Available online at www.sciencedirect.com
SCIENCE DIRECT
ACADEMIC PRESS
Journal of Functional Analysis 197 (2003) 110–139
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

The Dirac–Ramond operator on loops in flat space ☆

Mauro Spera[a] and Tilmann Wurzbacher[b,*]

[a] *Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Padova, Via Belzoni 7, I-35131 Padova, Italy*
[b] *Laboratoire de Mathematiques, Université de Metz et CNRS, Ile du Saulcy, F-57045 Metz, France*

Received 7 May 2001; revised 4 December 2001; accepted 13 December 2001

Dedicated to our teachers Sergio Doplicher and Alan Huckleberry on the occasion of their 60th birthdays

Abstract

In this paper, a rigorous construction of the S^1-equivariant Dirac operator (i.e., Dirac–Ramond operator) on the space of (mean zero) loops in $\mathbb{R}^d$ is given and its equivariant L^2-index computed. Essential use is made of infinite tensor product representations of the canonical anticommutation relations algebra.

Keywords: Spinors on loop spaces; Dirac–Ramond operator; Infinite tensor products; Equivariant L^2-index

0. Introduction

The (supersymmetric) path integral approach to the calculation of the index of a differential operator acting on sections of a vector bundle over a finite-dimensional manifold developed by theoretical physicists (see, e.g., [36]) led to the by now well-known probabilistic proof of the Atiyah–Singer index theorem for Dirac-type operators (see [6,19]). Similar considerations for a hypothetical Dirac operator on loop spaces indicate that their (equivariant) indices should be modular forms

☆ Supported by GNSAGA (MURST) and by IRMA.
*Corresponding author.
E-mail addresses: spera@dmsa.unipd.it (M. Spera), wurzbacher@poncelet.univ-metz.fr (T. Wurzbacher).

0022-1236/03/$ - see front matter © 2003 Elsevier Science (USA). All rights reserved.
doi:10.1016/S0022-1236(02)00178-7

(cf., e.g. [4,7,13,34–36]). Though much progress has been made in applying the Wiener measure approach (also employed in the above context) to this situation (see, e.g., [20]), for the time being the existence of the corresponding characteristic classes, namely the elliptic genus, seems to be established only by topological methods (cf. [16,21,22]) but apparently no construction yielding these genera as indices of differential operators is known (though for certain homogeneous spaces a representation-theoretic substitute is given in [17]).

A solution of this outstanding problem in full generality would have a considerable impact in differential geometry (see, e.g., [29]). In the present article, we construct—in the flat case, i.e., for loops in $\mathbb{R}^d$—a rotation equivariant Dirac operator and prove that its equivariant index is given by Euler's partitio numerorum function, closely related to Dedekind's η-function. Our approach basically consists in realizing the Clifford (or CAR) algebra on the space of spinors associated to a separable Hilbert space as infinite tensor product representations, acting on "incomplete direct product spaces" (in the von Neumann sense [32]). By repeated use of Nelson's analytic vector theorem, we construct the operator in question (acting on L^2-spinors on the loop space of $\mathbb{R}^d$), giving a rigorous meaning to the so-called Dirac–Ramond operator of string theory (see [2,14,26,33]). Though our approach is not probabilistic, we observe that the relevant measure might not be the Wiener measure, but the "string measure," closely related to the $H^{\frac{1}{2},2}$-metrics on functions on the circle and already discussed, e.g., in [8].

Our approach should be also compared to that in [31], where a Dirac operator on "infinitesimal loops" (i.e., on the normal bundle of M as the submanifold of constant loops in $\mathscr{L}M$, the loop space of M) is constructed and its index computed, in view of proving a rigidity conjecture of Witten—his construction being notably different from ours—and to [9], where a general formalism for second quantization of models with bosons and fermions is developed. Indeed, our Dirac–Ramond operator can be interpreted as the second quantization of a suitable operator on the appropriate one-particle space (boson–fermion exchanging supersymmetry). Also, there exist connections with the general framework developed by Arai (see [3] and references therein).

The present article is organized as follows. After a heuristic discussion illustrating the physicists' approach, meant to lend motivation to the subsequent developments (Section 1) we gather, in Section 2 (which is further divided into several subsections), some miscellaneous technical tools which will be needed in the sequel, in order to ease readability. The core of the paper is Section 3, where an analysis of what we call partial Dirac operators is given, proving their self-adjointness (on an appropriate domain) via Nelson's analytic vector theorem. A similar technique is then employed for the "full" Dirac–Ramond operator. Subsequently, in Section 4, we define and compute the equivariant index (in the L^2-sense) of the above operator, which turns out to be given by Euler's partitio numerorum generating function. In the final section we remark on the interpretation of the Dirac–Ramond operator as the second quantization of a one-particle space operator.

1. The Dirac–Ramond operator: a heuristic approach

The quantum mechanical Dirac operator can be interpreted as a supersymmetric version of a quantized Noether conserved current, induced by the basic supersymmetry transformation which exchanges boson and fermions (see, e.g., [1]). The vanishing of the index of the Dirac operator detects "spontaneous supersymmetry breaking" (absence of a SUSY-invariant vacuum), i.e., the impossibility of implementing supersymmetry at the quantum level (see, e.g., [33] for the physical background). Generalization to quantum field theory requires consideration of a rotation equivariant version of the theory, involving an "equivariant Dirac operator" (or Dirac–Ramond operator) acting on the "sections of the spinor bundle on $\mathscr{L}M$," the loop space of M, which first needs fulfillment of a topological condition on M (referred to as the existence of a "string structure"). We shall not delve any further into this specific problem since it is absent in the flat case.

Recall the physicists' formula for the Dirac–Ramond operator (see, e.g., [2,14,34]).

$$iD_K = \int_0^{2\pi} d\sigma \psi^\mu(\sigma)\left[-i\frac{D}{Dx^\mu(\sigma)} + g_{\mu,\nu}\frac{\partial x^\nu}{\partial\sigma}\right].$$

Here, the ψ's are the Clifford variables, and the formal expression for the differential operator is induced from the Levi–Civita connection on the (compact) Riemannian spin manifold (M,g) in question. The extra term accounts for the (Clifford) infinitesimal action of the rotation group. In the non-compact case of $\mathbb{R}^d$, replacing the "continuous sum" over $\sigma\in[0,2\pi]/\sim\,\cong S^1$ by a summation over real Fourier modes (and omitting the "zero-modes"), i.e., developing

$$x^\mu(\sigma) = \sum_{n\geqslant 1}\left(x_n^\mu\frac{\cos(n\sigma)}{\sqrt{\pi}} + y_n^\mu\frac{\sin(n\sigma)}{\sqrt{\pi}}\right)$$

for $\mu = 1,\dots,d = \dim_{\mathbb{R}} M$, yields,

$$D_K = \sum_{\nu,\mu=1}^{d}\sum_{n\geqslant 1}\left(\psi\left(\frac{\partial}{\partial x_n^\mu}\right)\left[\frac{\partial}{\partial x_n^\mu} + ig_{\mu,\nu}\cdot\ (ny_n^\nu)\right] + \psi\left(\frac{\partial}{\partial y_n^\mu}\right)\left[\frac{\partial}{\partial y_n^\mu} + ig_{\mu,\nu}\cdot(-nx_n^\nu)\right]\right).$$

(Note that the "spin connection induced from the Levi–Civita connection" is assumed to be flat in this case.) Up to a dualization from vector fields to one-forms by means of the metric tensor g of M, one arrives by formal manipulations at

$$\begin{aligned}D_K = {} & \sum_{\mu=1}^{d}\sum_{n\geqslant 1}\left(\psi\left(\frac{\partial}{\partial x_n^\mu}\right)\frac{\partial}{\partial x_n^\mu} + \psi\left(\frac{\partial}{\partial y_n^\mu}\right)\frac{\partial}{\partial y_n^\mu}\right)\\ & + i\cdot\sum_{\nu,\mu=1}^{d}\sum_{n\geqslant 1}\left(\psi\left(\frac{\partial}{\partial x_n^\mu}\right)g_{\mu,\nu}\ \cdot\ (ny_n^\nu) + \psi\left(\frac{\partial}{\partial y_n^\mu}\right)g_{\mu,\nu}\ \cdot(-nx_n^\nu)\right)\end{aligned}$$

$$= D^0 + i \cdot \psi \left(\sum_{\nu,\mu=1}^{d} \sum_{n \geqslant 1} \left[g_{\mu,\nu} \cdot (n y_n^{\nu}) \frac{\partial}{\partial x_n^{\mu}} + g_{\mu,\nu} \cdot (-n x_n^{\nu}) \frac{\partial}{\partial y_n^{\mu}} \right] \right)$$
$$= D^0 + i \cdot c(K),$$

where D^0 is the "plain Dirac operator" and $c(K)$ denotes Clifford multiplication by the vector field K on the loop space, induced from rotating the parameter of the loops. We are thus led to an infinite-dimensional version of a "equivariant Dirac operator." Suppressing the index μ, we can rewrite this as follows:

$$D_K = \sum_{n \geqslant 1} \tilde{D}_n.$$

Furthermore, we can interpret the summands as

$$\tilde{D}_n = \tilde{a}_n D_n^- + \tilde{a}_n^* D_n^+,$$

where the $\tilde{a}$'s and $\tilde{a}^*$'s (fermion annihilation and creation operators, respectively) are obtained from Clifford algebra variables in the standard manner and the ensuing differential operators $D_n^{\pm}$ called (left or right) "movers," see e.g. [31], properly encoding the rotation action, fulfill boson (canonical commutation relations or CCR) type commutation relations for annihilation and creation operators, respectively; the boson–fermion exchange action of D_K is then manifest (cf. Section 5 for the last-mentioned aspects of the Dirac–Ramond operator).

Of course, the above remarks are sketchy and completely formal. We are going to depict a rigorous and detailed portrait de novo in the following sections.

2. Algebraic and analytic preliminaries

For the sake of brevity, we refer to the term "self-adjoint" sometimes simply by "s.a." in the sequel.

2.1. Incomplete direct products

Let us very briefly recall von Neumann's original approach to incomplete direct products, in the case of a countable family of complex Hilbert spaces $\{(H_n, \langle\, , \,\rangle_n) \mid n \geqslant 1\}$ (see [32], cf. also [10]). Set first $C = \{f = (f_n)_{n \geqslant 1} \in \prod_{n \geqslant 1} H_n \mid \prod_{n \geqslant 1} \|f_n\|_{H_n}$ converges$\}$ and

$$M = \{m : C \to \mathbb{C} \mid \forall f \in C,\ \forall z \in \mathbb{C},\ \forall k \in \mathbb{N}^*,\ \forall f_k' \in H_k \text{ one has}$$
$$m((f_1, \ldots, f_{k-1}, z \cdot f_k, f_{k+1}, \ldots))$$

$$= \bar{z} \cdot m(f) \text{ and } m((f_1, \ldots, f_k + f'_k, \ldots)) = m((f_1, \ldots, f_k, \ldots)) + m((f_1, \ldots, f'_k, \ldots))\}$$

and fix $\Phi = (\varphi_1, \varphi_2, \ldots) \in \prod_{n\geqslant 1} H_n$, a sequence of unit vectors. One can then associate to each $\Psi = (\psi_1, \psi_2, \ldots)$, $\psi_n \in H_n$, such that $\psi_n = \varphi_n$ for all but a finite set of n's, a "multilinear functional," i.e., an element of M, m_Ψ via the position

$$m_\Psi(f) = \prod_{n\geqslant 1} \langle \psi_n, f_n \rangle_n.$$

The algebraic linear span of the set of these m_Ψ's in the vector space M will be denoted by $\otimes^{(\Phi)} H_n = \otimes^{(\Phi)}_{n\geqslant 1} H_n$. The space $\otimes^{(\Phi)} H_n$ is naturally a pre-Hilbert space upon introducing the inner product $\langle , \rangle$ of two sequences Ψ' and Ψ'' (such that their entries are equal to φ_n for all but a finite number of n's) as follows:

$$\langle \Psi', \Psi'' \rangle \coloneqq \prod_{n\geqslant 1} \langle \psi'_n, \psi''_n \rangle_n$$

and extending sesquilinearly (we assume linearity in the second variable for inner products throughout this article). Subsequent completion yields the "incomplete direct product" Hilbert space $H \coloneqq \otimes^{\Phi} H_n \coloneqq \otimes^{\Phi}_{n\geqslant 1} H_n$. Clearly, $||\Phi|| = 1$. (The incomplete direct product can also be defined in terms of inductive limits, see [11] for the details.)

If we denote, as usual, with $\hat{\otimes}$ the topological tensor product of Hilbert spaces, and $H \coloneqq \otimes^{\Xi} H_n$, $G \coloneqq \otimes^{\Omega} G_n$, with $\Xi = (\xi_1, \xi_2, \ldots)$, $\Omega = (\omega_1, \omega_2, \ldots)$ we have a canonical identification

$$H \hat{\otimes} G \cong \bigotimes_{n\geqslant 1}^{\Phi} (H_n \hat{\otimes} G_n)$$

with $\varphi_n = \xi_n \otimes \omega_n$. (We will in the sequel tacitly assume that all tensor products of Hilbert spaces are completed, without explicitly adding the symbol "ˆ.")

We would like to point out the following relationship between infinite direct products and product probability measures. Let us consider a countable family of probability spaces $(\Omega_n, \mathscr{F}_n, P_n)$, $n = 1, 2, \ldots, \Omega_n$ being a measure space, with σ-algebra $\mathscr{F}_n$, and probability measure P_n. Let $\Omega = \prod_{n\geqslant 1} \Omega_n$ with σ-algebra $\mathscr{F}$ generated by the cylindric sets on Ω and the usual countable infinite product measure P. Choosing for all n in $\mathbb{N}^*$, the function $\varphi_n \equiv 1$ in $L^2(\Omega_n, \mathscr{F}_n, P_n)$, one can prove the following

Proposition 2.1. *There exists a natural Hilbert space isomorphism*

$$L^2(\Omega, \mathscr{F}, P) \cong \bigotimes_{n\geqslant 1}^{\Phi} L^2(\Omega_n, \mathscr{F}_n, P_n).$$

In this situation, the following theorem due to Streit and Kraus–Polley–Reents (cf. [15,30]) is of central importance.

Theorem 2.2. (SKPR). *Let $H := \otimes^{\Phi} H_n$ and let $U_n(t)$, $n = 1, 2, \ldots$ be a family of strongly continuous one-parameter unitary groups on H_n, with self-adjoint infinitesimal generators A_n (i.e., $U_n(t) = \exp(itA_n)$). Then $U(t) = \otimes_{n \geqslant 1} U_n(t)$ exists as a strongly continuous one-parameter group on H if and only if there exists a vector $\Psi = (\psi_1, \psi_2, \ldots) \in H$ such that $\psi_n \in \mathscr{D}(A_n)$, the domain of A_n, and*

(i) $\sum_{n \geqslant 1} |\langle \psi_n, A_n \psi_n \rangle| < \infty$, *and*
(ii) $\sum_{n \geqslant 1} ||A_n \psi_n||^2 < \infty$.

In particular, (i) *and* (ii) *are automatically satisfied if the reference vector Φ is such that $\varphi_n \in Ker\, A_n$ for each n, and one takes $\Psi = \Phi$.*

We also point out the following crucial lemma:

Lemma 2.3. (i) *Let $\mathscr{H}$ be a Hilbert space and $A : \mathscr{D}(A) \subset \mathscr{H} \to \mathscr{H}$ be a self-adjoint operator. Then $Ker\, A = Ker\, A^2$ and A^2 is a positive operator.*

(ii) *Let $T : \mathscr{D}(T) \subset H \to H$ be a self-adjoint operator on $H = \otimes^{\Phi} H_n$, with $T = \sum_{n \geqslant 1} T'_n$, where T_n is self-adjoint and positive on H_n and $T'_n = Id_{H_1} \otimes \cdots \otimes Id_{H_{n-1}} \otimes T_n \otimes Id_{H_{n-1}} \otimes \cdots$. Then T is a positive operator. Furthermore, if for all $n \geqslant 1$, $T_n \varphi_n = 0$, then*

$$Ker\, T = \bigotimes_{n \geqslant 1}^{\Phi} Ker\, T_n.$$

Proof. Ad (i). This is a standard result, easily deduced, e.g., from Theorem X.25 in [24], Vol. II.

Ad (ii). Upon resorting to unitary one-parameter groups (via the Stone and SKPR theorems), the first assertion follows since T is a strong resolvent limit of obvious positive operators.

The assertion involving kernels is clear for finite tensor products (cf., e.g., [24], Vol. I), since a finite sum of essentially s.a. operators $\sum_k A'_k$ (defined analogously to the T'_n in the text of the lemma) is essentially s.a. on $\mathscr{D} := \otimes_k \mathscr{D}_k$, with $\mathscr{D}_k \subset H_k$ a domain of essential self-adjointness for A_k, and its spectrum is given by the formula $\sigma(\overline{\sum_k A'_k}) = \overline{\sum_k \sigma(\overline{A_k})}$. We shall check that it is true in the incomplete tensor product case by reducing to the above case.

First observe that, in general, for a s.a. operator A, φ is in $Ker\, A$ if and only if φ is a fixed point for the corresponding unitary group $U(t)$: one direction is clear from the very proof of Stone's theorem, the other follows immediately from observing that an element in the kernel is an analytic vector for A so the power series representation is valid for $U(t)\varphi$ (see, e.g., [24], Vol. II), yielding the fixed point property.

The inclusion

$$Ker\ T \supset \overset{\Phi}{\underset{n\geqslant 1}{\bigotimes}}\ Ker\ T_n$$

is obvious. For the reverse inclusion, let $\Psi \in Ker\ T$, i.e., let Ψ be a fixed point of $U(t)$. Thus, we may expand

$$\Psi = c_0 \Phi + \sum_{N\geqslant 1} \hat{\psi}_{(N)} \otimes \varphi_{N+1} \otimes \varphi_{N+2} \otimes \cdots$$

with $\hat{\psi}_{(N)} \in (\otimes_{n=1}^{N-1} H_n) \otimes H'_N$, where $H'_N = ((\varphi_N))^{\perp} \subset H_N$. (Note that the summands are mutually orthogonal.)

Applying $U(t)$ (and recalling its very definition, and taking into account that commutation with the series is allowed in view of strong continuity, and preservation of orthogonality), we get

$$\Psi = U(t)\Psi = c_0 \Phi + \sum_{N\geqslant 1} (U^{(N)}(t)\hat{\psi}_{(N)}) \otimes \varphi_{N+1} \otimes \varphi_{N+2} \otimes \cdots,$$

whence

$$U^{(N)}(t)\hat{\psi}_{(N)} = \hat{\psi}_{(N)}$$

for all $t \in \mathbb{R}$, that is, $\hat{\psi}_{(N)} \in Ker\ A^{(N)}$ where the superscript "(N)" refers to obvious truncated operators, as, e.g., $U^{(N)}(t) = U_1(t) \otimes \cdots \otimes U_N(t)$. An application of the corresponding result for a finite number of factors concludes the proof. $\square$

Note that in the above proof we tacitly switched from the infinite sequence notation to an infinite tensor product notation, since it proved to be more vivid. This will be done in the sequel as well.

2.2. *The rotation action on $\mathscr{L}_0\mathbb{R}$*

For convenience we set throughout the main text $d = 1$. At the end we will go back to the general case.

Let $\mathscr{L}\mathbb{R} := C^{\infty}(S^1, \mathbb{R})$ denote the space of smooth loops on $\mathbb{R}$, with $S^1 \cong [0, 2\pi]/\sim$ (in the obvious sense) and let $\mathscr{L}_0\mathbb{R}$ denote the space of loops with mean zero, i.e., those loops whose constant term in their Fourier expansion vanishes. We fix the following natural orthonormal basis of $H := L_0^2(S^1, \mathbb{R})$, the space of square-integrable real-valued functions of mean zero on the circle:

$$f_n(t) = \frac{1}{\sqrt{\pi}} \cos nt, \qquad g_n(t) = \frac{1}{\sqrt{\pi}} \sin nt \quad \text{for } n \geqslant 1.$$

Using the associated real Fourier decomposition of square integrable (or smooth) loops with values in $\mathbb{R}$,

$$\gamma(t) = \sum_{n \geqslant 1} x_n f_n(t) + y_n g_n(t),$$

we get the following orthogonal decomposition (with respect to the natural L^2-product), together with the natural inclusions:

$$\mathscr{L}\mathbb{R} = \mathbb{R} \oplus \mathscr{L}_0\mathbb{R} \subset L^2_{\mathbb{R}}(S^1, dt) = \mathbb{R} \oplus H = \mathbb{R} \oplus \hat{\bigoplus_{n=1}^{\infty}} H_n \subset \mathbb{R} \oplus \prod_{n=1}^{\infty} H_n,$$

where $H_n := (\mathbb{R}^2)_n := ((f_n, g_n))_{\mathbb{R}}$ and $H = \hat{\bigoplus}_{n \geqslant 1} H_n$. The summand $\mathbb{R}$ represents the constant loops (i.e., points in $\mathbb{R}$). In the sequel it will be discarded since we are interested in finding an equivariant index, and the rotation action (see below) will be trivial thereon.

We proceed to recall the explicit description of the action of S^1 on $\mathscr{L}_0\mathbb{R}$. Let θ, $t \in [0, 2\pi]/\sim$ and let $q = e^{i\theta}$ be in $Rot(S^1)$, the rotation group. The action ϑ_q on loops is defined (for $\gamma \in \mathscr{L}_0\mathbf{R}$) by

$$\vartheta_q(\gamma)(t) := \gamma(t + \theta).$$

The above action dualizes as follows to an action on $Map(\mathscr{L}_0\mathbb{R}, E)$, the space of functions on (mean zero) loops with values in a vector space E. For F in $Map(\mathscr{L}_0\mathbb{R}, E)$ and q in $Rot(S^1)$

$$\check{\vartheta}_q(F)(\gamma) := F((\vartheta_q)^{-1}(\gamma)) = F(\vartheta_{q^{-1}}(\gamma)).$$

Let us underline that for $n \geqslant 1$, x_n and y_n, as well as $z_n := x_n + iy_n$, should be considered as functions on $\mathscr{L}_0\mathbb{R}$, H, and H_n. We then have

Proposition 2.4. (i) *The S^1-action reads explicitly, on the function z_n:*

$$\check{\vartheta}_q(z_n) = q^n z_n.$$

(ii) *The (Killing) vector field K_n on H_n, defined for p a point of H_n and F a germ of smooth function near p by*

$$K_n(F)(p) := \frac{d}{d\theta}\bigg|_0 (\check{\vartheta}_{e^{-i\theta}} F)(p)$$

fulfills

$$K_n = n\left(y_n \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial y_n}\right).$$

Proof. Straightforward, by a few trigonometric manipulations, starting from

$$(\vartheta_q^{-1}(\gamma))(t) = \sum_{n\geqslant 1} x_n f_n(t-\theta) + y_n g_n(t-\theta). \qquad \square$$

2.3. *The string measure on $\mathscr{L}_0\mathbb{R}$*

In this section, we make contact with the string measure approach developed in [8]. Set first, $B_n = L^2_{\mathbb{C}}(H_n, \lambda^2_{(n)})$, where $\lambda^2_{(n)}$ denotes the two-dimensional Lebesgue measure λ^2 on the two-dimensional real vector space H_n. Consider, in B_n, the function (of norm one)

$$\varepsilon_n = \sqrt{\frac{n}{\pi}}\exp\left(-\frac{n}{2}\bar{z}_n z_n\right).$$

The position

$$P_n = \varepsilon_n^2 \lambda^2_{(n)}$$

on $\Omega_n = H_n$ yields a rotationally invariant probability measure such that multiplying an L^2-function by ε_n^{-1} yields an isomorphism $B_n = L^2_{\mathbb{C}}(H_n, \lambda^2_{(n)}) \to L^2_{\mathbb{C}}(H_n, \varepsilon_n^2\lambda^2_{(n)}) = L^2_{\mathbb{C}}(H_n, P_n)$. The ensuing product measure on $\Omega = \prod_{n\geqslant 1}\Omega_n$, which is formally given by

$$P = \left(\prod_{n\geqslant 1}\frac{n}{\pi}\right)\exp\left(-\sum_{n\geqslant 1} n|z_n|^2\right)\left(\prod_{n\geqslant 1}\lambda^2_{(n)}\right)$$

is the "string measure" [8], related to the $H^{\frac{1}{2},2}$ metric on loops (see, e.g., [23] or [28]).

2.4. *CAR algebra and spinors in infinite dimensions*

We briefly review, in the specific case we deal with, the infinite tensor product realization of the Fock representation of the CAR (and Clifford) algebras, in order to fix notations. We refer to [10], cf. also [27] and references therein, for more details.

Given a complex, separable Hilbert space $(K, \langle\, , \rangle)$, called the "one particle space" (whose scalar product we assume again to be linear in the second argument) the canonical anticommutation relations (CAR) algebra $A(K)$ is the universal unital C^*-algebra generated by "annihilation operators" $a(f)$ (depending linearly on f in K) and their adjoints $a(f)^*$, "creation operators" (depending antilinearly on f in K) fulfilling the following relations (CAR):

$$a(f)^*a(g) + a(g)a(f)^* = [a(f)^*, a(g)]_+ = \langle f, g\rangle \cdot \mathbf{1},$$

$$a(f)a(g) + a(g)a(f) = [a(f), a(g)]_+ = 0$$

for all elements f and g of K, and $\mathbf{1}$ being the identity element of $A(K)$ (slightly differing conventions are often used in the literature). The unitary group $U(K)$, i.e., the natural symmetry group of K, acts on $A(K)$ through C^*-automorphisms α_U defined by $\alpha_U(a(f)) := a(Uf)$ for $U \in U(K)$ and $f \in K$.

Similarly, let $(H, g = (,)_H)$ be a real Hilbert space and $H_{\mathbb{C}} = H \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, endowed with the $\mathbb{C}$-linear extension $B = g^{\mathbb{C}}$ of the metric and the canonical hermitian structure $\langle u, v \rangle = B(\bar{u}, v)$, where $\overline{h \otimes \lambda} = h \otimes \bar{\lambda}$ denotes the canonical conjugation of $H_{\mathbb{C}}$. The Clifford algebra $C(H, g)$ is the real (universal) Banach algebra with unit $\mathbf{1}$ generated by operators $\gamma(u)$, $u \in H$ fulfilling the anticommutation relation (u, $v \in H$)

$$[\gamma(u), \gamma(v)]_+ = -2g(u, v) \cdot \mathbf{1} = -2(u, v)_H \cdot \mathbf{1}.$$

The complex Clifford algebra $C(H_{\mathbb{C}}, B)$ is then by definition

$$C(H_{\mathbb{C}}, B) = C(H, g) \otimes_{\mathbb{R}} \mathbb{C}.$$

It turns out to be a unital C^*-algebra isomorphic to $A(W)$, where W is any B-isotropic subspace of $H_{\mathbb{C}}$ such that $\bar{W} = W^{\perp}$. We give explicit formulae tailored to our purposes immediately below.

Let

$$F_n = \mathbb{C}^2 \cong \Lambda^*(\mathbb{C}) = \Lambda^0\mathbb{C} \oplus \Lambda^1\mathbb{C} = ((e_n^+))_{\mathbf{C}} \oplus ((e_n^-))_{\mathbb{C}} = F_n^+ \oplus F_n^-$$

be the usual explicit realization of the space of complex spinors associated to $\mathbb{R}^2$ with the standard Euclidean metric. Taking

$$H_n = ((f_n, g_n))_{\mathbb{R}} \cong \mathbb{R}^2,$$

we consider the following representation ψ_n of the (real) Clifford algebra $C(H_n) \cong C(\mathbb{R}^2)$:

$$\psi_n(f_n) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \psi_n(g_n) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Indeed, one easily checks

$$[\psi_n(f_n), \psi_n(f_n)]_+ = [\psi_n(g_n), \psi_n(g_n)]_+ = -2I, \qquad [\psi_n(f_n), \psi_n(g_n)]_+ = 0.$$

We pass, as mentioned above, to the complex Clifford algebra

$$C(H_n) \otimes_{\mathbb{R}} \mathbb{C} \cong C(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} \cong A(\mathbb{C}),$$

i.e., “the one-particle space” W_n is isomorphic to $\mathbb{C}$ and it is generated inside F_n by

$$e_n^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(Slightly abusively we will identify in the sequel $W_n \subset H_n \otimes \mathbb{C}$ with $((e_n^-))_{\mathbb{C}} \subset F_n$ since F_n is—non-canonically—isomorphic to $\Lambda^* W_n$.) This is an explicit realization of the Fock representation, with vacuum vector

$$e_n^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and space of spinors $F_n = ((e_n^+, e_n^-))_{\mathbb{C}} \cong \mathbb{C}^2$. We also have

$$a_n = a_n(e_n^-) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{resp. } a_n^* = a_n(e_n^-)^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(keeping the suffixes n of the operators for the sake of clarity). The relationship between Clifford and CAR operators reads

$$\psi_n(f_n) + i\psi_n(g_n) = 2ia_n(e_n^-), \qquad \psi_n(f_n) - i\psi_n(g_n) = 2ia_n(e_n^-)^*.$$

We recall also the parity operator

$$V_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

acting on F_n, the nth copy of $\mathbb{C}^2$, being intrinsically defined by the requirements:

$$V_n^2 = I, \qquad V_n \psi_n(f_n) V_n^{-1} = -\psi_n(f_n), \qquad V_n \psi_n(g_n) V_n^{-1} = -\psi_n(g_n),$$
$$V_n e_n^+ = e_n^+.$$

Let us consider the Fock representation of $C(H) \otimes \mathbb{C}$ over the Majorana–Fock space

$$F = \bigotimes_{n \geqslant 1}^{e^+} F_n$$

(where e^+ denotes of course the sequence $(e_1^+, e_2^+, \ldots)$) defined via the position:

$$\tilde{\psi}(f_n) = V_1 \otimes V_2 \otimes \cdots V_{n-1} \otimes \psi_n(f_n) \otimes Id_{F_{n+1}} \otimes Id_{F_{n+2}} \otimes \cdots$$

and similarly for g_n. It is easily checked that the Clifford relations (or equivalently the CAR) are fulfilled, i.e.,

$$[\tilde{\psi}(f), \tilde{\psi}(g)]_+ = -2(f, g)_H \qquad \text{for all } f, g \in H$$

(the parity operators are designed precisely to achieve this, see, e.g., [10]). The fixed point subalgebra (under parity) is called the "even CAR algebra." Infinite tensor product realization of quasifree state representations play a major role in von Neumann algebra theory and quantum field theoretic problems. In the preceding treatment, we sacrificed full generality for the sake of definiteness and simplicity. We remark, in passing, that in general the full parity operator ("Fermion number operator") $\mathbf{V}_F$ on F (unitarily implementing—in the Fock representation—the parity automorphism α_{-I} stemming from the one particle unitary $-I$) can be explicitly represented by

$$\mathbf{V}_F = \prod_{n=1}^{\infty} (1 - 2a(h_n)^* a(h_n)),$$

where $\{h_1, h_2, \dots\}$ is any orthonormal basis in the fermionic one-particle space. The precise meaning to be ascribed to the above expression is the following: on considering the natural sequence of truncated products, it is easily verified that it gives rise, in the limit, to a bounded sesquilinear form, which, by Riesz's theorem, yields the desired operator, enjoying the same properties as the above V_n.

Remark. The appearance of the grading operators V_k in the formula for $\tilde{\psi}(f_n)$ (as well as in the definition of $\tilde{D}_n$ below in Section 3.2) can of course conceptually be understood in terms of graded tensor products. Without considering the topological aspects we would like to sketch the basic algebra involved.

Given two $\mathbb{Z}_2$-graded vector spaces W' and W'', the spaces $End(W')$, $End(W'')$ and $W' \otimes W''$ are naturally $\mathbb{Z}_2$-graded as well. Furthermore, the usual, ungraded, tensor product of the respective grading operators $V' = V_{W'}$ and $V'' = V_{W''}$ is the natural grading operator of $W' \otimes W''$. Finally, given two $\mathbb{Z}_2$-graded algebras $\mathscr{A}'$ and $\mathscr{A}''$, their vector space product $\mathscr{A}' \otimes \mathscr{A}''$ carries a natural structure of a $\mathbb{Z}_2$-graded algebra if we define the multiplication on tensor products of homogeneous elements as follows:

$$(A' \otimes A'') \cdot (B' \otimes B'') = (-1)^{|A''||B'|} (A' \cdot B') \otimes (A'' \cdot B''),$$

where $|A''|, |B'|$, etc. denotes the degree in $\mathbb{Z}_2$ of a homogeneous element. This "$\mathbb{Z}_2$-graded tensor product" is denoted by $\mathscr{A}' \hat{\otimes} \mathscr{A}''$, and one has the fundamental fact that the Clifford algebra of an orthogonal sum of two vector spaces is naturally isomorphic to the $\mathbb{Z}_2$-graded product of the respective Clifford algebras (see, e.g., [18]). Mutatis mutandis the same holds true for CAR-algebras.

Since these algebras are often represented on $\mathbb{Z}_2$-graded vector spaces, it is useful to have a formula for the action ϱ of $End(W') \hat{\otimes} End(W'')$ on $W' \otimes W''$ (W' and W'' being again $\mathbb{Z}_2$-graded). In order to comply with the formula of the multiplication in a graded tensor product one arrives at

$$(\varrho(T' \otimes T''))(w' \otimes w'') = (-1)^{|T''||w'|} T'(w') \otimes T''(w'')$$

for homogeneous elements. Let us replace the notation $\varrho(T' \otimes T'')$ by $T' \hat{\otimes} T''$ for the rest of this remark.

Considering $W' = F' = \otimes_{k=1}^{n} F_k$ and $W'' = F'' = \otimes_{k\geqslant n+1}^{e''+} F_k$ (with $e''^+ = (e_{n+1}^+, \dots)$), one has $F \cong F' \otimes F''$ (as ungraded vector spaces), $V_{F'} = V_1 \otimes \cdots \otimes V_k$ (tensor product of operators in the ungraded sense) and $V_{F''}$ is given analogously to V_F above. It follows that

$$\tilde{\psi}(f_n) = Id_{F_1} \hat{\otimes} \cdots \hat{\otimes} Id_{F_{n-1}} \hat{\otimes} \psi_n(f_n) \hat{\otimes} Id_{F''},$$

i.e., the "natural implementation of $\psi_n(f_n)$ on $F' \otimes F'' = F_1 \otimes \cdots \otimes F_n \otimes F''$ in the $\mathbb{Z}_2$-graded sense."

Recalling from Section 2.3 that $B_n = L^2_{\mathbb{C}}(H_n, \lambda^2_{(n)}) \cong L^2_{\mathbb{C}}(\mathbb{R}^2, \lambda^2)$, and setting $\varepsilon := (\varepsilon_1, \varepsilon_2, \dots)$ and $B := \otimes_{n\geqslant 1}^{\varepsilon} B_n$ we can give

Definition 2.5. The space of "L^2-sections of the spinor bundle over $\mathscr{L}_0\mathbb{R}$" is defined as

$$\mathscr{H} := \bigotimes_{n\geqslant 1}^{\varepsilon \otimes e^+} (B_n \otimes F_n) \cong (\otimes^{\varepsilon} B_n) \otimes (\otimes^{e^+} F_n) = B \otimes F.$$

Remark. The space B is canonically isomorphic to the space $L^2(\Omega, \mathscr{F}, P)$, with P the string measure considered in Section 2.3.

We notice the following:

Proposition 2.6. (i) *The full parity operator* $\mathbf{V}$ *is given by*

$$\mathbf{V} = \mathbf{1}_B \otimes \mathbf{V}_F.$$

(ii) *The reference vector* $\varepsilon \otimes e^+$ *is even.*
(iii) *The vectors of the form*

$$h \otimes (e_1^{\alpha_1} \otimes e_2^{\alpha_2} \otimes \cdots \otimes e_N^{\alpha_N} \otimes e_{N+1}^{+} \otimes \cdots)$$

with $\alpha_j \in \{+, -\}$ for $j = 1, 2, \dots, N$ *have parity* $\alpha_1 \cdot \dots \cdot \alpha_N$.

2.5. Positive energy representations of S^1

Let F be a complex, separable Hilbert space and let $\varrho : S^1 \to U(F)$ be a strongly continuous unitary representation of the circle group S^1. We consider those representations such that for all k in $\mathbb{Z}$, the multiplicity μ_k^F, i.e., the dimension of the kth isotypical summand $F_{(k)}$ (corresponding to the character $q \to q^k$ on S^1), is finite for each k and $\mu_k^F = 0$ for $k < k_0$, for some $k_0 \in \mathbb{Z}$. We call these representations "positive energy representations."

Let A be the abelian semiring consisting of all unitary equivalence classes of positive energy representations, and denote the associated ring by R_+. Denote by $\mathbb{Z}_L[[q]]$ the ring $(\mathbb{Z}[[q]])[q^{-1}]$ consisting of formal Laurent series, "having a pole at $q=0$," i.e., formal Laurent series of the form $\sum_{k \geqslant k_0} m_k q^k$ for some k_0 in $\mathbb{Z}$, and m_k in $\mathbb{Z}$. Recall, for future use that for two (genuine) positive energy S^1-representations E and F, fulfilling $E_{(r)} = \{0\}$ for $r < r_0$ and $F_{(s)} = \{0\}$ for $s < s_0$, one has

$$(E \otimes F)_{(p)} = \bigoplus_{l=r_0}^{p-s_0} E_{(l)} \otimes F_{(p-l)}$$

hence,

$$\sum_{p \geqslant r_0+s_0} \mu_p^{E \otimes F} q^p = \sum_{p \geqslant r_0+s_0} \left(\sum_{l=r_0}^{p-s_0} \mu_l^E \mu_{p-l}^F \right) q^p = \left(\sum_{r \geqslant r_0} \mu_r^E q^r \right) \cdot \left(\sum_{s \geqslant s_0} \mu_s^F q^s \right).$$

Upon taking rational coefficients, $\mathbb{Q}_L[[q]] = \mathbb{Q}_L[[q]] \otimes_{\mathbb{Z}} \mathbb{Q} = (\mathbb{Q}[[q]])[q^{-1}]$ becomes a field, which can be identified with the fraction field $\mathbb{Q}(q) = \{f/g \mid f, g \in \mathbb{Q}[q]\}$, i.e., with quotients of power series in the variable q with rational coefficients. Of course, the positive energy condition is crucial here. The following lemma is easy but useful.

Lemma 2.7. *The rings $\mathbb{Z}_L[[q]]$ and R_+ are naturally isomorphic, so R_+, after taking rational coefficients, is a field as well.*

Proof. The isomorphism is, of course, induced by the map

$$\sum_{k \geqslant k_0} m_k q^k \mapsto \hat{\bigoplus_{k \geqslant k_0}} \operatorname{sgn}(m_k)(\mathbb{C}^{|m_k|})_{(k)}$$

(here "sgn" denote the sign of an integer, being equal to, e.g., 0 for 0). $\square$

3. Construction of the Dirac–Ramond operator

3.1. Partial Dirac operators and their properties

We begin with the following

Proposition 3.1. *The partial Dirac–Ramond operator, acting a priori on $C^\infty(H_n, \mathbb{C}^2) \cong C^\infty(\mathbb{R}^2, \mathbb{C}^2)$ and denoted by D_n, is defined by*

$$D_n := D_n^0 + ic(K_n) := \psi_n(f_n)(\partial_{x_n} + iny_n) + \psi_n(g_n)(\partial_{y_n} - inx_n),$$

where $\partial_{x_n} = \frac{\partial}{\partial x_n}$, $\partial_{y_n} = \frac{\partial}{\partial y_n}$, $\partial_{z_n} = \frac{\partial}{\partial z_n}$, $\partial_{\bar{z}_n} = \frac{\partial}{\partial \bar{z}_n}$, $D_n^0 = \psi_n(f_n)\partial_{x_n} + \psi_n(g_n)\partial_{y_n}$ denote the plain Dirac operator on $C^\infty(H_n, \mathbb{C}^2)$ and $c(K_n) = c(n(y_n\partial_{x_n} - x_n\partial_{y_n})) = n(\psi_n(f_n)y_n -$

$\psi_n(g_n)x_n)$ is the Clifford action of the Killing vector field K_n on $H_n \cong \mathbb{R}^2$ and it reads, explicitly

$$D_n = \begin{pmatrix} 0 & D_n^- \\ D_n^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & i(2\partial_{z_n} - n\bar{z}_n) \\ i(2\partial_{\bar{z}_n} + nz_n) & 0 \end{pmatrix}.$$

Proof. This is just a routine calculation, after recalling that $2\partial_{\bar{z}_n} = \partial_{x_n} + i\partial_{y_n}$ and $2\partial_{z_n} = \partial_{x_n} - i\partial_{y_n}$. □

Observe that D_n is easily seen to be symmetric with respect to the L^2-scalar product on its initial domain $\mathscr{D}(D_n) := \mathscr{S}(\mathbb{R}^2, \mathbb{C}^2)$, the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^2$ with values in $\mathbb{C}^2$. (For convenience, we will identify in this section H_n with $\mathbb{R}^2$, carrying coordinates denoted by x_n and y_n, or in complex notation by $z_n = x_n + iy_n$.)

Remark. Let us point out that, on $\mathscr{S}(\mathbb{R}^2, \mathbb{C})$, the operators $D_n^\pm$ fulfill (up to a constant) the canonical commutation relations (CCR) for annihilation (+) and creation (−) operators:

$$[D_n^+, D_n^-] = D_n^+ D_n^- - D_n^- D_n^+ = 4nI.$$

Moreover, the above formula for D_n can be cast in the form

$$D_n = a_n(e_n^-)D_n^- + a_n(e_n^-)^* D_n^+$$

(cf. Section 2.4). The operators $D_n^\pm$ can be interpreted as "left-moving bosonic annihilation and creation operators." Upon exchanging the variables z and $\bar{z}$, we get "right-moving" analogues $\underline{D}_n^\pm$. More precisely, the nth-mode right mover boson creation operator is given as follows:

$$\underline{D}_n^- = i(2\partial_{\bar{z}_n} - nz_n).$$

Analogously, one defines $\underline{D}_n^+$. Furthermore, the a_n's and their adjoints yield their fermionic right-moving counterparts (cf. [31]).

We now proceed to show explicitly that D_n has a unique self-adjoint extension to $L^2(\mathbb{R}^2, \mathbb{C}^2)$. It is no surprise that our computation partially resembles the one given in the second volume of [24] concerning self-adjointness of position and momentum operators in the Schrödinger representation of quantum mechanics.

Set, for fixed $n \geqslant 1$:

$$H_\alpha^{m,l} = H^{m,l} \otimes e_n^\alpha = z_n^m \bar{z}_n^l \varepsilon_n(z_n) \otimes e_n^\alpha,$$

where $m, l \in \mathbf{N}$, $\alpha \in \{+, -\}$, and $\{e_n^+, e_n^-\}$ is the canonical basis for $\mathbb{C}^2$ already considered in Section 2.4. Furthermore, recall from Section 2.3 that

$$\varepsilon_n(z_n) = \sqrt{\frac{n}{\pi}} \exp\left(-\frac{n}{2}\bar{z}_n z_n\right).$$

Obviously, for all m, n and α, $H_\alpha^{m,l}$ is in $\mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)$, our initial domain of definition for D_n. Furthermore, $D_n(\mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)) \subset \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)$ and $D_n(H_\alpha^{m,l})$ is a finite linear combination in the family $H_{\alpha'}^{m',l'}$. (Similar statements hold for $D_n^\pm$ with initial domain $\mathcal{D}(D_n^\pm) = \mathcal{S}(\mathbb{R}^2, \mathbb{C})$ and $H^{m,l}$.) From now on we discard the suffix n, when no confusion can arise, provisionally setting, in particular $D := D_n$. Our aim is to compute the L^2-norm

$$||D^k H_\alpha^{m,l}||$$

in view of applying Nelson's analytic vector theorem. First, we note the following: from

$$D^2 = \begin{pmatrix} D^- D^+ & 0 \\ 0 & D^+ D^- \end{pmatrix}$$

we get

$$||D^{2n} H_+^{m,l}|| = ||(D^2)^n H_+^{m,l}|| = ||(D^- D^+)^n H_+^{m,l}||$$

and also

$$||D^{2n} H_-^{m,l}|| = ||(D^+ D^-)^n H_-^{m,l}||.$$

Likewise, using obvious notations, we get from

$$D^{2n+1}\begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = D^{2n} D \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = D^{2n}\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}\begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = D^{2n}\begin{pmatrix} D^- \varphi_- \\ D^+ \varphi_+ \end{pmatrix},$$

$$||D^{2n+1} H_+^{m,l}|| = ||D^{2n} D H_+^{m,l}|| = ||D^{2n}(D^+ H^{m,l} \otimes e^-)|| = ||(D^+ D^-)^n D^+ H_+^{m,l}||$$

and, similarly

$$||D^{2n+1} H_-^{m,l}|| = ||(D^- D^+)^n D^- H^{m,l}||.$$

The crucial estimate is the following:

Proposition 3.2. *If, with the above notations, $D^\sharp$ denotes either D^+ or D^-, there exists $\lambda > 0$ such that the following estimate holds:*

$$(*) \quad ||D^\sharp \ldots D^\sharp H^{m,l}|| \leqslant C_{m,l} \lambda^k k!$$

for some constant $C_{m,l}$ (independent of k, the number of factors).

Before proving Proposition 3.2, we note that it implies the following

Corollary 3.3. *The vectors $H_\alpha^{m,l}$ are analytic for all m, l, α.*

Proof. Let $t \in \mathbb{R}$ such that $|t| < \lambda^{-1}$. We wish to estimate

$$\sum_{k=0}^{\infty} \frac{|t|^k}{k!} ||D^k H_\alpha^{m,l}||$$

which, for $\alpha = +$, is equal to

$$\sum_{n \geqslant 0} \left\{ \frac{|t|^{2n}}{(2n)!} ||(D^- D^+)^n H^{m,l}|| + \frac{|t|^{2n+1}}{(2n+1)!} ||(D^+ D^-)^n D^+ H^{m,l}|| \right\}$$

whereas, for $\alpha = -$, equals

$$\sum_{n \geqslant 0} \left\{ \frac{|t|^{2n}}{(2n)!} ||(D^+ D^-)^n H^{m,l}|| + \frac{|t|^{2n+1}}{(2n+1)!} ||(D^- D^+)^n D^- H^{m,l}|| \right\}.$$

So it is enough to treat the first case. An application of Proposition 3.2 yields

$$\sum_{k=0}^{\infty} \frac{|t|^k}{k!} ||D^k H_\alpha^{m,l}|| \leqslant C_{m,l} \sum_{k=0}^{\infty} (|t|\lambda)^k,$$

a convergent series since $|t|\lambda < 1$. $\square$

So we are left with the task of proving Proposition 3.2. We need the following

Lemma 3.4. (i) *With the above notations, one has*

$$D^+ H^{r,s} = 2is H^{r,s-1}, \qquad D^- H^{r,s} = 2i(rH^{r-1,s} - nH^{r,s+1}).$$

(ii) *The norm squared of $H^{r,s}$ is given by the following expression*:

$$||H^{r,s}||^2_{L^2(\mathbb{R}^2,\mathbb{C})} = \frac{\pi}{n^{r+s+1}}(r+s)!.$$

Proof. Both assertions are easy, the second one being obtained via polar coordinates and the identity $\int_0^{+\infty} \xi^h e^{-\xi}\, d\xi = h!$. $\square$

Proof of Proposition 3.2. On setting

$$\tilde{\lambda} := \max\{2n, 2r, 2s\} \ (>0),$$

we find, by Lemma 3.4 (i), the norm estimates

$$||D^- H^{r,s}|| \leqslant \tilde{\lambda}(||H^{r-1,s}|| + ||H^{r,s+1}||), \qquad ||D^+ H^{r,s}|| \leqslant \tilde{\lambda}||H^{r,s-1}||$$

which, in turn, imply (again in conjunction with Lemma 3.4)

$$||D^\sharp \dots D^\sharp H^{m,l}|| \leqslant (2\tilde{\lambda})^k \max\{||H^{l,m}|| \mid 0 \leqslant l \leqslant r,\ 0 \leqslant m \leqslant s+k\}$$

$$\leqslant (2\tilde{\lambda})^k \left(\frac{\pi}{n}\right)^{\frac{1}{2}} [(r+s+k)!]^{\frac{1}{2}}.$$

Now, from $\lim_{k\to+\infty} [(r+s+k)!]^{\frac{1}{2}}/(k!) = 0$, one has, for some positive constant $\tilde{C}_{r,s}$,

$$[(r+s+k)!]^{\frac{1}{2}} \leqslant \tilde{C}_{r,s} k!$$

whence the above inequality implies the following:

$$||D^\sharp \dots D^\sharp H^{r,s}|| \leqslant (2\tilde{\lambda})^k \left(\frac{\pi}{n}\right)^{\frac{1}{2}} \tilde{C}_{r,s} k!$$

so, after redefining constants $\lambda := 2\tilde{\lambda}$, $C_{r,s} := (\frac{\pi}{n})^{\frac{1}{2}} \tilde{C}_{r,s}$ the sought for estimate $(*)$ follows, and Proposition 3.2 is proved. □

We are in a position to state the following

Theorem 3.5. *The operator D_n, defined on its initial domain $\mathscr{D}(D_n) = \mathscr{S}(\mathbb{R}^2, \mathbb{C}^2)$, is essentially self-adjoint.*

Proof. The set $\mathscr{D}_{\text{fin}} = \{\text{finite linear combinations of } H_\alpha^{r,s}\}$ is D_n-invariant and provides a total set of analytic vectors for the operator in question, so the result follows from Nelson's analytic vector theorem (see, e.g., [24], Vol. II). □

In the sequel, we shall denote the unique self-adjoint extension of the above operator, with its appropriate domain, either by $(\hat{D}_n, \mathscr{D}(\hat{D}_n))$ or simply by $\hat{D}_n$, or again by D_n, if no confusion can arise. We determine the kernel of $\hat{D}_n$, for future use. We begin by pointing out the following:

Lemma 3.6. *Let $f : \mathbb{R}^d \to E$ be a C^2-function with values in a Hilbert space E such that* (i) *f is harmonic, and* (ii) *$f \in L^2(\mathbb{R}^d, E)$. Then $f = 0$.*

Proof. The Fourier transformed Laplace equation reads $||\xi||^2 \hat{f} = 0$, whence $\hat{f} = 0$ a.e., so $0 = f$ in L^2, hence pointwise as well, since it is C^2. □

Corollary 3.7. *Let* $f:\mathbb{R}^2\to\mathbb{C}$ *be holomorphic or anti-holomorphic and in* $L^2(\mathbb{R}^2,\mathbb{C})$. *Then* f *is identically zero.*

Proposition 3.8. *The kernel of* $\hat{D}_n$ *(n fixed) is given by*

$$Ker\,\hat{D}_n = (Ker\,\hat{D}_n)^+ := \left\{ \begin{pmatrix} f_+ \\ 0 \end{pmatrix} \in \mathscr{D}(\hat{D}_n) \subset L^2(\mathbb{R}^2,\mathbb{C}^2) \,\middle|\, \hat{D}_n \begin{pmatrix} f_+ \\ 0 \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} h_+ \cdot \varepsilon_n \\ 0 \end{pmatrix} \in L^2(\mathbb{R}^2,\mathbb{C}^2) \,\middle|\, h_+ \text{ is holomorphic} \right\}.$$

In particular,

$$(Ker\,\hat{D}_n)^- := \left\{ \begin{pmatrix} 0 \\ f_- \end{pmatrix} \in \mathscr{D}(\hat{D}_n) \subset L^2(\mathbb{R}^2,\mathbb{C}^2) \,\middle|\, \hat{D}_n \begin{pmatrix} 0 \\ f_- \end{pmatrix} = 0 \right\} = \{0\}.$$

Proof. Let us drop the suffix "n" and let $(\hat{D},\mathscr{D}(\hat{D}))$ denote the s.a. extension of $(D,\mathscr{D})$ during this proof. A vector

$$f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \qquad \text{in } \mathscr{D}(\hat{D})$$

belongs to *Ker* $\hat{D}$ if and only if

$$0 = \langle \hat{D}f, g \rangle \qquad \forall g \in \mathscr{S}(\mathbb{R}^2,\mathbb{C}^2).$$

This is, in turn, equivalent to

$$0 = \langle f, Dg \rangle \qquad \forall g \in \mathscr{S}(\mathbb{R}^2,\mathbb{C}^2).$$

But, since $C_c^\infty(\mathbb{R}^2,\mathbb{C}^2)$, the space of $\mathbb{C}^2$-valued smooth functions with compact support is dense in $\mathscr{S}(\mathbb{R}^2,\mathbb{C}^2)$, one finds

$$Ker\,\hat{D} = \mathscr{D}(\hat{D}) \cap \{f \in (C_c^\infty(\mathbb{R}^2,\mathbb{C}^2))' \mid Df = 0\} \cap L^2(\mathbb{R}^2,\mathbb{C}^2)$$

$$\subset \{f \in (C_c^\infty(\mathbb{R}^2,\mathbb{C}^2))' \mid Df = 0\} \cap L^2(\mathbb{R}^2,\mathbb{C}^2),$$

i.e., one has first to look for distributional solutions to $Df = 0$ which are square-integrable. Setting

$$\begin{pmatrix} h_+ \\ h_- \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1} f_+ \\ \varepsilon f_- \end{pmatrix},$$

the (distributional) identity $Df = 0$ is equivalent to

$$\begin{cases} \partial_{\bar{z}} h_+ = 0, \\ \partial_z h_- = 0, \end{cases}$$

hence, h_+ (resp. h_-) must be a holomorphic (resp. anti-holomorphic) function, in view of elliptic regularity.

Now, the function $f_- = h_- \varepsilon^{-1}$ must be in L^2, so a fortiori, h_- itself belongs to L^2, so it vanishes by Corollary 3.7.

The condition

$$0 = \left\langle \begin{pmatrix} h_+ \varepsilon \\ 0 \end{pmatrix}, Dg \right\rangle \qquad \forall g \in C_c^\infty(\mathbb{R}^2, \mathbb{C}^2),$$

implies obviously

$$0 = \left\langle \begin{pmatrix} h_+ \varepsilon \\ 0 \end{pmatrix}, Dg \right\rangle \qquad \forall g \in \mathscr{S}(\mathbb{R}^2, \mathbb{C}^2) = \mathscr{D}(D).$$

Since D is essentially s.a. by Theorem 3.5, we have $\hat{D} = D^*$ and thus we conclude that $\binom{h_+\varepsilon}{0}$ is in $Ker\, \hat{D}$. The proposition is proved. $\square$

Let now

$$i_\pm : L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C}) \oplus L^2(\mathbb{R}^2, \mathbb{C}) \cong L^2(\mathbb{R}^2, \mathbb{C}^2),$$

respectively,

$$p_\pm : L^2(\mathbb{R}^2, \mathbb{C}^2) \cong L^2(\mathbb{R}^2, \mathbb{C}) \oplus L^2(\mathbb{R}^2, \mathbb{C}) \to L^2(\mathbb{R}^2, \mathbb{C})$$

denote the canonical injections and, respectively, projections. Setting $\hat{D}_n^+ = p_- \circ \hat{D}_n \circ i_+ : \mathscr{D}(\hat{D}_n^+) := i_+^{-1}[\mathscr{D}(\hat{D}_n) \cap (L^2(\mathbb{R}^2, \mathbb{C}) \oplus \{0\})] \to L^2(\mathbb{R}^2, \mathbb{C})$ and similarly for the superscript "–," we have

Lemma 3.9. *Let* $(\hat{D}_n^\pm, \mathscr{D}(\hat{D}_n^\pm))$ *be defined as above. Then,*

(i) $\mathscr{S}(\mathbb{R}^2, \mathbb{C}) \subset \mathscr{D}(\hat{D}_n^\pm) \subset L^2(\mathbb{R}^2, \mathbb{C})$.
(ii) $(\hat{D}_n^\pm, \mathscr{D}(\hat{D}_n^\pm))^* = (\hat{D}_n^\mp, \mathscr{D}(\hat{D}_n^\mp))$.

Proof. (i) This assertion is clear from $\mathscr{S}(\mathbb{R}^2, \mathbb{C}) \oplus \mathscr{S}(\mathbb{R}^2, \mathbb{C}) = \mathscr{S}(\mathbb{R}^2, \mathbb{C}^2) \subset \mathscr{D}(\hat{D}_n)$.

(ii) In order to show that the LHS includes the RHS consider $f \in \mathscr{D}(\hat{D}_n^-)$ and $g \in \mathscr{D}(\hat{D}_n^+)$. It suffices to show that

$$\langle \hat{D}_n^- f, g \rangle = \langle f, \hat{D}_n^+ g \rangle$$

with respect to the scalar product in $L^2(\mathbb{R}^2, \mathbb{C})$. Now,

$$\langle \hat{D}_n^- f, g \rangle = \left\langle \hat{D}_n \begin{pmatrix} 0 \\ f \end{pmatrix}, \begin{pmatrix} g \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ f \end{pmatrix}, \hat{D}_n \begin{pmatrix} g \\ 0 \end{pmatrix} \right\rangle = \langle f, D_n^+ g \rangle,$$

i.e., the LHS includes the RHS.

Consider now $g \in (\hat{D}_n^+, \mathscr{D}(\hat{D}_n^+))^* \subset (D_n^+, \mathscr{S}(\mathbb{R}^2, \mathbb{C}))^*$. For all $f \in \mathscr{S}(\mathbb{R}^2, \mathbb{C})$, the map

$$\mathscr{S}(\mathbb{R}^2, \mathbb{C}) \ni f \mapsto \langle g, D_n^+ f \rangle \in \mathbb{C}$$

is continuous (with respect to the pre-hilbertian structure given by the L^2-scalar product on $\mathscr{S}(\mathbb{R}^2, \mathbb{C})$), this being the same as the continuity of

$$\mathscr{S}(\mathbb{R}^2, \mathbb{C}) \ni f \mapsto \left\langle \begin{pmatrix} 0 \\ g \end{pmatrix}, D_n^+ \begin{pmatrix} f \\ 0 \end{pmatrix} \right\rangle \in \mathbb{C}.$$

But, by self-adjointness of $\hat{D}_n$, the map

$$\mathscr{S}(\mathbb{R}^2, \mathbb{C}) \oplus \mathscr{S}(\mathbb{R}^2, \mathbb{C}) \ni \begin{pmatrix} f \\ h \end{pmatrix} \mapsto \left\langle \begin{pmatrix} 0 \\ g \end{pmatrix}, D_n \begin{pmatrix} f \\ h \end{pmatrix} \right\rangle \in \mathbb{C}$$

is continuous as well and restricts to the former, whence

$$\begin{pmatrix} 0 \\ g \end{pmatrix} \in (D_n, \mathscr{S}(\mathbb{R}^2, \mathbb{C}) \oplus \mathscr{S}(\mathbb{R}^2, \mathbb{C}))^* = (D_n, \mathscr{D}(D_n))^* = (\hat{D}_n, \mathscr{D}(\hat{D}_n)),$$

since, for any essentially self-adjoint operator T, one has $\bar{T} = T^{**} = T^*$. This entails $g \in \mathscr{D}(\hat{D}_n^-)$. Thus, for each $f \in \mathscr{S}(\mathbb{R}^2, \mathbb{C})$,

$$\langle g, D_n^+ f \rangle = \left\langle \begin{pmatrix} 0 \\ g \end{pmatrix}, D_n \begin{pmatrix} f \\ 0 \end{pmatrix} \right\rangle = \left\langle D_n \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} f \\ 0 \end{pmatrix} \right\rangle = \langle D_n^- g, f \rangle,$$
$$\text{i.e.,} \quad \langle (D_n^+)^* g, f \rangle = \langle g, D_n^+ f \rangle = \langle D_n^- g, f \rangle,$$

showing that the LHS is included in the RHS and thus completing the proof of (ii). □

Corollary 3.10. *The operators $\hat{D}_n^\pm$ are closed and Ker $\hat{D}_n^\pm = (\mathrm{Im}\, \hat{D}_n^\mp)^\perp \cap \mathscr{D}(\hat{D}_n^\pm)$. Furthermore Ker $\hat{D}_n^\pm \cong (\mathrm{Ker}\, \hat{D}_n)^\pm$ (the latter "kernels" being defined in Proposition* 3.8).

Proof. The first assertion follows since the operators involved are each other's adjoints. The second follows from the fact that, for any densely defined, closed operator T, one has $Ker\, T = (\mathrm{Im}\, T^*)^\perp \cap \mathscr{D}(T)$ (see, e.g., [25] p. 363). The last part follows directly from inspecting the various definitions. □

Remark. (1) If no confusion arises, in the sequel we shall denote the s.a. extensions $\hat{D}_n^\pm$ of $D_n^\pm$ by $D_n^\pm$ as well.

(2) The triviality of $Ker\, D_n^-$ can be ascertained, at least formally, via the following computation: if $\varphi \in Ker\, D_n^-$, then

$$4n\varphi = [D_n^+, D_n^-]\varphi = -D_n^- D_n^+ \varphi,$$

which cannot hold unless $\varphi = 0$, since the operator in the last right-hand side is non-positive.

(3) Clearly, the above representation of the CCR is highly reducible: taking von Neumann's uniqueness theorem (see, e.g., [24]) into due account, its multiplicity is actually infinite, being governed by $Ker\, D_n^+$. The latter can be realized as $\overline{((\{(D_n^-)^k \varepsilon_n \mid k \geqslant 0\}))}$ using the "right movers" CCR algebra (commuting with the left movers one), see the remark after the proof of Proposition 3.1 and below in Section 5.

3.2. The Dirac–Ramond operator

In this section, we are going to construct the Dirac–Ramond operator. Let $\mathscr{H}_0$ denote the linear span of the vectors $\xi \in \mathscr{H}$ of the form

$$(**) \quad \xi_1 \otimes \cdots \otimes \xi_N \otimes \varphi_{N+1} \otimes \cdots$$

with $\xi_j = H_{\alpha_j}^{r_j, s_j}$ ($r_j, s_j \in \mathbb{N}$, $\alpha_j \in \{+, -\}$) for $j = 1, 2, \ldots, N$. Note that again, we are employing an infinite tensor product notation. Obviously, $\mathscr{H}_0$ is a dense subspace of $\mathscr{H}$. Let us define, for $n \geqslant 1$

$$\begin{aligned}\tilde{D}_n &:= \tilde{\psi}(f_n)(\partial_{x_n} + iny_n) + \tilde{\psi}(g_n)(\partial_{y_n} - inx_n) \\ &= V_1 \otimes \cdots \otimes V_{n-1} \otimes D_n \otimes \mathrm{Id}_{B_{n+1} \otimes F_{n+1}} \otimes \cdots \ .\end{aligned}$$

(Compare with the discussion in Section 2.4 for the insertion of $V_1, \ldots, V_{n-1}$ before D_n. In the preceding formula, V_k stands in fact for $Id_{B_k} \otimes V_{F_k}$, where V_{F_k} is the grading operator of F_k, formerly denoted by V_k.)

Also,

$$\tilde{D}_n = \tilde{a}(e_n^-) D_n^- + \tilde{a}(e_n^-)^* D_n^+$$

(where the $\tilde{a}$'s are obtained from the $\tilde{\psi}$'s in the standard manner). Set

$$D_K := \sum_{n \geqslant 1} \tilde{D}_n.$$

This operator is initially defined on $\mathscr{H}_0$ and symmetric thereon. We are going to show that, indeed, it is essentially s.a., whence it admits a unique s.a. extension, denoted in the same way. We again resort to Nelson's theorem.

First, setting

$$D_n' = \mathrm{Id}_{B_1 \otimes F_1} \otimes \cdots \otimes \mathrm{Id}_{B_{n-1} \otimes F_{n-1}} \otimes D_n \otimes \mathrm{Id}_{B_{n+1} \otimes F_{n+1}} \otimes \cdots,$$

we notice the following Lichnerowicz-type formula, which is straightforward to prove

Proposition 3.11. *On $\mathscr{H}_0$, one has*

$$D_K^2 = \sum_{n \geqslant 1} \tilde{D}_n^2 = \sum_{n \geqslant 1} (D_n')^2.$$

Proposition 3.12. *The vectors of the form $(**)$ above are analytic for D_K.*

Proof. First, we consider a $t>0$ and a ξ as in $(**)$,

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} ||D_K^k \xi|| = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} ||D_K^{2k} \xi|| + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} ||D_K^{2k} D_K \xi||.$$

The vector $\eta := D_K \xi$ is again in $\mathscr{H}_0$, i.e., it is a finite linear combination of vectors as in $(**)$. Thus, the preceding equality shows that, in order to prove analyticity of the vector ξ, we may restrict our attention to the first summand. For the sequel we set $D_K^2 = \Delta$, and remove suffixes from the partial operators. We deal with the case $N=2$, the general case following by induction or by a direct (slightly more involved computation). So, consider the series

$$\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} ||\Delta^k (\xi_1 \otimes \xi_2 \otimes \varphi_3 \otimes \cdots)||,$$

which, by virtue of the (crucial!) fact $D_n^2 \varphi_n = 0$, $n = 1, 2 \ldots$, also reads as

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \left|\left| \sum_{h=0}^{k} \binom{k}{h} \Delta^h \xi_1 \otimes \Delta^{k-h} \xi_2 \otimes \varphi_3 \otimes \cdots \right|\right| \\ &\quad \leqslant \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} 2^k \left(\sum_{h=0}^{k} ||\Delta^h \xi_1|| \cdot ||\Delta^{k-h} \xi_2|| \right) \end{aligned}$$

having used the crude estimate $\binom{k}{h} \leqslant 2^k$. Now, recalling that (for $j = 1, 2$)

$$||\Delta^h \xi_j|| \leqslant C_j \lambda_j^{2h} (2h)!$$

(with C_j independent of h), and letting $\lambda = \max(\lambda_1, \lambda_2)$, and $C = C_1 C_2$, we bound as follows:

$$C \sum_{k=0}^{\infty} (t\lambda\sqrt{2})^{2k} \sum_{h=0}^{k} \frac{(2h)! \cdot (2(k-h))!}{(2k)!} \leqslant C \sum_{k=0}^{\infty} (k+1)(t^2\lambda^2 2)^k$$

(since each term in the finite sum is bounded by 1), and the last series converges for $t\sqrt{2}\lambda < 1$. This completes the proof by observing that for real t with $|t| < (\sqrt{2}\lambda)^{-1}$, the series $\sum_{k=0}^{\infty}(\frac{t^k}{k!} D_K^k \xi)$ is absolutely convergent. □

Nelson's theorem then yields

Theorem 3.13. *We call the unique self-adjoint extension of D_K the "Dirac–Ramond operator" pertaining to $\mathscr{L}_0\mathbb{R}$ and denote it also by D_K. The associated "Hamiltonian (operator)" $\Delta = D_K^2$ is self-adjoint as well.*

Remark. (1) Recall that the (more general) normal bundle operators considered in [31] are always defined only on smooth sections. Also, we have discarded the manifold part since in the flat case it does not contribute to the equivariant index.

(2) The self-adjointness of Δ (an immediate consequence of spectral calculus), can be directly ascertained either by the analytic vector calculation above, or by using the Streit–Kraus–Polley–Reents theorem, recalled in Section 2.1. We finally note that, in view of positivity, Δ certainly admits, a priori, a self-adjoint extension, namely the Friedrichs' one (see again, e.g., [24]).

(3) Although it is not needed in this work, we record the explicit formula for D_n^2 (yielding, in turn, a formula for D_K^2):

$$D_n^2 = (-\Delta_n + n^2(x_n^2 + y_n^2)) \cdot I - 2iK_n \cdot I - 2n \cdot V_n,$$

where Δ_n is the two-dimensional Laplacian on the space $H_n \cong \mathbb{R}^2$ with coordinates x_n and y_n, I is the 2×2 identity matrix, and K_n and V_n are the generators of the rotation group and the parity operator, respectively (cf. Sections 2.2 and 2.4). The first summand is of course a two-dimensional quantum harmonic oscillator Hamiltonian. Similar formulae appear in Witten's approach to the Atiyah–Hirzebruch theorem (see [36]).

4. The S^1-equivariant L^2-index of the Dirac–Ramond operator

In this section, we are going to define and compute the rotation equivariant (L^2)-index of the Dirac–Ramond operator constructed in the previous section. For the general background concerning equivariant index theorems on finite-dimensional manifolds, we refer to, e.g., [5]. Nevertheless, our approach is essentially self-contained.

4.1. Kernels and indices of partial Dirac operators

Proposition 4.1. *The S^1-equivariant L^2-index of D_n, defined by*

$$ind^{S^1}(D_n)(q) := Tr(\varrho_n(q)|_{Ker\ D_n^+}) - Tr\,(\varrho_n(q)|_{Ker\ D_n^-})$$

is in R_+ and

$$ind^{S^1}(D_n)(q) = \sum_{m\geqslant 0} q^{nm} = \frac{1}{1-q^n},$$

where ϱ_n denotes the S^1-representation on $B_n = L^2_{\mathbb{C}}(H_n, \lambda^2_{(n)}) \cong L^2_{\mathbb{C}}(\mathbb{R}^2, \lambda^2)$ induced by multiplication by q^n on H_n.

Proof. First of all, note that we are justified in using the kernels of $D_n^{\pm}$ in the definition of the equivariant index since they are bona fide each other's adjoints by Corollary 3.10. Furthermore, recall that $Ker\,D_n^- = \{0\}$, so the "virtual" term vanishes, and $E_n := Ker\ D_n^+ = \{h\varepsilon_n \in L^2(\mathbb{R}^2, \mathbb{C}) \mid h \in \mathcal{O}(\mathbb{C})\}$, namely, it is a Bargmann space; this allows us to compute, for fixed n, the dimensions of the isotypical factors $(E_n)_{(k)}$, i.e., $\mu_k := dim(E_n)_{(k)}$, quite easily, in view of the explicit realization of the S^1-action discussed in Section 2.2 (it is enough to specify the latter on monomials $z_n^m := (z_n)^m$ in E_n):

$$\check{\vartheta}_q(z_n^m) = q^{nm} \cdot z_n^m$$

and recalling that

$$(E_n)_{(k)} = \{f \in E_n \mid \check{\vartheta}_q(f) = q^k \cdot f\}$$

it is easy to check that

$$\mu_k = \begin{cases} 1 & \text{if } n|k, \\ 0 & \text{otherwise,} \end{cases}$$

whence the conclusion. □

4.2. The Kernel and index of the Dirac–Ramond operator

Let $p(k)$ denote Euler's partitio numerorum, namely, the number of partitions of a whole number k into a sum of positive integers up to order (see, e.g., [12], p. 273). We are in a position to state and prove the following theorem, which is the goal of the present paper:

Theorem 4.2. *With the above notations*:

(i)

$$Ker\, D_K = \overset{\Phi}{\underset{n \geqslant 1}{\otimes}} Ker\, D_n.$$

(ii) *Moreover*, $Ker\, D_K \in R_+$ *and*

$$\mu_k = \begin{cases} p(k) & \textit{if } k \geqslant 1, \\ 1 & \textit{if } k = 0, \\ 0 & \textit{if } k < 0. \end{cases}$$

(iii) *The* S^1*-equivariant* L^2*-index of* D_K *is well defined as an element in* R_+ *and reads*

$$ind^{S^1}(D_K)(q) = 1 + \sum_{k=1}^{\infty} p(k)q^k = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

(iv) *The* S^1*-equivariant* L^2*-index of* D_K *on* $\mathscr{L}_0\mathbb{R}^d$ *is given by*

$$ind^{S^1}(D_K)(q) = \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right)^d.$$

Proof. Ad (i). Start from Lichnerowicz's formula $D_K^2 = \sum_n \tilde{D}_n^2 = \sum_n (D'_n)^2$. Now, since D_K^2 and D_n^2 are s.a. and positive, one has $Ker\, D_K = Ker\, D_K^2$ and $Ker\, D_n = Ker\, D_n^2$. Moreover, in view of Lemma 2.3, positivity implies

$$Ker\, D_K^2 = \overset{\Phi}{\underset{n \geqslant 1}{\otimes}} Ker\, D_n^2,$$

whence the conclusion. Furthermore, one has

$$Ker\, D_K = \overset{\Phi}{\underset{n \geqslant 1}{\otimes}} Ker\, D_n = \overset{\Phi}{\underset{n \geqslant 1}{\otimes}} Ker\, D_n^+ = Ker\, D_K^+$$

with $D_K^{\pm}$ denoting the respective decompositions of the full operator D_K with respect to the parity operator $\mathbf{V}$.

Assertion (ii) and the first equality in (iii) are proved by resorting to the previous isotypical factor computation, Proposition 4.1, and determination of the coefficients of powers of q. (Note that $\otimes^{\Phi} Ker\, D_n$ is a positive energy representation, having, notably, finite multiplicities pertaining to a given irreducible S^1-representation;

this being of course not true for an arbitrary countable incomplete direct product of positive energy representations.) The second equality in (iii) is just Euler's formula.

The last assertion (iv) is an immediate consequence of multiplicativity properties, in view of

$$\mathscr{L}\mathbb{R}^d = \mathscr{L}\mathbb{R} \otimes \mathbb{R}^d \qquad \text{and} \qquad \mathscr{L}_0\mathbb{R}^d = \mathscr{L}_0\mathbb{R} \otimes \mathbb{R}^d.$$

This concludes the proof. □

5. Remarks on supersymmetric second quantization

In this final section, we wish to interpret the Dirac–Ramond operator in the general formalism for the second quantization of models with bosons and fermions set up in [9]. Actually, our picture is slightly more complicated than the one developed there, since it involves a (highly) reducible representation of the CCR algebra. Our rigorous but informal discussion can be resumed as follows: after identifying the ($\mathbb{Z}_2$-graded) one-particle space, we show that the Dirac–Ramond operator stabilizes it but exchanges bosons and fermions, i.e., "the Dirac–Ramond operator is the second quantization of a supersymmetry transformation."

The one-particle space of the theory is the direct (orthogonal) sum of a boson (say U) and a fermion one-particle space (W). Each type, in turn, should be given as an orthogonal direct sum over nth-mode one-particle spaces. We shall then realize U and W inside $\mathscr{H} = B \otimes F$.

The bosonic part U is dealt with as follows. We first recall the formula for the nth-mode right mover boson creation operator, given in Section 3:

$$\underline{D}_n^- = i(2\partial_{\bar{z}_n} - nz_n).$$

The multiplicity of the representation of the CCR defined by $D_n^{\pm}$ is given by $Ker\ D_n^+$, which is then interpreted as the (nth-mode) "pure" boson vacuum space. The nth-mode boson one-particle space U_n is then

$$U_n = D_n^-(Ker\ D_n^+) \otimes ((e_n^+))_{\mathbb{C}}.$$

Hence,

$$U = \hat{\bigoplus_{n \geqslant 1}} U_n.$$

Observe that by a direct calculation it is easily checked that $\underline{D}_n^-$ stabilizes $Ker\ D_n^+$ and that we also have

$$U_n = \overline{((\{D_n^-(\underline{D}_n^-)^k \varepsilon_n \mid k \geqslant 0\}))} \otimes ((e_n^+))_{\mathbb{C}}.$$

The fermionic part is identified as

$$W = \hat{\bigoplus_{n \geqslant 1}} W_n,$$

where

$$W_n = (Ker\, D_n^+) \otimes ((e_n^-))_{\mathbb{C}}$$

(consistently with Section 2.4). The full $\mathbb{Z}_2$-graded one-particle space is of course $U \oplus W$.

Also, observe that the space B_n can be described as

$$B_n = \overline{((\{(D_n^-)^h (\underline{D}_n^-)^k \varepsilon_n \mid h, k \geqslant 0\}))}$$

(using both CCR algebras of left- and right-moving bosons).

Next, we show that the Dirac–Ramond operator leaves the one-particle space invariant. The images of the nth-mode one-particle states are realized in the infinite tensor product description in the following way. A typical nth-mode one-particle boson state $D_n^- \xi_n \otimes e_n^+$, with $\xi_n \in Ker\, D_n^+$, is represented in $\mathscr{H}$ as

$$(\varepsilon_1 \otimes e_1^+) \otimes \cdots \otimes (\varepsilon_{n-1} \otimes e_{n-1}^+) \otimes (D_n^- \xi_n \otimes e_n^+) \otimes (\varepsilon_{n+1} \otimes e_{n+1}^+) \otimes \cdots,$$

whereas (nth-mode) one-particle Fermi states can be realized as

$$(\varepsilon_1 \otimes e_1^+) \otimes \cdots \otimes (\varepsilon_{n-1} \otimes e_{n-1}^+) \otimes (\xi_n \otimes e_n^-) \otimes (\varepsilon_{n+1} \otimes e_{n+1}^+) \otimes \cdots$$

with $\xi_n \in Ker\, D_n^+$. It is then enough to show the claimed property of the Dirac–Ramond operator on vectors of the above kind, the general statement following by linearity (no problem arises with the parity operators, since the vacua are even). But this is clear, since (with slight abuses of notation)

$$(a_n D_n^- + a_n^* D_n^+)(D_n^- \xi_n \otimes e_n^+) = 4n \xi_n \otimes e_n^-$$

and

$$(a_n D_n^- + a_n^* D_n^+)(\xi_n \otimes e_n^-) = D_n^- \xi_n \otimes e_n^+.$$

The boson–fermion exchange action of the Dirac–Ramond operator is then manifest.

Acknowledgments

The authors thank the IRMA (Strasbourg) and the DMMMSA (Padua) where part of this work was carried out, for their hospitality. They also thank the referee for critical remarks aimed at improving our exposition and for directing our attention to Landweber's paper [17].

References

[1] O. Alvarez, Lectures on quantum mechanics and the index theorem, in: D. Freed, K. Uhlenbeck (Eds.), Geometry and Quantum Field Theory, American Mathematical Society, Providence, RI, 1995, pp. 271–322.

[2] O. Alvarez, T. Killingback, M. Mangano, P. Windey, String theory and loop space index theorems, Comm. Math. Phys. 111 (1987) 1–10.

[3] A. Arai, Dirac operators in boson–fermion Fock spaces and supersymmetric quantum field theory, J. Geom. Phys. 11 (1993) 465–490.

[4] D. Bennequin, En marge de l'apologie de Witten, Gaz. Math. 50 (1991) 71–93.

[5] N. Berline, E. Getzler, M. Vergne, Heat kernels and Dirac Operators, Springer, Berlin, 1992.

[6] J.M. Bismut, The Atiyah–Singer theorems. A probabilistic approach. I, J. Funct. Anal. 57 (1984) 56–99.

[7] J.L. Brylinski, Representations of loop groups, Dirac operators on loop space, and modular forms, Topology 29 (1990) 461–480.

[8] P. Gosselin, T. Wurzbacher, A stochastic approach to the Virasoro anomaly in quantization of strings in flat space, Proc. Steklov Math. Inst. 217 (1997) 230–252.

[9] H. Grosse, E. Langmann, A superversion of quasifree second quantization I: Charged particles, J. Math. Phys. 33 (1992) 1032–1046.

[10] A. Guichardet, Produits tensoriels infinis et représentations des relations d'anticommutations, Ann. Ecole. Norm. Sup. 3eme Sér. 83 (1966) 1–52.

[11] A. Guichardet, Produits tensoriels continus d'espaces et d'algèbres de Banach, Comm. Math. Phys. 5 (1967) 262–287.

[12] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, 5th Edition, Oxford University Press, Oxford, 1979.

[13] F. Hirzebruch, T. Berger, R. Jung, Manifolds and Modular Forms, Vieweg, Braunschweig, 1992.

[14] T. Killingback, World-sheet anomalies and loop geometry, Nucl. Phys. B 288 (1987) 578–588.

[15] K. Kraus, L. Polley, G. Reents, Generator of infinite direct products of unitary groups, J. Math. Phys. 18 (1977) 2166–2171.

[16] M. Kreck, S. Stolz, HP^2-bundles and elliptic homology, Acta Math. 171 (1993) 231–261.

[17] G.D. Landweber, Multiplets of representations and Kostant's Dirac operator for equal rank loop groups, arXiv:math.RT/0005057 v2, 9 November 2000.

[18] H.B. Lawson Jr., M.-L. Michelsohn, Spin Geometry, Princeton Univ. Press, Princeton, NJ, 1989.

[19] R. Léandre, Sur le théoreme d'Atiyah–Singer, Probab. Theory Related Fields 80 (1988) 119–137.

[20] R. Léandre, Analysis over loop space and topology, Preprint of the IEC Nancy No. 14, 2000.

[21] S. Ochanine, Sur les genres multiplicatifs définis par des intégrales elliptiques, Topology 26 (1987) 143–151.

[22] S. Ochanine, Elliptic genera, modular forms over KO_* and the Brown–Kervaire invariant, Math. Z. 206 (1991) 277–291.

[23] A. Pressley, G. Segal, Loop Groups, Oxford University Press, Oxford, 1986.

[24] M. Reed, B. Simon, "Methods of Modern Mathematical Physics I,II,III," Academic Press, New York/London, 1972–5;1980;1979.

[25] W. Rudin, Functional Analysis, McGraw–Hill, New York, 1973.

[26] A. Schellekens, N. Warner, Anomalies, characters and strings, Nucl. Phys. B 287 (1987) 317–361.

[27] M. Spera, T. Wurzbacher, Determinants, pfaffians, and quasi-free representations of the CAR-algebra, Rev. Math. Phys. 10 (1998) 705–721.

[28] M. Spera, T. Wurzbacher, Differential geometry of Grassmannian embeddings of based loop groups, Differential Geom. Appl. 13 (2000) 43–75.

[29] S. Stolz, A conjecture concerning positive Ricci curvature and the Witten genus, Math. Ann. 304 (1996) 785–800.

[30] L. Streit, Test function spaces for direct product representations of the canonical commutation relations, Comm. Math. Phys. 4 (1967) 22–31.

[31] C. Taubes, S^1-actions and elliptic genera, Comm. Math. Phys. 122 (1989) 455–526.

[32] J. von Neumann, On infinite direct products, in: Collected Works III, Pergamon Press, New York, 1961, pp. 323–399 (or: Comp. Math. 6 (1938) 1–77).
[33] E. Witten, Supersymmetry and Morse theory, J. Differential Geom. 17 (1982) 661–692.
[34] E. Witten, Global anomalies in String Theories, in: W.A. Bardeen, A.R. White (Eds.), Symposium on Anomalies, Geometry, Topology, World Scientific, Singapore, 1985, pp. 91–95.
[35] E. Witten, The index of the Dirac operator in loop space, in: P. Landweber (Ed.), Elliptic Functions and Modular Forms in Algebraic Topology, Lecture Notes in Mathematics, Vol. 1326, Springer, Berlin, 1988, pp. 161–181.
[36] E. Witten, Index of Dirac operators, in: P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, E. Witten (Eds.), Quantum Fields and Strings: A Course for Mathematicians, Vol. 1, American Mathematical Society, Providence, RI, 1999, pp. 475–511.