Valuations on Sobolev Spaces

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Let F be a space of real valued functions, for example, the Sobolev space $W^{1,1}(\mathbb{R}^n)$ and let $\mathbb{A}$ be an abelian semi-group. A function $\mathrm{z} : F \to \mathbb{A}$ is called a *valuation* if

$$\mathrm{z}(f \vee g) + \mathrm{z}(f \wedge g) = \mathrm{z}(f) + \mathrm{z}(g)$$

for all $f, g \in F$, where $f \vee g$ denotes the pointwise maximum and $f \wedge g$ the pointwise minimum of f and g. If the abelian semi-group is given as the set of convex bodies (compact convex sets) in $\mathbb{R}^n$ with Minkowski addition (defined by $K + L = \{x + y : x \in K, y \in L\}$), we talk about Minkowski valuations.

After a brief excursion to the history of valuations in geometry, we give a complete classification of affinely contravariant Minkowski valuations on $W^{1,1}(\mathbb{R}^n)$ and show that every such valuation z is given as

$$\mathrm{z}(f) = c\,\Pi\langle f\rangle$$

for all $f \in W^{1,1}(\mathbb{R}^n)$ with a suitable constant $c \geq 0$. Here the convex body $\Pi\langle f\rangle$ is defined via its support function as

$$h(\Pi\langle f\rangle, v) = \frac{1}{2}\int_{\mathbb{R}^n} |v \cdot \nabla f(x)|\, dx.$$

We discuss the connection of the convex bodies $\Pi\langle f\rangle$ and $\langle f\rangle$ (both introduced by Lutwak, Yang & Zhang) with the optimal norm in the sharp Sobolev inequality for a general norm, the affine Sobolev-Zhang inequality and the solution of the functional Minkowski problem.