THE INDEX OF THE DIRAC OPERATOR IN LOOP SPACE

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1. Introduction

Let M be a spin manifold and $\mathcal{L}M$ the free loop space of maps $S^1 \to M$. Although analysis on infinite dimensional spaces such as $\mathcal{L}M$ is a new subject in the mathematical literature, it has been in the last sixty years much studied by physicists in the framework of quantum field theory. In particular, certain quantum field theories can be interpreted as infinite dimensional analogues of structures which have an established mathematical significance in the finite dimensional case. For instance, I showed in [1] that the supercharge Q of the supersymmetric nonlinear sigma model in $1+1$ dimensions (with target space M) can be interpreted as a Dirac-like operator on $\mathcal{L}M$.

These lecture notes will be devoted to discussing formally the index of the Dirac operator on $\mathcal{L}M$. This subject is closely related to the recent work of Schellekens and Warner on anomalies [2], and it has interesting applications, as we will see, to a certain topological problem which has been discussed at this conference by Landweber and Ochanine. In essence we will see that the topological conjecture in question would follow from certain simple (conjectured) properties of the supersymmetic nonlinear sigma model. Since a cutoff version of the nonlinear sigma model would be adequate, there is a reasonable hope that the requisite properties of the sigma model can be proved within a few years. This application of the sigma model to topology was briefly discussed in [3], along with some speculations about applications to physics. Our considerations are also closely related to the use of fixed point theorems on the loop space of a group to obtain the Weyl-Kac character formula for affine Lie algebras; see [4] for an exposition of this subject.

2. The Dirac Index on $\mathcal{L}M$

Let M be an even dimensional spin manifold, and

$$D : \Gamma(S_+) \to \Gamma(S_-) \tag{1}$$

the Dirac operator (S_+ and S_- are the positive and negative spin bundles and $\Gamma(\cdots)$ denotes the

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smooth sections of a vector bundle). The index of the Dirac operator is defined as

$$\operatorname{index} D = \dim H_+ - \dim H_- \tag{2}$$

where H_+ and H_- are the kernel and cokernel of D. If G is a compact group that acts on M, and we pick a G-invariant metric on M, then H_+ and H_- become G modules. For $g \epsilon G$ we then define

$$F(g) = Tr_{H_+}\, g - Tr_{H_-}\, g. \tag{3}$$

The function $F(g)$ obviously depends only on the conjugacy class of $g \epsilon G$ and is known as the character-valued Dirac index. Evidently, $F(1)$ is the Dirac index in the sense of equation (2).

The index theorem leads to a formula that determines $F(g)$ in terms of the fixed points of G [5]. Given $g \epsilon G$, if the fixed point set M^g of g is a union of connected components M^g_α, the fixed point formula is of the general nature

$$F(g) = \sum_\alpha F_\alpha(g) \tag{4}$$

where $F_\alpha(g)$ depends on local data at M^g_α.

Let us specialize to the case $G = S^1$, which will be of particular interest for us. The general element of G is then

$$g = e^{\theta P} \tag{5}$$

with P a generator of S^1, and θ an angular variable, $0 \le \theta \le 2\pi$. The character-valued index becomes a function $F(\theta)$. Let us describe explicitly the fixed point formula for $F(\theta)$. Consider first the contribution of an isolated fixed point x.

In the tangent space T_x to M at x, one can find an orthonormal basis $e_1, \cdots, e_{2k}$ ($2k = \dim M$) in which P takes the form

$$P = \begin{pmatrix} & n_1 & & & & & \\ -n_1 & & & & & & \\ & & & n_2 & & & \\ & & -n_2 & & & & \\ & & & & \ddots & & \\ & & & & & & n_k \\ & & & & & -n_k & \end{pmatrix} \tag{6}$$

with integers n_i, which are non-zero as x is an isolated fixed point.

The e_i can be chosen so the n_i are all positive. Fixing an orientation ϵ of M, which induces an orientation ϵ_x of each T_x, let λ_x be +1 or -1 according to whether $e_1 \wedge e_2 \cdots \wedge e_{2k}$ is a positive or negative multiple of ϵ_x. The contribution of x to $F(\theta)$ is then

$$F_x(\theta) \;=\; \lambda_x \cdot \prod_{i=1}^{k} \frac{e^{in_i\theta/2}}{1 - e^{in_i\theta}} \tag{7}$$

and if all fixed points are isolated, then $F(\theta) = \Sigma_x F_x(\theta)$.

The generalization of (7) to the case of fixed points which may not be isolated is as follows. Let M_α be an arbitrary connected component of the fixed point set of S^1. The normal bundle N to M_α in M has a decomposition

$$N \;=\; \bigoplus_{\ell \neq 0} N_\ell \tag{8}$$

with P acting on N_ℓ as multiplication by $i\ell$ (the ℓ are integers).

For any vector bundle V and complex number t, we will let

$$\frac{1}{1-tV} \;=\; 1 \oplus tV \oplus t^2 S^2 V \oplus \cdots \oplus t^k S^k V + \cdots \tag{9}$$

where $S^k V$ denotes the k^{th} symmetric tensor power. We will also write $\frac{1}{1-tV}$ as $S_t V$.

Let $N_+ = \bigoplus_{\ell>0} N_\ell$, and let $\det N_+ \left(= \bigotimes_{\ell>0} \det N_\ell\right)$ denote the highest exterior power. Let n_ℓ be the dimension of N_ℓ. The generalization of (7) is then

$$F_\alpha(\theta) \;=\; \lambda_\alpha \left\langle \hat{A}(M_\alpha)\,\mathrm{ch}\left\{ \sqrt{\det N_+} \cdot \prod_{\ell>0} e^{i\theta\ell n_\ell/2} \cdot \bigotimes_{\ell>0} \frac{1}{1 - e^{i\ell\theta} N_\ell} \right\}, M_\alpha \right\rangle \tag{10}$$

(The combination $\sqrt{\det N_+} \cdot \prod_{\ell>0} e^{i\theta\ell n_\ell/2}$ should be viewed as the equivariant generalization of $\sqrt{\det N_+}$.) Here $\hat{A}(M_\alpha)$ is the total $\hat{A}$ class of M_α, and the sign λ_α is determined as before. The character-valued index of the Dirac operator is then $F(\theta) = \Sigma_\alpha F_\alpha(\theta)$. If $H_1(M_\alpha, Z_2) \neq 0$, then $\det N_+$ has various square roots and it is necessary to specify a consistent choice corresponding to a choice of spin structure on M.

All of this has the following generalization. Let V be a vector bundle on M to which the S^1 action on M has been lifted, and suppose $w_2(T) = w_2(V)$ (T being the tangent bundle of M). Then we can consider the character-valued index of a twisted Dirac operator

$$D_V \,:\, \Gamma(S_+ \otimes V) \longrightarrow \Gamma(S_- \otimes V) \tag{11}$$

The fixed point formula is now modified as follows. The restriction V_α of V to M_α will decompose as

$$V_\alpha = \bigoplus_\ell V_{\ell,\alpha} \tag{12}$$

where P acts on V_ℓ as multiplication by $i\ell$. (10) is then replaced by

$$\begin{aligned} F_\alpha^V(\theta) &= \lambda_\alpha \Big\langle \hat{A}(M_\alpha)\mathrm{ch}\Big\{\sqrt{\det N_+}\cdot\prod_{\ell>0} e^{i\theta\ell n_\ell/2}\cdot\bigotimes_{\ell>0}\frac{1}{1-e^{i\ell\theta}N_\ell}\Big\} \\ &\bigoplus_m e^{im\theta}V_{m,\alpha}\,,\, M_\alpha\Big\rangle \end{aligned} \tag{13}$$

Our goal is now to work out the formal analogues of (10) and (13) for the case of Dirac-like operators on the free loop space $\mathcal{L}M$ of maps $S^1 \to M$.

Regardless of the choice of M, $\mathcal{L}M$ always admits a natural S^1 action. Indeed, if σ is a standard angular parameter on S^1 (so $0 \le \sigma \le 2\pi$) then the translation $\sigma \to \sigma + c$, c a constant, induces an S^1 action on $\mathcal{L}M$. It is the character-valued index for this group action on $\mathcal{L}M$ that we will now study.

If X^i are coordinates for M, a map $S^1 \to M$ may be described explicitly in terms of functions $X^i(\sigma)$. To have a fixed point for the natural S^1 action on $\mathcal{L}M$ means that $X^i(\sigma)$ must be invariant under the translation $\sigma \to \sigma+c$; *i.e.* it must be a constant map $X^i(\sigma) = X_0^i$, with X_0^i the coordinates for a point in M. Thus, a fixed point of the S^1 action is simply a constant map $S^1 \to M$, and the fixed point set is just a copy of M, embedded in $\mathcal{L}M$. Our basic idea in these notes is to study M via this embedding in $\mathcal{L}M$, and we henceforth identify M with its image in $\mathcal{L}M$.

To describe the normal bundle to M in $\mathcal{L}M$, we note that an almost constant map $S^1 \to M$ is of the form

$$X^i(\sigma) = X_0^i + \sum_{n\neq 0} e^{in\sigma}\epsilon_n^i \tag{14}$$

Here, for each non-zero integer n, ϵ_n^i is a vector tangent to M at the point with coordinates X_0^i. Thus, the decomposition of the normal bundle of M in $\mathcal{L}M$ (analogous to (8)) is

$$N = \bigoplus_{\ell\neq 0} T_\ell \tag{15}$$

where the T_ℓ are all isomorphic to the tangent bundle T of M.

We may now readily determine the analogue of equation (10). As M is a spin manifold, and in particular orientable, $\det T$ is a trivial line bundle, and $\det N_+$, an infinite tensor product of trivial line bundles, can be dropped. ($\sqrt{\det N_+}$ is an arbitrary flat line bundle of order two, corresponding to a choice of spin structure on M.) The n_ℓ are all equal to $d = 2k = \dim M$. The analogue of $\Pi_{\ell>0} e^{i\theta\ell n_\ell/2}$ is thus

$$\left(\prod_{n=1}^{\infty} q^n\right)^{d/2} \tag{16}$$

where we have let $q = e^{i\theta}$. We interpret $\prod_{n=1}^{\infty} q^n = q^{\sum_{n=1}^{\infty} n}$ as $q^{\zeta(-1)} = q^{-1/12}$. Thus (16) becomes $q^{-d/24}$. The remaining factor in (10) is just $\bigotimes_{\ell=1}^{\infty} \frac{1}{1-q^\ell T} = \bigotimes_{\ell=1}^{\infty} S_{q^\ell} T$. Thus, the formal expression for the character-valued Dirac index on $\mathcal{L}M$ is

$$F(q) \ = \ q^{-d/24} \left\langle \hat{A}(M) \mathrm{ch} \bigotimes_{\ell=1}^{\infty} S_{q^\ell} T \, , \, M \right\rangle \tag{17}$$

The reason that the variable in (17) has been called q is as follows. It can be shown that (17) is of the form

$$\frac{\Phi(q)}{\eta(q)^d} \tag{18}$$

with $\eta(q) = q^{1/24} \prod_{\ell=1}^{\infty}(1-q^\ell)$ the Dedekind eta function and $\Phi(q)$ a modular form of weight $d/2$ for $SL(2,Z)$, provided $p_1(M) = 0$. The modular form $\Phi(q)$ is the so-called level one elliptic genus of M. The restriction on p_1 has to do with anomalies, and will be explained heuristically in the next section. The natural transformation law of (17) under $SL(2,Z)$, which may come as a surprise in the present exposition, has a simple explanation in terms of facts that are well known to physicists. Indeed, the Feynman path integral representation of $F(q)$ involves integration over maps $\Sigma \to M$, where Σ is an elliptic curve over C with complex structure determined by q; this representation leads naturally to an understanding of the behavior of (17) under modular transformations of q. I will not enter into this here (but see [6] for a qualitative discussion of an analogous question, namely the role of $SL(2,Z)$ in the theory of affine Lie algebras and in "monstrous moonshine"). A computational proof of the modular properties of (17) under the restriction $p_1(M) = 0$ can be given without reference to Feynman path integrals; see [2] and the article by Zagier in this volume.

We will now generalize (17) to a formal expression for the character-valued index of various twisted Dirac-like operators on $\mathcal{L}M$. The first case that we will consider is the signature operator on $\mathcal{L}M$. The signature operator is simply

$$D_S \, : \, \Gamma(S_+ \otimes S) \to \Gamma(S_- \otimes S) \tag{19}$$

where $S = S_+ \oplus S_-$ is the spin bundle. Thus we must construct the spin bundle S of $\mathcal{L}M$. Actually, since we are only dealing with fixed point formulas, we only need the restriction of S to $M \subset \mathcal{L}M$.

Let $\mathcal{T}$ be the tangent bundle of $\mathcal{L}M$, restricted to M. Then

$$\mathcal{T} = T \oplus N \tag{20}$$

with T the tangent bundle of M and N its normal bundle in $\mathcal{L}M$. For any real vector bundle A, let $\Delta(A)$ be the associated spinor bundle. Then $\Delta(A \oplus B) = \Delta(A) \otimes \Delta(B)$. Thus

$$S = \Delta(\mathcal{T}) = \Delta(T) \otimes \Delta(N) \tag{21}$$

We decompose N as in equation (15). It is convenient to combine T_ℓ and $T_{-\ell}$ together. So $N = \oplus_{\ell>0} K_\ell$ with $K_\ell = T_{-\ell} \oplus T_\ell$, and

$$\Delta(N) = \bigotimes_{\ell=1}^{\infty} \Delta(K_\ell) \tag{22}$$

In general, if $\bar{A}$ denotes the dual bundle of A, then

$$\Delta(A \oplus \bar{A}) = \sqrt{\det \bar{A}} \cdot \Lambda(A)$$

where $\Lambda(A) = 1 \oplus A \oplus \Lambda^2 A \oplus \cdots$ is the direct sum of the exterior powers. (For any vector space A, $\Lambda^k A$ will denote its k^{th} exterior power.) With $K_\ell = T_{-\ell} \oplus T_\ell$ (and $T_{\pm\ell}$ both isomorphic to T, so that $\sqrt{\det T_{-\ell}}$ is trivial), this shows that $\Delta(K_\ell) = \Lambda(T)$. We need, however, a more refined formula that keeps track of the S^1 action. With $K_\ell = q^{-\ell}T_{-\ell} \oplus q^\ell T_\ell$ this formula is

$$\Delta(K_\ell) = q^{-d\ell/2} \Lambda_{q^\ell} T \tag{23}$$

The spinor bundle of $\mathcal{L}M$, restricted to M, is thus

$$\Delta(\mathcal{T})|_M = q^{\frac{d}{24}} \Delta(T) \bigotimes_{\ell=1}^{\infty} \Lambda_{q^\ell} T \tag{24}$$

Here $\Lambda_t T$ represents $1 \oplus tT \oplus t^2 \Lambda^2 T \oplus \cdots$. More precisely, this is a description of what we *mean* by the spinor bundle of $\mathcal{L}M$; a priori, there are various conceivable choices. Putting (24) together with (13) and (17), we get a formal expression for the equivariant signature of $\mathcal{L}M$:

$$G(q) = \left\langle \hat{A}(M) \mathrm{ch} \bigotimes_{\ell=1}^{\infty} S_{q^\ell} T \bigotimes \Delta(T) \bigotimes_{m=1}^{\infty} \Lambda_{q^m} T \, , \, M \right\rangle \tag{25}$$

The Feynman path integral representation for (25) shows that it is of the form

$$G(q) = \Phi(q) \left(\frac{\eta(q^2)}{\eta^2(q)} \right)^d \tag{26}$$

where $\Phi(q)$ is a modular form for a level two subgroup Γ of $SL(2, Z)$ which has index three in $SL(2, Z)$. (There is no requirement here on $p_1(M)$.) The "level two" property arises because the

Feynman path integral representation of (25) involves integrating over maps $\Sigma \to M$ where now the elliptic curve Σ is endowed with a point of order two. As $[SL(2,Z) : \Gamma] = 3$, (25) can be transformed into two essentially different formulas by $SL(2,Z)$ transformations not in Γ. One of these is

$$H(q) = q^{-d/16} \left\langle \hat{A}(M) \bigotimes_{\ell=1}^{\infty} S_{q^\ell} T \bigotimes_{m=1/2,3/2,5/2,\cdots} \Lambda_{q^m} T \, , \, M \right\rangle \tag{27}$$

The other differs by $q^{1/2} \to -q^{1/2}$. (27) can be interpreted as the index of a sort of twisted version of the signature operator on $\mathcal{L}M$. The operator in question, while well known to physicists (it is the supercharge with right-moving Ramond and left-moving Neveu-Schwarz boundary conditions), does not have a finite dimensional analogue. Its existence is a characteristic feature of analysis on the infinite dimensional manifold $\mathcal{L}M$. I will now describe it briefly.

A point γ in $\mathcal{L}M$ is a loop

$$\gamma : S^1 \to M \tag{28}$$

Pulling back the tangent bundle T of M via γ gives a vector bundle $\gamma^* T$ over S^1. Let T_γ be the space of sections of $\gamma^* T$. The family $\{T_\gamma | \gamma \epsilon \mathcal{L}M\}$ form the fibers of an infinite dimensional vector bundle $\mathcal{T}$ over $\mathcal{L}M$; it is none other than the tangent bundle of $\mathcal{L}M$, which we discussed earlier.

Now, let ϵ be the Hopf bundle of S^1 —the unique non-trivial real line bundle over S^1. Let $\hat{T}_\gamma$ be the space of sections of $\epsilon \otimes \gamma^* T$. The family $\{\hat{T}_\gamma | \gamma \epsilon \mathcal{L}M\}$ are fibers of a vector bundle over $\mathcal{L}M$ which we may call $\hat{\mathcal{T}}$.

The restriction of $\hat{\mathcal{T}}$ to $M \subset \mathcal{L}M$ is particularly simple; it is

$$\hat{\mathcal{T}}|_M \simeq \bigoplus_{m \epsilon Z + \frac{1}{2}} q^m T_m \tag{29}$$

with T_m isomorphic to T for all $m \epsilon Z + \frac{1}{2}$. The factors q^m in (29) have been introduced to keep track of the S^1 action. From (29) one finds the analogue of (24):

$$\Delta(\hat{\mathcal{T}})|_M = q^{-\frac{1}{48}} \bigotimes_{m=1/2,3/2,5/2,\cdots} \Lambda_{q^m} T \tag{30}$$

From here one obtains (27) as the index of the Dirac operator on $\mathcal{L}M$ twisted by $\Delta(\hat{\mathcal{T}})$. The Feynman path integral gives a conceptual explanation, well known among string theorists, of the fact that the Dirac operator twisted by $\Delta(\mathcal{T})$ is related by $SL(2,Z)$ to that twisted by $\Delta(\hat{\mathcal{T}})$. The fact that (25) and (27) are related by $SL(2,Z)$ can also be verified computationally.

Now, I wish to consider some generalizations that depend on the choice of a vector bundle V on M. For reasons that will be discussed qualitatively in the next section, a good theory will emerge only if $w_2(V) = w_2(T)$ and $p_1(V) = p_1(T)$.*

Considering again a loop $\gamma : S^1 \to M$, we pull back V to $\gamma^* V$, and let V_γ denote the space of sections of $\gamma^* V$. Let $\mathcal{V}$ be the vector bundle over $\mathcal{L}M$ whose fiber at $\gamma \epsilon \mathcal{L}M$ is V_γ. The index of the Dirac operator on $\mathcal{L}M$ twisted by $\Delta(\mathcal{V})$ is

$$F_V(q) \;=\; q^{-d/24} q^{n/24} \left\langle \hat{A}(M)\mathrm{ch} \bigotimes_{\ell=1}^{\infty} S_{q^\ell} T \bigotimes \Delta(V) \bigotimes_{m=1}^{\infty} \Lambda_{q^m} V,\, M \right\rangle \tag{31}$$

as one can see by repeating the arguments that led to (25). Here n is the dimension of V. Equation (31) is of the form

$$\Phi(q) \cdot \frac{\eta(q^2)^d}{\eta(q)^{d+n}} \tag{32}$$

with $\Phi(q)$ a modular form of weight $d/2$ for a level two subgroup of $SL(2, Z)$, if $p_1(V) = p_1(T)$.

Finally, if V is even dimensional, then we can make a decomposition

$$\Delta(\mathcal{V}) \;=\; \Delta_+(\mathcal{V}) \oplus \Delta_-(\mathcal{V}) \tag{33}$$

analogous to the decomposition of the spin bundle in finite dimensions. Then we consider the Dirac index twisted by $\Delta_+(\mathcal{V}) \ominus \Delta_-(\mathcal{V})$ instead of $\Delta_+(\mathcal{V}) \oplus \Delta_-(\mathcal{V})$ to get

$$J_V(q) \;=\; q^{-\frac{(d-n)}{24}} \left\langle \hat{A}(M)\mathrm{ch} \bigotimes_{\ell=1}^{\infty} S_{q^\ell} T \bigotimes (\Delta_+(V) \ominus \Delta_-(V)) \bigotimes_{m=1}^{\infty} \Lambda_{-q^m} V,\, M \right\rangle \tag{34}$$

If $n = \dim V$ exceeds $d = \dim M$, this is zero. For $n = d$, $J_V(q)$ is independent of q and equals the Euler characteristic of V. For $n < d$, (34) is of the form

$$\frac{\Phi(q)}{\eta(q)^{d-n}} \tag{35}$$

with $\Phi(q)$ a modular form of weight $(d-n)/2$ for $SL(2, Z)$. In contrast to our previous examples, (34) is an unstable characteristic class which might be regarded as the analogue in elliptic cohomology of the Euler characteristic of a vector bundle. Despite being unstable, (34) is the subject of an interesting mathematical theory; the theorem that we will discuss in section 4 concerning the vanishing of equivariant characteristic numbers applies to (34) as well as (31).

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* For a real vector bundle (or virtual bundle such as $T \ominus V$) with $w_2 = 0$, there is a natural way to define $p_1/2$ (*i.e.*, there is a characteristic class λ with $2\lambda = p_1$), and the equality $p_1(V) = p_1(T)$ must be taken to include this twofold refinement.

3. Anomalies

In this section we will indicate a few aspects of the relation between anomalies and the Dirac operator in loop space. The whole subject of anomalies is quite vast, and we will content ourselves with drawing attention to a few relevant points. Our main purpose is to indicate why the (untwisted) Dirac operator on $\mathcal{L}M$ only makes sense if $p_1(M) = 0$. This was originally discovered at the rational level in [7]. In [1], global anomalies were used to obtain the restriction on p_1 (or actually $p_1/2$) as an *integral* cohomology class. In my brief remarks here, I will follow a viewpoint proposed by Killingback [8]; I refer to his paper for more detail.

Let M be a manifold of finite dimension n. The structure group of the tangent bundle T is $SO(n)$. The spinor bundle $\Delta(T)$, if it exists, has structure group Spin (n), this being the simply connected double cover of $SO(n)$:

$$0 \to Z_2 \to \mathrm{Spin}(n) \to SO(n) \to 0 \tag{36}$$

As Z_2 is a discrete group, the group extension in (36) is trivial at the Lie algebra level, and shows up only globally. The obstruction to lifting from $SO(n)$ to Spin(n) involves a two dimensional cohomology class, $w_2(M)$, which must vanish if $\Delta(T)$ is to exist.

Now let us consider the analogous issues on $\mathcal{L}M$, the loop space of the n dimensional manifold M. The structure group of the tangent bundle $\mathcal{T}$ of $\mathcal{L}M$ is naturally $\mathcal{L}\,SO(n)$ — the loop group of $SO(n)$. The fundamental difference between the finite and infinite dimensional cases arises when one attempts to construct the spinor bundle $\Delta(\mathcal{T})$. Constructing $\Delta(\mathcal{T})$ will of course involve a central extension of $\mathcal{L}\,SO(n)$, as in (36). The essential novelty of loop space is that the central extension that arises is by the Lie group $U(1)$ rather than the discrete group Z_2:

$$0 \to U(1) \to \mathcal{G} \to \mathcal{L}\,SO(n) \to 0 \tag{37}$$

The Lie algebra corresponding to $\mathcal{G}$ is the so-called affine Lie algebra $\widehat{SO}(n)$; up to normalization of the central generator it is the unique non-trivial central extension of the Lie algebra of $\mathcal{L}SO(n)$. It is $\mathcal{G}$ which is the structure group of the spinor bundle $\Delta(\mathcal{T})$ of $\mathcal{L}M$, as this spinor bundle was described in the last section.

By way of explaining that last statement, I will only observe the following. We have already described in equation (24) the restriction of $\Delta(\mathcal{T})$ to $M \subset \mathcal{L}M$:

$$\Delta(\mathcal{T})\,|_M \;=\; q^{d/24}\,\Delta(T) \bigotimes_{\ell=1}^{\infty} \Delta_{q^\ell} T \tag{38}$$

From the construction of $\Delta(\mathcal{T})|_M$, one might expect that the Lie algebra of $\mathcal{L}\,SO(n)$ would act naturally on $\Delta(\mathcal{T}\,|_M)$. Instead, in trying to implement in the infinite dimensional context standard formulas for the spin representation, one finds that the Lie algebra that acts naturally on the module (38) is the central extension $\widehat{SO}(n)$ [9]. Indeed, (38) is a description of the fundamental spin representation of $\widehat{SO}(n)$; it represents in a way the most elementary construction of a highest weight module for any affine Lie algebra.

Anyway, the fact that the extension in (37) is an extension of $\mathcal{L}SO(n)$ by $U(1)$ means that the obstruction to the existence of $\Delta(\mathcal{T})$ is not a torsion element as in finite dimensions but a cohomology class with $U(1)$ coefficients. In fact, the projective spin bundle $P\Delta(\mathcal{T})$ (the bundle whose fiber at $\gamma\epsilon\mathcal{L}M$ is the complex projective space consisting of lines through the origin in the fiber at γ of $\Delta(\mathcal{T})$) always exists, regardless of the topology of M. Given $P\Delta(\mathcal{T})$, the obstruction to constructing $\Delta(\mathcal{T})$ is a three dimensional cohomology class on $\mathcal{L}M$. This in turn is related by transgression to a four dimensional cohomology class on M, namely $\frac{1}{2}p_1(M)$. Vanishing of $p_1(M)$ permits one to define the spinor bundle $\Delta(\mathcal{T})$ and thus the Dirac operator on $\mathcal{L}M$.* For elucidation of this I refer again to [8].

Now, let us briefly consider some of the twisted Dirac operators that figured in section 2. First, we consider the signature operator. On a finite dimensional manifold M, the signature operator σ can be defined if and only if M is orientable. Thus, to discuss the signature operator on $\mathcal{L}M$, we must know whether $\mathcal{L}M$ is orientable. I argued in [1] that $\mathcal{L}M$ should be considered orientable precisely if M is a spin manifold. This in any case is definitely the criterion that is relevant in quantum field theory. The quantum field theory that leads to the index (25) makes sense if and only if M is spin, and thus even though the formula (25) for the so-called Jacobi elliptic genus makes sense whenever M is orientable, any property of this formula that is proved using quantum field theory will require a spin condition.

In (31) we generalized the index of the signature operator to an index of a Dirac operator twisted by $\Delta(\mathcal{V})$, $\mathcal{V}$ being a bundle on $\mathcal{L}M$ obtained from an underlying vector bundle V on M. In this construction it is the tensor product of $\Delta(\mathcal{T})$ with the dual of $\Delta(\mathcal{V})$ that must exist, so the necessary requirement is $p_1(V) = p_1(T)$. The same is true for the "Euler characteristic" (34). Thus, our basic statement in the next section about equivariant constancy of (31) and (34) will require $p_1(V) = p_1(T)$.

Finally, with an eye toward the next section, we must discuss the equivariant generalization of the above, to a situation in which a group G is acting on M. We have then a fibration

$$\begin{array}{ccc} M & \longrightarrow & X \\ & & \downarrow \; \pi \\ & & BG \end{array}$$

over the classifying space BG. Let T_G be the tangent bundle to the fibers of this fibration.

It is possible to define a spin bundle $\Delta(T)$ and a Dirac equation on the manifold M if $w_2(T)$, the second Stiefel-Whitney class of the tangent bundle T, vanishes (one often denotes $w_2(T)$ as $w_2(M)$).

$\star$ It is conceivable but not likely that these can be defined if the following somewhat weaker condition is obeyed. If $X \subset M$ is a four dimensional cycle that can be fibered over a circle, then $< p_1(M), X >$ should vanish.

Given a group G that acts on a spin manifold M, it is not necessarily possible to choose on M a spin structure that admits an action of G, and thus G cannot necessarily act as a symmetry of the Dirac equation. The condition for being able to choose a G-equivariant spin bundle is that $w_2(T)$ must vanish in the equivariant sense. In fact, one needs $w_2(T_G) = 0$, with T_G the vector bundle described in the previous paragraph. More generally, if we are given a manifold M and a vector bundle V, we can define a Dirac equation for spinors twisted by V if and only if $w_2(T) = w_2(V)$. And in an equivariant situation, with a group G acting on M, T, and V, the Dirac equation for spinors twisted by V can be chosen to admit a G action if $w_2(T_G) = w_2(V_G)$. The definition of V_G is analogous to that of T_G : given the vector bundle V over M with G action, there is a natural vector bundle V_G over $X \to BG$ whose restriction to each fiber is isomorphic to V.

In the next section, we will study the Dirac operator on $\mathcal{L}M$ in a situation in which a group G is acting on M. Thus, we will need the generalization to $\mathcal{L}M$ of the above remarks. We already know that the Dirac operator on $\mathcal{L}M$ makes sense only if $p_1(T) = 0$ (or $p_1(T) = p_1(V)$, if one has twisted by a vector bundle $\Delta(\mathcal{V})$ derived from V as in the previous section). The obvious equivariant generalization of that restriction would be to say that the Dirac equation on $\mathcal{L}M$ should admit a G action only if $p_1(T_G) = 0$ (or more generally $p_1(T_G) = p_1(V_G)$). In physical terms, this restriction arises because if $p_1(T) = p_1(V)$, so that the Dirac equation in loop space makes sense, but $p_1(T_G) \neq p_1(V_G)$, then the G symmetry is violated by anomalies in instanton amplitudes, that is, the contributions to the Feynman path integral from homotopically non-trivial maps $X : \Sigma \to M$ (Σ is an elliptic curve) do not respect the G symmetry.†

Therefore, in formulating in the next section a theorem about vanishing of certain equivariant characteristic numbers, we will require a hypothesis that $p_1(T_G) = p_1(V_G)$. A natural and general way to obey this restriction is to pick $V = T$. This amounts to studying the signature operator on $\mathcal{L}M$, whose equivariant index was given in equation (25). However, our results will be valid for any V such that $p_1(T_G) = p_1(V_G)$.

4. Circle Actions and Characteristic Numbers

In this section, I will sketch the application of Dirac operators in loop space —or in other words of supersymmetric nonlinear sigma models —to a topological problem involving the vanishing of certain equivariant characteristic numbers.

Let M be an n-dimensional spin manifold that admits the action of a compact connected Lie group G. In what follows, it will be sufficient to consider the case $G = S^1$, as the generalization of our statements to arbitrary G follows by considering suitable S^1 subgroups of G.

† It is conceivable that the restriction $p_1(T_G) = p_1(V_G)$ can be relaxed slightly along the lines of the previous footnote.

Let K be the vector field that generates the S^1 action, and L_K the corresponding Lie derivative. Let $S_\pm$ be the spin bundles and T the tangent bundle of M. For any representation R of Spin(n), let T_R be the corresponding associated bundle derived from T; thus, T_R is simply T if R is the n dimensional real vector representation of Spin(n).

Picking an S^1 invariant metric on M, we consider then the Dirac operator

$$D_R : \Gamma(S_+ \otimes T_R) \longrightarrow \Gamma(S_- \otimes T_R) \tag{39}$$

where $\Gamma(V)$ denotes the smooth sections of a vector bundle V. The index of D_R is the dimension of its kernel minus that of its cokernel.

Since D_R and L_K commute, the operator D_R can be restricted to eigenspaces of L_K. For any vector bundle V with S^1 action, let $\Gamma_k(V)$ denote the space of sections ψ of V with eigenvalue k of L_K. The values of k that appear will be integers or half-integers if the S^1 action is "even" or "odd". Let $D_R^{(k)}$ be the restricted Dirac operator

$$D_R^{(k)} : \Gamma_k(S_+ \otimes T_R) \longrightarrow \Gamma_k(S_- \otimes T_R) \tag{40}$$

and let $c_{R,k}$ be its index. We will be discussing some vanishing theorems for the $c_{R,k}$. It is often convenient to combine the $c_{R,k}$ in a function

$$f_R(\theta) = \Sigma\, e^{ik\theta} c_{R,k} \tag{41}$$

If R is the trivial representation, then the $c_{R,k}$ are zero for all k, by a theorem of Atiyah and Hirzebuch [10]. It seems that the trivial representation is the only representation with this property. If, however, R is the spin representation, then D_R is essentially the signature operator, whose zero eigenvalues are harmonic differential forms, and it is a relatively elementary result that $c_{R,k} = 0$ for $k \neq 0$. One is naturally led to ask whether there are other representations R with that property. Some years ago, I conjectured [11] that the $c_{R,k}$ for $k \neq 0$ all vanish if R is the vector representation of Spin(n).‡ In trying to understand this conjecture, Landweber and Stong generalized it [12] to a certain infinite series of representations, which turned out, through the efforts of Ochanine, the Chudnovskys, and Zagier, to be naturally described in terms of certain elliptic modular functions [13-15]. It has emerged that the series of representations in question correspond precisely to the signature operator in $\mathcal{L}M$ (free loop space) described in section 2. I would now like to sketch how the vanishing of $c_{R,k}$ for $k \neq 0$, for the relevant R, follows formally from simple properties of the Dirac operator on $\mathcal{L}M$ or, if you will, from certain standard conjectures about quantum field theory.

‡ This result was by no means desirable at the time, since it frustrated certain efforts to explain parity violation in weak interactions.

First, let us state a precise conjecture. We define a sequence R_n of representations of Spin(n) by the formula

$$\sum_{n=0}^{\infty} q^n R_n \;=\; \Delta(T) \bigotimes_{\ell=1}^{\infty} S_{q^\ell}(T) \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(T) \tag{42}$$

Here T and $\Delta(T)$ are symbols that represent, respectively, the vector and spinor representations of Spin(n); one is to formally expand the right-hand side in powers of q, the coefficient of q^n being R_n. These representations are the ones that were found in section 2 to arise in the study of the signature operator in loop space. The S^1 equivariant signature of loop space is according to equation (25) precisely

$$G(q) \;=\; \sum_{n=0}^{\infty} q^n \,\mathrm{index}\,(R_n) \tag{43}$$

where index (R_n) denotes the index of the Dirac operator acting on sections of $S_+ \otimes R_n$. We consider here the signature operator rather than one of the other operators on $\mathcal{L}M$ considered in section 2 because, as sketched in the last section, the signature operator is canonically free of all anomalies.

We wish to consider now a situation in which we are given a non-trivial S^1 action on M, generated by a vector field K. We refine the quantity index(R_n) to an equivariant index, as indicated in our introductory discussion. Thus, we let

$$c_{n,k} \;=\; (\mathrm{index}\, D_{R_n})_{L_K=k}\,. \tag{44}$$

The integer $c_{n,k}$ is thus simply the index of the n^{th} Dirac operator D_{R_n} restricted to the k^{th} eigenspace of L_K, $\Gamma_k(S_+ \otimes R_n)$. The conjecture that we wish to discuss asserts that $c_{n,k} = 0$ for $k \neq 0$.

It is useful to form the generating functional

$$G(q,\theta) \;=\; \Sigma c_{n,k}\, q^n\, e^{ik\theta}\,. \tag{45}$$

Here $G(q,\theta)$ can be regarded as the $S^1 \times S^1$ equivariant signature of $\mathcal{L}M$, the first S^1 being the universal circle action on $\mathcal{L}M$ and the second one being the circle action on $\mathcal{L}M$ induced from the S^1 action on M generated by K. At $\theta = 0$, $G(q,\theta)$ reduces to $G(q)$ as defined earlier.

In what follows, the series on the right-hand side of (45) can be viewed as a purely formal series, whose convergence is not relevant. We will be making statements about the integers $c_{n,k}$ that were concretely defined in equation (44).

The statement that $c_{n,k} = 0$ for $k \neq 0$ follows from two properties. One of these

$$c_{n,k} \;=\; 0 \qquad \text{for} \quad n < 0 \tag{46}$$

is implicit in (42) and reflects at a more fundamental level the fact that the energy of the supersymmetric nonlinear sigma model is non-negative. The second property we will require is that

$$c_{n,k} = c_{n-k,k}. \tag{47}$$

This will follow from simple considerations involving the signature operator on a twisted loop space. From (47) one observes that $c_{n,k} = c_{n-tk,k}$ for any integer t, and choosing t so $|tk| > n$ it then follows from (46) that $c_{n,k} = 0$ for $k \neq 0$.

To prove (47) we introduce the "twisted loop space." Looking at S^1 as $R/2\pi Z$, R being the real line, we regard maps $S^1 \to M$ as maps $R \to M$ that are invariant under 2π translation. Thus, a map $\sigma \to X(\sigma)$ of $R \to M$ (σ and X denote points in R and M, respectively) induces a map $S^1 \to M$ precisely if

$$X(\sigma + 2\pi) = X(\sigma) \tag{48}$$

Now, for $g \epsilon S^1$, we define the twisted loop space $(\mathcal{L}M)^g$ to consist of maps $R \to M$ such that

$$X(\sigma + 2\pi) = gX(\sigma) \tag{49}$$

$(\mathcal{L}M)^g$ can be viewed as the space of sections of a certain M bundle

$$\begin{array}{ccc} M & \longrightarrow & W \\ & & \downarrow \\ & & S^1 \end{array} \tag{50}$$

over S^1.

It is easy to see that $(\mathcal{L}M)^g$ is a manifold, unlike —say —the naive quotient M/S^1 to which it is in some ways similar. In fact, the fiber bundle in (50) is trivial (but has no canonical trivialization), and as a manifold $(\mathcal{L}M)^g$ is isomorphic (but not canonically) to $\mathcal{L}M$.

Let us compare $(\mathcal{L}M)^g$ to $\mathcal{L}M$ from the standpoint of the group action that we are considering. Let P generate the rotation of S^1(thus$P \sim d/d\sigma$). Thus on $\mathcal{L}M$ or $(\mathcal{L}M)^g$, P generates the transformation $X(\sigma) \to X(\sigma + c)$, c being a constant. On $\mathcal{L}M$ it follows from (48) that

$$\exp 2\pi P = 1 \tag{51}$$

but on $(\mathcal{L}M)^g$ it follows from (49) that

$$\exp 2\pi P = g. \tag{52}$$

On $\mathcal{L}M$, P and K generate a group F isomorphic to $S^1 \times S^1$. On $(\mathcal{L}M)^g$, P and K generate a two dimensional abelian group F_g which is isomorphic to F but with no canonical choice of an isomorphism. We can therefore reasonably hope to learn something essentially new by studying equivariant index problems on $(\mathcal{L}M)^g$. Actually, it turns out that it is crucial to study the one parameter family $(\mathcal{L}M)^g$, $g \epsilon S^1$.

A formal expression for the equivariant index of the signature operator on $(\mathcal{L}M)^g$ could be written down as in section 2. However, we will not need this formula. It will suffice to observe the following. Let $g = e^{2\pi\alpha K}$, $0 \leq \alpha \leq 1$. Thus, α is an angular parameter on the S^1 group generated by K. At $\alpha = 0$, (51) asserts that the eigenvalues of P are integers. At $\alpha \neq 0$, this is not so. In an eigenspace of K (or more precisely of L_K) with eigenvalue k, (52) asserts that the eigenvalues of P are of the form $n + \alpha k$, n being an integer. Thus, the generalization of (45) to an equivariant index on $(\mathcal{L}M)^g$ at $g = \exp 2\pi\alpha K$ is of the general form

$$G_\alpha(q,\theta) \;=\; \Sigma\, c_{n,k}(\alpha)\, q^{n+\alpha k}\, e^{ik\theta}\,. \tag{53}$$

Now, in giving a sound mathematical basis for the supersymmetric nonlinear sigma model (even in a cutoff version), one would expect to prove that the signature operator on loop space has formal properties analogous to the properties of an elliptic operator in finite dimensions. In particular, the spectrum should vary smoothly with parameters such as α. This implies that the $c_{n,k}(\alpha)$ must be independent of α, since they arise in the solution of an equivariant index problem, and continuity of the spectrum implies that the index of an operator is invariant under continuous change of parameters.

Given that the $c_{n,k}(\alpha)$ are independent of α, we observe next that the index problem at $\alpha = 1$ is the same as the index problem at $\alpha = 0$. Setting $G_1(q,\theta) = G_0(q,\theta)$, we learn that

$$c_{n+k,k} \;=\; c_{n,k} \tag{54}$$

and as discussed earlier, this implies that $c_{n,k} = 0$ for $k \neq 0$.

The above may perhaps be clarified by the following considerations. On the infinite dimensional manifold $\mathcal{L}M$, if one works with the naive de Rham complex of differential forms of finite order, there is no analogue of Poincaré duality. The supersymmetric nonlinear sigma model corresponds rather to a theory in which one works near the "middle dimension" of the de Rham complex of $\mathcal{L}M$, with "semi-infinite" forms. The use of semi-infinite forms is implicit in the choice (24) of what we mean by the spin bundle of $\mathcal{L}M$. With the definitions as usually made in quantum field theory, there is in the supersymmetric nonlinear sigma model a transformation $*$ of the spin complex* which has the formal properties of Poincaré duality. In particular, the character-valued signature is

$$G(q,\theta) \;=\; Tr(*q^P\, e^{i\theta L_K})\,, \tag{55}$$

the trace running over the Hilbert space of harmonic sections of the signature complex —*i.e.*, zero energy states of the supersymmetric nonlinear sigma model.

* To physicists it is the operator $(-1)^{F_L}$ that changes the sign of left-moving fermions but commutes with right movers.

Implicit in (55) is the statement that $G(q,\theta)$ exists as a function in some range of q and θ where the trace on the right-hand side converges. In our previous discussion, we treated $G(q,\theta)$ as a formal series. Indeed, one of the fundamental tasks in gaining mathematical understanding of the nonlinear sigma model is to show that the so called partition function converges absolutely for $|q| < 1$. As $G(q,\theta)$ is bounded above by the partition function, it should converge absolutely for $|q| < 1$, uniformly in θ.

In our previous discussion, we observed that the twisted loop space $(\mathcal{L}M)^g$ is isomorphic to $\mathcal{L}M$ but not canonically isomorphic to it. The obstruction to finding a natural identification of $(\mathcal{L}M)^g$ with $\mathcal{L}M$ really lies in the existence of a certain natural automorphism $\tau : \mathcal{L}M \to \mathcal{L}M$ which has been hidden in the previous discussion and which we will now make explicit. For a point $\gamma \epsilon \mathcal{L}M$, represented by a loop $\sigma \to X(\sigma)$, let $\tau\gamma$ be the loop

$$\sigma \longrightarrow (\exp \sigma L_K)\, X(\sigma)\,. \tag{56}$$

One readily sees that the transformation τ so defined obeys

$$\begin{aligned} \tau^{-1} L_K \tau &= L_K \\ \tau^{-1} P \tau &= P + L_K \end{aligned} \tag{57}$$

(The latter comes from the fact that P is essentially $d/d\sigma$.)

In finite dimensions, we would now pick a τ invariant metric in the definition of the Hodge dual operator $*$, and claim

$$\begin{aligned} G(q,\theta) &= Tr * q^P e^{i\theta L_K} \\ &= Tr * \tau^{-1}(q^P e^{i\theta L_K})\tau \\ &= Tr * q^{P+L_K} e^{i\theta L_K} \\ &= G(q, \theta - i\ell nq)\,. \end{aligned} \tag{58}$$

(58) is in fact valid; it is easily seen to be equivalent to the relation (54) from which we originally extracted the fact that $c_{n,k} = 0$ for $k \neq 0$. However, in infinite dimensions the argument in (58) is too facile. Although we may well pick a K invariant metric on M, which will induce a τ invariant metric on $\mathcal{L}M$, the quantum field theory construction of "elliptic operators on $\mathcal{L}M$" involves additional terms, described in [1], which are not τ invariant. The supercharge D of the nonlinear sigma model (*i.e.* the Dirac operator in loop space) is transformed by τ into, say, $D^\tau = \tau^{-1} D \tau$. To justify (58), we must know that D and D^τ have the same equivariant index. This is so, since the family $D_x = xD + (1-x)D^\tau$, $0 \le x \le 1$, gives an equivariant interpolation from D to D^τ, and the D_x can be seen formally to be well-behaved operators on $\mathcal{L}M$. Consideration of the family D_x, with $D_{x=1}$ unitarily equivalent (via τ) to $D_{x=0}$, is an equivalent but perhaps illuminating way to re-express the spectral flow argument that we gave originally for our basic result on the $c_{n,k}$.

Together with the obvious fact $G(q,\theta) = G(q,\theta+2\pi)$, (58) states that for fixed q, $G(q,\theta)$ is a doubly periodic function on the complex plane with periods 2π and $-i\ell nq$. In other words, $G(q,\theta)$ is an elliptic function on the elliptic curve $\Sigma = \mathcal{C}/\Lambda$, $\mathcal{C}$ being the complex plane and Λ the lattice generated by 2π and $-i\ell nq$. The doubly periodic character of $G(q,\theta)$ was first observed (and proved) by Ochanine, who defined $G(q,\theta)$ for real θ as a graded sum of character-valued indices of the operators D_{R_n}, and then showed (from the fixed point formula for the index of D_{R_n}) that the analytic continuation of $G(q,\theta)$ to complex θ is doubly periodic.

Actually, Feynman path integrals give an attractive conceptual explanation for the doubly periodic nature of $G(q,\theta)$. The "ordinary" equivariant signature $G(q,0)$ of loop space has a representation in terms of integrals over maps $\Sigma \to M$, Σ being the elliptic curve with periods 2π, $-i\ell nq$. The refinement $G(q,\theta)$ has an analogous representation in terms of integration over the space of sections of a twisted bundle,

$$\begin{array}{ccc} M & \longrightarrow & W \\ & & \downarrow \\ & & \Sigma \end{array} \tag{59}$$

The twisted bundles of interest are flat bundles, constructed by choosing a homomorphism $\pi_1(\Sigma) \to G$, G being a compact group that acts on M. With $G = S^1$, the choice of a homomorphism $\pi_1(\Sigma) \to G$ amounts to a choice of a point θ on the Jacobian of Σ. The function $G(q,\theta)$ that we have been discussing can be obtained as an integral over sections of the bundle (59), and its various properties (holomorphic in q and θ, doubly periodic in θ for fixed q, natural transformation law under a congruence subgroup of $SL(2,Z)$) follow from this representation by arguments that are fairly standard once one has developed a certain amount of quantum field theory machinery.

If one can show that the double periodic function $G(q,\theta)$ has no poles as a function of θ for fixed q, then, since an elliptic function without poles must be constant, $G(q,\theta)$ must be independent of θ. That $G(q,\theta)$ is independent of θ is precisely our desired result $c_{n,k} = 0$, $k \neq 0$. From the standpoint of quantum field theory, the absence of poles in $G(q,\theta)$, as a function of θ for fixed q, follows from the fact that the series defining $G(q,\theta)$ is absolutely convergent for $|q| < 1$, being dominated by the partition functions of supersymmetric nonlinear sigma models on the twisted loop spaces $(\mathcal{L}M)^g$. This reasoning might give a somewhat different approach to using quantum field theory to prove the result about the $c_{n,k}$.

We have here concentrated on the signature operator on $\mathcal{L}M$ because this is in many ways the most canonical case in which the anomaly criterion of section 3 is obeyed. However, if V is any vector bundle obeying the criterion $p_1(V_G) = p_1(T_G)$ discussed in section 3, then the arguments that we have given generalize immediately to the "Dirac operator with values in V" and the "elliptic Euler characteristic of V"; the relevant formulas were given in equations (31) and (34). In each case one gets $c_{n,k} = 0$ for $k \neq 0$.

5. Complex Manifolds

So far in this paper, we have been working in the context of real differential geometry. However, the supersymmetric nonlinear sigma model has particularly rich properties for special classes of manifolds —almost complex manifolds, Kahler manifolds, and hyper-Kahler manifolds. It is natural to expect that the theorem on equivariant characteristic numbers that we discussed in the last section would have analogues for those special classes of manifolds. The purpose of this section is to make a start in this direction. In particular, we will sketch a generalization to elliptic genera of certain results of Hattori [16].

To begin with, let M be a manifold of dimension d, and let W be a complex vector bundle over M of complex dimension k, endowed with a hermitian metric. Let $\overline{W}$ be the dual of W. Then $W \oplus \overline{W} = V_C$ is naturally the complexification of a $2k$ dimensional real vector bundle V. Let L be the canonical line bundle $L = \Lambda^k W$. We will be interested in a situation in which for some integer $N > 1$, L has an N^{th} root ξ. In other words, we will assume the existence of a line bundle ξ with $\xi^N = L$. We will describe constructions that associate to this data modular forms for the subgroups $\Gamma_0(N)$ of $SL(2,Z)$. For $N = 2$ the constructions do not require the complex structure of V and reduce to those of section 2.

A section of V is a pair $(f, \bar{f})$, with f a section of W and $\bar{f}$ its complex conjugate. If ζ is a complex number of modulus one, then $(\zeta f, \bar{\zeta}\bar{f})$ is likewise a section of V; this gives an action of the group H of complex numbers of modulus one on sections of V.

As in section 2, let $\gamma : S^1 \to M$ be a loop in M, that is, a point in $\mathcal{L}M$. Let $\gamma^* V$ be the pullback of V from M to S^1 via γ, and let V_γ be the space of sections of $\gamma^* V$. Then H acts on V_γ in an obvious way, *i.e.* $(f(\sigma), \bar{f}(\sigma)) \to (\zeta f(\sigma), \zeta^{-1}\bar{f}(\sigma))$. As γ varies, the V_γ are fibers of a vector bundle $\mathcal{V}$ over $\mathcal{L}M$. We already described in equation (31) the spinor bundle $\Delta(\mathcal{V})$ restricted to $M \subset \mathcal{L}M$:

$$\Delta(\mathcal{V})|_M = \Delta(V) \bigotimes_{m=1}^{\infty} \Lambda_{q^m} V \tag{60}$$

We now wish to refine this to keep track of the action of complex numbers of modulus one. We write symbolically $V = \zeta W \oplus \zeta^{-1}\overline{W}$, and observe that

$$\Lambda_{q^m} V = \Lambda_{q^m}(\zeta W \oplus \zeta^{-1}\overline{W}) = \Lambda_{q^m}(\zeta W) \otimes \Lambda_{q^m}(\zeta^{-1}\overline{W}). \tag{61}$$

Also, from standard facts about the spin representation, it follows that

$$\begin{aligned} \Delta(V) &= \Delta(\zeta W \oplus \zeta^{-1}\overline{W}) \\ &= \zeta^{k/2} L^{1/2} \Lambda(\zeta^{-1}\overline{W}) \end{aligned} \tag{62}$$

with $L^{1/2}$ a square root of L. The refinement of (60) to keep track of the action of H is thus

$$\Delta(\mathcal{V})|_M = \zeta^{k/2} L^{1/2}\, \Lambda(\zeta^{-1}\overline{W}) \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\zeta W) \bigotimes_{r=1}^{\infty} \Lambda_{q^r}(\zeta^{-1}\overline{W}) . \tag{63}$$

Now, we consider the Dirac index for spinors on $\mathcal{L}M$ twisted by $\Delta(\mathcal{V})$. More precisely, we wish to consider the equivariant Dirac index, equivariant under the natural circle action on $\mathcal{L}M$ and the action of H. The generalization of equation (31) is simply

$$\begin{aligned} F_V(q,\zeta) = q^{-\frac{(d-2k)}{24}} \Big\langle \hat{A}(M) \mathrm{ch} \bigotimes_{\ell=1}^{\infty} S_{q^\ell} T \otimes \zeta^{k/2} L^{1/2} \Lambda(\zeta^{-1}\overline{W}) \\ \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(\zeta W) \bigotimes_{r=1}^{\infty} \Lambda_{q^r}(\zeta^{-1}\overline{W}) , M \Big\rangle \end{aligned} \tag{64}$$

For (64) to be the equivariant index of an operator on $\mathcal{L}M$, it is necessary to impose certain restrictions on characteristic classes, which in our previous discussion were $w_2(V) = w_2(T), p_1(V) = p_1(T)$. The main novelty is that the requirement $p_1(T) = p_1(V)$ must be imposed equivariantly with respect to the action of H. (T is the tangent bundle of M.) This condition is in fact obeyed for the whole group $H \cong U(1)$ if and only if L is trivial. If, however, L is not trivial, but is the N^{th} power of a line bundle ξ, then (64) is an equivariant index provided that $(-\zeta)^N = 1$. In this case, (64) is a modular form for a congruence subgroup of $SL(2, Z)$ conjugate to $\Gamma_1(N)$. For special cases, the restriction $(-\zeta)^N = 1$ and the statement that (64) gives modular forms of $\Gamma_0(N)$ have already entered in section 2. For $N = 1$, $\zeta = -1$, this is the statement that the "Euler characteristic of a vector-bundle" given in equation (34) is a modular form for $\Gamma_1(1) = SL(2, Z)$. For $N = 2$, $\zeta = 1$, this is the statement that the "signature of a vector bundle" (31) is a modular form of level two. The physical origin of the restriction on ζ is that if $-\zeta$ is of order N, then instanton amplitudes are invariant under the twist by ζ only if the first Chern class of L is divisible by N or in other words only if L has an N^{th} root. I will not attempt here to elucidate the connection of this statement with the equivariant p_1.

Since (64) leads to modular forms of level N, by making transformations by $SL(2, Z)$ matrices that are not in $\Gamma_1(N)$, (64) can be related to various other and essentially equivalent formulas. Let $-\zeta$ be a primitive N^{th} root of 1, and let s be any positive integer in the range $1 \le s \le N-1$ which is relatively prime to N. Then a suitable $SL(2, Z)$ transformation will transform (64) into the form

$$\begin{aligned} \tilde{F}_V(q,s) = q^{-\frac{d}{24} - \frac{(1-3(1-2s/N)^2)k}{24}} \Big\langle \hat{A}\, \mathrm{ch} \bigotimes_{n=1}^{\infty} S_{q^n} T \\ \bigotimes L^{-\frac{1}{2}+\frac{s}{N}} \bigotimes_{m=0}^{\infty} \Lambda_{-q^{m+s/N}} W \bigotimes_{m=0}^{\infty} \Lambda_{-q^{m+1-s/N}} \overline{W} , M \Big\rangle . \end{aligned} \tag{65}$$

(65) is the equivariant index of a Dirac operator on $\mathcal{L}M$ twisted by a certain generalization of the bundle $\Delta(\hat{\mathcal{T}})$ of equation (30). (Essentially, one uses a complex line bundle over S^1 of order N, a

notion that makes sense equivariantly with respect to the rotation of S^1, in place of the Hopf bundle ϵ that was used in constructing $\Delta(\hat{T})$.) In any case, since (64) and (65) are related by $SL(2,Z)$, any true statement about (64) is equivalent to a true statement about (65).

Now, we assume that M admits the action of a compact connected Lie group G, which we may as well take to be $G = S^1$. Suppose that the G action lifts to an action on W and that in the G-equivariant sense $p_1(T_G) = p_1(V_G)$. Then we can use the arguments of section 4 to show that the G-index generalization of (64) or (65) to a character of G gives in fact a multiple of the trivial character.

A natural way to obey $p_1(T_G) = p_1(V_G)$ is to suppose that M is an almost complex manifold, and pick $V = T$, with the G action on T and V induced from that on M.* In this case, the statement that the G-index generalization of (64) and (65) is a multiple of the trivial G-character gives an interesting restriction on almost complex manifolds. For instance, if we look at (65) and specialize to the lowest power of q, we learn that the G-index of spinors on M twisted by $L^{-\frac{1}{2}+\frac{s}{N}}$, or in other words the G-equivariant extension of

$$\left\langle \hat{A}(M)\mathrm{ch}\, L^{-\frac{1}{2}+\frac{s}{N}}\, ,\, M \right\rangle \tag{66}$$

is a multiple of the trivial G character for $s = 1, 2, \cdots, N-1$. Hattori actually proved that this vanishes. Our results are thus analogues of his for the higher powers of q.

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* More generally, M may be stably almost complex, and then we take $V = T \oplus \theta \oplus \cdots \oplus \theta$, with θ a trivial line bundle. In this case the lifting of G to act on V may be any such that $p_1(V_G) = p_1(T_G)$.

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