline{\partial}_b + \overline{\partial}_b^*$. Then let P_i^+ be the 'Atiyah-Patodi-Singer' type projection, the projection onto the span of the strictly positive eigenvalues of $\overline{\partial}_b + \overline{\partial}_b^*$. Then we consider the boundary problems (for n even)

$$(9)\quad (\overline{\partial} + \overline{\partial}^*) : \left\{u \in \mathcal{C}^\infty(M_i; \Lambda^{0,\mathrm{even}}); (S_i + P_i^+ + T_i)u_i = 0\right\} \longrightarrow \mathcal{C}^\infty(M_i; \Lambda^{0,\mathrm{odd}}).$$

This map is always an isomorphism (essentially we have forced this) and so has zero index.

We need, for a reason that will become clearer below, to 'stabilize' (9). This can be thought of as taking the product with a 2-torus. We do not actually want to do this, since it would destroy the contact structure on the boundary. What we really want is just the extra factor of $\mathbb{R}^2 = \mathbb{C}$ in the tangent bundle. Since $\Lambda^*(X \times T) \equiv \Lambda^* X \otimes \Lambda^* T$ we get the desired effect much 'cheaper' by simply taking the tensor product with $\mathbb{C}^2 = \Lambda^{0,*}\mathbb{C}$. That is, we simply 'twist' the Dirac operator in (9) by a trivial complex 2-plane bundle, it remains an isomorphism with everything acting diagonally.

Now we have the two 'ends' set up. We also make a 'middle cylinder.' We can do this from $M = X_1 \times [0,1]_t$. This 'looks' just like a collar around $X_1 = S^*Y_1$ so it has a complex structure. This however is not the one we want. There is a speacial complex line bundle corresponding to ∂_t, along the interval, and the 'Reeb' vector field T which pairs non-trivially with the contact form on X_1. We want to 'turn ∂_t' around between $t = 0$ and $t = 1$. Of course we cannot do this, even dropping the integrability condition on the complex structure. We can do it however if 'stabilize' by adding the extra real 2-plane to the tangent bundle. We get ∂_t to turn around, so that $J\partial_t = T$ at $t = 0$ and $J\partial_t = -T$ at $t = 1$ by twisting in the extra two directions. This gives a Clifford module structure which looks just like the one we have constructed above on M_1, from the outside, near each end of M. We may further deform it so that it looks like the ψ image of the Clifford module structure on M_2 (again from the outside) near $t = 1$. Now we take the associated Dirac operator on M and take as boundary conditions on M the complementary boundary conditions for X_1 at $t = 0$ and the ψ-conjugate of the complementary boundary conditions on X_2 at $t = 1$. Essentially because of the complementary nature the boundary conditions give a Fredholm problem – it is not elliptic.

The boundary conditions at the end $t = 1$, with the operator may be deformed back to those at $t = 0$. However, this involves a non-trival relative index of the boundary projections, i.e. it cannot be done through projections and hence not through Fredholm problems. As discussed above this can be readily interpreted to shown that the relative index of the boundary conditions, and hence the index of the boundary problem on M, is the relative index in (7). Finally, because we have arranged the boundary conditons on the ends of M to be the complements of the corresponding ends of the M_i, we may glue the three manifolds together to get Z' (the middle sections makes no contribution diffeomorphically) to get the index for the Dirac operator associated to the stable almost complex structure.

This completes the brief outline of the proof of (7).

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