18.199 Spring 2024
Analysis on Loop space

Richard Melrose

Department of Mathematics, MIT
Email address: rbm@math.mit.edu

Abstract.

Contents

Preface

This is the Graduate Analysis Seminar in Spring 2024. As you see, the topic is 'Analysis on Loop space'. 'The' loop space of a manifold X consists of the maps $\lambda : \mathbb{S} \longrightarrow X$ from the circle to the manifold where there is some (deliberate) ambiguity as to the regularity of these curves – we will consider several different regularities as circumstances dictate but really consider these all to be manifestations of the same 'space'. For definiteness you can think about continuous maps, or smooth maps.

The basic problem is that analysis on loop space does not really exist. The intention of this seminar is to get to the level of what is currently known by the end – or as far as time permits. Why is analysis on loop space hard? The fundamental difficulty is that this is an infinite dimensional space and as a result a lot of the concepts we would like to rely on are absent or at least require serious thought.

The speed at which we progress through the material (which will have to be drawn from quite a few different sources) depends on the participants. I really mean this to be a seminar, so I don't plan to speak unless called upon to do so.

Grades: Will be based on one presentation chosen (at the conclusion) by the student. So this should not be taken too seriously. However, the whole thing will not work without a degree of commitment from the participants.

My aim: I have serveral different sets of notes that I would like to draw together to cover all this material. I am hoping that this seminar will settle the level at which these are (re-)written.

Preliminaries: I would like to speak, either by zoom or in person (I will not be at MIT until 30 Jan) with any prospective participants. I would not expect to discourage anyone, I think understanding even part of the material in this course would be worthwhile. Rather I want to set the early level so we can proceed in a deliberate fashion.

Now, about this being hard. Really the problem is that it is broad. The immediate goal is to understand 'string structures' on a manifold. What are strings? You will have to ask a Physicist, but one interpretation is that a string is a curve up to reparameterization. This is a 'warning' that we need to consider the action of the (orientation-preserving) diffeomorphisms on the circle as a 'gauge' group action on loop space. Everything is supposed to be 'equivariant' with respect to this action. However, a string structure on a manifold is a definite concept. It is topologically the 'next structure up' in a sequence

$$\text{Orientation} < \text{Spin structure} < \text{String structure}.$$

This is the basic issue. We need a mixture of geometry, topology, algebra and analysis to sort all this out.

First aim:

THEOREM 1 (Stolz-Teichner 2005 [7]). *The spin structures on a manifold are in 1-1 correspondence with the fusion orientations of the loop space.*

I decided to give a very brief overview of what I hope we can cover this semester. My plan is to use this seminar to (re-)write notes on loop space and string structures.

Thus one of the main aims is to understand the notion of a *string structure* on a manifold. Let's assume that we are talking about a compact connected manifold. There are three (and in principle more) successively 'finer' structures which a manifold may admit. So a manifold may be:

$$\text{Orientable} < \text{Spin} < \text{String} < \text{`five-brane'}$$

and I don't really know what the last one is.

Each of these is 'obstructed' by a cohomology class; in these cases w_1, w_2 and ∂p_1, for the moment just names. There is a direct relationship between these structures and the *Whitehead*[1] *tower* for the orthogonal group $\mathrm{O}(n)$. Namely there is a sequence of groups

$$\mathrm{O}(n) \hookleftarrow \mathrm{SO}(n) \twoheadleftarrow \mathrm{Spin}(n) \twoheadleftarrow \mathrm{String}(n)\dots \quad n \geq 4 \tag{1}$$

What this sequence does is 'kill' precisely the leading homotopy group at each stage. Thus the first step passes to the connected component of the identity, 'losing' the other component this 'kills' $\Pi_0(\mathrm{O}(n)) = \mathbb{Z}_.$ In the second case $\Pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$, $n > 2$, and the spin group is a double cover 'killing' this group. The third step, which is where we want to get to, kills the next homotopy, group – the bottom one for $\mathrm{Spin}(n)$ which is $\Pi_3 = \mathbb{Z}$.

Clearly we need to go through quite a bit of material to understand all this. What are the corresponding 'structures'. They come about by considering the tangent bundle of the manifold, M, and associated principal bundles – starting from the frame bundle. Giving M a Riemann structure, which we always can, means that we can define the orthonormal frame bundle F_{O}. Then we can ask for successive 'refinements' of this

$$\begin{tikzcd}
\mathrm{String}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{String}} \arrow[d] \\
\mathrm{Spin}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{Spin}} \arrow[d] \\
\mathrm{SO}(n) \arrow[r, dash] \arrow[d, hook] & F_{\mathrm{SO}} \arrow[d] \\
\mathrm{O}(n) \arrow[r, dash] & F_{\mathrm{O}} \arrow[d] \\
& M
\end{tikzcd} \tag{2}$$

Spin stuctures are heavily studied and we will need to do some of that. Of particular significance is the Dirac operator associated to a spin stucture and its index.

[1] George Whitehead, a former colleague of mine here at MIT

There is a big gap between spin and string structures mainly because everything related to string structures must be infinite-dimensional. This is where the loop space of a manifold comes in. The smooth loop space,

$$\mathcal{L}(M) = \mathcal{C}^\infty(\mathbb{S}; M) \tag{3}$$

is also an infinite-dimensional manifold but it is a very geometric one. One class of results about loop space is concerned with 'trangression'. At a basic level this arises from the evaluation map

$$\mathrm{ev} : \mathcal{L}(M) \times \mathbb{S} \longrightarrow M,\ (\lambda, \theta) \longmapsto \lambda(\theta). \tag{4}$$

If for instance u is a cohomology class on M then it can be pulled back under ev and then pushed forward, courtesy of the orientation of the circle to define the topological transgression map

$$\tau : H^k(M) \longrightarrow H^{k-1}(\mathcal{L}M).$$

This map is lossy – in general neither injective nor surjective. We are really interested in using the loop space to study M and to this end we can 'elevate' this transgression map to an isomorphism by changing the cohomology on the range space by adding 'fusion conditions'.

The primary relevance of loop space is a more general transgression result:

Theorem 2. *A string structure on a manifold is equivalent to a fusive spin structure on the loop space.*

To come to grips with this we need to understand loop groups and their central extensions. The hope, and it is still no more than that, is that we can emulate the 'extraction of information' from a spin structure via the Dirac operator with an analogous treatment of the Dirac-Ramond operator, on the looop space of a string manifold. This story is very far from complete.

Although we have not defined the manifold structure on $\mathcal{L}M$ we can identify the tangent space – so making us think that it should be a manifold. One point to note about smooth maps and infinite-dimensional spaces is that it is relatively easy to understand smoothness of maps from a finite-dimensional manifold *into* and infinite-dimensional one but much harder to understand smoothness of maps *from* an infinite-dimensional manifold into a finite-dimensional one. This includes the identity of the smooth functions on such an infinite-dimensional manifold, and hence bundles over it etc. We do need to go there!

Claim 1. *For any $\mathcal{C}^\infty$ manifold there is a natural identification*

$$T\mathcal{L}M = \mathcal{L}TM. \tag{5}$$

This is a claim rather than a proposition since we have not yet agreed that $\mathcal{L}M$ is a manifold so the left side of (5) is not actually defined. It is a (convincing I hope) plausibility argument.

However for $\mathcal{L}M$ we can quickly decide on the meaning of $\mathcal{C}^\infty((-1,1); \mathcal{L}M)$. Indeed for maps from any finite-dimensional manifold N into $\mathcal{L}M$ we take the natural definition

$$\mathcal{C}^\infty(N; \mathcal{L}M) = \mathcal{C}^\infty(M \times \mathbb{S}; M). \tag{6}$$

For curves, $N = (-1,1)$, we then see that a smooth curve in $\mathcal{L}M$, $Q \in \mathcal{C}^\infty((-1,1) \times \mathbb{S}; M)$ we can fix $\theta \in \mathbb{S}$ and see that

$$Q(\cdot, \theta) : (-1,1) \longrightarrow M$$

is a curve in M. One definition of the tangent space at a point $m \in M$ is as an equivalence class of smooth curves, $q : (-1, 1) \longrightarrow M$, through m, i.e. with $q(0) = m$ at parameter $t = 0$. In fact we think of the image as being the derivative $q'(0) \in T_{q(0)}M$ of the curve at $t = 0$. So this means we can define from Q a map

$$Q'(0, \cdot) : \mathbb{S} \ni \theta \longmapsto Q'(0, \theta) \in T_{Q(0,\theta)}M. \tag{7}$$

So this is indeed a smooth map and hence a loop

$$Q'\big|_{t=0} : \mathbb{S} \longrightarrow TM,\ Q'\big|_{t=0} \in \mathcal{L}TM. \tag{8}$$

Now we have a projection map, coming from $\pi : TM \longrightarrow M$ and this generates a projection map (where we just reuse the name)

$$\pi : \mathcal{L}TM \longrightarrow \mathcal{L}M,\ Q'\big|_{t=0} \longmapsto Q\big|_{t=0} \in \mathcal{L}M.$$

This is justification of (5) but we need to think about coordinate patches for the $\mathcal{C}^\infty$ structure on $\mathcal{L}M$.

Why are we interested in this? Really there are two reasons, the second being the more significant. The choice of a Riemann metric on a compact $\mathcal{C}^\infty$ manifold, M, fixes the expoential map at each point. This is the smooth map

$$\exp_m : T_mM \longrightarrow M \tag{9}$$

which maps each non-zero tangent vector $v \in T_mM$ the the point along the geodesic starting at m with tangent vector $\hat{v} = v/|v|$ to the point at distance $|v|$. It is a diffeomorphism from some ball $\{v \in T_mM; |v| < \epsilon\}$ (always in terms of the Riemannian metric as a Euclidean on T_mM where $\epsilon > 0$ can be taken independent of m by compactness. The image is the ball in M of point at Riemannian distance no more than ϵ. Each such ball is then a coordinate patch around m.

The first reason that this is useful is because of convexity. These balls are geodesically convex – any two point in the ball around any m are connected by a unique 'short' geodesic which lies in the ball. Again by compactness, M is covered by a finite number of such balls, all of radius ϵ, and this is a *good cover*. Namely each non-empty intersection of any collection of the elements in this cover of M are contractible. Just take any point in the intersection and contract the open set of point in the intersection to this point along the geodesics formed by the exponential map from that point.

Why is this useful? It means that we can compute the Čech cohomology of M using any such cover and you will see why this is handy.

The second, more directly significant, reason the exponential map is useful for us is that taken togther all the exponential maps give a smooth map

$$\exp : TM \longrightarrow M \times M,\ (m, v) \longrightarrow (m; \exp_m(v)). \tag{10}$$

This is again smooth and is a diffeomorphism of the 'tube' $|v| < \epsilon$ in the tangent bundle onto a neigbhourhood of the diagaonal in M^2.

Suppose $l \in \mathcal{L}M = \mathcal{C}^\infty(\mathbb{S}; M)$ is a smooth loop. The consider the set

$$\Gamma(l) = \{l' \in \mathcal{L}M; l'(\theta) \in \exp_{l(\theta)}(\{v \in T_{l(\theta)}M; |v| < v\}). \tag{11}$$

So this is the collection of loops such that the Riemannian distance $d(l(\theta); l'(\theta)) < \epsilon$ for all $\theta \in \mathbb{S}$.

Need a picture!

THEOREM 3. *The sets* $\Gamma(l) \subset \mathcal{L}M$ *are open and any non-empty intersection of any finite collection of them is contractible. If* M *is oriented they given coordinate patches on* $\mathcal{L}M$ *modeled on open subsets of* $\mathcal{C}^\infty(\mathbb{S};\mathbb{R}^n)$, *where* $n = \dim M$ *and contain a countable good cover of* $\mathcal{L}M$.

CHAPTER 1

Preliminary material

In this chapter brief introductions to some of the 'classical' topics that we need are included. Clearly these cannot properly cover the material required as background but are intended to fix notation and as an indication of what will be assumed below. Thanks are due to the participants in the Analysis Seminar in Spring 2024 for guidance as to what should be included.

1. Manifolds

For a more complete discussion see for example [4].

One objective of these notes is to explore the notion of, and consequences of the existence of, a string structure on a finite-dimensional manifold. The central object of study to (try to) do this is the infinite-dimensional manifold, the loop space, consisting of the loops in the manifold, i.e. maps from the circle into the manifold. So we start with a brief review of finite-dimensional manifolds but with an eye to viewing the loop space as a (very special) infinite-dimensional manifold.

A topological manifold with corners of dimension n, X, is a topological space, with a metrizable and second-countable (i.e. separable) topology and a covering by n-dimensional coordinate patches. Such a coordinate patch is a homeomorphism $F : U \longrightarrow U'$, from an open set $U \subset X$ to an open set $U' \subset [0,\infty)^n$. To encode smoothness on the manifold we require that there be an *atlas*, meaning an open cover by *compatible* coordinate patches $F_a : U_a \longrightarrow U'_a$

$$X = \bigcup_a U_a,$$
$$U_a \cap U_b \neq \emptyset \Longrightarrow F_b \circ F_a^{-1} : F_a(U_a \cap U_b) \longrightarrow F_b(U_a \cap U_b) \text{ be smooth.}$$

By 'smooth' here one can take $\mathcal{C}^k$ (and other) regularity as long as it behaves properly. Since corners are allowed in the coordinate patches the appropriate notion of smoothness is needed. For our purposes us we will take $k = \infty$, so all manifolds are by default '$\mathcal{C}^\infty$'. Conventionally, the '$\mathcal{C}^\infty$structure' on X is identified with a 'maximal atlas'. Given a $\mathcal{C}^\infty$ atlas there is a unique maximal atlas consisting of all the compatible coordinate patches.

So the term 'manifold' below will mean a smooth finite-dimensional manifold, unless otherwise stated. We will also generally assume that manifolds are connected.

The fundamental object on a manifold is the space, $\mathcal{C}^\infty(X)$, of smooth functions – either real or complex valued depending on context but typically real-valued. A function $f : X \longrightarrow \mathbb{R}$ is smooth, so an element of $\mathcal{C}^\infty(X)$, if all its 'local representatives'

$$f \circ F_a^{-1} : U'_a \longrightarrow \mathbb{R} \text{ are } \mathcal{C}^\infty \tag{1}$$

for all elements of the maximal atlas (or equivalently for any one compatible atlas). The function in (1) is typically written $f(x_1, \dots, x_n)$ where x_i are the coordinates on $\mathbb{R}^n$, where the 'the local coordinate functions' $x_i \circ F_a$ are defined on U_a.

An open subset of a manifold is a manifold. The product of two manifolds is a manifold with product coordinate patches.

The definition of a function is extend to maps between manifolds, so $E : X \longrightarrow Y$ is smooth, for two smooth manifolds X and Y if and only if it is continuous and the localized map

$$E_{ab} = G_b \circ F \circ F_a^{-1} : G^{-1}(V_b) \cap U_a \longrightarrow \mathbb{R}^m$$

is smooth for any two smooth coordinate patches $F_a : U_a \longrightarrow U'_a$ on X and $G_b : V_b \longrightarrow V'_b$ on X and Y respectively. As always it suffices to check this in atlases on each manifold, i.e. covers by compatible coordinate patches of the two manifolds.

The composite $E' \circ E$ of smooth maps $E : X \longrightarrow Y$ and $E : Y \longrightarrow Z$ is smooth. A map $F : X \longrightarrow Y$ is smooth if and only if the pull-back map on functions, defined by $F^* f = f \circ F$ for $f : Y \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) maps smooth functions to smooth functions

$$F^* : \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(X). \tag{2}$$

From smooth maps we can construct the two basic bundles over a smooth manifold. Namely the cotangent and tangent bundles; the spaces at each point can be defined by

(3)

$$T_p^* X = \mathcal{I}_p / \mathcal{I}_p^2,\ \mathcal{I}_p = \{f \in \mathcal{C}^\infty(X); f(p) = 0\},\ \mathcal{I}_p^2 = \{\sum_{\text{finite}} f_i g_i,\ f_i,\ g_i \in \mathcal{I}_p\}.$$

$$T_p X = \{c : (-1,1) \longrightarrow X;\ \text{smooth } c(0) = p\} / \simeq,\ c \simeq d \Longrightarrow (c^* f - d^* f)'(0) = 0.$$

In the second case the derivative is well-defined for functions on $(-1,1)$. You can take your pick as to which is more fundamental. By reference to local coordinates both are linear spaces of dimension n and they are canonically dual to each other with the pairing arising from the map

$$\{c : (-1,1) \longrightarrow X;\ \text{smooth } c(0) = p\} \times \mathcal{I}_p \ni (c, f) \longmapsto (f \circ c)'(0) = (c^* f)'(0). \tag{4}$$

The pull-back operation on functions, for a smooth map, descends to a

$$F^* : T^*_{F(x)} Y \longrightarrow T^*_x\ \forall\ x \in X \tag{5}$$

and the dual is the 'differential' of F

$$F_* : T_x Y \longrightarrow T_{F(x)}\ \forall\ x \in X. \tag{6}$$

2. Fibre bundles

The disjoint unions

$$T^* X = \bigcup_{p \in X} T^* X,\ TX = \bigcup_{p \in X} T_p X \tag{1}$$

are manifolds and also vector bundles, special cases of fibre bundles. The pull-back map (5) extends to a a smooth map

$$F^* : T^* Y \longrightarrow T^* X.$$

A triple consisting of a smooth map $\pi : X \longrightarrow M$ is a fibre bundle 'with typical fibre a manifold, F,' if π is surjective and locally trivial in the sense that each point

$m \in M$ is contained in an open set $O \subset M$ and there is a diffeomorphism T giving a commutative diagramme

$$\begin{tikzcd} F \times O \arrow[rr, "T"] \arrow[dr, "\pi"'] & & F \times O \arrow[dl, "\pi_2"] \\ & O & \end{tikzcd}$$

On the right the map is just the projection onto the second factor. If X, and hence M, is compact then this is equivalent to the pointwise (everywhere) condition that the differential of π is surjective. Note that if M is not connected, which we are usually assuming, then it is perfectly possible to have different model fibres over the different components.

A vector bundle is a fibre bundle where the fibres of the map π have a linear structure, the model space, F, is a linear space and the trivialization map T is fibre-linear.

3. Riemann metrics

3.1. Exponential map. Geodesic on a Riemann manifold can be defined in several ways, most directly here as parallel curves for the Levi-Civita connection. The exponential map, defined by geodesics, is a smooth map for each p (which depends smootly on p as well)

$$\mathrm{Exp}_p : T_pM \longrightarrow M. \tag{1}$$

The main points we want are that Exp_p is a diffeomorphism from sufficiently small balls around $0 \in T_pM$ (in terms of the fibre metric) to their images, which are the metric balls in M around p of the same radius. One of the important properties is that for small open balls the exponential map around *any point* defines a contraction. This is really the statement that small balls are geodesically convex. In consequence

PROPOSITION 1. *Small balls in a Riemann manifold (of fixed small radius if the manifold is compact) form a 'good' open cover, one in which all non-trival intersections are contractible.*

4. Sobolev and Hölder spaces

5. Lie groups

A Lie group G is a smooth (finite-dimensional) manifold with a smooth group structure. More precisely, $e \in G$ is a unit elements for a commutative product where every element has an inverse and such that the map

$$G \times G \ni (g, h) \longrightarrow gh^{-1} \in G \text{ is smooth.} \tag{1}$$

Left multiplication by an element $g \in G$, $g\cdot : G \ni h \longmapsto gh \in G$, is therefore a diffeomorphism. The tangent space at the identity $\mathfrak{g} = T_eG$ is then mapped isomorphically by the left action

$$g_* : \mathfrak{g} \longrightarrow T_gG.$$

It follows that the tangent bundle is globally trivial,

$$TG = G \times \mathfrak{g}.$$

In particular every element $v \in \mathfrak{g}$ extends uniquely to a left-invariant vector field

$$L_v(g) = g_* v \tag{2}$$

and conversely any left-invariant vector field arises this way.

The commutator of two left-invariant vector fields is left-invariant making $\mathfrak{g}$ into a finite-dimensional Lie algbra (in particular the Jacobi identity

$$[v_1, [v_2, v_3] + [v_2, [v_3, v_1] + [v_3, [v_2, v_1] = 0, \ \forall \ v_i \in \mathfrak{g}$$

holds. Conversely each such Lie algebra arises from a Lie group.

The integration of a left-invariant vector field generates a 1-parameter subgroup $\exp(tv) \in G$ with tangent vector $\exp(tv)_* v$ at parameter value t. The resulting exponential map

$$\exp : \mathfrak{g} \longrightarrow G \tag{3}$$

is a diffeomorphism of a neighbourhood of 0 to a neighbourhood of e.

We are particularly interested in the matrix groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ and there subgroups

$$\begin{aligned} \mathrm{SO}(n) \subset \mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R}) \\ \mathrm{SU}(n) \subset \mathrm{U}(n) \subset \mathrm{GL}(n, \mathbb{C}). \end{aligned} \tag{4}$$

Here $\mathrm{O}(n)$ and $\mathrm{U}(n)$ are the subgroups preserving the standard Euclidean and Hermitian inner products respectively. The 'special' subgroups consist of the elements of determinant one. All four subgroups act transitively on the corresponding sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ or $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$. This leads to short exact sequences

$$\begin{aligned} \mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n) \longrightarrow \mathbb{S}^{n-1} \\ \mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \longrightarrow \mathbb{S}^{2n-1} \end{aligned} \tag{5}$$

and hence corresponding long exact sequences in homotopy, for instance

$$\ldots \pi_{k+1}(\mathbb{S}^{n-1}) \longrightarrow \pi_k(\mathrm{SO}(n-1)) \longrightarrow \pi_k(\mathrm{SO}(n)) \longrightarrow \pi_k(\mathbb{S}^{n-1}) \ldots . \tag{6}$$

Since $\pi_k(\mathbb{S}^l) = \{0\}$ for $k < l$ the stability of the homotopy groups folows:

$$\pi_k(\mathrm{SO}(n)) \text{ is independent of } n \text{ for } n > k + 2. \tag{7}$$

We can check directly that $\mathrm{SO}(n) = \mathbb{RP}^3$ is the quotient of the 3-sphere by the antipodal map and from this we find

$$\begin{gathered} \pi_1(\mathrm{SO}(2)) = \mathbb{Z}, \ \pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2 \ n > 2, \\ \pi_2(\mathrm{SO}(n)) = \{0\} \ \forall \ n \\ \pi_3(\mathrm{SO}(3)) = \mathbb{Z}, \ \pi_3(\mathrm{SO}(4)) = \mathbb{Z}^2, \ \pi_3(\mathrm{SO}(n)) = \mathbb{Z}, \ n > 4. \end{gathered} \tag{8}$$

6. Principal bundles

7. Čech cohomology

See for example Bott and Tu, [1, §5, §10].

The spaces we consider, namely manifolds, path and loop spaces, all have bases of open sets forming a good open cover. In view of this, any open cover has a good open cover and to compute Čech theory with values in a presheaf it suffices to do so over any one good open cover.

8. Orientation

The first 'topological condition' on a manifold (always connected below) M is whether or not it is orientable. Perhaps the most useful characterization of an orientation is that it is equivalent to the existence of a (continuous or smooth) nowhere vanishing form of maximal degree – a volume form.

For us it is better stated in terms of the frame bundle, or the orthonormal frame bundle F_{O} (for a choice of Riemann metric) on M. This is a principal bundle

$$\begin{tikzcd} \mathrm{O}(n) \arrow[r, dash] & F_{\mathrm{O}} \arrow[d] \\ & M. \end{tikzcd} \tag{1}$$

As noted above, the orthogonal group has two components, labeled by the determinant $\det : \mathrm{O}(n) \longrightarrow \mathbb{Z}_2$. It follows that the fibres of (1) have two components and an orientation on M can be identified with a consistent choice of component – so constant in a local trivialization.

Somewhat pedantically, but we are going somewhere with this, an orientation can therefore be identified with a continuous map

$$o : F_{\mathrm{O}} \longrightarrow \mathbb{Z} \tag{2}$$

which takes both signs on each component. Of course continuous is equivalent to locally constant when the image space is discrete.

We see straight away that if o_1 and o_2 are two orientations then $o_1 o_2^{-1} = o_1 o_2 : F_{\mathrm{O}} M \longrightarrow \mathbb{Z}_2$ is constant on each fibre. It therefore descends to a continuous function $M \longrightarrow \mathbb{Z}_2$. Since we are assuming that M is connected, this reduces to an element of $\mathbb{Z}_2$; so if there is an orientation there are precisely two, o and o^{-1}.

The frame bundle is locally trivial (in fact it is trivial over any coordinate neighbourhood) so if we take a cover of M by small geodesic balls, O_i, then we can make a choice of component over each O_i, and hence a choice of $o_i : F_{\mathrm{O}}\big|_{O_i} \longrightarrow \mathbb{Z}_2$. There is no reason why these should be consistent, so on the overlap of any two balls we get a map

$$o_{ij} = o_i o_j^{-1} : O_i \cap O_j \longrightarrow \mathbb{Z}_2 \tag{3}$$

which takes the value 1 if the two choices are the same and -1 if they are different – it is constant on fibres of F_{O}. This is a *Čech* cocyle – directly from the definition the product over a triple intersection

$$o_{ij} o_{jk} o_{ki} = 1 \text{ on } O_i \cap O_j \cap O_k$$

which is one reason we need to think about Čech cohomology.

The class so defined is $w_1 \in \check{H}^1(M; \mathbb{Z}_2)$ called the first Stiefel-Whitney class. If we choose the open balls small enough that they give a good open cover, then to say this class vanishes is to say that there is a continuous function for each i, $e_i : O_i \longrightarrow \mathbb{Z}_2$ such that

$$o_{ij} = e_i e_j^{-1} \text{ here of course } e_i^{-1} = e_i.$$

Then we use these functions to change our initial choice of o_i to $o_i e_i^{-1}$ the new choice is globally well-defined and continuous, so gives us an orientation.

PROPOSITION 2. *A connected manifold has an orientation if and only if the first Stiefel-Whitney class vanishes and then there are two orientations.*

Here the class w_1 is taken as defined by the procedure!

9. Complex line bundles

We show next that complex line bundles, up to bundle isomorphism, are classified by $\check{H}^2(M;\mathbb{Z})$. The main step is to define the Chern class of such a line bundle, in Čech cohomology.

PROPOSITION 3. *For a complex line bundle, L, over a compact manifold, M, local trivializations over a good open cover $\mathcal{U}$ define a class* $\mathrm{Ch}(L) \in \check{H}^2(\mathcal{U};\mathbb{Z})$*; every such class arises this way and two line bundles have the same Chern class if and only if they are isomorphic.*

PROOF. If $\mathcal{U} = \{U_i\}$ is a good open cover then L is trivial over each U_j. Namely a non-vanishing section $u_j \in \mathcal{C}^\infty(U_j;L)$ can be constructed using parallel transport along the curves giving a contraction of U_j to a point. On the overlaps the two sections are related by a smooth map

$$Z_{ij} : U_{ij} = U_i \cap U_j \longrightarrow \mathbb{C}^*,\ u_i(x) = Z_{ij}(x)u_j(x).$$

The Z_* define a Čech class in $\mathcal{C}^1(\mathcal{U};\underline{\mathbb{C}^*})$, for the sheaf of smooth functions with values in $\mathbb{C}^*$. From the definition, this is a cocycle since

$$Z_{ij}Z_{jk}Z_{ki}u_i = u_i \Longrightarrow Z_{ij}Z_{jk}Z_{ki} = 1 \text{ on } U_{ijk} = U_i \cap U_j \cap U_k. \tag{1}$$

Since the non-empty sets U_{ij} are contractible there are smooth logarithms

$$\Theta_{ij} : U_{ij} \longrightarrow \mathbb{C},\ Z_{ij}(x) = \exp\left(2\pi i \Theta_{ij}(x)\right). \tag{2}$$

The choice of the logarithm is not unique, up to addition of an integer and it follows from (1) that the boundary of this as a chain with values in the sheaf of complex-valued functions

$$\kappa_{ijk} = (\delta\Theta_*)_{ijk} \in \mathbb{Z} \tag{3}$$

is a Čech chain in $\mathcal{C}^2(\mathcal{U};\mathbb{Z})$. In fact it is closed since since $\delta\kappa = \delta(\delta\Theta_*) = 0$ has the same meaning for functions and constants. This is the Chern class of L

$$\mathrm{Ch}(L) = [\kappa] \in \check{H}^2(\mathcal{U};\mathbb{Z}) = \check{H}^2(M;\mathbb{Z}) \tag{4}$$

where we use the fact that the Čech cohomology with respect to a good open cover is the full cohomology.

To see that this is independent of choices, observe first that changing the choice of logarithms Θ_* changes κ_* by a boundary in $\delta\mathcal{C}^1(\mathcal{U};\mathbb{Z})$. Similarly changing the initial choice of sections changes the Z_* by a boundary and choosing logarithms changes κ by the Čech differential of an integral class. Thus the class in (4) is determined by L and the cover.

If $\mathrm{Ch}(L) = 0$, i.e. $\kappa = \delta\lambda$ is exact with $\lambda \in \mathcal{C}^1(\mathcal{U};\mathbb{Z})$, then replacing Θ_* by $\Theta'_* = \Theta_* - \lambda_*$ gives a closed element of $\mathcal{C}(\mathcal{U};\underline{\mathbb{C}})$, for the sheaf of complex-valued functions on the U_{ij} such that

$$Z_{ij} = \exp(2\pi i \Theta'_{ij}). \tag{5}$$

Choose a partition of unity on M subordinate to the open cover $\mathcal{U}$. For each i set

$$\theta_i(x) = \sum_{j\neq i} \rho_j(x)\Theta'_{ij} \in \mathcal{C}^\infty(U_i) \tag{6}$$

where the sum makes sense since the support of ρ_j restricted to U_i lies within $U_i \cap u_j$ where Θ'_{ij} is defined. Thus $\theta_* \in \mathcal{C}^0(\mathcal{U};\underline{\mathbb{C}}$ is a Čech class with coefficients in the complex-valued $\mathcal{C}^\infty$ functions.

Computing the Čech differential

$$\begin{aligned}(\delta\theta_*)_{kl} = \theta_k - \theta_l = \sum_{p\neq k} \rho_p \Theta'_{pk} - \sum_{p\neq l} \rho_p\Theta'_{pl} \\ = \rho_l\Theta'_{lk} - \rho_k\Theta'_{kl} + \sum_{p\neq k,l} \rho_p(\Theta'_{pk} - \Theta'_{pl}) = \Theta'_{kl}\end{aligned} \tag{7}$$

where we use the fact that Θ'_* is closed and antisymmetry.

Thus

$$Z_* = \delta\exp(2\pi i\theta_*) \tag{8}$$

where we are working in the sheaf of functions with values in $\mathbb{S} = \mathbb{C}^*$ so the operations are written multiplicatively. Now, replacing the original sections of the bundle by the new ones over each U_i

$$u'_i = \exp(-2\pi i\theta_i)u_i \tag{9}$$

gives a global non-vanishing sections. Thus if $\mathrm{Ch}(L) = 0$ then L is trivial.

The fact that every class $[\kappa] \in \check{H}^2(\mathcal{U};\mathbb{Z})$ arises this way is similar. As in (6) we use a partition of unity to 'lift' the cocycle κ to a class in the sheaf of complex-valued functions

$$\tau_{ij} = \sum_{k\neq i,j} \rho_k\kappa_{kij} \text{ on } U_{ij} \tag{10}$$

which makes sense as before since ρ_k has support in U_k. Then

$$\delta\tau = \kappa \tag{11}$$

is constant with integral values. It follows that

$$\zeta_{ij} = \exp(2\pi i\tau_{ij}) \in \mathcal{C}^1(\mathcal{U};\underline{\mathbb{S}}) \tag{12}$$

is a cocycle, $\delta\zeta_* = 0$.

Then we construct a line bundle from κ by taking the product bundle $U_j \times \mathbb{S}$ over each of the open sets and using the ζ_{ij} as transition maps

$$L = \bigsqcup_j (U_j \times \mathbb{S})/\simeq,\ (x,z) \simeq (x,z') \text{ iff } x \in U_i \cap U_j \text{ and } z' = \zeta_{ij}z \text{ for some } i,j. \tag{13}$$

Proceeding through the construction above recovers $[\kappa]$ as the Chern class of this bundle. □

You may amuse yourself checking the functorial properties of the Chern class – that under smooth maps

$$f : N \longrightarrow M \Longrightarrow \mathrm{Ch}(f^*L) = f^*\,\mathrm{Ch}(L),\ \mathrm{Ch}(L_1 \otimes L_2) - \mathrm{Ch}(L_1) + \mathrm{Ch}(L_2) \text{ etc.} \tag{14}$$

10. Čech and Bockstein

It is not highly relevant for us, but the argument in the preceding section can be extended to identify Čech cohomology with coefficients in $\mathbb{Z}$ with the cohomology for the sheaf of continous, or smooth, maps into the circle.

Start by noting the idea of Bockstein[1]. If U is a contractible open set in a manifold then there is a short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}^\infty(U;\mathbb{R}) \xrightarrow{\exp(2\pi i\cdot)} \mathcal{C}^\infty(U;\mathbb{S}) \longrightarrow 1. \tag{1}$$

From this we get a correspond short exact sequence of the Čech classes for a fixed open cover $\mathcal{U}$ of a manifold

$$0 \longrightarrow \mathcal{C}^k(\mathcal{U},\mathbb{Z}) \longrightarrow \mathcal{C}^k(\mathcal{U},\underline{\mathbb{R}}) \xrightarrow{\exp(2\pi i\cdot)} \mathcal{C}^k(\mathcal{U},\underline{\mathbb{S}}) \longrightarrow 1. \tag{2}$$

For such an exact sequence of chain spaces for a cohomology theory there is a corresponding long exact sequence

$$\begin{tikzcd}
\check{H}^0(\mathcal{U},\mathbb{Z}) \arrow[r] & \check{H}^0(\mathcal{U},\underline{\mathbb{R}}) \arrow[r, "\exp(2\pi i\cdot)"] & \check{H}^0(\mathcal{U},\underline{\mathbb{S}}) \arrow[dll, rounded corners] \\
\cdots \arrow[drr, rounded corners] & & \\
\check{H}^k(\mathcal{U},\mathbb{Z}) \arrow[r] & \check{H}^k(\mathcal{U},\underline{\mathbb{R}}) \arrow[r, "\exp(2\pi i\cdot)"] & \check{H}^k(\mathcal{U},\underline{\mathbb{S}}) \arrow[dll, rounded corners] \\
\check{H}^{k+1}(\mathcal{U},\mathbb{Z}) \arrow[r] & \check{H}^{k+1}(\mathcal{U},\underline{\mathbb{R}}) \arrow[r, "\exp(2\pi i\cdot)"] & \check{H}^{k+1}(\mathcal{U},\underline{\mathbb{S}}) \ldots
\end{tikzcd} \tag{3}$$

The chain maps in (2) commute with the Čech differential so map closed to closed and exact to exact classes giving the maps on the lines of (3). The connecting homomorphism is constructed by starting from a cocycle $\kappa \in \mathcal{C}^k(\mathcal{U},\underline{\mathbb{S}})$. This must have a preimage $\alpha \in \mathcal{C}^k(\mathcal{U};\underline{\mathbb{R}})$ and then $\delta\alpha \in \mathcal{K}^{k+1}(\mathcal{U};\underline{\mathbb{R}})$ has trivial image in $\mathcal{C}^{k+1}(\mathcal{U};\underline{\mathbb{S}})$ and so must have a preimage $\beta \in \mathcal{C}^{k+1}(\mathcal{U};\mathbb{Z})$. Again by commutativity this must be Čech closed. Following the ambiguity in this construction the class of β only depends on the class of κ giving the connecting homorphism. Of course one should go through the standard argument that (3) is an exact sequence.

The arguments using partitions of unity above generalizes to show

PROPOSITION 4. *The Čech cohomology with coefficients in sections of a vector bundle W over M (connected) vanishes except in dimension* 0

$$\check{H}^0(M;W) = \mathcal{C}^\infty(M;W),\ \check{H}^k(M;W) = \{0\},\ k \geq 1. \tag{4}$$

In particular $\check{H}^k(M;\underline{\mathbb{R}})$ vanishes in positive dimension. In fact

$$\check{H}^{k+1}(M;\mathbb{Z}) = \check{H}^k(M;\underline{\mathbb{S}},\ k \geq 0. \tag{5}$$

For $k = 0$ this is follows directly starting from the sequence (2).

[1]Meyer Bockstein, 1913-1990

Corollary 1. *The 1-dimensional integral cohomology of a compact manifold can be realized as the collection of homotopy classes of smooth (or continous) maps to* $\mathbb{S}$.

Remark 1. This translates to the statement that $\mathbb{S} = K(1, \mathbb{Z})$ is an Eilenberg[2]-MacLane[3] space for 1-dimensional integral cohomology. Also, it is a classifying space for the goup $\mathbb{Z}$.

11. Čech and deRham

Here we briefly describe the isomorphism between deRham cohomology and Čech cohomology with real coefficients. More detail can be found in the book of Bott and Tu [1].

We know that for a contractible open set U in a manifold of dimension n the deRham complex

$$\mathcal{C}^\infty(U) \xrightarrow{d} \mathcal{C}^\infty(U, \Lambda^1) \xrightarrow{d} \dots \qquad \mathcal{C}^\infty(U, \Lambda^n) \xrightarrow{d} 0 \tag{1}$$

is exact.

Starting with a closed k-form and a good cover $\mathcal{U}$ we can follow a zig-zag across the double complex formed by the Čech spaces for the Λ^* :

(2)

$$\begin{tikzcd}
 & & v_*^2 \in \check{\mathcal{C}}^0(\Lambda^{k-2}) \arrow[r, "\delta"] \arrow[d, "d"] & u_*^2 \in \check{\mathcal{C}}^1(\Lambda^{k-2}) \arrow[r, "\delta"] \arrow[d, "d"] & 0 \\
 & v_*^1 \in \check{\mathcal{C}}^0(\Lambda^{k-1}) \arrow[r, "\delta"] \arrow[d, "d"] & u_*^1 \in \check{\mathcal{C}}^1(\Lambda^{k-1}) n\delta \arrow[r, "b"] \arrow[d, "d"] & 0 & \\
u \in \mathcal{C}^\infty(M, \Lambda^k) \arrow[r, "\delta"] \arrow[d, "d"] & u_* \in \check{\mathcal{C}}^0(\Lambda^k) \arrow[r, "\delta"] \arrow[d, "d"] & 0 & & \\
0 & 0. & & &
\end{tikzcd}$$

Here the vertical sequences are deRham and the horizontal are Čech. We can keep on 'moving' to the right and up, using the exactness of deRham on contractible sets and applying the Čech differential to the primitives. When we reach the 'top'

[2]Samuel Eilenberg, 1913-1998

[3]Saunders MacLane (or Mac Lane), 1909-2005

where the form degree is 0 it looks like this

$$
\begin{tikzcd}
 & & u^k_* \in \check{\mathcal{C}}^k(\mathbb{R}) \arrow[r, "\delta"] \arrow[d, hook, "d"] & 0 \\
 & v^k_* \in \check{\mathcal{C}}^{k-1}(\Lambda^0) \arrow[r, "\delta"] \arrow[d, "d"] & u^k_* \in \check{\mathcal{C}}^k(\Lambda^0) \arrow[r, "\delta"] \arrow[d, "d"] & 0 \\
v^{k-1}_* \in \check{\mathcal{C}}^{k-2}(\Lambda^1) \arrow[r, "\delta"] \arrow[d, "d"] & u^{k-1}_* \in \check{\mathcal{C}}^{k-1}(\Lambda^1) \arrow[r, "\delta"] \arrow[d, "d"] & 0 & \\
u^{k-2}_* \in \check{\mathcal{C}}^{k-2}(\Lambda^2) \arrow[r, "\delta"] & 0. & &
\end{tikzcd}
\tag{3}
$$

The zig-zag construction gets us to the second row from the top with a Čech exact collection of 0-forms on the $(k+1)$-fold intersections of our open cover. These are individually closed, so each is constant and so this class is in $\check{\mathcal{C}}^k(\mathcal{U},\mathbb{R})$. As such it is still Čech closed but need not be the Čech image of an element of $\check{\mathcal{C}}^{k-1}(\mathcal{U},\mathbb{R})$. It is straightforward to see that different choices along the zig-zag change the output u^k by an exact class and hence we do have a linear map

$$H^k_{\mathrm{dR}}(M) \longrightarrow \check{H}^k(\mathcal{U},\mathbb{R}) \text{ for each } k. \tag{4}$$

THEOREM 4. *For a good open cover of a manifold M there is a natural isomorphism* (4)*.*

PROOF. The construction can be reversed using the exactness of the rows. Lots to check! Maybe it would be better to treat this as a spectral sequence argument and apply some heavier machinery! □

12. Simplicial spaces

Fibre products are examples of simplicial spaces, so let me discuss this at least briefly here. From a primitive point of view a simplicial space is a sequnce of topological spaces – manifolds for us – which we can denote $\mathcal{Q}_k$ where $k \in \mathbb{N}$. [I should warn you that my numbering convention is a little non-standard.] Then for each $k \in \mathbb{N}$ and each $1 \le j \le k$ there are 'forgetful' maps – usually called face maps –

$$\sigma^{[k]}_j : \mathcal{Q}_k \longrightarrow \mathcal{Q}_{k-1}. \tag{1}$$

There are also 'degeneracy maps' going the other way. The maps in (1) are required to satisfy the identities

$$\sigma^{[k-1]}_q \circ \sigma^{[k]}_p = \sigma^{[k-1]}_{p-1} \circ \sigma^{[k]}_q \ \forall\ q < p. \tag{2}$$

For fibre products you can check this easily. The maps are actually the restrictions of the projections $X^k \longrightarrow X^{k-1}$. Then (2) just comes from renumbering, on the right if you drop the qth factor first then the pth factor slips to position $p-1$.

Now, suppose that A is an abelian topological group, so we can consider continuous maps $\mathcal{Q}_k \longrightarrow A$ forming $\mathcal{C}(\mathcal{Q}_k; A)$. Writing the group composition additively, we can define the 'simplicial differential'

$$\delta u = \sum_{j=1}^{k} (-1)^j \sigma_j^* u,\ \ \delta : \mathcal{C}(\mathcal{Q}_k; A) \longrightarrow \mathcal{C}(\mathcal{Q}_{k+1}; A). \tag{3}$$

LEMMA 1. *For any simplicial space*

$$\delta^2 = 0. \tag{4}$$

PROOF. The definition gives for $u \in \mathcal{C}(\mathcal{Q}_k; A)$ gives

$$\delta^2 = \sum_{l=1}^{k+1} (-1)^l \sum_{j=1}^{k} \sigma_l^* \sigma_j^* u = \sum_{j<l} (-1)^{l+k} \sigma_l^* \sigma_j^* u + \sum_{j \geq l} (-1)^{l+k} \sigma_j^* \sigma_{l-1}^* u = 0 \tag{5}$$

using (2). □

Note that a simplicial space is better formalized as a functor. Consider the small (in the technical, but also the casual sense) with objects the integers, realized as the sets $J(n) = \{1, \dots, n\}$ and morphisms the increasing (i.e. non-decreasing) maps $J(m) \longrightarrow J(n)$. Then the simplical space is a *contravariant* functor into the category of topological spaces and continuous maps.

The face maps $\sigma_j : \mathcal{Q}_k \longrightarrow \mathcal{Q}_{k-1}$ are the images of the maps $J(k-1) \longrightarrow J(k)$ which are stricly increasing and 'miss' j. The degeracy maps are the images of the maps $J(k) \longrightarrow J(k)$ which are stricly decreasing except for one pair j and $j+1$ which are mapped to $j \in J(k-1)$.

EXERCISE 1. Show that the fibre products form a simplicial space with the face maps discussed above and degeneracy maps $X^{[k-1]} \longrightarrow X^{[k]}$ are the restrictions of the diagonal maps $X^{k-1} \ni (x_1, \dots, x_j, \dots) \longrightarrow (x_1, \dots, x_j, x_j, \dots)$.

13. Spin structures

For an oriented Riemann manifold, M, of dimension $n \geq 3$, the bundle of oriented orthonormal frames, F_{SO}, is a princpal $\mathrm{SO}(n)$ bundle and a spin structure is a double cover which is a $\mathrm{Spin}(n)$ principal bundle with compatible actions

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, dash] \arrow[d, "\pi"'] & F_{\mathrm{Spin}} \arrow[d, "\pi"] \\ \mathrm{SO}(n) \arrow[r, dash] & F_{\mathrm{SO}} \arrow[d] \\ & M. \end{tikzcd} \tag{1}$$

Thus, for $s \in \mathrm{Spin}(n)$ and $f \in F_{\mathrm{Spin}}$ we require

$$\pi(sf) = \pi(s)\pi(f). \tag{2}$$

THEOREM 5. *A spin structure exists if and only if the second Stiefel-Whitney class $w_2 \in H^2(M; \mathbb{Z}_2)$ vanishes and then the structures up to isomorphism are identified with $H^1(M; \mathbb{Z}_2)$.*

Let me go quickly through the proof, as much as a guide to things to come. Of course if you don't know what w_2 is this just says there is such a class. I am thinking of this result as a precursor of the 'lifting bundle gerbe' which I want to introduce soon.

Take a good open cover of M – so this can be by small geodesic balls; we can take the cover to be finite if M is compact, or in general countable but it really does not matter in the argument. Over each of these sets, B_i, the frame bundle is trivial, which is to say it has a section given by parallel transport along radial geodesics from the central point

$$\phi_i : B_i \longrightarrow F_{\mathrm{SO}},\ \pi\phi_i = \mathrm{Id}\,. \tag{3}$$

On the intersection of pairs of sets there is a transition map between the sections

$$\theta_{ji} : B_i \cap B_j \longrightarrow \mathrm{SO}(n),\ \phi_j(x) = \theta_{ji}\phi_i(x)\ \forall\ x \in B_i \cap B_j. \tag{4}$$

Everything here is continuous, but can be taken to be smooth. Again, since $B_i \cap B_j$ is contractible we can lift these maps

$$\begin{tikzcd} B_i \cap B_j \arrow[r, "\tilde\theta_{ji}"] \arrow[dr, "\theta_{ji}"'] & \mathrm{Spin}(n) \arrow[d] \\ & \mathrm{SO}(n). \end{tikzcd} \tag{5}$$

Over triple intersections we know, from the definition, that

$$\theta_{ik}\theta_{kj}\theta_{ji} = \mathrm{Id} \tag{6}$$

so for the lifts we find

$$\mu_{kji} = \tilde\theta_{ik}\tilde\theta_{kj}\tilde\theta_{ji} : B_i \cap B_j \cap B_k \longrightarrow \mathbb{Z}_2. \tag{7}$$

This is a Čech cochain with values in $\mathbb{Z}_2$. In fact it is a cocycle, the Čech differential vanishes. The Čech differential is

$$(\partial\mu)_{lkji} = \mu_{lkj}\mu_{lki}^{-1}\mu_{lji}\mu_{kji}^{-1} = 1. \tag{8}$$

This follows by inserting (7) and noting that $\mathbb{Z}_2$ lies in the centre of $\mathrm{Spin}(n)$.

At some point I will get around to writing out a brief treatment of Čech theory.

Exercise 2. Write out the proof.

In fact since we are working with a good cover the class $w_2 = [\mu] \in H^2(M;\mathbb{Z}_2)$ vanishes if and only if μ is exact. A primitive of μ can be used to modify the $\tilde\theta_{ji}$ so that the new μ in (7) vanishes. This means that these $\tilde\theta_{ji}$ form a cocycle and hence we can define a principal bundle by taking these as the transition map and

$$F_{\mathrm{Spin}} = \bigsqcup_i (\mathrm{Spin}(n) \times B_i)/\simeq \tag{9}$$

is a double cover of F_{SO} and (1) follows.

You should also check that that two different choices of primitive for μ lead to a class in $H^1(M;\mathbb{Z}_2)$ which vanishes if and only if the resulting bundles are isomorphic as principal bundles.

This completes the cursory discussion of spin structures provided we identify w_2 as the second Stiefel-Whitney class.

14. Clifford algebra and Spin group

We are interested in Clifford algebras for several reasons, but the primary one is the role they play in the definition of Dirac operators. In two dimension we can consider the operator, acting on say $\mathcal{C}^\infty(\mathbb{R}^2;\mathbb{C}^2)$ given in terms of the Cauchy-Riemann operator by

$$\eth = \begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix}. \tag{1}$$

Its square is $\mathrm{Id}(\partial_x^2+\partial_y^2)$ so it is a square-root of the Laplacian. Dirac showed the same was possible in dimension 4 (well, for the wave operator). In fact there is such an operator in any dimension thanks to the properties of the Clifford algebra. To see where this might come from we can look for a linear differential operator acting on functions with values in some vector space

$$\eth = \sum_{i=1}^{n} \gamma^i \partial_i \Longrightarrow \eth^2 = \sum_i \gamma_i^2 \partial_i^2 + \sum_{i<j} (\gamma_i\gamma_j + \gamma_j\gamma_i)\partial_i\partial_j. \tag{2}$$

This will look like the identity matrix times the Laplacian if we can arrange that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\,\mathrm{Id},\ \forall\ i,j. \tag{3}$$

In the 2×2 case such γ_i are called Pauli matrices, but their existence is much older and comes from the ideas of Clifford[4].

Since it is the case of primary interest here we will consider a real vector space V with a positive-definite quadratic form q, i.e. a Euclidean vector space. The full tensor algebra of V is the sum

$$\mathcal{T}(V) = \bigoplus_k V^{\otimes k} \tag{4}$$

So an element of $\mathcal{T}(V)$ is a finite sum of elements of the $V^{\otimes k}$ and the product is the tensor product coming from

$$V^{\otimes j} \otimes V^{\otimes k} = V^{\otimes(j+k)}.$$

This is a graded algebra, infinite-dimensional of course, and $\mathrm{GL}(V)$, the group of invertible linear transformations orthogonal acts on $\mathcal{T}(V)$ by acting on each factor.

Then the more invariant form of (3) is obtained by looking at the ideal generated by these conditions, namely

$$\mathcal{I}(q) = \mathrm{sp}\{\alpha\,(v\otimes w + w\otimes v + 2q(v,w)\,\mathrm{Id})\otimes\beta,\ \alpha,\ \beta\in\mathcal{T}(V)\}. \tag{5}$$

This is invariant under the action of $\mathrm{O}(V)$, the orthogonal group preserving q. The quotient is the Clifford algebra

$$\mathrm{Cl}(V) = \mathcal{T}(V)/\mathcal{I}(V). \tag{6}$$

Lemma 2. *There is a natural linear (not multiplicative) bijection*

$$\Lambda^* V \longrightarrow \mathrm{Cl}(V) \tag{7}$$

and as an algebra $\mathrm{Cl}(V)$ *inherits the odd/even grading and the action of* $\mathrm{O}(V)$ *from* $\mathcal{T}(V)$ *and is generated by the subspace* V.

[4]William Kingdon Clifford (1845–1879)

PROOF. The elements of V, together with 1 generate $\mathcal{T}(V)$ and from (5) $v\otimes v = -1$ for any vector in V of unit length, so V alone generates $\mathrm{Cl}(V)$.

The subspace $\Lambda^* V$ consists of the finite sums of totally antisymmetric elements in each $V^{\otimes k}$, in particular it is trivial in degrees greater than $\dim V$. An element of $V^{\otimes k}$ is a sum of products $v_1\otimes v_2\cdots\otimes v_k$ for elements $v_i \in V$. Modulo terms in degrees at most $k-2$ this can be replaced, up to terms in $I(q)$, by $cv_1 \wedge v_2 \cdots \wedge v_k$, which vanishes if $k > \dim V$. Thus, proceeding inductively from the top degree down, any element of $\mathcal{T}(V)$ is equal, up to a term in $\mathcal{I}(q)$, to an element of $\Lambda^* V$. This gives a two-sided inverse to the map (7) defined by inclusion of $\Lambda^* V \subset \mathcal{T}(V)$. The elements with only terms of odd or even degree are preserved by these operations and the product in $\mathcal{T}(V)$, so this $\mathbb{Z}_2$ grading descends to $\mathrm{Cl}(V)$. Since the action of $\mathrm{O}(V)$ fixes the ideal $I(q)$ it descends to $\mathrm{Cl}(V)$. $\square$

The action of $\mathrm{O}(V)$ on the Clifford algebra allows us to reduce to the model case that $V = \mathbb{R}^n$ with the Euclidean metric which we will denote simple $\mathrm{Cl}(n)$. Even here the structure of these algebras is somewhat complicated. However we are really interested in the complexified Clifford algebras

$$\mathbb{C}\mathrm{l}(n) = \mathrm{Cl}(n) \otimes \mathbb{C}. \tag{8}$$

Note that this is the same as the Clifford algebra for $\mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}$ but with the complexified inner product, complex linear in both variables, not the Hermitian inner product.

We can check the low-dimensional cases. Thus $\mathrm{Cl}(1)$ is spanned, as a vector space, by 1 and a unit vector $e \in V = \mathbb{R}$. In the complexified algebra the two elements

$$E_\pm = \frac{1}{2}(1 \pm ie) \text{ satisfy } E_\pm^2 = E_\pm,\ E_+E_- = E_-E_+ = 0 \tag{9}$$

as follows by easy computation. These span $\mathbb{C}\mathrm{l}(1)$ as a vector space and the map

$$\mathbb{C} \oplus \mathbb{C} \ni (z_+, z_-) \longrightarrow z_+E_+ + z_-E_- \in \mathbb{C}\mathrm{l}(1) \tag{10}$$

is mutliplicative and identifies $\mathbb{C}\mathrm{l}(1)$ with the direct sum of two copies of $\mathbb{C}$.

There is a similar decomposition for $n = 2$. The four elements given by the standard basis of $\mathbb{R}^2$,

$$1, e_1, e_2, e_1e_2 \text{ span } \mathrm{Cl}(2). \tag{11}$$

The four elements

$$F_{11} = \frac{1}{2}(1+ie_1e_2),\ F_{22} = \frac{1}{2}(1-ie_1e_2),\ F_{12} = \frac{1}{2}(e_1+ie_2),\ F_{21} = \frac{1}{2}(e_1-ie_2) \tag{12}$$

also form a basis and satisfy

$$F_{ij}F_{kl} = \delta_{jk}F_{il}. \tag{13}$$

It follows that

$$\sum_{i,j} a_{ij}F_{ij} \cdot \sum_{p,q} a_{pq}F_{pq} = \sum_{i,j}(\sum_{l} a_{il}b_{lj})F_{ij} \tag{14}$$

for any complex constants a_{ij} and b_{ij}. This map identifies $\mathbb{C}\mathrm{l}(2)$ with $M(2;\mathbb{C})$, the algebra of 2×2 complex matrices.

As for the exterior algebra, there are important 'volume elements' in the Clifford algebras. Namely the products

$$Z_n = i^{n(n-1)/2}e_1e_2\dots e_n \in \mathbb{C}\mathrm{l}(n). \tag{15}$$

The normalization here is to ensure that

$$Z_n^2 = \mathrm{Id}\,. \tag{16}$$

LEMMA 3. *The volume element in* (15) *is invariant under the action of* $\mathrm{SO}(n)$.

So these are well-defined if V is oriented.

PROOF. Since $\mathrm{SO}(n)$ is connected it suffices to show invariance under the infinitesmal action of the Lie algebra. This is spanned by the elementary antisymmetric matrices $T_{ij}e_k = 0$, $k \neq i$ or j, $T_{ij}e_i = e_j$, $T_{ij}e_j = -e_i$. Taking $i < j$, the infinitesmal action

$$\frac{d}{dt}\Big|_{t=0} \exp(tT)Z_n = \\ i^{n(n-1)/2}e_1e_2\dots e_{i-1}e_je_{i+1}\dots e_j\dots e_n - i^{n(n-1)/2}e_1e_2\dots e_{j-1}e_ie_{j+1}\dots e_n = 0. \tag{17}$$

□

These volume elements behave somewhat differently in the odd and even cases. Namely

$$\begin{gathered} Z_{2n+1} \text{ is in the centre of } \mathbb{C}\mathrm{l}(2n+1) \\ Z_{2n}(\alpha^+ + \alpha^-) = (\alpha^+ - \alpha^-)Z_{2n} \end{gathered} \tag{18}$$

in terms of the odd/even decomposition.

PROPOSITION 5. *The complexified Clifford algebra exhibits 'Bott periodicity'*

$$\mathbb{C}l(2n+2) = \mathbb{C}l(2n) \otimes \mathbb{C}l(2),\ \mathbb{C}l(2n+1) = \mathbb{C}l(2n) \oplus \mathbb{C}l(2n). \tag{19}$$

As a direct corollary

$$\mathbb{C}\mathrm{l}(2n) = M(2^n, \mathbb{C}),\ \mathbb{C}\mathrm{l}(2n+1) = M(2^n, \mathbb{C}) \oplus M(2^n, \mathbb{C}). \tag{20}$$

PROOF. First consider $\mathbb{C}\mathrm{l}(2n+1)$ and the subalgebras $\mathbb{C}\mathrm{l}(1)$ generated by e_1 and $\mathbb{C}\mathrm{l}(2n)$ generated by the e_i with $i > 1$. Then we generalize the elements in (9) by inserting the volume element for $\mathbb{C}\mathrm{l}(2n)$

$$E_\pm = \frac{1}{2}(1 \pm iZ_{2n}e) \Longrightarrow E_\pm^2 = E_\pm,\ E_+E_- = E_-E_+ = 0. \tag{21}$$

The identities follow from the fact that e_1 and Z_{2n} commute (since the $2n$ factors in Z_n are orthogonal to e_1). In place of (10) consider

$$\mathbb{C}\mathrm{l}(2n) \oplus \mathbb{C}\mathrm{l}(2n) \ni (\alpha, \beta) \longmapsto \alpha E_+ + \beta E_-. \\ = \frac{1}{2}(\alpha + \beta) + \frac{i}{2}(\alpha - \beta)Z_{2n}e_1 \in \mathbb{C}\mathrm{l}(2n+1). \tag{22}$$

Since $\mathbb{C}\mathrm{l}(2n+1) = \mathbb{C}\mathrm{l}(2n) + \mathbb{C}\mathrm{l}(2n)e_1$ as a vector space, it is clear that (22) is surjective. It is also multiplicative since the combination $Z_{2n}e_1$ commutes with all elements of $\mathbb{C}\mathrm{l}(2n)$.

Similarly, for $\mathbb{C}\mathrm{l}(2n+2)$ for $n \geq 2$ consider the subalgebras $\mathbb{C}\mathrm{l}(2)$ generated by e_1 and e_2 and the subalgebra $\mathbb{C}\mathrm{l}(2n)$ generated by the e_i for $i > 2$. These have volume elements

$$Z_{12} = ie_1e_2 \text{ and } Z_{2n} \in \mathbb{C}\mathrm{l}(2n). \tag{23}$$

Again we generalize (12) to

$$F_{11} = \frac{1}{2}(1 + ie_1e_2),\ F_{22} = \frac{1}{2}(1 - ie_1e_2),$$
$$F_{12} = \frac{1}{2}Z_{2n}(e_1 + ie_2),\ F_{21} = \frac{1}{2}Z_{2n}(e_1 - ie_2) \tag{24}$$

and the identities (13) again hold. Each F_{ij} commutes with all elements of $\mathbb{C}\mathrm{l}(2n)$ so (14) now holds for elements a_{ij}, b_{ij} in $\mathbb{C}\mathrm{l}(2n)$. □

In fact it is helpful to have a more direct proof of particularly the even case of (20). We can identify $\mathbb{R}^{2n}$ as the real vector space underlying $\mathbb{C}^n$. Since we will ultimately want to take the Clifford algebra on the cotangent fibres of a Riemann manifold we will denote the basis of $\mathbb{C}^n$ as differentials,

$$dz_j = dx_j + idy_j,\ j = 1, \dots, n \tag{25}$$

where dx_j, dy_j give the corresponding basis of (the dual of) $\mathbb{R}^{2n}$. Then consider the complete complex exterior algebra

$$\Lambda^* = \bigoplus_{k=0}^{n} \Lambda^k,\ \Lambda^k \text{ spanned by } dz_{i_1} \wedge \dots \wedge dz_{i_k},\ 1 \le i_1 < i_2 < \dots < i_k \le n. \tag{26}$$

There are standard operations

$$\omega_j : \Lambda^k \ni \gamma \longmapsto dz_j \wedge \alpha \in \Lambda^{k+1},\ \iota_j : \Lambda^k \ni \alpha \longmapsto \alpha(\partial_j, \dots) \in \Lambda^{k-1}. \tag{27}$$

Here $\partial_j = \partial_{z_j}$ is the dual basis to dz_j corresponding to the choice of Euclidean inner product. These satisfy the identities

$$\omega_j^2 = \iota_j^2 = 0,\ \omega_j\iota_k + \iota_k\omega_j = \delta_{jk}. \tag{28}$$

Now we consider the complexified Clifford algebra on the dual of $\mathbb{R}^{2n}$ which is generated by the dx_i and dy_l. We map these to operators on Λ^* by

$$\mathrm{cl}(dx_j) = \omega_i + \iota_j,\ \mathrm{cl}(dy_j) = i(\omega_j - \iota_j). \tag{29}$$

Then all the Clifford identities holds as maps on Λ^* :

$$\begin{gathered}\mathrm{cl}(dx_j)\,\mathrm{cl}(dx_k) + \mathrm{cl}(dx_k)\,\mathrm{cl}(dx_j) = \mathrm{cl}(dy_j)\,\mathrm{cl}(dy_k) + \mathrm{cl}(dy_k)\,\mathrm{cl}(dy_j) = \delta_{jk}\,\mathrm{Id},\\ \mathrm{cl}(dx_j)\,\mathrm{cl}(dy_k) + \mathrm{cl}(dy_k)\,\mathrm{cl}(dx_j) = 0.\end{gathered} \tag{30}$$

Any assignment of linear maps on a vector space to the elements of $\mathbb{R}^{2n}$, with the identity assigned to 1, extends uniquely to a multiplicative map for the tensor algebra to the homomorphism algebra of the vector space. The identities (30) ensure that this descends to a multiplicative map

$$\mathbb{C}\mathrm{l}(2n) \longrightarrow \hom(\Lambda^*). \tag{31}$$

Either from (20), or directly, it follows that this must be an isomorphism of algebras. Conventionally this is thought of as the spin representation of the Clifford algebra.

Within the real Clifford algebra $\mathrm{Cl}(n)$ we can consider two groups

$$\mathrm{Spin}(n) = \{\prod_i v_i \dots v_{2k},\ |v_i| = 1\} \subset \mathrm{Pin}(n) = \{\prod_i v_i \dots v_k,\ |v_i| = 1\}. \tag{32}$$

The inverse of $v_1 \dots v_k$ is $((-1)^k)v_k) \dots v_1$ so these are indeed groups. Their smoothness is less obvious from this definition since the representation of an element is by no means unique. To examine this consider the map

$$\mathrm{Pin}(n) \longrightarrow \hom(\mathbb{R}^n),\ v_1 \dots v_k(w) = v_1 \dots v_k w v_k \dots v_1. \tag{33}$$

The product is in Cl($2n$) but the result is in the underlying vector space as follows from the fact that decomposing $w = av + w'$, $w' \perp v$

$$vwv = avvv + vw'v = av - w'. \tag{34}$$

Thus the action of a single vector v gives the linear map on $\mathbb{R}^{2n}$ which is reflection in the plan perpendicular to v. This is an orthogonal transformation so (33) gives a group homomorphism

$$\mathrm{Pin}(n) \longrightarrow \mathrm{O}(n). \tag{35}$$

Now a standard fact about orthogonal transformations is that they can always be written as a product of such reflections so (35) is surjective. Similarly the elements of $\mathrm{SO}(n) \subset \mathrm{O}(n)$ are those which can be written as a product of an even number of reflections (since each reflection reverses the orientation). Thus

$$\mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \text{ is surjective.} \tag{36}$$

This map is the non-trivial double cover, justifying the identification with the spin group.

Thus we have defined the spin representation

$$\mathrm{Spin}(2n) \longrightarrow \mathrm{GL}(\Lambda^*). \tag{37}$$

LEMMA 4. *The spin representation* (37) *is the direct sum of two irreducible representations.*

PROOF. □

We can approach the question of whether or not a given oriented manifold has a spin structure in much the same way as an orientation.

The bundle or oriented orthonormal frames, F_{SO}, corresponding to a choice of Riemann structure is trivial over small Euclidean balls, i.e. has a smooth section. This can be constructed using the exponential map.

Let B_i be a good covering of the manifold, X, by such balls and let $e^i : B_i \longrightarrow F_{\mathrm{SO}}B_i$ be the chosen section so an orthonormal basis e^i_k, $k = 1, \ldots, k$ of TB_i. Thus any other (smooth always) section is of the form $(A^i(x)e^i(x))_l = \sum\limits_l (A^i)_{lk} E^i_k$ for a smooth map $A^i : B_i \longrightarrow \mathrm{SO}(n)$. So on overlaps the transition map

$$T^{ij} : B_{ij} = B_i \cap B_j \longrightarrow \mathrm{SO}(n),\ e^i = T^{ij} e^j \tag{38}$$

Here the indices of the bases and in $\mathrm{SO}(n)$ have been suppressed.

The non-empty B_{ij} are contractible under the exponential map based at any point. This allows the transition maps to be lifted to smooth maps

$$\hat{T}^{ij} : B_{ij} \longrightarrow \mathrm{Spin}(n),\ \pi\hat{T}^{ij} = T^{ij}. \tag{39}$$

The lift is unique up to the choice of preimage in $\mathrm{Spin}(n)$ of the value of T^{ij} at the chosen base point, so the lift is uniquen up to a map into the centre, $\mathbb{Z}_2$, of $\mathrm{Spin}(n)$ and hence by discreteness to a constant element of $\mathbb{Z}_2 \subset \mathrm{Spin}(n)$.

Over each B_i we can find a 'local' spin structure. Using the section of $F_{\mathrm{SO}}B_i$ this can be defined explictly as

$$F^i_{\mathrm{Spin}} = (F_{\mathrm{SO}}B_i \times \mathrm{Spin}(n))/ \simeq \tag{40}$$

where the equivalence relation is the identification

$$(e^i, B) \equiv (\pi(B')e^i, B'B)\ \forall\ B' \in \mathrm{Spin}(n). \tag{41}$$

Check that this does indeed carry a consistent action of $\mathrm{Spin}(n)$ giving a spin structure.

We can try to patch these local spin structures together using the $\hat{T}^{ij}$ identifying

$$(e^i, B) = (T^{ij} e_j, \hat{T}^{ij} B) \text{ over } B_{ij}$$

however this will not be consistent, i.e. 'equality' here will not be an equivalence relation, unless

$$\hat{T}^{ij}\hat{T}^{jk}\hat{T}^{ki} = \mathrm{Id} \text{ over } B_{ijk} = B_i \cap B_j \cap B_k \tag{42}$$

for all non-empty triple intersections.

By the construction of the T^{ij} we do have the corresponding cocycle condition for F_{SO} itself

$$T^{ij}T^{jk}T^{ki} = \mathrm{Id} \text{ over } B_{ijk} = B_i \cap B_j \cap B_k$$

and it follows that the failure of (42) is 'mild':

$$\hat{T}^{ij}\hat{T}^{jk}\hat{T}^{ki} = w^{ijk} \in \mathbb{Z}_2 \text{ over } B_{ijk} = B_i \cap B_j \cap B_k. \tag{43}$$

So this 'failure' is encoded in $w \in \check{C}^2(\mathcal{B}; \mathbb{X}_2)$

Now, a little computation shows that this forms a (multiplicatively written) Čech cocycle in that

$$\delta w = 0. \tag{44}$$

The corresponding element is the *second Stieffel-Whitney class* $w_2 \in \check{H}(X; \mathbb{Z}_2)$.

To proceed we need to assume that $w_2 = 0$. This means that $w = \delta t$ for a Čech class $t^{ij} \in \mathbb{Z}_2$ for each non-empty B_{ij}. We can use this to modify our initial choice of lift of the T^{ij} and take instead

$$S^{ij} = tij\hat{T}^{ij} \text{ over } B_{ij}. \tag{45}$$

Now the modified local spin bundles as in (40) can be patched consistently, replacing (14) by $(e^i, B) = (T^{ij} e_j, S^{ij} B)$ which now gives an equivalence relation.

Theorem 6. *A spin structure on an oriented manifold X exists if and only if the second Stieffel-Whitney class $w_2 \in \check{H}(X; \mathbb{Z}_2)$ vanishes and then the spin structures are in 1-1 correspondence with $\check{H}(X; \mathbb{Z}_2)$.*

I would also like us to approach the existence of a spin structure on an oriented manifold X through a '$\mathbb{Z}_2$-gerbe'. This perhaps seems like overkill, but the point is that it is a model for what we need to do for string structures later.

At this stage we have understoond the notion of an orientation and so we know that if we take a Riemann structure on it then X comes equipped with its oriented orthonormal frame bundle

$$\begin{tikzcd} \mathrm{SO}(n) \arrow[r, dash] & F_{\mathrm{SO}} \arrow[d] \\ & X. \end{tikzcd} \tag{46}$$

We have also seen (we now assume $n \geq 3$) that $\mathrm{SO}(n)$ has a double cover in $\mathrm{Spin}(n)$. We can therefore ask whethere there is a principal bundle for the spin group which projects naturally to the orthonormal frame bundle

$$
\begin{tikzcd}
\mathrm{Spin}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{Spin}} \arrow[d] \\
\mathrm{SO}(n) \arrow[r, dash] & F_{\mathrm{SO}} \arrow[d] \\
 & X.
\end{tikzcd}
\tag{47}
$$

What does project naturally mean? The map from F_{Spin} to F_{SO} is supposed to be a double cover, a 2-1 map. The map from $\mathrm{Spin}(n)$ to $\mathrm{SO}(n)$ is of course the double cover. Then the action of $A \in \mathrm{Spin}(n)$ is required to preserve the double cover. So if we write π for the two projection maps from the top row in (47) to the second row we want

$$\pi(As) = \pi(A)\pi(s), \ \forall\ A \in \mathrm{Spin}(n) \text{ and } s \in F_{\mathrm{Spin}}. \tag{48}$$

In general there is no such 'refinement', i.e. no spin structure.

The approach via gerbes, or simplicial spaces, is to look at the fibre product of F_{SO} with itself. This is a bundle over X with fibre at each point which is two copies of the fibre of F_{SO} at that point. It is ofter written

$$F_{\mathrm{SO}}^{[2]} = F_{\mathrm{SO}} \times_X F_{\mathrm{SO}}. \tag{49}$$

It can be constructed by lookint at the full product $F_{\mathrm{SO}} \times F_{\mathrm{SO}}$ which is a bundle over X^2. Just look at the preimage of the diagonal, $X \hookrightarrow X^2$.

Now, this fibre product is a principal bundle for $\mathrm{SO}(n)^2$, one factor acting on each factor of F_{SO}. More significantly for us the there is a 'shift map'

$$\sigma : F_{\mathrm{SO}} \times_X F_{\mathrm{SO}} \longrightarrow \mathrm{SO}(n),\ \sigma(s,s') = A \Longleftrightarrow As = s'. \tag{50}$$

Note that this only makes sense for the fibre product, because s and s' are in the same fibre of F_{SO}.

What this map constructs for us is a $\mathbb{Z}_2$ cover of $F_{\mathrm{SO}}^{[2]}$

$$
\begin{tikzcd}
\sigma^* \mathrm{Spin}(n) \arrow[d] \\
F_{\mathrm{SO}}^{[2]}
\end{tikzcd}
\tag{51}
$$

as the pull-back under σ of the $\mathbb{Z}_2$ bundle which is $\mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$.

To say this prosaically, the two points in $\sigma^* \mathrm{Spin}(n)$ above a point $(f_1, f_2) \in F_{\mathrm{SO}}^{[2]}$ are the two elements of $\mathrm{Spin}(n)$ which map to $A \in \mathrm{SO}(n)$ where $f_2 = Af_1$.

Proposition 6. *A spin structure on X is a is a principal $\mathbb{Z}_2$-bundle (and hence double cover) $F' \longrightarrow F_{\mathrm{SO}}$ such that the tensor product of the pull-backs to $F_{\mathrm{SO}}^{[2]}$ under the two projection maps is isomorphic to $\sigma^* \mathrm{Spin}(n)$.*

As I say this is overkill but humour me for the moment (including my spelling). Certainly we have a $\mathbb{Z}_2$ cover of F_{SO}, we need a little more to see that it is actually has a compactible principal action of $\mathrm{Spin}(n)$. So given $B \in \mathrm{Spin}(n)$ and $s \in F'$ where is $Bf \in F'$? If $A = \pi(B) \in \mathrm{SO}(n)$ and $\pi(s) = f \in F_{\mathrm{SO}}$ then $(f, Af) \in F_{\mathrm{SO}}^{[2]}$ and B determines a point in $\sigma^* \mathrm{Spin}(n)$ above (s, Af). If we look at the two points in F' above $Af \in F_{\mathrm{SO}}$, call them s' and s'', then (s,s') and (s,s'') define the two points in $\sigma^*\operatorname{Spin}$ above (f,Af) using the identification of the tensor product (over $\mathbb{Z}_2$) of the two copies of F' with $\sigma^*\operatorname{Spin}(n)$. One of these is B and that defines the action of $B \in \operatorname{Spin}(n)$.

15. $\operatorname{Spin}_{\mathbb{C}}$ structures

The loop-spin structures we will construct on string manifolds below are more akin to a generalization of spin structures, namely $\operatorname{Spin}_{\mathbb{C}}$ structures. These are principal bundles for the group

$$\operatorname{Spin}_{\mathbb{C}}(n) = \mathrm{U}(1) \times \operatorname{Spin}(n)/\{(\mathrm{Id},\mathrm{Id}) = \{-\mathrm{Id},-\mathrm{Id})\} \xrightarrow{\pi_{\mathrm{SO}}} \mathrm{SO}(n). \tag{1}$$

In terms of the complexified Clifford algebra $\mathbb{C}\mathrm{l}(n)$ on $\mathbb{R}^n$ this may be identified with

$$\operatorname{Spin}_{\mathbb{C}}(n) = \{zv_1v_2 \cdot v_{2k} \in \mathbb{C}\mathrm{l}(n),\ z \in \mathrm{U}(1),\ v_i \in \mathbb{R}^n,\ |v_i| = 1\} \tag{2}$$

so $\operatorname{Spin}(n) \subset \operatorname{Spin}_{\mathbb{C}}(n)$ is the intersection with the real part, $\mathrm{Cl}(n)$, of the Clifford algebra. The projection to $\mathrm{SO}(n)$ is the quotient the image of $\mathrm{U}(1) \times \{\mathrm{Id}\}$ which lies in the centre.

Definition 1. A $\operatorname{Spin}_{\mathbb{C}}$ structure on an oriented Riemann n-manifold is a principal $\operatorname{Spin}_{\mathbb{C}}(n)$ bundle $F_{\operatorname{Spin}_{\mathbb{C}}}$ giving a commutative diagram just as in (47)

$$\begin{tikzcd} \operatorname{Spin}_{\mathbb{C}}(n) \arrow[r, dash] \arrow[d] & F_{\operatorname{Spin}_{\mathbb{C}}} \arrow[d] \\ \mathrm{SO}(n) \arrow[r, dash] & F_{\mathrm{SO}} \arrow[d] \\ & X. \end{tikzcd} \tag{3}$$

Notice that $F_{\operatorname{Spin}_{\mathbb{C}}}$, if it exists, must be a circle bundle over F_{SO} as opposed to the double cover, a principal $\mathbb{Z}_2$ bundle, given by a spin structure.

If X has a Spin structure with principal bundle F_{Spin} then this induces a $\operatorname{Spin}_{\mathbb{C}}$ structure by taking

$$F_{\operatorname{Spin}_{\mathbb{C}}} = \mathrm{U}(1) \times F_{\mathrm{Spin}}/\simeq \tag{4}$$

where the equvalence relation is the quotient by the subgroup as in (1). So any obstruction to the existence of a $\operatorname{Spin}_{\mathbb{C}}$ structure must vanish if $w_2(X) = 0$.

To attack the question of the exustence of a $\operatorname{Spin}_{\mathbb{C}}$ strucure one can proceed very much as for spin structure. So take a good open cover of X consisting of Riemannian balls $B_a = B(a,\epsilon)$ of radius ϵ and trivialize F_{SO} over each ball by choosing an orthonormal frame e_a at the central point and extending it, by parallel transport along the radial geodesics, to give a smooth section

$$e_a : B(a,\epsilon) \longrightarrow F_{\mathrm{SO}}.$$

This trivializes F_{SO} over each B_a

$$F_{\mathrm{SO}}\big|_{B_a} \longleftrightarrow \mathrm{SO}(n) \times B_a,\ (f,x) \longmapsto (S,x),\ f = Se_a. \tag{5}$$

Then one can choose the $\operatorname{Spin}_{\mathbb{C}}$ bundle over B_a to be

$$\operatorname{Spin}_{\mathbb{C}} \times B_a \longrightarrow \mathrm{SO}(n) \times B_a.$$

On the overlap of two balls B_a and B_b the two trivializations of F_{SO} as in (15) are related by a transition map

$$e_b = \tau_{b,a} e_a \text{ over } B_{b,a} = B_b \cap B_a,\ \tau_{b,a} \in \mathcal{C}^\infty(B_{b,a}, \mathrm{SO}(n)). \tag{6}$$

Over triple overlaps these satisfy the cocycle condition

$$\tau_{b,a}\tau_{a,c}\tau_{c,b} = \mathrm{Id} \text{ over } B_{b,a,c} = B_{b,a} \cap B_{a,c} \cap B_{c,b} = B_a \cap B_b \cap B_c \tag{7}$$

Using the contractibility of $B_{b,a}$, we can lift the maps $\tau_{b,a}$ to

$$\tilde{\tau}_{b,a} \in \mathcal{C}^\infty(B_{b,a}, \mathrm{Spin}_{\mathbb{C}}(n)),\ \pi_{\mathrm{SO}}\tilde{\tau}_{b,a} = \tau_{b,a}. \tag{8}$$

Namely, picking a point $\bar{q} \in B_{b,a}$ the restriction of $\tau_{b,a}$ to the radial geodesics from $\bar{q}$ is of the form

$$\tau_{b,a}(\bar{q})\exp(\mu_{\bar{q}}),\ \mu_{\bar{q}} \in \mathcal{C}^\infty(B_{\bar{y},a}, \mathfrak{so}(n)). \tag{9}$$

The Lie algebra of $\mathrm{Spin}_{\mathbb{C}}(n)$ contains the Lie algebra of $\mathrm{SO}(n)$ so $\mu_{\bar{q}}$ may be interpreted as a map $\tilde{\mu}_{\bar{q}}$ to the Lie algebra of $\mathrm{Spin}(n)$ and then, using the exponential map for $\mathrm{Spin}(n)$,

$$\tilde{\tau}_{b,a} = \tilde{\tau}_{b,a}(\bar{q})\exp(\tilde{\mu}_{\bar{q}}) \tag{10}$$

is such a lift, for any lift $\tilde{\tau}_{b,a}(\bar{q})$ of $\tau_{b,a}(\bar{q})$ to $\mathrm{Spin}_{\mathbb{C}}(n)$.

Now π_{SO} is a group homomorphism so

$$\pi_{\mathrm{SO}}(\tilde{\tau}_{b,a}\tilde{\tau}_{a,\bar{z}}\tilde{\tau}_{c,b}) = \mathrm{Id} \text{ in } \mathrm{SO}(n)$$

and hence

$$\tilde{\tau}_{b,a,c} = \tilde{\tau}_{b,a}\tilde{\tau}_{a,\bar{z}}\tilde{\tau}_{c,b} \in \mathcal{C}^\infty(B_{b,a,c}, \mathrm{U}(1)) \tag{11}$$

takes values in the centre $\pi_{\mathrm{SO}}^{-1}(\mathrm{Id})$ of $\mathrm{Spin}_{\mathbb{C}}(n)$. It is therefore a Čech class for the sheaf $\mathrm{U}(1)$.

Moreover $\tilde{\tau}_{b,a,c}$ is a Čech cocycle and hence induces a cohomology class

$$W_3(X) \in \check{H}^3(X;\mathbb{Z}). \tag{12}$$

Tracing back through the construction it can be see that changing choices shifts $\tilde{\tau}_{b,,a,c}$ by an exact term.

Proposition 7. *An oriented n-manifold has a* $\mathrm{Spin}_{\mathbb{C}}$ *structure if and only if the cohomology class* W_3 *vanishes and then the* $\mathrm{Spin}_{\mathbb{C}}$ *structures are in 1-1 correspondence with* $\check{H}^2(X,\mathbb{Z})$.

So, when they exist, $\mathrm{Spin}_{\mathbb{C}}$ structures are determined up to a circle bundle on the manifold.

Proof. Since we are working on a good open cover, the vanishing of W_3 implies that

$$\tilde{\tau} = \delta\gamma \tag{13}$$

for a Čech class $\gamma_{b,a} \in \mathcal{C}^\infty(B_{b,a}, \mathrm{U}(1))$. Using this to modify the initial choice of lifts to

$$\tilde{\tau}'_{b,a} = \gamma_{b,a}^{-1}\tilde{\tau}_{b,a} \tag{14}$$

gives maps into $\mathrm{Spin}_{\mathbb{C}}(n)$ which satisfy the cocycle condition and are still lifts of the $\tau_{b,a}$. These determine a principal bundle

$$F_{\mathrm{Spin}_{\mathbb{C}}} = \left(\bigcup_a \mathrm{Spin}_{\mathbb{C}}(n) \times B(a,\epsilon)\right) / \equiv \tau' \tag{15}$$

where the equivalence relation over each $B_{b,a}$ identifies □

16. Dirac operators

This is a finite-dimensional model for the Dirac-Ramond operator.

Only the even-dimensional case is considered in this section.

A spin structure, of more generally a $\mathrm{spin}_{\mathbb{C}}$ structure, on a manifold induces a spinor bundle. One of the analytic consequences of this is that there is an associated Dirac operator, a first-order differential operator acting on section of the spinor bundle, i.e. spinors.

As noted above, Dirac's idea was that electrons should be described by a 'square-root' of the operator describing ordinary particles (now they are called baryons as opposed to fermions which include electrons). In his case the ordinary operator corresponds to the the wave equation on Minkowski space, for us it is some variant of the Lapacian as briefly discussed in § 14.

The term 'Dirac operator' is used with increasing generality (but here on a Riemann manifold):

- Spin and $\mathrm{spin}_{\mathbb{C}}$ operators
- Twisted versions of these
- Dirac operators associated to a Clifford module
- Generalized Dirac operators.

Let's start at the end of the list and work backwards.

A first order linear differential operator acting on sections of a smooth (complex) vector bundle $E \longrightarrow M$ over a manifold is a linear operator

$$D : \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; E) \tag{1}$$

which in local coordinates can be written in terms of first derivatives. In particular it is *local* so if $u \in \mathcal{C}^\infty(M; E)$ vanishes in an open set then Du vanishes in that set.

The *symbol* of D can be defined as follows. By working in local coordinates it can be seen that if $f \in \mathcal{C}^\infty(M)$ and $u \in \mathcal{C}^\infty(M; E)$ then

$$D(e^{itf}u) = e^{itf}(t\sigma(D, f)u + Du). \tag{2}$$

The point is that the right side is a first-order polynomial in t and $\sigma(D, f) \in \mathcal{C}^\infty(M; \hom(E))$. In fact $\sigma(D, f)$ at a point of M only depends on the differential of f at that point – again this is easily seen in local coordinates – and is linear. Thus the symbol can be viewed invariantly as a smooth bundle map

$$\sigma(D) : T^*M \longrightarrow \hom(E). \tag{3}$$

If two linear differential operators as in (1) have the same symbol if and only if they differ by a term of order zero, a bundle map for E.

So Dirac's idea that D becomes

Definition 2. A generalized Dirac operator is a first order differential operator (1) such that

$$\sigma(D)^2(\xi) = |\xi|^2 \operatorname{Id}_E . \tag{4}$$

Here the length of a covector is determined by, and determines, the Riemann metric on the manifold.

Examples are pretty easy to find. Take the 'Hodge' operator on a Riemann manifold

$$d + d^* : \mathcal{C}^\infty(M, \Lambda^*) \longrightarrow \mathcal{C}^\infty(M; \Lambda^*) \tag{5}$$

where d^* is the (formal) adjoint of d using the inner product on Λ^* induced by the Riemann metric. Then

$$\sigma(d + d^*)(\xi) = \xi \wedge -\iota(\xi),\ \xi \in T^*M, \tag{6}$$

where $\iota(\xi)$ is 'contraction with the dual vector to ξ'. It suffices to prove (4) for unit covectors and this is straightforward in Riemannian normal coordinates with firs coordinate corresponding to ξ.

Definition 3. A Clifford module on a Riemann manifold is a vector bundle $E \longrightarrow M$ with a non-trivial action of the complexified Clifford algebra giving a smooth bundle map

$$\mathrm{cl} : \mathbb{C}\mathrm{l}(M) \longrightarrow \hom(E). \tag{7}$$

A Clifford module is Hermitian if it has a Hermitian inner product $(,)_E$ such that the real generators of the Clifford module satisfy

$$(\mathrm{cl}(\xi)u, v)_E = (u, \mathrm{cl}(\xi)v)_E. \tag{8}$$

In the even-dimensional case, M^{2n}, to which we are restricting attention here, the Clifford algebra at a point is isomorphic, as an algebra, to the $2^n \times 2^n$ matrix algebra. It follows from the simplicity of this algebra (it has no non-trivial ideals) that an action on a vector space is either trivial – maps to a multiple of the identity – or is injective.

Lemma 5. *The symbol of a generalized Dirac operator acting on a bundle $E \longrightarrow M$ induces a Clifford module structure on M.*

Proof. The (fibre) linearity of the symbol in its dependence on the covector means that $\sigma(D)(\xi + \eta) = \sigma(D)(\xi) + \sigma(\eta)$ for any ξ, $\eta \in T^*_mM$. Then expanding out

$$\sigma(D)(\xi + \eta)^2 = |\xi_\eta|^2 \Longrightarrow \sigma(D)(\xi)\sigma(D)(\eta) + \sigma(D)(\eta)\sigma(D)(\xi) = 2\langle \xi, \eta\rangle. \tag{9}$$

The only issue here is the sign, otherwise this is the in the ideal defining the Clifford algebra, so

$$\mathrm{cl}(\xi) = i\sigma(D)(\xi) \tag{10}$$

extends multiplicatively to give a Clifford module structure (7) on E. □

Going one step backwards in the lift of types of Dirac operators we now know what a Clifford module is.

Definition 4. A unitary connection on an Hermitian Clifford module is a Clifford connection if it distributes over the Clifford action in the sense that

$$\nabla_v(\mathrm{cl}(\alpha)u) = \mathrm{cl}(\nabla^{\mathrm{LC}}_v \alpha)u + \mathrm{cl}(\alpha)\nabla_v u. \tag{11}$$

where ∇^{LC} is the Levi-Civita connection on $\mathbb{C}\mathrm{l}(M)$.

PROPOSITION 8. *Every Clifford module admits an Hermitian structure for which the Clifford action is unitary and then admits a Clifford connection.*

An Hermitian Clifford moule with Clifford connection is sometimes called a Dirac structure.

PROOF. Go for it – do it all locally and then patch with a partition of unity. □

Any connection defined a first order differential operator

$$\nabla : \mathcal{C}^\infty(M, E) \longrightarrow \mathcal{C}^\infty(M, T^*M \otimes E). \tag{12}$$

The Clifford action restricted to the cotangent bundle gives a smooth linear bundle map $T^*M \longrightarrow \hom(E)$ and this can be restated as a linear map

$$\mathrm{cl} : T^*M \otimes E \longrightarrow E. \tag{13}$$

DEFINITION 5. If E is an Hermitian Clifford module over M with Clifford connection then the composite of (12) and (13)

$$\eth u = \mathrm{cl} \circ \nabla : \mathcal{C}^\infty(M, E) \longrightarrow \mathcal{C}^\infty(M, E) \tag{14}$$

is the associated Dirac operator.

LEMMA 6. *The Dirac operator defined by an Hermitian Clifford module with Clifford connection is a generalized Dirac operator which is essentially self-adjoint and commutes with the action of the volume element of $\mathbb{C}l(M)$ and is therefore a $\mathbb{Z}_2$-graded operator for*

$$E = E^+ \oplus E^-. \tag{15}$$

Getzler's rescaling originates with the fact that the identification of the homomorphisms of the spinor bundle at a point with the Clifford algebra at that point transfers the filtration of the Clifford algebra to a filtration $\hom^{(k)}(S) \subset \hom(S)$. This can be extended naturally to a fitration of the exterior homomorphism bundle $\mathrm{Hom}(S)$ near the diagonal.

PROPOSITION 9. *For a spin manifold, the filtration of the Clifford algebra extends to a multiplicative filtration of the exterior homomorphism bundle of the spinor bundle where $\mathrm{Hom}^{(k)}_{(x,y)}(S)$ is given by parallel translation of $\mathbb{C}l^{(k)}_y$ from y to x and then for any three points*

$$\mathrm{Hom}^{(k)}_{(x,y)}(S) \circ \mathrm{Hom}^{(l)}_{(y,z)}(S) \subset \mathrm{Hom}^{(k+l)}_{(x,z)}(S). \tag{16}$$

PROOF. Parallel translation in S backwards along a given curve from y to x identifies $\mathrm{Hom}_{(x,y)}(S)$ with $\mathbb{C}\mathrm{l}_y$ and so induces a filtration $\mathrm{Hom}^{(k)}_{(x,y)}$ from the filtration of the Clifford algebra. Thus $A \in \mathrm{Hom}^{(k)}_{(x,y)}(S)$ if there exists $A' \in \mathbb{C}\mathrm{l}^{(k)}_y$ such that As is the parallel translate along the given curve of $A's \in S_y$. This is the same as applying the parallel translate of A' in the Clifford algebra to the parallel translate of s. Parallel translation in the Clifford algebra preserves the filtation, since this arises from the order filtration in the tensor algebra. So the induced filtration is independent of the curve even though the identification of $\mathrm{Hom}_{(x,y)}(S)$ with $\mathbb{C}\mathrm{l}_y$ does depend on the choice of curve.1

By concatenation of curves it follows that for any three points (16) holds. □

This filtration of $\mathrm{Hom}(S)$ leads to a rescaled semiclassical calculus. Namely if W is any smooth complex vector bundle on M^2 we define a corresponding space of semiclassical (smoothing) kernels by

$$\Psi_{\mathrm{sl}}^{-\infty}(M,W) = \epsilon^{-n}\{A \in \mathcal{C}^\infty(M[2,\mathrm{sl}]; W \otimes \Omega_R) A \equiv 0 \text{ at } \beta^\# H_0\}. \tag{17}$$

Here $M[2,\mathrm{sl}]$ is the semiclassical space obtained by blowing up $\{0\} \times \mathrm{Diag}(M)$ in $[0,\infty] \times M^2$ where the interval is radially compactified and H_0 is the boundary hypersurfaces $\{\epsilon = 0\}$.

Proposition 10. *The rescaled semiclassical smoothing calculus on the spinor bundle consisting of the kernels*

$$\Psi_{\mathrm{rsl}}^{-\infty}(M,S) = \bigoplus_{k=0}^{n} \epsilon^k \Psi_{\mathrm{sl}}^{-\infty}(M, \mathrm{Hom}^{(k)}) \tag{18}$$

is an algebra with a short exact symbol sequence

$$\epsilon\Psi_{\mathrm{rsl}}^{-\infty}(M,S) \hookrightarrow \Psi_{\mathrm{rsl}}^{-\infty}(M,S) \xrightarrow{\sigma_{\mathrm{rsl}}} \dot{\mathcal{C}}^\infty(\overline{T^*M}; \Lambda^*). \tag{19}$$

Proposition 11. *For an even-dimensional spin manifold the supertrace extends from the smoothing operarors in $\epsilon > 0$ to a continuous linear map*

$$\mathrm{STr} : \Psi_{\mathrm{rsl}}^{-\infty}(M,S) \longrightarrow \mathcal{C}^\infty([0,\infty]) \text{ with } \mathrm{STr}(A)(0) = \int_{T^*M} \sigma_{\mathrm{rsl}}(A). \tag{20}$$

Proposition 12. *The heat kernel* $\exp(-\epsilon^{\frac12}\eth^2)$ *for the Spin Dirac operator is an element of* $\Psi_{\mathrm{rsl}}^{-\infty}(M,S)$ *with*

$$\sigma_{\mathrm{rsl}}(\exp(-\epsilon^{\frac12}\eth^2) = \tag{21}$$

x

17. Bundle gerbes

Gerbes come in several flavours but their origin lies in more-or-less geometric representations of integral cohomology three classes. I will mostly consider *bundle gerbes*, due to Murray [5], and eventually bigerbes, from [**empty citation**] due to Chris Kottke and myself. For today just bundle gerbes.

By back-construction, line bundles or principal circle bundles, are regarded as '0-gerbes' and $\mathbb{Z}$-equivariant covers ($\mathbb{Z}$-torsors) as (-1)-gerbes. They represent respectively integral 2- and 1-classes in the cohomology of the base.

Notice that there is a close relationship between (always principal) circle bundles and $\mathbb{Z}$-torsors. Take a circle bundle

$$\begin{tikzcd} \mathbb{T} \arrow[r, dash] & Y \arrow[d] \\ & M \end{tikzcd} \tag{1}$$

which in the case of compact manifolds is the same as a free action of $\mathbb{T}$ on Y – local triviality is automatic. Now we can associate to this the fibre product

$$\begin{tikzcd} Y^{[2]} \arrow[r, "\sigma"] \arrow[d] & \mathbb{T} \\ M \end{tikzcd} \tag{2}$$

Thus the fibre $Y^{[2]}_m$ of $Y^{[2]}$ above $m \in M$ is $Y_m \times Y_m$, where Y_m is the fibre of Y. A point $(p,q) \in Y^{[2]}$ is therefore necessarily of the form (p, zp) where $z \in \mathbb{T}$ is the principal circle action on Y_m, so transitive and free. This defines the 'fibre shift map' σ in (2), namely

$$\sigma(p,q) = z,\ q = zp.$$

This works for any principal G-bundle

$$\begin{tikzcd} G \arrow[r, dash] & P \arrow[d] & P^{[2]} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, "\sigma"] \arrow[dl] & G \\ & M & & \end{tikzcd} \tag{3}$$

Returning to the case of a circle bundle, we can construct a $\mathbb{Z}$-cover, a torsor, from the map σ by considering all the possible values of $\frac{1}{2\pi i}\log\sigma$ at each point. I leave you to check the details.

The second thing to recall, is that a circle bundle defines an integral 2-class, its Chern class $c_1(Y) \in H^2(M;\mathbb{Z})$. This can be seen readily in terms of Čech cohomology. Take a good open cover, U_i, of M, so over each U_i the circle bundle is trivial – it has smooth (or just continuous) section

$$\mu_i : U_i \longrightarrow Y,\ \pi \circ \mu_i = \mathrm{Id}\,. \tag{4}$$

Over the overlaps this gives

$$\gamma_{ij} I_i \cap U_j \longrightarrow \mathbb{T},\ \gamma_{ij}(m) = \sigma(\mu_i(m), \mu_j(m)). \tag{5}$$

In patricular $\gamma_{ji} = (\gamma_{ij})^{-1}$. This is a Čech cocycle, since over triple intersections

$$(\partial\gamma)_{ijk} = \gamma_{ij}\gamma_{jk}\gamma_{ki} = 1 \tag{6}$$

since it maps $\mu_i(m)$ successively to $\mu_k(m)$, to $\mu_j(m)$ and back to $\mu_i(m)$.

Thus we have defined the class

$$c_1(Y) = [\gamma] \in H^1(M;\mathbb{T}). \tag{7}$$

First check that it is independent of the choice of the sections μ_i. Indeed, new sections are related by maps into the circle, $\mu_i' = Z_i\mu_i$ with $Z_i : U_i \longrightarrow \mathbb{T}$. This shifts the new Čech class to

$$\gamma_{ij}' = \sigma(\mu_i'(m), \mu_j'(m)) = Z_i^{-1} Z_j \gamma_{ij}$$

which changes it by a Čech boundary. In principle we should also check what happens under change of the covering.

Note that the Bockstein homomorphism, corresponding to change of coefficients, is an isomorphism in this case

$$\beta : H^k(M;\mathbb{T}) \longrightarrow H^{k+1}(M;\mathbb{Z}). \tag{8}$$

So we have the more conventional view that $c_1(Y) \in H^2(M;\mathbb{Z})$. The homomorphism arises by taking smooth choices for $\lambda_{ij} \frac{1}{2\pi i}\log\gamma_{ij}$. The Čech boundary of these real-valued functions in then intervalued on triple overlaps precisely because of (6). This defines $c_1(Y) \in H^2(M;\mathbb{Z})$.

The triviality of c_1 means (because we have a good cover) that $\gamma_{ij} = \tau_i^{-1}\tau_j$ is exact, where $\tau_i : U_i \longrightarrow \mathbb{T}$ are smooth. These can be used to 'correct' the original sections to $\tau_i\mu_i$ which now combine to give a global section of Y over M. Thus the vanishing of c_1 implies that Y is trivial.

EXERCISE 3. Show that the Chern class in $H^2(M;\mathbb{Z})$ is additive under tensor product and hence that two circle bundles with the same Chern class are isomorphic.

I have gone through this, presumably familiar, discussion of the Chern character for circle bundles to motivate the discussion of the corresponding invariant for bundle gerbes.

Before passing on, note another naturality property of the Chern class. If $\phi : N \longrightarrow M$ is a smooth map (just continuous works) and Y is a circle bundle over M then the pull-back

$$(\phi^*Y)_n = Y_{\phi(n)} \tag{9}$$

is a circle bundle over N and

$$c_1(\phi^*Y) = \phi^*c_1(Y) \in H^2(N;\mathbb{Z}). \tag{10}$$

This is straightforward to check.

Let's apply this to $\pi : Y \longrightarrow M$ as a map. So Y pulls back to a circle bundle over its own total space. Note that the total space of π^*Y is naturally identified with $Y^{[2]}$ with the projection to Y being the corresponding face map $Y^{[2]} \longrightarrow Y$. However π^*Y is clearly a trivial circle bundle since it has the global section given by the 'diagonal' in $Y^{[2]}$ in which $p \in Y$ is mapped to $(p,p) \in Y^{[2]}$ (this is the degeneracy map of the simplicial structure). From this we learn what can be checked readily in other ways, that

$$\pi^*c_1(Y) = 0 \text{ in } H^2(Y;\mathbb{Z}). \tag{11}$$

Now, to bundle gerbes. The basic idea is that we want to 'add another level' to a circle bundle, really by analogy with the circle bundle and a principal $\mathbb{Z}$-bundle. Anyway, we start by considering a space mapping to M :

$$\begin{tikzcd} X \arrow[d, "\beta"] \\ M. \end{tikzcd} \tag{12}$$

I will assume that this is a fibre bundle – let me not be quite specific at the moment but this will cover all the cases we want. In any case we suppose that there are local continuous sections of β over sufficiently small contractible sets in M.

The idea, although you will not see this for a while, is that the 3-class in $H^3(M;\mathbb{Z})$ that we are trying to 'represent' should vanish when pulled back to X. This is a small problem compare to 2-classes since, whilst it is always possible as we shall see, it may not be possible with X a finite-dimensional bundle over M.

Anyway, the idea is that we consider again the simplicial space defined by the fibre products of X with respect to β :

$$\begin{tikzcd} X \arrow[d] & X^{[2]} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & X^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[dll] & X^{[4]} \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \arrow[dlll] \\ M \end{tikzcd} \tag{13}$$

Over $X^{[2]}$ we suppose there is a given circle bundle

$$\begin{tikzcd} Y \arrow[d] \\ X^{[2]} \arrow[d] \\ M \end{tikzcd} \tag{14}$$

and we suppose that this is *simplicial* in the following sense. First we can define a 'simplicial differential' on circle bundles, modelled on that on functions with values in $\mathbb{T}$. Namely we combine the three face maps from $X^{[3]}$ to $X^{[2]}$ and define

$$\delta Y = f_1^* Y \otimes f_2^* Y^{-1} \otimes f_3^* Y \tag{15}$$

where $f_i : (p_1, p_2, p_3)$ 'forgets' p_i. The inverse means the dual circle bundle. We can do the same for all levels of the simplicial. For instance if $Z \longrightarrow X$ is a circle bundle then

$$\delta Z = f_1^* Z \otimes f_2 Z^{-1} \longrightarrow X^{[2]}.$$

Recalling that if $f_i^{[k]}$ are the face maps from $X^{[k]}$ to $X^{[k-1]}$ then

$$f_j^{[k-1]} f_l^{[k]} = f_l^{[k-1]} f_j^{[k]} \text{ if } j < l,\ f_j^{[k-1]} f_l^{[k]} = f_l^{[k-1]} f_{j-1}^{[k]} \text{ if } j \le l \tag{16}$$

we can see that $\delta^2 Z$ is naturally trivial – has a canonical section defined by choosing any point in the fibre of Z. This actually holds at any level. So *if* $Y = \delta Z$ then Y is trivial. What we shall assume is that

$$\delta Y \longrightarrow X^{[3]} \text{ is trivial with section } s. \tag{17}$$

18. K-theory

Although it is does not appear very much in the sequel here is a brief treatement of topological (i.e. complex) K-theory. This is very relevant for index theory and some variant of it *should* be involved in the index of the Dirac-Ramond operator.

The standard introduction is through even K-theory arising as the Grothendieck group from pairs of vector bundles over a space. We will get to that below but first consider the approach via odd K-theory. The odd K-theory of a manifold can be constructed from smooth or continuous maps

$$f : X \longrightarrow \mathrm{GL}(N, \mathbb{C}). \tag{1}$$

Since the topology of $\mathrm{GL}(N, \mathbb{C})$ is quite complicated part of the equivalence relation on such maps is stability, mapping $\mathrm{GL}(N, \mathbb{C}) \hookrightarrow \mathrm{GL}(N + M, \mathbb{C})$ as block submatrices. Instead we will stabilize $\mathrm{GL}(N, \mathbb{C})$ directly.

Consider the 'infinite dimensional matrix algebra' consisting of the double sequences

$$\Psi^{-\infty}(\mathbb{N}) = \{a : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C},\ \sup_{k,l} (1 + j + k)^p |a_{kl}| < \infty\ \forall\ p\}. \tag{2}$$

This is a Fréchet space and algebra with the product

$$(ab)_{jk} = \sum_l a_{jl} b_{lk}. \tag{3}$$

It contains the algebra of $N \times N$ matrices as finite sequences.

These are compact, in fact trace class, operators on the Hilbert space $l^2(\mathbb{N})$ with the obvious action

$$l^2(\mathbb{N}) \ni u \longmapsto au \in l^2(\mathbb{N}),\ (au)_j = \sum_k a_{jk}u_k. \tag{4}$$

Appending the identity gives the ring $\mathrm{Id} + \Psi^{-\infty}$ in which the product is

$$(\mathrm{Id} + a)(\mathrm{Id} + b) = \mathrm{Id} + a + b + ab. \tag{5}$$

Interpreting the coefficients in Fourier series identifies $\Psi^{-\infty}(\mathbb{N})$ with $\mathcal{C}^\infty(\mathbb{S}\times\mathbb{S})$

$$\Psi^{-\infty}(\mathbb{N}) \ni a \longmapsto \sum_{j,k} a_{jk}\exp(ij\theta - ik\theta') \in \mathcal{C}^\infty(\mathbb{S}\times\mathbb{S})\Psi^{-\infty}(\mathbb{S})$$

as the algebra of smoothing operators on the circle. Using similar identifications $\Psi^{-\infty}(\mathbb{N})$ may be identified with the algebra $\Psi^{-\infty}(M)$ of smoothing operators on any compact manifold or $\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)$, the algebra of Schwartz-smoothing operators on $\mathbb{R}^n$.

The group

$$G^{-\infty}(\mathbb{N}) = \{\mathrm{Id} + a;\ \text{invertible}\} \tag{6}$$

contains $\mathrm{GL}(N)$ as the block elements

$$\begin{pmatrix} A & 0 \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{7}$$

Many of the properties of $\mathrm{GL}(N)$ carry over to $G^{-\infty}$. In particular the Fredholm determinant is a multiplicative function defined on the ring

$$\det : \mathrm{Id} + \Psi^{-\infty} \longrightarrow \mathbb{C},\ G^{-\infty} = \{\mathrm{Id} + a; \det(\mathrm{Id} + a) \neq 0\}. \tag{8}$$

In fact det is an entire function on $\Psi^{-\infty}$ and on $G^{-\infty}$ satisfies the differential equation

$$d\det(\mathrm{Id} + a) = \det(\mathrm{Id} + a)\,\mathrm{tr}((\mathrm{Id} + a)da). \tag{9}$$

Definition 6. We make the direct assertion that $G^{-\infty}$ is a classifying space for odd K-theory by setting

$$K^{-1}(X) = \mathcal{C}^\infty_{\mathrm{c}}(X; G^{-\infty})/\text{homotopy} \tag{10}$$

for any manifold; here compact support means that the image of the complement of a compact subset should be the identity.

It follows that if $F : X \longrightarrow Y$ is a smooth map of manifolds then there is an induced pull-back map

$$F^* : K^{-1}(Y) \longrightarrow K^{-1}(X). \tag{11}$$

Proposition 13. *For the product $\mathbb{S}\times X$ the inclusion i of X as $\{1\}\times X$ induces a short exact sequence*

$$K^{-2}(X) \longrightarrow K^{-1}(\mathbb{S}\times X) \xrightarrow{i^*} K^{-1}(X) \tag{12}$$

where

$$K^{-2}(X) = \mathcal{C}^\infty_c(X; \mathcal{L}_{\#0}G^{-\infty})/\mathit{homotopy} \tag{13}$$

and

$$\mathcal{L}_{\#0}G^{-\infty} = \{g : \mathbb{S} \longrightarrow G^{-\infty}; g \equiv \mathrm{Id}\ \ \mathit{at}\ 1 \tag{14}$$

is the 'pointed flat loop-group'.

PROOF. □

As noted above, $K^0(X)$ may be defined directly in terms of pairs of complex vector bundles over the manifold X. The equivalence relation is stable isomorphism. That is for smooth complex vector bundles

$$K^0(X) = \{(E_+, E_-)\}/\simeq,\ (E_+, E_-) \simeq (F_+, F_-) \iff \exists\ H \text{ and a bundle isomorphism } E_+ \oplus F_- \oplus H \equiv E_- \oplus F_+ \oplus H. \tag{15}$$

Each element of the group $\mathcal{L}_{\#0}G^{-\infty}$ in (14) defines a bounded multiplication operator on the Hilbert space

$$L^2(\mathbb{S}; l^2(\mathbb{N})). \tag{16}$$

The Toeplitz projection H on $l^2(\mathbb{S})$ extends to act on this same space and this allows $g \in \mathcal{L}_{\#0}G^{-\infty}$ to be quantized to

$$HgH \text{ bounded on } L^2(\mathbb{S}; l^2)). \tag{17}$$

Now, extending a little the discussion in §of Toeplitz quantization we can expect that the difference

$$HgHhG - HghH = HaH,\ a \in \Psi^{-\infty}(\mathbb{S} \times \mathbb{N}) \tag{18}$$

should be a Toeplitz-smoothing operator. So we define

$$\mathcal{R} = \{H(g+a)H;\ g \in \mathcal{L}_{\#0}G^{-\infty}),\ a \in \Psi^{-\infty}(\mathbb{S} \times \mathbb{N}). \tag{19}$$

THEOREM 7 (Bott-Periodicity). *Toeplitz quantization,* (17)*, defines a ring $\mathcal{R}$ of Fredholm operators on $L^2(\mathbb{S}; l^2(\mathbb{N})$ and any smooth family in $F \in \mathcal{C}^\infty_c(X, \mathcal{R})$ is homotopic to a family with smooth null space with the equivalence class of the pair $((F), (F^*)$ a well-defined element of $K^0(X)$ giving an isomorphism*

$$K^{-2}(X) \longrightarrow K^0(X). \tag{20}$$

PROOF. □

19. Toeplitz algebra

CLAIM 2. *The group of orientation-preserving diffeomorphisms of the circle has a natural central extension giving a short exact sequence of groups*

$$\mathbb{S} \hookrightarrow \widehat{\mathrm{Dff}}^+(\mathbb{S}) \twoheadrightarrow \mathrm{Dff}^+(\mathbb{S}). \tag{1}$$

This is of course a Theorem, I just have not defined the terms in it. Note the meaning of 'central extension' – elements of the circle group commute with all other elements.

QUESTION 1. The question arises as to why? Why the central exists I guess is answered by the construction. Why is it important? I don't have a good answer to this. As I mentioned earlier, we want a theory on the loop space, coming from a string structure on a manifold, which is 'equivariant' for the group $\mathrm{Dff}^+(\mathbb{S})$ acting as a reparameterization group on loops. It turns out that it is the central extension (1) which acts on the objects we will construct on the loop space, covering the action of $\mathrm{Dff}^+(\mathbb{S})$ on loops. Why is maybe to deep a question. The construction below, in terms of Hardy space, generalizes to the construction of central extensions on the loop group, such as $\mathcal{C}^\infty(\mathbb{S};\mathrm{Spin})$ and just as $\mathrm{Dff}^+(\mathbb{S})$ acts on this loop group, $\widehat{\mathrm{Dff}}^+(\mathbb{S})$, the Bott-Virasoro group, acts on the central extension.

We identify the circle as the quotient

$$\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}. \tag{2}$$

Then there is a bijections

$$\theta : [0, 2\pi) \longrightarrow \mathbb{S}.$$

Any continuous map $f : \mathbb{S} \longrightarrow \mathbb{S}$ has a unique lift under the covering map corresponding to the quotient (2) to a continuous map

$$f : \mathbb{S} \longrightarrow \mathbb{R},\ f(0) \in [0, 2\pi).$$

Using the coordinate θ this in turn lifts to a unique continous map

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R} \text{ s.t. } \tilde{f}(0) \in [0, 2\pi),\ \tilde{f}(\theta + 2\pi) = \tilde{f}(\theta) + 2\pi w$$

where $w \in \mathbb{Z}$ is the winding number of f. Then f is smooth if and only if $\tilde{f}$ is smooth.

If $f \in \mathrm{Dff}(\mathbb{S})$ is a diffeomorphism then $\tilde{f}' \neq 0$ and $\tilde{f}$ is 1-1, so either $f' > 0$ and $w = 1$ of $f' < 0$ and $w = -1$. Thus the group of orientation-preserving diffeomorphism can be written as a product

$$\mathrm{Dff}^+(\mathbb{S}) = \mathbb{S} \times \left\{ g \in \mathcal{C}^\infty(\mathbb{S}),\ g > 0,\ \int_{\mathbb{S}} g d\theta = 2\pi \right\} \tag{3}$$

where the map takes f to $(f(0), f')$.

Lemma 7. *The group* $\mathrm{Dff}^+(\mathbb{S})$*is smoothly contractible to the circle* $\mathbb{S}$.

Proof. Take the map on the second factor in (3)

$$g_t = (1 - t)g + 2\pi t,\ t \in [0, 1]$$

which contracts it to be constant, $t = 1$. □

We are interested in the action of $\mathrm{Dff}^+(\mathbb{S})$ on functions, by pull-back,

$$f^* u = u \circ f,\ u \in \mathcal{C}^\infty(\mathbb{S}).$$

This extends to a bounded operator on $L^2(\mathbb{S})$ but

$$\int_{\mathbb{S}} |u \circ f|^2 d\theta = \int_{\mathbb{S}} |u|^2 \delta d\theta'$$

where $\delta = d\theta/d\theta'$ with $\theta = \tilde{f}(\theta)$. So in general this is not a unitary operator. To 'correct' this we consider what is really the action on half-densities but we can think of it as acting on functions via

$$T_f : L^2(\mathbb{S}) \longrightarrow L^2(\mathbb{S}),\ T_f u = (f')^{\frac{1}{2}} u \circ f,\ \|T_f\| = 1. \tag{4}$$

This is norm-preserving and is an action by the chain rule.

Now we have a group of unitary operators on $L^2(\mathbb{S})$ but, as always with diffeomorphism groups, the map

$$\mathrm{Dff}^+(\mathbb{S}) \ni f \longmapsto T_f \in \mathrm{U}(L^2(\mathbb{S})) \tag{5}$$

is *not* norm continuous. It is easily seems to be strongly continuous in terms of the topology coming from (3), with the $\mathcal{C}^\infty$ topology on the second factor; this is just something to watch out for!

Next we think about the Hardy[5] space(s) – since they come with various amounts of regularity. One approach to this is through Fourier series. If $u \in L^2(\mathbb{S})$ then

$$u = \sum_{k\in\mathbb{Z}} u_k e^{ik\theta},\ \sum_k |u_k|^2 < \infty \tag{6}$$

with convergence in $L^2(\mathbb{S})$ (and pointwise convergence a.e if you want to try harder ...). The Sobolev spaces on $\mathbb{S}$ are similarly identified with the sequences

$$u \in H^s(\mathbb{S}) \Longleftrightarrow \sum_{k\in\mathbb{Z}} (1+k^2)^{\frac{s}{2}} |u_k|^2 < \infty$$

where for $s < 0$ convergence is in the sense of distributions, in $\mathcal{C}^{-\infty}(\mathbb{S})$.

In terms of (6), the Toeplitz projection is the map

$$Hu = \sum_{k\geq 0} u_k e^{ik\theta}. \tag{7}$$

There is always an issue as to what to do with the constant term $k = 0$ but here at least we will keep it. I will denote the range of H, the Hardy space simply as

$$\mathcal{H} = HL^2(\mathbb{S}) \subset L^2(\mathbb{S}). \tag{8}$$

It clear that H extends to a bounded operator on each of the Hilbert spaces $H^s(\mathbb{S})$, so there are Hardy spaces of all Sobolev regularities.

A fundamental fact about H is that it 'almost commutes' with multiplication by smooth functions. This is ultimately connected to the fact that there are two components to infinity in $T^*\mathbb{S}$. Hardy space and its orthocomplement correspond to these two pieces. Anyway, concretely

PROPOSITION 14. *If $v \in \mathcal{C}^\infty(\mathbb{S})$ acts on $L^2(\mathbb{S})$ as a multiplication operator then*

$$[H, v] = Hv - vH$$

is a smoothing operator *on $L^2(\mathbb{S})$, an integral operator with a smooth kernel*

$$u \longmapsto \int_{\mathbb{S}} A(\theta,\theta')u(\theta')d\theta',\ A \in \mathcal{C}^\infty(\mathbb{S}\times\mathbb{S}).$$

This, I believe, is due to Toeplitz[6]. You might amuse yourself by looking into the compactness properties of the commutator for v of lower regularity.

PROOF. The expansion in Fourier series, (6), identifies $L^2(\mathbb{S})$ with $l^2(\mathbb{Z})$. Thus

$$vu = \sum_k u_k(v(\theta)e^{ik\theta}). \tag{9}$$

The Fourier coefficients are given by

$$u_k = \frac{1}{2\pi}\int_{\mathbb{S}} u(\theta)e^{-ik\theta}$$

and applying this to (9)

$$(vu)_j = \sum_k V_{jk}u_k,\ V_{jk} = \frac{1}{2\pi}\int_{\mathbb{S}} v(\theta)e^{i(k-j)\theta}d\theta = v_{j-k}.$$

[5]Godfrey Hardy, usually G.H Hardy, 1877-1947

[6]Otto Toeplitz, 1881-1940

Thus the action of v in terms of Fourier series is given (not surprisingly) by convolution. From this we can easily compute the form of the commutator. Namely

$$\begin{aligned}((Hv-vH)u)_j &= (H(vu))_j - (v(Hu))_j\\ &= \begin{cases}\sum\limits_{k<0} v_{j-k}u_k & j\ge 0\\ -\sum\limits_{k\ge 0} v_{j-k}u_k & j<0.\end{cases}\end{aligned}$$

Thus

$$((Hv-vH)u)_j = \sum_k \beta_{jk}u_k \text{ where } \beta_{jk} = \begin{cases} v_{j-k} & j\ge 0,\ k<0\\ -v_{j-k} & j<0,\ k\ge 0\\ 0 & \text{otherwise.}\end{cases}$$

Since j and k occur with opposite signs here it follows that for every N there is a constant C_N such that

$$|\beta_{jk}| \le C_N(1+|j|+|k|)^{-N}$$

is a rapidly decreasing double sequence. The kernel of the operator $Hv-vH$ is therefore the function

$$A(\theta,\theta') = \frac{1}{2\pi}\sum_{j,k}\beta_{jk}e^{ij\theta-ik\theta'} \in \mathcal{C}^\infty(\mathbb{S}\times\mathbb{S}).$$

Note that there is no issue with the convergence of the series anywhere here. □

This commutation result is for multiplication operators, but there is a similar one for pull-back. We can think of this in terms of the *Toeplitz operator*

$$S_f = HT_fH \text{ on } \mathcal{H}\subset L^2(\mathbb{S}). \tag{10}$$

PROPOSITION 15. *For any $f\in \mathrm{Dff}^+(\mathbb{S})$, S_f is a Fredholm operator on $\mathcal{H}$ of index zero, with parametrix $S_{f^{-1}}$ up to (Toeplitz) smoothing errors.*

PROOF. We only need think about the commutator $[H,f^*]$ as before to see that

$$[H,T_f] \text{ is a smoothing operator.} \tag{11}$$

Indeed,

$$[H,T_f] = [H,(f')^{\frac12}]f^* + (f')^{\frac12}[H,f^*]$$

and the composites of a smoothing operator with a multiplication or pull-back operator are smoothing.

Clearly

$$f^*e^{ik\theta} = \exp(ikf(\theta))$$

so

$$[H,f^*]e^{ik\theta} = (Hf^*)e^{ik\theta} - f^*He^{ik\theta} = \begin{cases} He^{ikf(\theta))} - e^{ikf(\theta)} & k\ge 0\\ He^{ikf(\theta)} & k<0.\end{cases}$$

The Fourier components of this are therefore

$$\big([H,f^*]e^{ik\theta}\big)_j = \begin{cases} -(e^{ikf(\theta)})_j & j<0,\ k\ge 0\\ (e^{ikf(\theta)})_j & j\ge 0,\ k<0\end{cases}$$

and zero otherwise. Since $f(\theta)$ is smooth and the derivative $kf'(\theta)-j > \epsilon(|k|+|j|)$, $\epsilon > 0$, in all cases where the coefficient is non-zero, integration by parts in

$$(e^{ikf(\theta)})_j = \frac{1}{2\pi}\int_{\mathbb{S}} e^{i(kf(\theta)-j\theta)}d\theta$$

shows that these coefficients are again rapidly decreasing and hence $[H, f^*]$ is a smoothing operator. Thus (11) holds and it follows that $[H, S_f]$ is always a smoothing operator.

Now consider the composite

$$S_{f^{-1}}S_f = HT_{f^{-1}}HT_fH = HT_{f^{-1}}T_fH - HT_{f^{-1}}[T_f, H]H = H - R_1. \tag{12}$$

Here R_1 is a smoothing operator. It follows that $S_{f^{-1}}$ is indeed a parameterix modulo smoothing errors as an operator on $\mathcal{H}$

$$\begin{aligned} S_{f^{-1}}S_f &= \operatorname{Id}_{\mathcal{H}} - R_1 \\ S_fS_{f^{-1}} &= \operatorname{Id}_{\mathcal{H}} - R_2. \end{aligned} \tag{13}$$

Since smoothing operators are compact it follows that S_f is a Fredholm operator on $\mathcal{H}$.

Next we check that its index is zero. Certainly if f is a rotation then it commutes with H and $S_f = T_fH = FT_f$ is invertible. In general it follows from Lemma 7. This gives a smooth curve f_t connecting f to a rotation. Although this is only strongly continuous, the error tems $R_{1,t}$ and $R_{2,t}$ are actually smooth also as functions of t. They are then continuous, in t, as families of trace-class operators on $\mathcal{H}$.

There is a formula (due I believe to Calderón[7]) for the index

$$\operatorname{ind}(S_{f_t}) = \operatorname{Tr}(R_{1,t}) - \operatorname{Tr}(R_{2,t}). \tag{14}$$

It follows that the index is continuous, but being an integer, is constant. □

It is a general fact that Fredholm operators of index zero can be perturbed to be invertible. Thus S_f has finite-dimensional null space and finite-dimensional complement to the range, acting on $\mathcal{H}$, of the same dimension. It follows from (13) that these are (can be taken to be in the case of the complement of the range) spanned by smooth functions. There is therefore a finite-rank smoothing operator B such that

$$S_f + B \text{ is invertible on } \mathcal{H}. \tag{15}$$

To see that we can choose a smoothing operator B so that $S_f + B$ is unitary, recall that T_f is unitary, i.e. its adjoint on $L^2(\mathbb{S})$ is $T_{f^{-1}}$. It follows that the adjoint of S_f acting on $\mathcal{H}$ is indeed $S_{f^{-1}}$. Once we have arranged (15) it follows that

$$(S_f + B^*)(S_f + B) = \operatorname{Id}_{\mathcal{H}} + E$$

is self-adjoint and positive. Taking its square-root, $(\operatorname{Id} + E)^{\frac{1}{2}} = \operatorname{Id} + F$ it follows that F is also a smoothing operator on $\mathcal{H}$ and then

$$(S_f + B)(\operatorname{Id} + F)^{-} \text{ is unitary}$$

and still of the same form.

Definition 7. Let $\mathcal{G}$ be the collection of unitary operators of the form $S_f + B$ on $\mathcal{H}$ with B a smoothing operator.

[7]Alberto Calderón, 1920-1998, at MIT for some years

Clearly $\mathcal{G}$ is a group since the composite of a smoothing operator with S_f is smoothing.

PROPOSITION 16. *There is a short exact sequence of groups*

$$\mathcal{G}^0 \hookrightarrow \mathcal{G} \xrightarrow{\sigma} \mathrm{Dff}^+(\mathbb{S}) \tag{16}$$

where $\mathcal{G}^0$ is the group of unitary operators on $\mathcal{H}$ of the form $\mathrm{Id} + B$ *with B smoothing.*

Here of course $\mathrm{Id} = H$ on $\mathcal{H}$.

PROOF. The main issue here is the definition of the 'symbol map' which recovers f. This is again an asymptotic statement. Consider

$$s_k(\theta) = (f')^{-\frac{1}{2}} e^{-ikf(\theta)} (S_f + B) e^{ik\theta}. \tag{17}$$

This is a sequence in $\mathcal{C}^\infty(\mathbb{S})$ and we will show that

$$s_k \to 1 \text{ uniformly as } k \to \infty. \tag{18}$$

This shows that σ is defined since

$$|f'|^{\frac{1}{2}} = \lim_{k\to\infty} |s_k|$$

shows that we can recover f' from $S_f + B$. This determines f up to a rotation and then

$$\arg(|f'|^{\frac{1}{2}} e^{-kf(\theta)} s_k) - k \to f(\theta).$$

□

Next we compute the form of the Lie algebra of $\widehat{\mathrm{Dff}}^+(\mathbb{S})$, the Virasoro algebra. The Lie algebra of $\mathrm{Dff}^+(\mathbb{S})$ consists of the (real) smooth vector fields on the circle. Since we are dealing with an infinite-dimensional group here, we need to check that the Lie algebra really makes sense. Recall that the topology we have comes from (3). This also writes it as the product of $\mathbb{S}$ and an open subset ($g > 0$) of the hypersurface $\int_{\mathbb{S}} g d\theta = 2\pi$. Subtracting 1 from g makes it into an open subset of the closed linear space L of mean-zero functions on the circle, a linear space with Fréchet space. In any case the tangent space of a linear space is easy to interpret as the linear space itself, so the tangent space to $\mathrm{Dff}^+(\mathbb{S})$ is $\mathbb{R} + L$. This is naturally identified with $\mathcal{C}^\infty(\mathbb{S})$ where the first term is the integral of the function.

In particular if $\phi \in \mathcal{C}^\infty(\mathbb{S})$ (always real-valued here) then it defines a curve through the identity given by

$$E(t;\phi) = (\exp(it\int_{\mathbb{S}}\phi), 1 + t\psi) \tag{19}$$

which has tangent vector ϕ at the identity. The definition of the Lie bracket involves taking two such and computing the quadratic part in t of the composite

$$E(t,\phi_1)E(t,\phi_2)E(-t,\phi_1)E(-t,\phi_2) = L(\phi_1,\phi_2). \tag{20}$$

LEMMA 8. *The Lie bracket from* (20) *is the commutator bracket of vector fields on* $\mathbb{S}$.

We take the basis of the of the complexified Witt algebra

$$v_p = e^{i\theta p}\frac{d}{d\theta},\ p \in \mathbb{Z}. \tag{21}$$

In terms of this the structure constants of the Lie algebra are

$$[v_q, v_p]e^{ik\theta} = (q-p)v_{p+q} \tag{22}$$

as can be seen by applying it to $e^{ik\theta}$ for each k.

The extension cocycle for the group constructed from $\mathrm{Dff}^+(\mathbb{S})$ by Toeplitz projection is then given by the trace of

$$\gamma = [Hv_qH, Hv_pH] - H[v_q, v_p]H. \tag{23}$$

Diagonal terms for the action on $\mathcal{C}^\infty(\mathbb{S})$ only arise if $p+q=0$ and then, assuming that $p \geq 0$ and $k \geq 0$,

$$\begin{gathered}
[v_{-p}, v_p]e^{ik\theta} = -2pke^{ik\theta} \\
Hv_{-p}HV_pHe^{ik\theta} = -(p+k)ke^{ik\theta} \\
Hv_pHV_{-p}He^{ik\theta} = \begin{cases} -(-p+k)ke^{ik\theta} & k > p \\ 0 & 0 \leq k \leq p. \end{cases}
\end{gathered}$$

Thus, for $q = -p$ and $p \geq 0$

$$\gamma e^{ik\theta} = \begin{cases} 0 & k \geq p \\ pk - k^2 & 0 \leq k \leq p. \end{cases} \tag{24}$$

Thus,

$$\mathrm{Tr}(\gamma) = \sum_{0 \leq k \leq p} pk - k^2 = \frac{1}{2}p^2(p+1) - \frac{1}{6}p(p+1)(2p+1) = \frac{1}{6}\left(p^3 - p\right).$$

This is odd in p, as is γ so the same formula holds for all p.

The Lie algebra of the Virasoro algebra therefore satisfies

$$[v_q + aw, v_p + bw] = (q-p)v_{p+q} + \frac{1}{6}w(p^3 - p)\delta_{p+q=0} \tag{25}$$

where w is generates the centre.

A corresponding group cocycle is

$$\Gamma(f,g) \equiv -\frac{1}{48\pi}\int_0^{2\pi} \log\left[f'\big(g(x)\big)\right]\frac{g''(x)}{g'(x)}\mathrm{d}x.$$

The real Bott-Virasoro group is then $\mathrm{Dff}^+(S^1) \times \mathbb{R}$ with group product

$$(f,\alpha)\cdot(g,\beta) = \big(f\circ g, \alpha + \beta + \Gamma(f,g)\big).$$

20. Bargmann decomposition

Although it is at the moment unmotivated, the 'Bargmann decomposition' of $L^2(\mathbb{R}^2)$ plays an important rôle below; for the origins see [**MR157250**] and [**MR1178936**]. The elements

$$\phi_{k,j} = z^k\overline{z}^j\exp(-\frac{1}{2}|z|^2) \in \mathcal{S}(\mathbb{R}^2),\ z = x+iy,\ k,j \in \mathbb{N}_0, \tag{1}$$

have finite span dense in $L^2(\mathbb{R}^2)$ as follows from the Hermite expansion, the eigenfunction expansion for the harmonic oscillator. That they are orthogonal can be seen by integration by parts using the the Cauchy-Riemann operator

$$\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y),\ \partial_{\overline{z}} z = 0,\ \partial_{\overline{z}}\overline{z} = 1. \tag{2}$$

Namely, it suffices to show that

$$\int_{\mathbb{R}^2} z^k \overline{z}^j \exp(-|z|^2) dxdy = 0 \text{ if } k > j. \tag{3}$$

Now

$$\partial_{\overline{z}} \exp(-|z|^2) = -z \exp(-|z|^2),\ \partial_z \exp(-|z|^2) = -\overline{z}\exp(-|z|^2).$$

So if $k > l > 0$ integration by parts, justified since all terms are in $\mathcal{S}(\mathbb{R}^2)$, implies

$$\begin{aligned}\int_{\mathbb{R}^2} z^k \overline{z}^j \exp(-|z|^2) dxdy &= -\int_{\mathbb{R}^2} \overline{z}^j \partial_{\overline{z}} \left(z^{k-1}\exp(-|z|^2)\right) dxdy \\ &= j\int_{\mathbb{R}^2} z^{k-1}\overline{z}^{j-1}\exp(-|z|^2) dxdy.\end{aligned} \tag{4}$$

Repeating this $l+1$ times gives (3).

Rather than orthonormalize the $\phi_{k,j}$ we start from the Bargmann space,

$$B = \{u \in L^2(\mathbb{R}^2),\ \partial_{\overline{z}}(u \exp(\frac{1}{2}|z|^2)) = 0 \text{ in distributions on } \mathbb{R}^2\} \tag{5}$$

which we shall see is the closure in $L^2(\mathbb{R}^2)$ of the span of the $z^k \exp(-\frac{1}{2}|z|^2)$. Alternatively B is the closure in L^2 of the null space of the 'isotropic' operator (which we will meet as an entry in the Dirac-Ramond operator for the torus)

$$\not{\partial}^+ = i(2\partial_{\overline{z}} + z) : \mathcal{S}(\mathbb{R}^2) \longrightarrow \mathcal{S}(\mathbb{R}^2). \tag{6}$$

We proceed to analyze $\not{\partial}^+$ and its formal adjoint

$$\not{\partial}^- = i(2\partial_z - \overline{z}),\ [\not{\partial}^+, \not{\partial}^-] = 4, \tag{7}$$

as unbounded operators on L^2. The commutation condition here is closely related to the Clifford algebra.

Since

$$\int_0^\infty r^{2k}\exp(-r^2)dr = \frac{1}{2}\int_0^\infty s^{k-\frac{1}{2}}\exp(-s)ds = \frac{1}{2}\Gamma(k+\frac{1}{2}) \tag{8}$$

the functions

$$\beta_{k,0} = (\Gamma(k+\frac{1}{2}))^{-\frac{1}{2}} z^k \exp(-\frac{1}{2}|z|^2) \tag{9}$$

are orthonormal with respect to the normalized Lebesgue measure

$$\nu = \frac{1}{\pi} dz d\overline{z}. \tag{10}$$

The functions

$$\beta_{0,k} = \overline{\beta_{k,0}} \tag{11}$$

are orthonormal and by (3) are orthogonal to the $\beta_{j,0}$ for $j > 0$.

By direct computation the $\phi_{k,j}$ in (1) satisfy

$$
\begin{aligned}
\eth^+\phi_{k,j} &= (i(2\partial_{\overline{z}} + z)(z^k\overline{z}^j\exp(-\frac{1}{2}|z|^2)) = 2ij\phi_{k,j-1},\\
\eth^-\phi_{k,j} &= (i(2\partial_z - \overline{z})(z^k\overline{z}^j\exp(-\frac{1}{2}|z|^2)) = 2ik\phi_{k-1,j} - 2i\phi_{k,j+1}.
\end{aligned}
\tag{12}
$$

Definition 8. For $p \in \mathbb{Z}$ let L_p be the closure in $L^2(\mathbb{R}^2)$ of the span of the $\phi_{k,j}$ for $k - j = p$.

From (12)

$$
\eth^-\phi_{k,j} \in L_{k-j-1}. \tag{13}
$$

Lemma 9. *For $k, j \geq 0$ the*

$$
\beta_{k,j} = (2i)^{-j}(j!)^{-\frac{1}{2}}\left(\eth^-\right)^j\beta_{k,0} \tag{14}
$$

form an orthonormal basis of $L^2(\mathbb{R}^2)$, consistent with (11), and the $\beta_{k,j}$ with $k-j = p$ give an orthonormal basis for L_p.

Proof. From (12) each $\beta_{0,j}$ defined by (14) is a real multiple of $\phi_{0,j}$ so (14) is consistent with (11). In general, for $k > 0$, $j > 0$, each $\beta_{k,j}$ is a linear combination of the $\phi_{k-l,j-l}$ for $l \leq j$ with a non-zero projection onto $\phi_{k,j}$. Thus the $\beta_{k,j}$ with $k - j = p$ span a dense subspace of L_p.

Orthogonality of $\beta_{p+l,l}$ and $\beta_{p+l',l'}$ for $l \neq l'$ follows from the usual 'creation-annihilation' computation. Thus if $p + l' > p + l > 0$ with $l \geq 0$

$$
\begin{aligned}
\langle(\eth^-)^{l'}\beta_{p+l',0}, (\eth^-)^l\beta_{p+l,0}\rangle_{L^2} &= \langle(\eth^-)^{l'-1}\beta_{p+l',0}, \eth^+(\eth^-)^l\beta_{p+l,0}\rangle_{L^2}\\
&= 4l\langle(\eth^-)^{l'-1}\beta_{p+l',0}, (\eth^-)^{l-1}\beta_{p+l,0}\rangle_{L^2} = 4^l l!\langle(\eth^-)^{l'-l}\beta_{p+l',0}, \beta_{p+l,0}\rangle_{L^2}.
\end{aligned}
\tag{15}
$$

Iterating shows that the pairing vanishes since $\phi_{p+l,0}$ is annihilated by $\eth^+$. $\square$

It follows that

$$
L^2(\mathbb{R}^2) = \bigoplus_{p\in\mathbb{Z}} L_p. \tag{16}
$$

The 'Laplacians'

$$
A^+ = \eth^-\eth^+ = -(2\partial_{\overline{z}} + z)(2\partial_z - \overline{z}) \text{ and } A^- = \eth^+\eth^- = A^+ + 4 \tag{17}
$$

are diagonalized in this basis,

$$
A^+\beta_{k,j} = 4j\beta_{k,j},\ A^-\beta_{k,j} = (4 + 4j)\beta_{k,j}. \tag{18}
$$

So all the eigenspaces have infinite multiplicity. The inverse of A^- and the generalized inverse of A^+

$$
B^+\beta_{k,j} = (4j)^{-1}\beta_{k,j},\ j > 0,\ B^+\beta_{k,0} = 0,\ (A^-)^{-1}\beta_{k,j} = (4j+4)^{-1}\beta_{k,j} \tag{19}
$$

give generalized inverses

$$
\begin{aligned}
Q^- = \eth^-B^+,\ Q^-\eth^+ = \operatorname{Id} - \Pi_B,\ \eth^+Q^- = \operatorname{Id}\\
Q^+ = \eth^-(A^-)^{-1},\ Q^+\eth^- = \operatorname{Id},\ \eth^-Q^+ = \operatorname{Id} - \Pi_B
\end{aligned}
\tag{20}
$$

where both $Q^\pm$ are bounded and map into the domain.

The domain of both the essentially self-adjoint operators $A^{\pm}$ is the dense subspace of $L^2(\mathbb{R}^2)$

$$J^{0,2}=\left\{\sum_{k,j}c_{k,j}\beta_{k,j},\ \sum_{k,j}(j+1)^2|c_{k,j}|^2<\infty\right\}. \tag{21}$$

Thus we have shown

Proposition 17. *The operators $\eth^+$ and $\eth^-$ are closed operators on $L^2(\mathbb{R}^2)$, are adjoints of each other, with the common domain in terms of the orthonormal basis $\beta_{k,j}$:*

(22)

$$\operatorname{Dom}(\eth^+)=\operatorname{Dom}(\eth^-)=J^{0,1}=\left\{\sum_{k,j}c_{k,j}\beta_{k,j};\ \sum_{k,j}(j+1)|c_{k,j}|^2<\infty\right\}\subset L^2(\mathbb{R}^2)$$

$$\eth^+\beta_{k,j}=2j^{\frac12}\beta_{k-1,j-1},\ \eth^-\beta_{k,j}=2(j+1)^{\frac12}\beta_{k+1,j+1}.$$

As operators from the Hilbert space defined by (22) both operators are semi-Fredholm, the one surjective with null space $B\subset J^{0,1}$, the other injective with range the orthocomplement of B.

For any non-negative intergers we can define 'Bergmann-Sobolev' spaces

$$J^{s,t}(\mathbb{R}^2)=\left\{u\in L^2(\mathbb{R}^2); u=\sum_{k,j}c_{k,j}\beta_{k,j};\ \sum_{k,j}(j+1)^s(k+1)^t|c_{k,j}|^2<\infty\right\}. \tag{23}$$

These are Hilbert spaces, which decrease with increasing s or t and it is straightforward to see from (22) that

$$\eth^{\pm}:J^{s,t}(\mathbb{R}^2)\longrightarrow J^{s-1,t}(\mathbb{R}^2),\ s\ge 1 \tag{24}$$

are semi-Fredholm, the null space of $\eth^+$ is the Bergmann space $B\cap J^{s,t}=B\cap J^{s,0}$.

Note that there is a basic asymmetry, 'favouring' the holomorphic Bargmann space. We can consider the (negatives of) the complex conjugate operators

$$\tilde{\eth}^+=i(2\partial_z+\overline{z}),\ \tilde{\eth}^-=i(2\partial_{\overline{z}}-z). \tag{25}$$

These commute with the initial choices

$$[\tilde{\eth}^{\pm},\eth^{\pm}]=0.$$

Computing directly,

$$\tilde{\eth}^+\phi_{k,0}=2ik\phi_{k-1,0},\ \tilde{\eth}^+\phi_{k,0}=-2i\phi_{k+1,0}.$$

From the definition (14) it follows that

$$\tilde{\eth}^+\beta_{k,j}=?\beta_{k-1,j},\ \tilde{\eth}^-\beta_{k,j}=?\beta_{k+1,j}. \tag{26}$$

It follows that the $\tilde{\eth}^{\pm}$ are 'sideways' shift operators, i.e. changing k rather than j and are semi-Fredholm operators

$$\tilde{\eth}^{\pm}:J^{s,t}\longrightarrow J^{s-1,t}. \tag{27}$$

The derivatives and multiplication operators ∂_x, ∂_y, x and y are all linear combinations of the $\tilde{\not{p}}^{\pm}$ and $\not{p}^{\pm}$ so are bounded from $J^{s,s}$ to $J^{s-1,s-1}$, $s \geq 1$, and from this and the properties of the harmonic oscillator

$$\mathcal{S}(\mathbb{R}^2) = \bigcap_{s,t} B^{s,t}(\mathbb{R}^2). \tag{28}$$

Indeed, the definitions can be extended to all real numbers s, t (not giving subsets of $L^2(\mathbb{R}^2)$ if one is negative) with (24) and (27) still valid and

$$\mathcal{S}'(\mathbb{R}^2) = \bigcup_{s,t} B^{s,t}(\mathbb{R}^2).$$

CHAPTER 2

Geometry of mapping spaces

The finite-dimensional $\mathcal{C}^\infty$ manifolds form a category 'Man' – which is a convenient way to organize things. The objects in this category are the manifolds and then the morphisms (or arrows) between two manifolds are the $\mathcal{C}^\infty$ maps so this is a subcategory of 'Set' (sets and maps) or 'Top' (topological spaces and continuous maps). The arrows from X to Y form the mapping space, which is precisely what we are interested in studying. We will work under the assumption that X is a compact manifold with corners and Y is a compact manifold without boundary – it is generally Y that we are most interested in. In fact we need to consider maps with less regularity than $\mathcal{C}^\infty$ so we denote classes of maps from X to Y by $\mathcal{M}_{\text{reg}}(X, Y)$ where 'reg' is a measure of regularity. With the subscript omitted altogether $\mathcal{M}(X, Y)$ is the space of $\mathcal{C}^\infty$, or smooth, maps. Other examples are reg=0 for continuous, reg='s' for Sobolev of order s (almost always $s > \frac{1}{2}\dim X$) and maybe reg=k, α for Schauder classes. These can all be thought of as infinite-dimensional manifolds, Banach, Hilbert of Fréchet depending on the regularity (see for instance [3]). Whilst they *are*, this rather misses the special bundle-like properties that they have.

The most important cases for us are $X = \mathbb{S}$ and $X = [a, b] \subset \mathbb{R}$, giving loop and path spaces respectively. We occasionally need to iterate these spaces, so X might also be a product of circles and intervals. If $Y = G$ is a Lie group then $\mathcal{M}(X, G)$ can be thought of as a 'gauge group' where the group structure comes from pointwise composition in G. Loop groups are particularly important below.

1. Continuous maps

Suppose that Y is equipped with a Riemann metric, so the Riemann distance between points is a metric on Y. For elements $F_1,\ F_2 \in \mathcal{M}_0(X, Y)$ set

$$D(F_1, F_2) = \inf\{d(F_1(x), F_2(X); x \in X\}. \tag{1}$$

This turns $\mathcal{M}_0(X; Y)$ into a complete and separable metric space. To see the separability we can use the fact that any two Riemann metrics on Y induce equivalent metrics on both Y and $\mathcal{M}_0(X, Y)$ for any compact X. Taking an embedding of $Y \longrightarrow \mathbb{R}^N$ into some Euclidean space we may therefore use the induced metric from $\mathbb{R}^N$. This makes

$$\mathcal{M}_0(X, Y) \subset \mathcal{C}^0(X; \mathbb{R}^N) \tag{2}$$

a closed subset. The separability of $\mathcal{C}^0(X, Y)$ therefore implies the separability of $\mathcal{M}_0(X, Y)$. Thus $\mathcal{M}_0(X, Y)$ has the basic topological properties of a manifold.

We are assuming that Y is connected but $\mathcal{M}_0(X, Y)$ can have several components, corresponding to the homotopy classes of continuous maps from X to Y.

Of course if X is contractible, then $\mathcal{M}_0(X,Y)$ is not only connected but is also contractible. For each component of $\mathcal{M}_0(X,Y)$ the bundles over X

$$F^*TY = \bigsqcup_{x\in X} T_{F(x)}Y \longrightarrow X \tag{3}$$

are isomorphic under a continuous map. As we see below $\mathcal{M}_0(X,Y)$ is an infinite dimensional manifold ($\mathcal{C}^\infty$ not just topological) where the local model for each component is

$$\mathcal{C}^0(X; F^*TY) \tag{4}$$

for F in that component.

To see this, take $\epsilon > 0$ to be less the infimum over Y of the injectivity radius of the given Riemann metric and consider the sets

$$\Gamma(F,\epsilon) = \{F' \in \mathcal{M}_0(X,Y); \sup d_Y(F(x), F'((x)) < \epsilon\} \tag{5}$$

consisting of the maps in an 'ϵ-tube' around F.

Proposition 18. *For $\epsilon > 0$ sufficiently small, so that all balls in Y of radius ϵ are geodesically convex, the open sets $\Gamma(F,\epsilon)$, form a good open cover of $\mathcal{M}_0(X,Y)$.*

Proof. For each point $x \in X$ the exponential map at $F(x) \in Y$ gives a diffeomorphism

$$\exp_{F(x)} : T_{F(x)}Y \supset \{|v|_g < \epsilon\} \longrightarrow B(F(x),\epsilon) \subset Y \tag{6}$$

which allows $B(F(x),\epsilon) \ni \exp(v)$ to be radially retracted to $F(x)$ through

$$[0,1] \times B(F(x),\epsilon) \ni \exp_{F(x)}(v) \longrightarrow \exp_{F(x)}((1-t)v) \in B(F(x),\epsilon). \tag{7}$$

The exponential map depends smoothly on the base point so this induces a smooth retraction

$$\Gamma(F,\epsilon) \ni F' \longrightarrow \exp((1-t)v(x)),\ F'(x) = \exp(v(x)). \tag{8}$$

Of course, any finite number of these tubes $\Gamma(F_i,\epsilon)$, $i = 1,\dots,N$, intersect if and only if there is a continuous map $F \in \Gamma(F_i,\epsilon)$ for all i. Then

$$F(x) \in \bigcap_i B(F_i(x),\epsilon) \Longrightarrow \bigcap_i B(F_i(x),\epsilon) = \exp_{F(x)}(Q),\ Q \subset T_{F(x)}Y$$

where Q is an open set containing the origin and the radial line through each of its points. Thus the radial contraction onto F shows that all intersections are contractible. $\square$

If we consider two open tubes which intersect

$$\Gamma(F_1,\epsilon) \cap \Gamma(F_2,\epsilon) \neq \emptyset \tag{9}$$

then

$$D_{12}(x) = \exp^{-1}_{F_1(x)}(B(F_1(x),\epsilon) \cap B(F_2(F_2(x),\epsilon) \subset T_{F_1(x)}Y \tag{10}$$

is open and

$$\exp^{-1}_{F_2(x)} \exp_{F_1(x)} : D_{12} \longrightarrow D_{21} \subset T_{F_2(x)}Y \tag{11}$$

is a diffeomorphism which depends smoothly on the point $x \in X$. These are the transition maps for the coordinate cover of $\mathcal{M}(X,Y)$ by the $\Gamma(F,\epsilon)$. One thing that makes $\mathcal{M}(X,Y)$ very special, compared to an abstractly defined manifold based on some infinite-dimensional Banach, Hilbert or Frèchet space is that these transition maps are formed by a family of smooth finite-dimensional maps.

2. The spaces $\mathcal{M}_s(X,Y)$

We next examine the mapping spaces of finite Sobolev regularity, $\mathcal{M}_s(X,Y)$. Similar considerations apply to other regularities, such as Hölder spaces but we concentrate on the Sobolev case, especially $s=1$, the 'energy' space.

For $s\in\mathbb{R}$ the standard real-valued Sobolev spaces $H^s(X)$ can be defined by localization to the Euclidean case. Thus $H^s(\mathbb{R}^d)\subset\mathcal{S}'(\mathbb{R}^d)$ is the space of Schwartz distributions u with Fourier transform locally integrable and in a weghted L^2 space

$$\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{u}(\xi)|^2<\infty. \tag{1}$$

By Sobolev's embedding theorem, for $s>\frac{1}{2}d$, $H^s(\mathbb{R}^d)$ consists of bounded continuous functions. For all s it is a module over $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ and this allows $H^s(X)$ to be defined for a compact manifold without boundary by use of a partition of unity subordinate to a coordinate cover. In particular these spaces are defined on open sets and form a fine sheaf over X.

In the case of a compact manifold with corners there are two extreme choices for the Sobolev spaces and the space denoted $H^s(X)$ here consists of the 'extendible' Sobolev distributions. These may be defined by embedding $X\longrightarrow\tilde{X}$ into a manifold without boundary of the same dimension and taking

$$H^s(X)=H^s(\tilde{X})\big|_{X\setminus\partial X}. \tag{2}$$

In all cases one can pass to the spaces $H^s(X,V)$, with values in a finite dimensional vector space V by demanding that for each $v\in V'$ the composite $v\circ u\in H^s(X)$. Again by localization this extends to give corresponding spaces of (distributional) sections of any smooth vector bundle over a compact manifold.

To define the corresponding spaces of maps into another compact manifold, Y, without boundary one can localize in coordinate charts of X or alternatively use an embedding

$$Y\longmapsto\mathbb{R}^N \tag{3}$$

as above. Then

$$\mathcal{M}_s(X,Y)=\{u\in H^s(X,\mathbb{R}^N);u:X\longrightarrow Y\},\ s>\frac{1}{2}\dim X. \tag{4}$$

Since these are all continuous maps

$$\mathcal{M}_s(X,Y)\subset\mathcal{C}^0(X,Y),\ s>\frac{1}{2}\dim X\text{ is a closed subspace.} \tag{5}$$

For finite s (1) defines a Hilbert space structure on $H^s(\mathbb{R}^d)$ and by localization $H^s(X,V)$ is also a Hilbert space (or Hilbertable, since it does not have a canonical choice of inner product).

Again by Sobolev's embedding theoren

$$\bigcap_s H^s(X)=\mathcal{C}^\infty(X). \tag{6}$$

The space $\mathcal{C}^\infty(X,V)$ is a Fréchet space, with the countably many norms from the $H^k(X,V)$ or equivalently the $\mathcal{C}^k$ norms. Again, using an embedding (3)

$$\mathcal{M}(X,Y) = \mathcal{C}^\infty(X,Y) \subset \mathcal{C}^\infty(X,\mathbb{R}^N) \text{ is closed.} \tag{7}$$

In all cases the spaces $H^s(X;V)$ are separable and it follows that

$$\mathcal{M}_s(X,Y) \text{ is a separable metrizable space if } \frac{1}{2}\dim X < s \le \infty.$$

Lemma 10. *For $\frac{1}{2}\dim X < s < s' \le \infty$ the inclusions*

$$\mathcal{M}_{s'}(X,Y) \hookrightarrow \mathcal{M}_s(X,Y) \hookrightarrow \mathcal{M}_0(X,Y) \tag{8}$$

have dense ranges.

Remark 2. Such density does not have an analogue in finite dimensions. We will treat the spaces $\mathcal{M}_s(X,Y)$ as being essentially one space, thinking of them as different 'thickenings' of the smooth space $\mathcal{M}(X,Y)$. For instance a function $f:\mathcal{M}(X,Y)$ extends to a continuous function on $\mathcal{M}_s(X,Y)$ if and only if it is continuous on $\mathcal{M}(X,Y)$ with respect to the induced (metric) topology from $\mathcal{M}_s(X,Y)$. The existence of such a continuous represents a regularity condition on the function. We consider related differentiability conditions ('8llitheness') below.

The sentiment in Remark 2 is reinforced by the fact that all the $\mathcal{M}_s(X,Y)$, always for $s > \frac{1}{2}\dim X$, have consistent coordinate patches with the same transition maps. Namely the 'same' tubular domains as in (5).

3. Hardy-Bargmann decomposition

Let $H^{\frac{1}{2}}(\mathbb{T}_+)$ denote the $\frac{1}{2}$ order Sobolev-Hardy space, were the notation pretends it is the Sobolev space on the 'Hardy half-circle'. As a space it is

$$\Pi_+ H^{\frac{1}{2}}(\mathbb{T}). \tag{1}$$

The constants need to be treated separately so we consider the Hilbert norm

$$\|u\|^2_{\frac{1}{2}} = |a_0|^2 + \sum_{k\ge 0} k|a_k|^2. \tag{2}$$

Although as yet unmotivated we consider Hilbert spaces obtained by completion of function spaces on the 'mean-zero' functions on the circle which may be identified with the orthocomplement of the constants in the (necessarily complex-valued) smooth Hardy space:

$$\mathcal{C}^\infty_0(\mathbb{S};\mathbb{R}) = \mathcal{C}^\infty_0(\mathbb{T}_+). \tag{3}$$

Let $z_p = x_p + iy_p$ be the pth Fourier coefficients, representing the functions

$$\frac{1}{\sqrt{2}} x_p \cos(p\theta) + i\frac{1}{\sqrt{2}} y_p \sin(p\theta) \tag{4}$$

We denote the rescaled version of this variable as

$$Z_p = \sqrt{p} z_p. \tag{5}$$

Consider the set of maps

$$\Sigma = \{\sigma : \mathbb{N} \longrightarrow \mathbb{N}^2; \operatorname{supp}(\sigma) \text{ is finite.}\} \tag{6}$$

where the support is the set of $p \in \mathbb{N}$ such that $\sigma(p) \neq (0,0)$. For each $\sigma \in \Sigma$ consider the function on $\mathcal{C}_0^\infty(\mathbb{T}_+)$

$$\gamma_\sigma = \left(\prod_{p\in\operatorname{supp}(\sigma)} \beta_{\sigma(p)}(Z_p, \overline{Z}_p)\right) \exp(-\frac{1}{2}\sum_{k\geq 0}|Z_k|^2). \tag{7}$$

Here the Bargmann basis β_{ij} is defined in §20.

Each such function is in the $\frac{1}{2}$-lithe Schwartz space and we define a Hilbert space completion by identifying the finite span as the finite sequences in $l^2(\Sigma)$. Thus formally an element of this space is identified as an infinite series

$$\mathcal{H}(H_+) = \{\sum_{\sigma\in\Sigma} c_\sigma\gamma_\sigma,\ \sum_{\sigma\in\Sigma}|c_\sigma|^2 < \infty\}. \tag{8}$$

For the moment at least we leave the questions of the convergence to a function on $\mathcal{C}_0^\infty(\mathbb{T}_+)$ unresolved, but by definition it has a dense subspace of such functions.

We can also consider subbspaces corresponding to weighted versions of the l^2 norms. For $s = (s_1, s_2, s_3)] \in [0,\infty)^3$ consider the weight

$$m_s(\sigma)(p) = p^{s_1}(k(\sigma(p)+1)^{s_2}(1+j(\sigma(p))^{s_3} \geq 1 \tag{9}$$

defined by $\sigma(p) = (k(\sigma(p)), j(\sigma(p)))$. Then set

$$\mathcal{H}^s(H_+) = \{\sum_{\sigma\in\Sigma} c_\sigma\gamma_\sigma,\ \sum_{p,\sigma\in\Sigma} m_s(\sigma)^2|c_\sigma|^2 < \infty\}. \tag{10}$$

Again these are Hilbert spaces in which the finite sums are dense and clearly

$$\mathcal{H}^s(H_+) \subset \mathcal{H}(H_+). \tag{11}$$

LEMMA 11. *Each of the rescaled operators*

$$\not{b}_j^+ = i(\partial_{\overline{Z}_j} + Z_j),\ \not{b}_j^- = i(\partial_{Z_j} - \overline{Z}_j) \tag{12}$$

extend to bounded operators

$$l^r\not{b}_l^\pm : \mathcal{H}^s(H_+) \longrightarrow \mathcal{H}^{s'}(H_+),\ s' = (s_1 - r, s_2, s_3 - \frac{1}{2}),\quad \textit{if } s_1 \geq r,\ s_3 \geq \frac{1}{2}. \tag{13}$$

These operators are by no means closed by the domains of their closures are easily computed. We introduce these spaces for the treatment of the Dirac-Ramond operator on the torus in Chapter 5 below.

CHAPTER 3

Trangression, Regression and Fusion

1. Orientation of the loop space

We examine the transgression of a spin structure on an oriented manifold to the loop space.

LEMMA 12. *For $n \geq 3$ the loop group $\mathcal{L}\,\mathrm{Spin}(n)$ is a double cover of the identity component $\mathcal{L}\,\mathrm{SO}(n)$.*

The regularity of the loops is not an issue here as long as it is the same for both groups.

PROOF. A continuous path in $\mathrm{SO}(n)$ has two lifts to continuous paths in $\mathrm{Spin}(n)$ just determined by the choice between the two preimages of the initial point. It follows that a loop in $\mathrm{SO}(n)$ has two lifts to paths in $\mathrm{Spin}(n)$ and these are either loops, i.e. the end-points are the same as the initial points, or the end-points are the opposite lifts of the intial points and they are not loops. In the former case, since $\mathrm{Spin}(n)$ is simply-connected, a lifted loop is contractible and hence the image loop is also contractible in $\mathcal{L}\,\mathrm{SO}(n)$ so lies in the component of the identity of $\mathcal{L}\,\mathrm{SO}(n)$. This shows that the projection from $\mathcal{L}\,\mathrm{Spin}(n)$ to $\mathcal{L}\,\mathrm{SO}(n)$ is surjective and hence a 2-1 cover. □

As shown in *** above, the loop space of a principal bundle for a connected Lie group is a principal bundle over the loop space of the base with structure bundle the loop group acting pointwise. For an oriented manifold, M, with its oriented orthonormal frame bundle, $F_{\mathrm{SO}}M$, corresponding to a choice of Riemann metric, this gives the principal bundle

$$\begin{tikzcd} \mathcal{L}\,\mathrm{SO}(n) \arrow[r, dash] & \mathcal{L}F_{\mathrm{SO}}M \arrow[d] \\ & \mathcal{L}M. \end{tikzcd} \tag{1}$$

Since $\mathcal{L}\,\mathrm{SO}(n)$, $n \geq 3$, has two components, each fibre of (1) has two components.

By close analogy with the finite-dimensional case we define the notion of an orientation of the loop space of a manifold:

DEFINITION 9. An orientation on the loop space of a connected, oriented manifold, is a continuous map

$$o : \mathcal{L}F_{\mathrm{SO}}M \longrightarrow \mathbb{Z}_2 \tag{2}$$

which takes both signs on each fibre of (1).

Of course this is not the same as the notion of an orientation for a finite-dimensional manifold, but since the latter does not make sense in the infinite-dimensional context so there should be no confusion. However there are quite substantial differences, apart from the fact that the dimension is now infinite. In particular $\mathcal{L}M$ is in general far from connected, its set of components being identified with $\pi_1(M)$. So the orientation in (2) is really defined component-by-component and you might, correctly, think that there is something inadequate here. We will get to that.

PROPOSITION 19 (Atiyah/Witten). *A spin structure on a connected oriented manifold induces an orientation on the loop space.*

PROOF. As discussed above, a spin structure on a connected manifold (with a Riemann structure but the choice here does not matter) is a double cover F_{Spin} of $F_{\mathrm{SO}}M$ giving a commutative diagram

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{Spin}} \arrow[d] \\ \mathrm{SO}(n) \arrow[r, dash] & F_{\mathrm{SO}} \arrow[d] \\ & M. \end{tikzcd} \tag{3}$$

Taking loops everywhere we get a corresponding diagramme (because the groups are connected)

$$\begin{tikzcd} \mathcal{L}\,\mathrm{Spin}(n) \arrow[r, dash] \arrow[d] & \mathcal{L}F_{\mathrm{Spin}} \arrow[d] \\ \mathcal{L}\,\mathrm{SO}(n) \arrow[r, dash] & \mathcal{L}F_{\mathrm{SO}} \arrow[d] \\ & \mathcal{L}M. \end{tikzcd} \tag{4}$$

We have seen in Lemma 12 above that $\mathcal{L}\,\mathrm{Spin}(n)$ is a double cover (coming from the initial point) of the identity component of $\mathcal{L}\,\mathrm{SO}(n)$. So the image of the looped spin bundle in $\mathcal{L}F_{\mathrm{SO}}$ is one of the components in each fibre over $\mathcal{L}M$. This determines an orientation of $\mathcal{L}M$ in the sense of Definition 9, taking o in (2) to be 1 on the component in the image and -1 on the other. □

Stated succinctly, the orientation of the loop space determined by a spin structure is

$$o(\lambda) = \pm 1 \text{ as } \lambda \in \mathcal{L}F_{\mathrm{SO}} \text{ is or is not of the form } \pi(\lambda') \text{ for } \lambda' \in \mathcal{L}F_{\mathrm{Spin}}. \tag{5}$$

It is clear from the fact that $\mathcal{L}M$ will have many orientations when $\pi_1(M)$ is large that in general an orientation on $\mathcal{L}M$ does not come from a spin structure on M. As one might guess, the orientation of the loop space coming from a spin structure has an additional property. To see this we proceed to discuss *fusion*.

2. Fusion

We can break any loop, at the midpoint, into two paths. The (continuous) path space consists of maps

$$\mathcal{P}M = \{p : [0,\pi] \longrightarrow M;\ \text{continuous}\}.$$

Then if $\gamma : \mathbb{S} \longrightarrow M$ is a loop and we write $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$

$$p_1(s) = \gamma(s),\ p_2(s) = \gamma(2\pi - s),\ s \in [0,\pi] \tag{1}$$

are both paths which clearly determine γ. We have reversed the second path so that the two paths have the same initial and end points

$$p_1(s) = p_2(s),\ s = 0,\ \pi.$$

Conversely if p_1 and p_2 are two continuous paths with this property then there is a continuous loop defined so that (1) holds.

We can write this in a more formal way. There are two maps from the path space to M (well there are more ..) corresponding to the initial and terminal points

$$e : \mathcal{P}M \ni p \longmapsto e(p) = (p(0), p(\pi)) \in M^2. \tag{2}$$

If M is connected, which we are assuming, then this map is surjective and is, in an appropriate sense, a fibration. In any case we can define the fibre product of two copies of the path space with respect to this map

$$\mathcal{P}M \times_e \mathcal{P}M = \{(p_1,p_2); e(p_1) = e(p_2)\}. \tag{3}$$

What we have just observed is that, for continuous paths at least, there is a natural 'join' isomorphism

$$J : \mathcal{P}M \times_e \mathcal{P}M \longrightarrow \mathcal{L}M,\ J(p_1,p_2) = l,\ l(s) = \begin{cases} p_1(s),\ s \in [0,\pi] \\ p_2(2\pi - s),\ s \in [\pi, 2\pi]. \end{cases} \tag{4}$$

We can pass to higher versions of the fibre product and shorten the notation to

$$\mathcal{P}^{[k]}M = \{(p_1,\dots,p_k) \in (\mathcal{P}M)^k; e(p_1) = e(p_2) = \cdots = e(p_k)\}. \tag{5}$$

Of course we can do this for any map between two sets! These are examples of *simplicial spaces.*

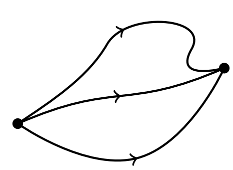

FIGURE 1. Fusion of three paths to three loops

For the moment just observe that there are maps for each $i, l \in \{1,2,3\}$

$$J_{il} : \mathcal{P}^{[3]}M \longrightarrow \mathcal{L}M,\ J_{il}(p_1,p_2,p_3) = J(p_i,p_l). \tag{6}$$

If $i = l$ this gives an 'out-and-back' path. We are mainly interested in the cases where $i \neq l$.

DEFINITION 10. A function $f : \mathcal{L}Q \longrightarrow G$ from a loop space into a group is *fusion* if

$$J_{13}^* f = J_{12}^* f \cdot J_{23}^* f \text{ on } \mathcal{P}^{[3]}Q. \tag{7}$$

3. Loop-orientations

DEFINITION 11. An orientation $o : \mathcal{L}F_{\mathrm{SO}} \longrightarrow \mathbb{Z}_2$ on the loop space of an oriented Riemann manifold is called a loop-orientation if o satisfies the fusion condition (7) with $G = \mathbb{Z}_2$.

THEOREM 8 (Stolz-Teichner). *There is a 1-1 corresponence between spin structures (in dimension $n \geq 3$) on an oriented manifold and fusion orientations of the loop space.*

PROOF. First we need to check the already intimated refinement of Proposition 19, namely that the orientation on the loop space coming from a spin structure on the manifold satisfies the fusion condition, (7), for $G = \mathbb{Z}_2$.

Consider the lifting of paths, from F_{SO} to F_{Spin}. This is a double cover and it follows that any path in F_{SO} has a unique lift to F_{Spin} once we choose as initial point one of the two preimages of the initial point in F_{SO}. Thus

$$\mathcal{P}F_{\mathrm{Spin}} \longrightarrow \mathcal{P}F_{\mathrm{SO}}$$

is again a double cover.

Now an element of $\mathcal{P}^{[2]}F_{\mathrm{SO}}$ is the join of two paths in F_{SO} with the same initial and terminal points. The indivial paths can be lifted to elements of $\mathcal{P}F_{\mathrm{Spin}}$ with the same intial point, but they may not have the same terminal point. In fact we can see from the discussion above that they have the same terminal point if and only if the join of the paths to a loop in $\mathcal{L}F_{\mathrm{SO}}$ has a lift to a loop in F_{Spin}. So in the two cases in (5) the loop in $\mathcal{L}F_{\mathrm{SO}}$ is the image of a loop in F_{Spin}, and has loop-orientation 1 or it doesn't have such a lift and has loop-orientation -1.

Extending this, if we take a triple in $\mathcal{P}^{[3]}F_{\mathrm{SO}}$ then each of the three constituent paths has a lift to a path, all with the same initial point, in F_{Spin} and there are three possibilities, either all three have the same end-point or two are the same and one is different. This means that of the three loops in $\mathcal{L}F_{\mathrm{SO}}$ appearing in (7) either all have orientation $+1$ or two have orientation -1 and one has orientation $+1$. In all cases the multiplicative condition (7) holds, so the orientation induced by a spin structure is fusion, i.e. is a loop-orientation.

So, this gives the 'transgression' of the spin structure to a loop-orientation. We need to see that 'regression' is possible and that a loop-orientation defines a spin structure and that these operations are inverses of each other.

First think about recovering F_{SO} form $\mathcal{P}F_{\mathrm{SO}}$. We can do this by choosing a point arbitrarily in F_{SO} and considering the pointed path space $\mathcal{P}_{\#}F_{\mathrm{SO}}$, so consisting of all the paths with this initial point. Then the 'far' end-point map is still surjective and we can formally recover

$$F_{\mathrm{SO}} = \mathcal{P}_{\#}F_{\mathrm{SO}}/\simeq \tag{1}$$

where the equivalence relation is $p \simeq q$ iff they have the same endpoint value at π.

Now consider the finer relation

$$p \simeq_o q \text{ in } \mathcal{P}_{\#}F_{\mathrm{SO}} \Longleftrightarrow \\ p \text{ and } q \text{ have the same terminal point and } o(J(p,q)) = 1 \tag{2}$$

using a given orientation of the loops space. The main observation is that this is an equivalence relation if and only if o satisfies the fusion condition.

To check that $p \simeq_o p$ for all $p \in \mathcal{P}_\# F_{\mathrm{SO}}$ take the triple (p,p,p) in $\mathcal{P}^{[3]} F_{\mathrm{SO}}$ for any path. All three loops in (7) are then the same, so $o(J(p,p)) = 1$. Similarly for any two paths with the same endpoint in $\mathcal{P}^\# F_{\mathrm{SO}}$, $o(J(p,q)) = o(J(q,p))$ so $p \simeq_o q$ implies that $q \simeq_o p$. To prove transitivity take three paths $(p_1,p_2,p_3) \in \mathcal{P}^{[3]}_\# F_{\mathrm{SO}}$, so all having the same endpoint. Then (7) tells us

$$o(J(p_1,p_2))o(J(p_2,p_3)) = o(J(p_1,p_3)) \tag{3}$$

so if the first two are $+1$ so is the third and we see

$$p_1 \simeq_o p_2,\ p_2 \simeq_o p_3 \Longrightarrow p_1 \simeq_o p_3. \tag{4}$$

Thus, provided o is fusion, the quotient

$$F_{\mathrm{Spin}} = \mathcal{L}_\# F_{\mathrm{SO}} / \simeq_o \tag{5}$$

is well-defined.

From the preceding reconstruction of F_{SO} from $\mathcal{P}_\# f_{\mathrm{SO}}$ there is clearly a map

$$F_{\mathrm{Spin}} \longrightarrow F_{\mathrm{SO}} \tag{6}$$

since the equivalence condition for the latter is stronger. That this a double cover of F_{SO} follows from the fact that o takes both signs.

The action of $\mathrm{Spin}(n)$ on F_{Spin} defined by (5) follows from the action of the pointed path space

$$\mathcal{P}_0 \,\mathrm{SO}(n) = \{p : [0,\pi] \longrightarrow \mathrm{SO}(n); p(0) = \mathrm{Id}\}$$

on $\mathcal{P}_\# F_{\mathrm{SO}}$. Given paths p, $q \in \mathcal{P}_\# F_{\mathrm{SO}}$ with $p \simeq q$ and $\lambda \in \mathcal{P}_0 \,\mathrm{SO}(n)$, $\lambda p \simeq \lambda q$ (in the sense of (1)) since they have the same end-point and the equivalence class of λp only depends on the end-point, in $\mathrm{SO}(n)$, of λ. This recovers the action of $\mathrm{SO}(n)$ on F_{SO}. For the more refined equivalence $\simeq_o$, and two paths $\lambda_i \in \mathcal{P}_0 \,\mathrm{SO}(n)$, with the same end-point, $\lambda_1 p \simeq_o \lambda_2 p$ if and only if $o(J(\lambda_1 p, \lambda_2 p))) = 1$. This requires $J(\lambda_1,\lambda_2)$ to be in the identity component, i.e. to be contractible, so the end-points of the lifts of λ_i to paths in $\mathcal{P}_0 \,\mathrm{Spin}(n)$ are the same and this gives the action of $\mathrm{Spin}(n)$ on F_{Spin} in (5). Local triviality of F_{Spin} follows from that of F_{SO} so this is indeed a spin structure on M.

Thus, we have constructed a map in each direction. That these are inverses of each other follows readily. If a loop orientation is constructed from a spin bundle the the regression construction (5) reduces to the analogue of (1) to reconstruct the spin bundle. Similarly if a spin principal bundle is constructed from a loop-orientation then it induces that orientation on $\mathcal{L} F_{\mathrm{SO}}$. □

The argument here is relatively straightforward, because $\mathbb{Z}_2$ is discrete. For example, being locally constant, o is invariant under the action of a connected group such as $\mathrm{Dff}^+(\mathbb{S})$. We need to work harder in other cases below!

4. Line bundles and holonomy

[**Holonomy**]

As a second example of a transgression/regression pair we consider a principal circle bundle, hereafter just called a circle bundle. This is really the same thing as a (complex) line bundle with Hermitian structure where the circle bundle consists of the unit vectors. The line bundle can be recovered from the circle bundle as the associated bundle to the representation of the circle as U(1) acting on $\mathbb{C}$; since this action is unitary it induces an Hermitian structure on the line bundle. So we can freely go back and forth between these two pictures. In this identification, a connection on the circle bundle induces and is induced by a unique Hermitian, i.e. unitary, connection on the line bundle.

In this section the trangression of a circle bundle to the holonomy of a connection is discussed extensively as an object lesson in characterizing objects over a manifold in terms of objects over its loop space.

Let $\pi : L \longrightarrow M$ be such a circle bundle, with connection as a principal $\mathrm{U}(1) = \mathbb{S} = \mathrm{SO}(2)$ bundle. The fact that we can lift paths in M to be 'horizontal' paths in L allows us to consider the notion of *holonomy*.

Recall that a connection on L is a choice of a smooth subbundle $H \subset TL$ such that at each point $q \in L$

$$\pi_* : H(q) \longrightarrow T_{\pi(q)}M \text{ is an isomorphism}$$

and which is equivariant for the action of U(1) on L :

$$z_* : H(q) \longrightarrow H(zq) \ \forall\ q \in L,\ z \in \mathrm{U}(1). \tag{1}$$

In this case the connection form on L is really a form on the total space of L if we identify the Lie algebra of U(1) with $\mathbb{R}$ (it is actually more usual to identify the Lie algebra of U(1) with $i\mathbb{R}$). Namely there is a unique form $\alpha \in \mathcal{C}^\infty(L;\Lambda^1)$ determined by the connection through

$$\alpha(q)\big|_{H(q)} = 0,\ \alpha(q)(\partial_\theta) = 1. \tag{2}$$

Here the vector field ∂_θ is the tangent at q to the curve through q, $\theta \longrightarrow \exp^{i\theta} q$; so the circle is identified with $\mathbb{R}/2\pi\mathbb{Z}$, which is why the Lie algebra is $\mathbb{R}$.

The invariance of the connection spaces $H(q)$ under the circle action becomes

$$z^*\alpha = \alpha,\ z \in \mathrm{U}(1) \text{ acting on } L. \tag{3}$$

Conversely α determines H as its null space.

Lemma 13. *Two connections on a circle bundle S differ by the pull-back of a smooth 1-form on the base and conversely.*

Proof. If $\beta \in \mathcal{C}^\infty(S : \Lambda^1)$ is a second connection form on S then it is determined by its restriction to the horizontal subspaces of the first connection α, the value in the 'vertical' direction being fixed by the second part of (2). This means that at each point $q \in S$ β determines, and is determined by a 1-form $\gamma \in \Lambda^1_{\pi q}M$ by $\beta_q\big|_{H(q)} = \pi^*\gamma$. The equivariance property (3) means that γ only depends on the base point. Smoothness of γ follows and so

$$\beta = \alpha + \pi^*\gamma. \tag{4}$$

Conversely this is indeed a connection form on S if α is. □

Remark 3. In particular it follows that the space of connections is contractible to any one connection.

Given a smooth (but much lower regularity works too) path $\psi \in \mathcal{P}M$ and a point $q \in S_{l(0)}$ in the fibre above $\psi(0) \in M$ there is a unique smooth path $\tilde\psi_q \in \mathcal{P}S$ which starts at q, $\tilde\psi_q(0) = q$ and is horizontal in the sense that for every s,

$$\tilde\psi_q'(s) \in H(\tilde\psi(s)),\ \pi_*(\tilde\psi_q'(s)) = \psi_q'(s)$$

The endpoint $\tilde{\psi}_q(\pi) \in S_{l(\pi)}$ is therefore determined, using the given connection, by q and ψ. The condition of being horizontal is equivalent to

$$\tilde{\psi}^* \alpha = 0. \tag{5}$$

From this and (3) it follows that $z\tilde{\psi}$ is the unique horizontal lift of ψ starting at zq for any $z \in \mathrm{U}(1)$. Thus in fact parallel transport along a given curve $\psi \in \mathcal{P}M$ gives an identification

$$E_\psi : S_{\psi(0)} \longrightarrow S_{\psi(\pi)} \text{ commuting with the action of } \mathrm{U}(1). \tag{6}$$

We return to this identification below but first consider the case $l \in \mathcal{L}M$ is a loop. A horizontal lift $\tilde{l}$ of l will only be a path in S but will have endpoint $\tilde{l}(\pi) \in S_{l(0)=l(\pi)}$ in the fibre over the initial point.

Definition 12. The identification (6) gives an element of U(1) which is the *holonomy* of S along the loop

$$\eta(l) = z \in \mathrm{U}(1),\ \tilde{l}(\pi) = z\tilde{l}(0). \tag{7}$$

Note that we cannot expect to completely recover the connection form from the holonomy since if we add to one connection the pull-back from the base of a 1-form with periods in $2\pi\mathbb{Z}$ then the holonomy is unchanged but this is the only issue. This is connected to differential cohomology.

Lemma 14. *The holonomy of any connection on a circle bundle determines the connection up to addition of a closed form on the base with periods in* $2\pi\mathbb{Z}$.

Proof. This is only say that the two connections have the same holonomy if and only if they differ by such a 1-form. We know that two connections differ by the pull-back of a 1-form α from the base and then the two holonomies are related by

$$\eta_2(\gamma) = \eta_1(\gamma) \exp(i \int_0^{[} 2\pi\gamma^*\alpha). \tag{8}$$

So the holonomies are the same if and only if

$$\int_0^{[} 2\pi\gamma^*\alpha \in 2\pi\mathbb{Z}\ \forall\ \gamma \in \mathcal{L}M. \tag{9}$$

By stokes theorem this implies that α is closed. □

5. Fusive property

Proposition 20. *The holonomy of a circle bundle with connection is a fusion function*

$$\eta : \mathcal{L}M \longrightarrow \mathrm{U}(1) \tag{1}$$

and is invariant under the reparameterization action of $\mathrm{Dff}^+(\mathbb{S})$.

Definition 13. A function on the loop space is said to be *fusive* if it satisfies the fusion condition of Definition 10 and is also invariant under the action of the reparameterization group $\mathrm{Dff}^+(\mathbb{S})$.

Note that in the introduction the loop-orientation is described as fusive, rather than just having the fusion property. The loop-orientation takes values in the discrete group $\mathbb{Z}_2$ and it follows that it is invariant under the (continuous) action of the connected group $\mathrm{Dff}^+(\mathbb{S})$, so it is fusive as a function on $\mathcal{L}F_{\mathrm{SO}}$.

To show this it is convenient to consider non-horizontal lifts of a given path $\psi \in \mathcal{P}M$. Fixing a point $q \in S_{\psi(0)}$ let $\tilde{\psi}$ be the horizontal lift of ψ as discussed above. Any other lift of ψ therefore takes the form

$$\mu(s) = \exp(i\theta(s))\tilde{\psi}(s),\ \theta : [0,\pi] \longrightarrow \mathbb{R} \text{ continuous.} \tag{2}$$

Here $\theta(s)$ is not unique because of the ambiguity of the initial choice with $\mu(0) = \exp(i\theta(0))q = \exp(i\theta(0)0\tilde{\psi}(0)$ but is unique up to addition of an element of $2\pi\mathbb{Z}$; it also follows that $\theta(s)$ is as smooth as μ. Since $\tilde{\psi}$ is horizontal we see that

$$\mu^*\alpha = \theta'(s)ds. \tag{3}$$

Lemma 15. *For a general lift $\mu \in \mathcal{P}L$ of $\psi \in \mathcal{P}M$ the parallel transport identification* (6) *is given by*

$$E_\psi : S_{\psi(0)} \ni \mu(0) \longrightarrow \exp(-i\int_0^\pi \psi^*\alpha ds)\psi(\pi). \tag{4}$$

Proof of Proposition 20. Using Lemma 15 we can compute the holonomy of the circle bundle with connection along any lift of $l \in \mathcal{L}M$ to a loop $\tau \in \mathcal{L}S$ with $\pi \circ \tau = l$. Namely

$$\eta(l) = \exp(i\int_0^{2\pi} \tau^*\alpha(s)ds) \in \mathrm{U}(1). \tag{5}$$

So now consider a triple of paths $(\psi_1, \psi_2, \psi_3) \in \mathcal{P}^{[3]}M$. The fusion condition on η requires that

$$\eta(J(\psi_1,\psi_2))\eta(J(\psi_2,\psi_3)) = \eta(J(\psi_1,\psi_3)). \tag{6}$$

For a loop obtained by joining two paths $(\psi_i, \psi_j) \in \mathcal{P}^{[2]}M$ we see from (5) that, by changing variable,

$$\begin{aligned} \eta(J(\psi_i,\psi_j)) &= \exp(-i\int_0^\pi \psi_i^*\alpha(s)ds)\exp(-i\int_\pi^0 \psi_j^*\alpha(s)ds) \\ &= \exp(-i\int_0^\pi \psi_i^*\alpha(s)ds)\exp(i\int_0^\pi \psi_j^*\alpha(s)ds). \end{aligned} \tag{7}$$

This gives (6). $\square$

6. Weak regularity

We also need to think a little about the regularity of η leading into the general question of the meaning of regularity for functions on loop/path spaces. For such an infinite-dimensional space there are several notions of regularity. We start with what is sometimes called 'diffeological' regularity – which I will call 'weak' – regularity because we do not need to get into the whole regime of diffeological spaces. Because these spaces are defined in terms of maps we can easily agree on what smoothness, and other regularity, means for a map from a (finite-dimensional) manifold Z *into* the loop space.

Definition 14. For a map from a manifold Z into a path or loop space $\mathcal{P}M$ or $\mathcal{L}M$

$$(1)\qquad \begin{gathered} F : Z \longrightarrow \mathcal{P}M \text{ is smooth iff } \exists\ \tilde F \in \mathcal{C}^\infty(Z \times [0,\pi]; M),\ \tilde F(\zeta, s) = F(\zeta)(s) \\ F : Z \longrightarrow \mathcal{L}M \text{ is smooth iff } \exists\ \tilde F \in \mathcal{C}^\infty(Z \times \mathbb{S}; M),\ \tilde F(\zeta, z) = F(\zeta)(z). \end{gathered}$$

A map $\eta : \mathcal{L}M \longrightarrow X$ with values in a finite-dimensional manifold is weakly smooth if all composites $\eta \circ F$ for smooth maps $F : Z \longrightarrow \mathcal{L}M$ are smooth.

Lemma 16. *The holonomy of a smooth connection on a principal circle bundle is a weakly smooth function on* $\mathcal{L}M$.

Proof. □

Suppose

$$(2)\qquad f : \mathcal{L}M \longrightarrow S$$

is a weakly smooth map from the loop space to a smooth manifold S.

Lemma 17. *For a weakly smooth map* (2) *the derivative f' at each point $\gamma \in \mathcal{L}M$ is well-defined as a linear map*

$$(3)\qquad T_\gamma \mathcal{L}M = \mathcal{C}^\infty(\mathbb{S}; \gamma^* TM) \longrightarrow T_{f(\gamma)} S.$$

Proof. □

7. Regression from pointed holonomy

Now we observe that such 'holonomy functions' on the loop space give a transgression/regression pair with circle bundles with connection, up to an appropriate notion of equivalence, on the manifold.

Theorem 9. *Every homotopy class of weakly smooth fusive functions $\eta : \mathcal{L}M \longrightarrow \mathrm{U}(1)$ determines a circle bundle on the manifold up to smooth isomorphism with holonomy in this homotopy class.*

So the question remains, is *every* holonomy function the holonomy of a circle bundle? Probably we need to add some further regularity conditions.

Proof. We start by considering how to reconstruct a circle bundle from the holonomy of a connection on it.

The connection gives trivializations of the circle bundle, L, by parallel transport. For simplicity, choose a base point $m_0 \in M$. We know that a neighbourhood, B_m, of any point $m \in M$ can be 'parametrized' by a smooth family of paths starting at m_0

$$(1)\qquad \chi_m : B_m \longrightarrow \mathcal{P}M \text{ smooth and such that} \qquad \chi(m')(0) = m_0,\ \chi(m')(\pi) = m'\ \forall\ m' \in B_m.$$

Parallel transport along these paths gives a smooth identificaton

$$(2)\qquad E_{\chi(m')} : L_{m_0} \longrightarrow L(m'),\ m' \in B_m.$$

This is a trivialization of L over B_m as soon as we turn L_{m_0} into a circle by choosing $q_0 \in L_{m_0}$.

Suppose B_{m_1} and B_{m_2} are two such balls with their families of curves χ_{m_i}, $i = 1, 2$, all starting from the same base point $m_0 \in M$. Then the transition map over the intersection $B_{12} = B_{m_1} \cap B_{m_2}$, assuming it is non-empty, is

$$S\big|_{B_{12}} \xrightarrow{E} B_{12} \times \mathbb{S} \overset{E_{\chi_1} \circ E_{\chi_2}}{\longrightarrow} B_{12} \times \mathbb{S} \xleftarrow{E} S\big|_{B_{12}} \tag{3}$$

From (7) we see that this means the transition maps for these trivializations of L are given by the holonomy around the loops formed by the joins of the two sets of paths to points in B_{12}.

We can think about this a little more 'globally'. The end-point map

$$e_\pi : \mathcal{P}_\# M \longrightarrow M \tag{4}$$

can be used to pull back the circle bundle S to $\mathcal{P}_\# M$. Then parallel transport along paths, as discussed above, gives a global trivialization

$$e_\pi^* S \simeq \mathcal{P}_\# M \times S_{m_0}. \tag{5}$$

Of course it should not be surprising that there is such a trivialization since $\mathcal{P}_\# M$ is contractible, by path-shortening, so any bundle over it will be trivial.

Then the recovery of s from the holonomy η takes the form

$$(\mathcal{P}_\# M \times S_{m_0} / \simeq_\eta) \simeq S \text{ over } M. \tag{6}$$

where the equivalence relation is

$$\begin{gathered}\mathcal{P}_\# M \times S_{m_0} \ni (\gamma, z) \simeq_\eta (\gamma', z') \in \mathcal{P}_\# M \times S_{m_0} \\ \text{iff } e_\pi(\gamma) = e_\pi \gamma' \text{ and } z' = \eta(J(\gamma, \gamma'))z.\end{gathered} \tag{7}$$

This is an equivalence relation precisely because of the fusion conditions on η which imply that, for paths with the same endpoint,

$$\eta(J(\gamma', \gamma))\eta(J(\gamma', \gamma)) = 1, \;\; \eta(J(\gamma_1, \gamma_3)) = \eta(J(\gamma_1, \gamma_2))\eta(J(\gamma_2, \gamma_3)).$$

Now, if we start with a general holonomy function η, without knowing that it is the holonomy of a circle bundle, we can construct a circle bundle over M in this way. Namely

$$S_\eta = (\mathcal{P}_\# M \times \mathbb{S}) / \simeq_\mu \xrightarrow{e_\pi} M \tag{8}$$

is a (locally trivial) circle bundle, smooth if η is weakly smooth. This is our 'regression' construction.

Having constructed the bundle (8) we can choose a connection α on it. As discussed above, parallel transport on S_η using this connection (which has nothing to do with η) gives a trivialization

$$E_\alpha : e_\pi^* S_\eta \simeq \mathcal{P}_\# M \times \mathbb{S} \tag{9}$$

since in this case the fibre at m_0 is identified with $\mathbb{S}$. We can compare this with the trivialization that is in the definition (8) so we have a function

$$\lambda : \mathcal{P}_\# \longrightarrow \mathbb{S} \tag{10}$$

which reduces the trivialization (9) to the defining one.

However, since $\mathcal{P}_\# M$ is contractible, λ has a continouous logarithm, which will be a smooth as E_α,

$$\lambda(\gamma) = \exp(2\pi i \beta(\gamma)), \;\; \beta : \mathcal{P}_\# M \longrightarrow \mathbb{R}. \tag{11}$$

Since this relates the two trivializations we see that the original holonomy function η and the holonomy η' of the connection chosen on L_η are related by

$$\eta'(J(\gamma,\gamma') = \eta'((J(\gamma,\gamma'))\exp(2\pi i\beta(\gamma))\exp(-2\pi i\beta(\gamma')),\ e_\pi\gamma = e_\pi\gamma'. \tag{12}$$

This shows that η restricted to loops with initial point m_0 is homotopy equivalent, to the holonomy of a connection – just replacing β by $t\beta$.

Except for the restriction to pointed loops this is the claim of the Theorem.

8. Regression to a simplicial bundle

To discuss general paths and loops we start again from a circle bundle S over M with connection and consider the circle

$$S^{-1} \boxtimes S \text{ over } M^2. \tag{1}$$

The tensor product makes sense here for abelian groups or we can think of the associated Hermitian line bundle L_S and consider the circle bundle $L_S^{-1} \otimes L_S$ over M^2. This has an induced Hermitian structure.

We can lift this bundle to the full path space under the double end-point map

$$e : \mathcal{P}M \longrightarrow M^2,\ e^*(S^{-1} \boxtimes S) \longrightarrow \mathcal{P}M. \tag{2}$$

The path space is contractible to M by path-shortening to the midpoint and since $S^{-1} \boxtimes S$ is trivial over the diagonal the pull-back should again be trivial.

Indeed parallel transport for the connection gives an identification

$$E_\lambda : S_{\lambda(0)} \longrightarrow S_{\lambda(\pi)} \tag{3}$$

which is the same thing as an element of $S^{-1}_{\lambda(0)} \otimes S_{\lambda(\pi)}$ and so a trivialization

$$e^*(S^{-1} \boxtimes S) \simeq \mathcal{P}M \times \mathrm{U}(1). \tag{4}$$

As before we can recover the bundle from the holonomy via the equivalence relation on $\mathcal{P}M \times \mathrm{U}(1)$

$$(\lambda, z) \simeq_\eta (\lambda', z') \Longleftrightarrow e(\lambda) = e(\lambda'),\ z' = \eta(J(\lambda,\lambda'))z. \tag{5}$$

Proceeding much as before, starting from a weakly smooth function $\zeta : \mathcal{L}M \longrightarrow \mathrm{U}(1)$ which satisfies the fusion condition, we can construct a circle bundle Σ over M^2 using the same relation, (5), with η replaced by ζ. It is locally trivial and smooth. The claim is that this is a 'simplicial bundle'. That is, for the pull this back to M^3 then there is a bundle isomorphism

$$\tau : \pi_{12}^*\Sigma \otimes \pi_{23}^*\Sigma \equiv \pi_{13}^*\Sigma. \tag{6}$$

We want this to be 'simplicial' in the sense that we saw for gerbes, that over M^4

9. Figure of eight

To see where the isomorphism (6) comes from we again consider a configuration of loops. We can decompose a loop into the join of to paths which leads to the map to the initial and midpoints

$$(i, m) : \mathcal{L}M \ni l \longmapsto (l(0, l(\pi)) \in M^2. \tag{1}$$

Two loops l_1, $l_2$form a figure-of-eight if

$$m(l_1) = i(l_2). \tag{2}$$

That is the pair is an element of the fibre product

$$\mathcal{L}M \times_{m=i} \mathcal{L}M. \tag{3}$$

By reparameterizing the paths involved we can map this back into the loop space

$$\mathcal{L}M \times_{m=i} \mathcal{L}M \ni (l_1, l_2) = l \in \mathcal{L}M,\ l_i = J(\lambda_i, \lambda_i'),\ i = 1, 2, \tag{4}$$

$$l(s) = \begin{cases} \lambda_1(2s) & s \in [0, \pi/2] \\ \lambda_2(2s - \pi) & s \in [\pi/2, \pi] \\ \lambda_2'(-2s + 3\pi) & s \in [\pi, 3\pi/2] \\ \lambda_1'(-2s + 4\pi) & s \in [3\pi/2, 2\pi]. \end{cases}$$

Lemma 18. *If $\zeta : \mathcal{L}M \longrightarrow \mathrm{U}(1)$ is fusive, weakly smooth and invariant under the action of $\mathrm{Dff}^+(\mathbb{S})$ and (l_1, l_2) is a figure-of-eight configuration with combination l as in (4) then*

$$\zeta(l) = \zeta(l_1)\zeta(l_2). \tag{5}$$

Proof. □

□

10. Transgression of Čech cohomology

From the discussion in §9 we know that circle bundles, up to isomorphism, are classified by the cohomology group $\check{H}^2(M; \mathbb{Z})$. On the other hand, Theorem 9 identifies these equivalence classes with homotopy classes of fusive functions on the loop space with values in $\mathbb{S}$. Recall from Remark 1 that $\mathbb{S}$ is a classifying space for 1-dimensional integral cohomology. So, without the fusion conditions this would be $\check{H}^1(\mathcal{L}M; \mathbb{Z})$. The usual topological meaning of 'transgression' is exactly this sort of map on cohomology; it is defined using the push-forward map for a bundle with oriented fibres from the evaluation map

$$\Theta : \mathcal{L}M \times \mathbb{S} \ni (\gamma, \theta) \longmapsto \gamma(\theta) \in M$$

$$\begin{tikzcd} H^k(\mathcal{L}M \times \mathbb{S}; \mathbb{Z}) \arrow[d, "(\pi_1)_!"'] & H^k(M; \mathbb{Z}) \arrow[l, "\Theta^*"'] \arrow[dl, "\tau"] \\ H^{k-1}(M; \mathbb{Z}). \end{tikzcd} \tag{1}$$

In general τ is neither injective nor surjective.

This suggests that we can 'transgress' Čech cohomology to an appropriate form of 'fusive Čech cohomology' on the loop space; this is done in [2].

To define the target fusive Čech cohomology on $\mathcal{L}M$ we start with the good open cover by tubular domains coming from the exponential map on M

$$\mathcal{L}M = \bigcup_{\gamma \in \mathcal{L}M} \Gamma(\gamma),\ \Gamma(\gamma) = \{\gamma' \in \mathcal{L}M; \gamma'(\theta) \in B(\gamma(\theta, \epsilon) \ \forall\ \theta \in \mathbb{S}\}. \tag{2}$$

Where $\epsilon > 0$ corresponds to the cover of M by geodesically convex balls

$$M = \bigcup_{m \in M} B(m, \epsilon). \tag{3}$$

CHAPTER 4

Loop-spin structures

We now proceed to discuss loop-spin structures on a compact oriented spin manifold M; these are a form of spin structure on the loop space.

1. Loop spin group

A spin structure on a comoact oriented manifold is a principal bundle

$$\begin{tikzcd} \mathcal{L}\operatorname{Spin}(n) \arrow[r, dash] & \mathcal{L}F_{\operatorname{Spin}} \arrow[d] \\ & \mathcal{L}M \end{tikzcd}, \quad n > 3. \tag{1}$$

We have analysed the first few homotopy groups of $\operatorname{Spin}(n)$, which is 2-connected, i.e. connected, simply connected and with trivial homotopy group $\Pi_2(\operatorname{Spin}(n))$. On the other hand $\Pi_3(\operatorname{Spin}(n)) = \mathbb{Z}$.

Lemma 19. *The loop group $\mathcal{L}\operatorname{Spin}(n)$ is connected and has*

$$\Pi_1(\mathcal{L}\operatorname{Spin}(n)) = \{0\},\ \Pi_2(\mathcal{L}\operatorname{Spin}(n)) = \mathbb{Z}. \tag{2}$$

Proof. First consider the semi-direct product decomposition. Inclusion of the pointed loop group and the map to the initial point gives the short exact sequence

$$\mathcal{L}_{\#}\operatorname{Spin}(n) \hookrightarrow \mathcal{L}\operatorname{Spin}(n) \twoheadrightarrow \operatorname{Spin}(n). \tag{3}$$

This splits, by incusion of $\operatorname{Spin}(n)$ in $\mathcal{L}\operatorname{Spin}(n)$ as the constant loops, giving the semidirect product decomposition

$$\mathcal{L}\operatorname{Spin}(n) = \mathcal{L}_{\#}\operatorname{Spin}(n) \ltimes \operatorname{Spin}(n). \tag{4}$$

Here $\operatorname{Spin}(n)$ acts on the pointed subgroup $\mathcal{L}_{\#}\operatorname{Spin}(n)$ by conjugation, so the product in $\mathcal{L}\operatorname{Spin}(n)$ reduces to $(l_1, S_1)\cdot(l_2, S_2) = (l_1 S_1 l_2 S_1^{-1}, S_1 S_2)$. Topologically this is the direct product so (2) reduces to the same statements for $\mathcal{L}_{\#}\operatorname{Spin}(n)$.

Then the short exact sequence

$$\mathcal{L}_{\#}\operatorname{Spin}(n) \longrightarrow \mathcal{P}_{\#}\operatorname{Spin}(n) \xrightarrow{e(2\pi)} \operatorname{Spin}(n) \tag{5}$$

leads to (2) through the long exact homotopy sequence and the contractibility of $\mathcal{P}_{\#}\operatorname{Spin}(n)$.

To see this very special case of Serre's theorem, let us note explicitly how to construct the connecting homomorphism

$$\Pi_3(\operatorname{Spin}(n)) \longrightarrow \Pi_2(\mathcal{L}_{\#}\operatorname{Spin}(n)) \tag{6}$$

although this argument applies starting from $\Pi_k(\operatorname{Spin}(n))$ for any $k \geq 1$ (and any connected compact Lie group).

Consider the smooth path space of the 3-sphere and fix the South Pole as origin. Then there is a smooth map

$$\mathbb{S}^3 \setminus \{N\} \longrightarrow \mathcal{P}_{\#}\mathbb{S}^3 \tag{7}$$

assigning to each point p the great circle path from S to p rescaled to have parameter length 2π. This does not extend smoothly up to N but does so if we 'blow up $\mathbb{S}^3$ at N.' In this case we can interpret this in term of the smooth map from the closed 3-ball in the tangent space to $\mathbb{S}^3$, $\mathbb{R}^3$, at S given by the (slightly rescaled) exponential map

$$E : \mathbb{B}^3 = \{|v| \leq 2\pi\} \ni \lambda(v) \in \mathcal{P}_{\#}\mathbb{S}^3,\ \ \lambda(v)(s) = \exp_S(sv/2\pi),\ \ s \in [0, 2\pi]. \tag{8}$$

Clearly this maps the boundary of the 3-sphere to the point N.

Now, if $f : \mathbb{S}^3 \longrightarrow \operatorname{Spin}(n)$ is a continuous map, representing an element of $\Pi_3(\operatorname{Spin}(n))$, then the composite is a continuius map

$$f \circ E : \mathbb{B}^3 \longrightarrow \mathcal{P}_{\#} \operatorname{Spin}(n). \tag{9}$$

The restriction to the bounding 2-sphere is therefore a continuous map with image lying in the fibre over $f(N)$, i.e. all curves have endpoint $f(N)$. Choosing one point in $\mathbb{S}^2$ as base point, the fibre action of $\mathcal{L}_{\#} \operatorname{Spin}(n)$ gives a continuous map

$$f' : \mathbb{S}^2 \longrightarrow \mathcal{L}_{\#} \operatorname{Spin}(n). \tag{10}$$

It is straightforward to check that different choices of base point, and different choices of the map f up to homotopy, give homotopic maps f'.

This constructs the connecting homorphism (after checking multiplicativity)

$$\Pi_3(\operatorname{Spin}(n)) \longrightarrow \Pi_2(\mathcal{L}_{\#} \operatorname{Spin}(n)). \tag{11}$$

In this case this map is an isomorphism. Injectivity follows from the fact that if f' is homotopic to a constant map then the map (9) can be deformed to constant so its projection, f, can be deformed to a constant map. Similarly surjectivity follows by noting that a continuous map $g : \mathbb{S}^2 \longrightarrow \mathcal{L}_{\#} \operatorname{Spin}(n)$ viewed as a map into $\mathcal{P}_{\#} \operatorname{Spin}(n)$ becomes a map $\mathbb{S}^2 \times [0, 2\pi] \longrightarrow \operatorname{Spin}(n)$ in which the initial and terminal sets $\mathbb{S}^2 \times \{0\}$ and $\mathbb{S}^2 \times \{2\pi\}$ are mapped to the identity. Thus it can be deformed to a continuous map $f : \mathbb{S}^3 \longrightarrow \operatorname{Spin}(n)$. Following through the discussion of the connecting homomorphism it is clear that the image of f is homotopic to g. Thus (11) is an isomorphism. □

REMARK 4. A little more diligence shows that $\Pi_3(\mathcal{L} \operatorname{Spin}(n)) = \mathbb{Z}$, but we do not need this.

From (2) it follows, from Hurewitz theorem, that

$$H^2(\mathcal{L} \operatorname{Spin}(n), \mathbb{Z}) = \mathbb{Z},\ H^3(\operatorname{Spin}(n), \mathbb{Z}) = \mathbb{Z}. \tag{12}$$

Below we recall that these statements too are connected by transgression.

Thus we know that there is a non-trivial circle bundle over $\mathcal{L} \operatorname{Spin}(n)$ with Chern class generating the cohomology group (12).

PROPOSITION 21. *There is a central extension of* $\mathcal{L} \operatorname{Spin}(n)$

$$\mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}} \operatorname{Spin}(n) \longrightarrow \mathcal{L} \operatorname{Spin}(n) \tag{13}$$

with the Chern class of the circle bundle a generator of $H^2(\mathcal{L} \operatorname{Spin}(n), \mathbb{Z})$.

We construct this central extension below and show that it is *fusive*, which notion we need to define by extending the notion of fusion for functions now combined with *equivariance* with respect to the action of $\mathrm{Dff}^+(\mathbb{S})$ on $\mathcal{L}\,\mathrm{Spin}(n)$.

This central extension is what leads to the notion of a loop-spin structure on M. This corresponds to a 'spin structure on $\mathcal{L}M$' meaning a lifting of the principal bundle (1) to

(14)
$$\begin{tikzcd}
\widehat{\mathcal{L}}\,\mathrm{Spin}(n) \arrow[r, dash] \arrow[d] & F_{\widehat{\mathcal{L}}\,\mathrm{Spin}} \arrow[d] \\
\mathcal{L}\,\mathrm{Spin}(n) \arrow[r, dash] & \mathcal{L}F_{\mathrm{Spin}} \arrow[d] \\
 & \mathcal{L}M.
\end{tikzcd}$$

This, of course, is not enough. We need to impose fusive conditions on the circle bundle $F_{\widehat{\mathcal{L}}\,\mathrm{Spin}} \longrightarrow \mathcal{L}F_{\mathrm{Spin}}$ to ensure that this 'comes from M.' The sense in which this is the case is that the existence of such a 'loop-spin structure' is equivalent to the existence of a string structure on M, all of which notions are discussed below.

2. Central extensions of $\mathcal{L}\,\mathrm{Spin}(n)$

We proceed to construct a central extension using Toeplitz compression as we have done for the Bott-Virasoro extension of $\mathrm{Dff}^+(\mathbb{S})$ and in so doing get an action of the latter on the former extending pull-back.

We start by constructing a central extension of the component of the identity in $\mathcal{L}\,\mathrm{SO}(n)$, $\mathcal{L}^0\,\mathrm{SO}(n)$. We know that $\mathcal{L}\,\mathrm{Spin}(n)$ is a double cover of $\mathcal{L}^0\,\mathrm{SO}(n)$. The Lie algebra of $\mathcal{L}\,\mathrm{SO}(n)$ is naturally identified with loops in the Lie algebra, $\mathfrak{so}(n)$, of $\mathrm{SO}(n)$

$$T_{\mathrm{Id}}\mathcal{L}^0\,\mathrm{Spin}(n) = \mathcal{C}^\infty(\mathbb{S};\mathfrak{so}(n)). \tag{1}$$

We can expand in Fourier series and so consider

$$T_{\mathrm{Id}}\mathcal{L}^0\,\mathrm{SO}(n) \ni \nu = \sum_k \tau_k e^{ik\theta},\ \tau_k \in \mathfrak{so}(n)\otimes\mathbb{C} \tag{2}$$

where as usual we must allow complex coefficients but the sum is actually real. Thus if E_α is a basis of $\mathfrak{so}(n)$ then the elements

$$J_{k\alpha} = e^{ik\theta}E_\alpha \tag{3}$$

form a 'basis' of $T_{\mathrm{Id}}\mathcal{L}^0\,\mathrm{SO}(n)$.

It is then straightforward to compute the form of the Lie algebra of $\mathcal{L}^0\,\mathrm{SO}(n)$ (and hence of $\mathcal{L}\,\mathrm{Spin}(n)$) as

$$[J_{k,\alpha}, J_{l\beta}] = e^{i(k+l)\theta}[E_\alpha, E_\beta] \tag{4}$$

which can be written as a combination of the $J_{(k+l)\gamma}$ using the structure constants of $\mathfrak{so}(n)$

$$\begin{gathered}[E_\alpha, E_\beta] = \sum_\gamma C^\gamma_{\alpha\beta}E_\gamma \Longrightarrow \\ [J_{k,\alpha}, J_{l\beta}] = \sum_\gamma C^\gamma_{\alpha\beta}J_{k+l,\gamma}.\end{gathered} \tag{5}$$

Now there is a pointwise action

$$\mathcal{L}\,\mathrm{SO}(n) \ni \lambda : \mathcal{C}^\infty(\mathbb{S}, \mathbb{C}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{S}; \mathbb{C}^n),\ \lambda \cdot u(\theta) = \lambda(\theta)u(\theta). \tag{6}$$

As bounded operators on $L^2(\mathbb{S}; \mathbb{C}^n)$ the elements of $\mathcal{L}\,\mathrm{SO}(n)$ are unitary.

Then consider the vector-valued Hardy space. This is simply n copies of the Hardy space

$$\mathcal{H}_n = H \otimes \mathbb{C}^n. \tag{7}$$

Thus $\mathcal{H}_n$ is the image of the Toeplitz projection $\Pi : \mathcal{C}^\infty(\mathbb{S}) \longrightarrow H$ extended as a diagonal matrix to give a projection onto $\mathcal{H}_n$ which we will continue to denote by Π.

Then consider the compressions

$$\lambda_H = \Pi\lambda\Pi : \mathcal{H}_n \longrightarrow \mathcal{H}_n,\ \lambda \in \mathcal{L}^0\,\mathrm{SO}(n). \tag{8}$$

Proposition 22. *The compressions λ_H of the elements of $\mathcal{L}^0\,\mathrm{SO}(n)$ to the Hardy space are Fredholm operators of index zero.*

Proof. The commutator $[\Pi, \lambda]$ is a smoothing operator on $L^2(\mathbb{S}; \mathbb{C}^n)$ since it is an $n \times n$ matrix of commutators of Π with smooth multiplication operators. From this it follows that

$$\mu_H\lambda_H = (\mu\lambda)_H + E(\mu, \lambda),\ E(\mu, \lambda) \in \Psi^{-\infty}(\mathbb{S}; \mathbb{C}^n)\ \forall\ \mu, \lambda \in \mathcal{L}^0\,\mathrm{SO}(n). \tag{9}$$

In particular $(\lambda^{-1})_H$ is a parametrix, modulo smoothing operators, for λ_H so λ_H is Fredholm. The connectedness of $\mathcal{L}^0\,\mathrm{SO}(n)$ means that these operators are homotopic to the identity through a norm continuous family of Fredholm operators, so have index zero. □

Since they have index zero, the operator λ_H has finite dimensional null space and finite dimensional complement to its range, realized as the null space of the adjoint, both contained in $\mathcal{C}^\infty(\mathbb{S}; \mathbb{C}^n)$. Thus there is a perturtion by by finite rank smoothing operators giving an invertible operstor. It follows that there is a group of invertible operators

$$\mathcal{G} = \{\lambda_H + E,\ \text{invertible}; \lambda \in \mathcal{L}^0\,\mathrm{SO}(n),\ E = \Pi E \Pi \in \Psi^{-\infty}(\mathbb{S}; \mathbb{C}^n)\}. \tag{10}$$

The composite

$$(\lambda_H + E)^*(\lambda_H + E) = \mathrm{Id} + F,\ F \in \Psi^{-\infty}(\mathbb{S}, \mathbb{C}^n) \tag{11}$$

is strictly positive and so has a unique positive square-root with

$$(\mathrm{Id} + F)^{-\frac{1}{2}} \in \mathrm{Id} + \Psi^{-\infty}(\mathbb{S}; \mathbb{C}^n). \tag{12}$$

Thus in place of the group (10) we may consider the unitary subgroup

$$\mathcal{U} = \{(\mathrm{Id} + F)^{-\frac{1}{2}}(\lambda_H + E)(\mathrm{Id} + F)^{-\frac{1}{2}} = \lambda_H + E',\ \lambda_H + E \in \mathcal{G}\}. \tag{13}$$

Proposition 23. *There is a short exact sequence of groups*

$$\mathcal{U}_H^{-\infty} \longrightarrow \mathcal{U} \longrightarrow \mathcal{L}^0\,\mathrm{SO}(n) \tag{14}$$

where $\mathcal{U}_H^{-\infty}$ is the group of unitary smoothing perturbations of the identity acting on $\mathcal{H}_n$.

Proof. □

The central extension we seek is then

$$\widehat{\mathcal{L}}^0\,\mathrm{SO}(n) = \mathcal{U}/\{g \in \mathcal{U}_H^{-\infty}; \det g = 1\}. \tag{15}$$

The circle bundle in (15) is a form of the determinant bundle, realizing the possible values of an extension of the determinant to $\mathcal{L}^0\,\mathrm{SO}(n)$.

PROPOSITION 24. *The Lie algebra of $\widehat{\mathcal{L}}^0\,\mathrm{SO}(n)$ is the affine Lie algebra with basis $\{J_{k\alpha}\} \cup \{\zeta\}$ and Lie bracket*

$$[J_{k\alpha} + c_1\zeta, J_{l\beta} + c_1\zeta] = \sum_{\gamma} C^{\gamma}_{\alpha\beta} J_{(k+l)\gamma} - k\delta_{k+l=0}\,\mathrm{Tr}(E_\alpha E_\beta)\zeta. \tag{16}$$

PROOF. □

LEMMA 20. *Under the 2-1 cover $\mathcal{L}\,\mathrm{Spin}(n) \longrightarrow \mathcal{L}^0\,\mathrm{SO}(n)$ the circle bundle defining $\widehat{\mathcal{L}}^0\,\mathrm{SO}(n)$ pulls back to define a central extension $\widehat{\mathcal{L}}\,\mathrm{Spin}(n)$ as a 2-1 cover of $\widehat{\mathcal{L}}^0\,\mathrm{SO}(n)$.*

PROOF. □

3. $\widehat{\mathcal{L}}\,\mathrm{Spin}(n)$ is fusive

Both conditions making functions on the loop space fusive, namely $\mathrm{Dff}^+(\mathbb{S})$-invariance and the fusion condition, need to be reinterpreted to apply to the circle bundle forming the central extension of $\mathcal{L}\,\mathrm{Spin}(n)$.

DEFINITION 15. A circle bundle S over $\mathcal{L}M$ is *fusive* if it admits an action of $\widehat{\mathrm{Dff}}^+(\mathbb{S})$ equivariant with respect to the reparametrization action of $\mathrm{Dff}^+(\mathbb{S})$ on $\mathcal{L}M$ and satisfies the fusion condition that the simplicial image $\delta^* S$ has a trivialization over $\mathcal{P}^{[3]}M$ consistent with the canonical trivialization of $(\delta^2)^* S$.

Of course we need to check regularity as well.

PROPOSITION 25. *There is an action of $\widehat{\mathrm{Dff}}^+(\mathbb{S})$ on $\widehat{\mathcal{L}}\,\mathrm{Spin}(n)$ equivariant with respect to the reparametrization action of $\mathrm{Dff}^+(\mathbb{S})$ on $\mathcal{L}\,\mathrm{Spin}(n)$.*

PROPOSITION 26. *The simplicial image of the circle bundle defining $\widehat{\mathcal{L}}\,\mathrm{Spin}(n)$ to $\mathcal{P}^{[3]}\,\mathrm{Spin}(n)$ has a trivialization inducing the canonical trivialization over $\mathcal{P}^{[4]}\,\mathrm{Spin}(n)$.*

To get a clearer picture of the circle bundle defining $\widehat{\mathcal{L}}\,\mathrm{Spin}(n)$ it is useful to construct a connection on it. Consider the invertible operator $A = \frac{1}{i}\partial_\theta + 1$ on the Hardy space. It has eigenvalues $k+1$, $k \geq 0$, with respect to the Fourier basis. The complex powers A^{-z} are well-defined, unbounded (for $\operatorname{Re} z < 0$) normal operators on H with eigenvalues $(k+1)^{-z}$. Then

LEMMA 21. *If for any $\lambda \in \mathcal{C}^\infty(\mathbb{T}; \mathrm{GL}(b,\mathbb{C})$ the composite $A^{-z}\lambda_H$ is in the trace ideal for $\operatorname{Re} z > 1$ and for any smoothing operator κ on $\mathcal{H}_n$*

$$D(z) = \mathrm{Tr}(A^{-z}(\lambda_H + \kappa)) \tag{1}$$

is holomorphic in $\operatorname{Re} z < -1$ with a meromorphic extension to the complex plan which is regular at $z = 0$.

We define the *regularized trace* on Toeplitz operators to be

$$\overline{\mathrm{Tr}}(T_\lambda + \kappa) = D(0) = -\frac{1}{2}\int \mathrm{tr}\,\lambda + \mathrm{Tr}(\kappa). \tag{2}$$

the value of the analytic continuation.

PROOF. Since the composite $A^{-z}\kappa$ is in the trace class ideal for $z \in \mathbb{C}$ and has entire trace it follows that

$$\overline{\mathrm{Tr}}(T_\lambda + \kappa) = \overline{\mathrm{Tr}}(T_\lambda) + \mathrm{Tr}(\kappa) \tag{3}$$

From (**??**) it follows that the matrix entries of the regularized operator are

$$(A^{-z}T_\lambda)_{kj,li} = (k+1)^{-z}\frac{1}{2\pi}\lambda_{k-j,li}. \tag{4}$$

Thus the diagonal entries are multiples of $\lambda_{0,li}$ and

$$(A^{-z}T_\lambda)_{kj,li} = \left(\sum_{k\geq 0}(k+1)^{-z}\right)\frac{1}{2\pi}\,\mathrm{tr}(\lambda_0) = \zeta(z)\,\mathrm{tr}(\lambda_0) \tag{5}$$

is given by a multiple of the Riemann zeta function. Since $\zeta(0) = -\frac{1}{2}$, (2) follows. □

The previous computation now gives us

LEMMA 22 (Trace-defect formula). *The regularized trace on Toeplitz operators satisfies*

$$\overline{\mathrm{Tr}}([T_\lambda + \kappa, T_\gamma + \mu]) = -\frac{1}{2\pi}\int_{\mathbb{S}} \mathrm{tr}(\lambda(\theta)\cdot\frac{1}{i}\partial_\theta\gamma(\theta))d\theta \tag{6}$$

for λ, $\gamma \in \mathcal{C}^\infty(\mathbb{T}; \mathrm{GL}(n,\mathbb{C})$ *and any smoothing operators* κ, μ *on* H.

PROOF. This follows by combining (2), (**??**) and the fact that

$$\overline{\mathrm{Tr}}(T_{[\lambda,\gamma]}) = 0. \tag{7}$$

□

Now recall that the central extension discussed above, of the component $\mathcal{L}_0G$ of the identity for $G = \mathrm{U}(n)$ or $\mathrm{SO}(n)$, is obtained by considering the invertible perturbations $\mathcal{G} = \{T_\lambda + \kappa;\ \text{invertible}\}$ where κ is a smoothing operator (projected onto H) and then taking the quotient $\widehat{\mathcal{L}}_0G = \mathcal{G}/G_1^{-\infty}$, by the normal subgroup $G_1^{-\infty} = \{\mathrm{Id} + \kappa \in G^{-\infty}; \det(\mathrm{Id} + \kappa) = 1\}$. The Maurer-Cartan form $g^{-1}dg$ is well-defined on the tangent space to $\mathcal{G}$ and we can define the complex-valued 1-form

$$\alpha = \overline{\mathrm{Tr}}(g^{-1}dg) \text{ in } \mathcal{G}. \tag{8}$$

In fact it descends to a 1-form on $\widehat{\mathcal{L}}_0G$. Namely if $h \in G_1^{-\infty}$ then

$$\begin{gathered}(hg)^{-1}d(hg) = g^{-1}dg + g^{-1}(h^{-1}dh)g \Longrightarrow \\ \alpha_{hg} = \overline{\mathrm{Tr}}((hg)^{-1}d(hg)) = \overline{\mathrm{Tr}}(g^{-1}dg) + \mathrm{Tr}(g^{-1}(h^{-1}dh)g)\end{gathered} \tag{9}$$

since $h^{-1}dh = (\mathrm{Id} + \kappa)^{-1}d\kappa$ is a smoothing operator. Moreover $\det h = 1$ on $G_1^{-\infty}$ so

$$\mathrm{Tr}(g^{-1}(h^{-1}dh)g) = \mathrm{Tr}(h^{-1}dh) = d\log\det h = 0 \Longrightarrow \alpha_{hg} = \alpha_g.$$

Thus α is constant on the orbits of $G_1^{-\infty}$ so descends to $\widehat{\mathcal{L}}_0G$.

LEMMA 23. *The 1-form* $\alpha = \overline{\mathrm{Tr}}(g^{-1}dg)$ *is a connection form on* $\widehat{\mathcal{L}}_0G$ *as a circle bundle over* $\mathcal{L}_0G$.

PROOF. The circle action on $\widehat{\mathcal{L}}_0 G$ is the quotient of the multiplicative action of $G^{-\infty}$. Thus, taking a map $\mathbb{T} \longrightarrow h_t \in G^{-\infty}$ such that $\det(h_t)(e^{it} = \exp(it)$ – for instance complex rotation around any basis vector – the same formula (9) shows that

$$\alpha_g(\partial_t) = \alpha_g \tag{10}$$

□

We can also compute the lie algebra cocyle corresponding to the central extension. Recall that we choose a decomposition over the identity

$$\begin{aligned} &T_{\mathrm{Id}}\widehat{\mathcal{L}}_0 G = \mathbb{R} + T_{\mathrm{Id}}\mathcal{L}_0 \text{ by setting the image of } T_{\mathrm{Id}}\mathcal{L}_0 = \mathrm{Nul}(\alpha) \\ &\qquad \Longrightarrow \mathcal{L}_0 G \ni \lambda \longmapsto T_\lambda + \kappa \in T_{\mathrm{Id}}\widehat{\mathcal{L}}_0 G,\ \overline{\mathrm{Tr}}(T_\lambda + \kappa) = 0. \end{aligned} \tag{11}$$

Then the 2-form on $\mathcal{L}_0 G$ corresponding (and determining) the central extension is

$$\omega(\lambda, \gamma) = \overline{\mathrm{Tr}}([T_\lambda + \kappa, T_\gamma + \mu]) = \overline{\mathrm{Tr}}([T_\lambda, T_\gamma]) = -\frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{Tr}(\lambda \frac{d\gamma}{d\theta}) d\theta. \tag{12}$$

Thus, we are back at the 2-form we 'guessed' earlier in (**??**).

We can see that this is just the differential of α at Id, i.e. the curvature 2-form of the central extensio, since

$$\begin{aligned} &d\alpha = -\overline{\mathrm{Tr}}(g^{-1}dg \wedge g^{-1}dg) \Longrightarrow \\ &\qquad (d\alpha)_{\mathrm{Id}}(\lambda, \gamma) = -\overline{\mathrm{Tr}}([T_\lambda, T_\gamma]) = -\frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{Tr}(\lambda \frac{d}{d\theta}\gamma) d\theta. \end{aligned} \tag{13}$$

CHAPTER 5

The torus

The Dirac-Ramond operator on the $2n$-torus can be analysed rather directly. This is essentially an extension of the work of Spera and Wurzbacher, [6], who computed an L^2 version of the U(1)-equivariant index of the Dirac-Ramond operator on Euclidean space.

1. Loops in the torus

The torus has universal covering space $\mathbb{R}^n$

(1)
$$\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n.$$

As a Lie group the tangent bundle is trivial and is naturally identified with $\mathbb{T}^n \times \mathbb{R}^n$. The smooth loop space

$$\mathcal{L}\mathbb{T}^n = (\mathcal{C}^\infty(\mathbb{S}, \mathrm{U}(1)))$$

is the product of n copies of $\mathcal{L}\,\mathrm{U}(1)$ so each loop $l \in \mathcal{L}(\mathbb{T}^n)$, therefore has a multi-winding number. For smooth loops

(2)
$$\omega(l) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{dl}{d\theta} \in \mathbb{Z}^n = \pi_1(\mathbb{T}^n)$$

which labels the components of the loop space.

Proposition 27. *The loop space of the torus*

(3)
$$\mathcal{L}(\mathbb{T}^n) = \bigsqcup_{\omega\in\mathbb{Z}^n} \mathcal{L}_\omega(\mathbb{T}^n) = \mathbb{Z}^n \times \mathcal{L}(\mathbb{R}^n) \longrightarrow \mathbb{Z}^n \times \mathbb{T}^n$$

fibres over $\mathbb{T}^n \times \mathbb{Z}^n$ with fibre $\mathcal{L}_0(\mathbb{R}^n)$, the subspace of $\mathcal{L}(\mathbb{R}^n)$ consisting of the loops of mean value zero.

Proof. Each loop in $\mathbb{T}^n$ has a $\mathbb{Z}^n$ of lifts to a smooth map $[0, 2\pi] \longrightarrow \mathbb{R}^n$ so the space

(4)
$$\{\tilde{l} : [0, 2\pi] \longrightarrow \mathbb{R}^n; \tilde{l}(2\pi) - \tilde{l}(0) = 2\pi\omega \in \mathbb{Z}^n\}$$

is a $\mathbb{Z}^n$ cover of $\mathcal{L}_\omega(\mathbb{T}^n)$. Then each such $\tilde{l}$ has a unique extension to a smooth map $\hat{l} : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $\hat{l}(t + 2\pi) - \hat{l}(t) = 2\pi\omega(l)$.

For the component of winding number zero, consisting of the contractible loops in $\mathbb{T}^n$, the lifts are of the form $\tilde{l}(t) + 2\pi\eta$, $\eta \in \mathbb{Z}^n$ and $\tilde{l} \in \mathcal{L}(\mathbb{R}^n)$ i.e. periodic. A loop in $\mathbb{R}^n$ has mean value

$$\mu(\hat{l}) = \frac{1}{2\pi}\int_0^{2\pi} \hat{l} \in \mathbb{R}^n.$$

We conclude that

$$(5)\quad \mathcal{L}_{\omega=0}(\mathbb{T}^n) = \mathbb{T}^n \oplus \mathcal{L}_0(\mathbb{R}^n),\ \ \mathcal{L}_0(\mathbb{R}^n) = \{\hat{l} \in \mathcal{L}(\mathbb{R}^n); \int_0^{2\pi} \hat{l}(\theta)d\theta = 0\},$$

being the Euclidean loops of mean value 0.

For non-zero winding number we choose the base loop $l_\omega : \theta \longmapsto \prod_{j=1^n} \exp(2\pi i\omega_j\theta)$. Then a general a loop with this winding number is of the form

$$(6)\qquad l(\theta) = l_\omega \cdot \check{l},\ \check{l} \in \mathcal{L}_{\omega=0}(\mathbb{R}^n).$$

This allows each component to be identified as

$$(7)\qquad \mathbb{T}^n \oplus \mathcal{L}_0(\mathbb{R}^n).$$

□

If we take the flat metric on $\mathbb{T}^n$ corresponding to the standard Euclidean metric on $\mathbb{R}^n$ then the orthormal frame bundle is naturally trivial

$$(8)\qquad \mathcal{F}(\mathbb{T}^n) = \mathbb{T}^n \times \mathrm{SO}(n)$$

and hence the loop frame bundle is a product

$$(9)\qquad \mathcal{L}(\mathcal{F}(\mathbb{T}^n)) = \mathcal{L}(\mathbb{T}^n) \times \mathcal{L}\,\mathrm{SO}(n)$$

and there is a product spin structure $\mathcal{F}_{\mathrm{Spin}}(\mathbb{T}^n) = \mathbb{T}^n \times \mathrm{Spin}(n)$.

The loop orientation corresponding to this product spin structure is therefore

$$(10)\qquad \mathcal{L}(\mathcal{F}_{\mathrm{Spin}}(\mathbb{T}^n)) = \mathcal{L}(\mathbb{T}^n) \times \mathcal{L}\,\mathrm{Spin}(n)$$

and there is always (for $n \geq 3$) a flat loop spin structure identified as

$$(11)\qquad \mathcal{L}(\mathbb{T}^n) \times \widehat{\mathcal{L}}\,\mathrm{Spin}(n)$$

for the basic central extension of $\mathcal{L}\,\mathrm{Spin}(n)$.

The associated loop spinor bundle is also naturally trivial for the Euclidean metric on $\mathbb{T}^{2n}$.

2. Complex structure on $\mathcal{L}\mathbb{R}^{2n}$

The tangent bundle to

$$\mathcal{L}\mathbb{T}^{2n} \simeq \bigsqcup_{\omega\in\mathbb{Z}^{2n}} \mathbb{T}^{2n} \times \mathcal{L}_0\mathbb{R}^{2n} \cdot \tilde{\omega}$$

is trivial with the fibre identified as

$$(1)\qquad T_l\mathcal{L}\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathcal{L}_0\mathbb{R}^{2n},\ \forall\, l \in \mathcal{L}\mathbb{R}^{2n}.$$

It follows that $\mathcal{L}\mathbb{T}^{2n}$ is a complex manifold with the assigment of the complex structure $\mathbb{R}^{2n} = \mathbb{C}^n$ and in terms of the Fourier expansions

$$(2)\qquad \begin{aligned} \mathcal{L}_0\mathbb{R}^{2n} \ni \tau &= \sum_{j\in\mathbb{Z}\setminus\{0\}} \sum_{p=1,\dots,2n} c_{j,p}e^{ij\theta}e_p \\ J\tau &= \sum_{j\in\mathbb{Z}\setminus\{0\}} \sum_{p=1,\dots,2n} \operatorname{sgn} j\, c_{-j,p}e^{ij\theta}e_p \end{aligned}$$

where e_p is the standard basis of $\mathbb{R}^{2n}$.

Thus $z_k = x_k + i_{k+n}$ on $\mathbb{R}^{2n}$ and $\zeta_{j,p} = c_{j,p} + ic_{-j,p}$ are complex coordinates on $\mathcal{L}\mathbb{T}^{2n}$. Fock space, in the sense of the spin representation of $\mathcal{L}\,\mathrm{Spin}(2n)$, is the infinite tensor product of the four dimesnial spaces

$$\Lambda\mathbb{C}_k \text{ and } \Lambda\mathbb{C}_{j,p},\ k = 1, \ldots, n,\ p = 1, \ldots, 2n, j \in \mathbb{N}. \tag{3}$$

Following Spera and Wurzbacher, the Dirac-Ramond operator decomposes as a (formal) infinite dirct sum of operators acting on the tensor factors in (3). Namely each of the factors $\mathbb{C}_{j,p}$

Bibliography

[1] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*. Vol. 82. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982, pp. xiv+331. ISBN: 0-387-90613-4.

[2] Chris Kottke and Richard B. Melrose. "Loop-fusion cohomology and transgression". In: *Math. Res. Lett.* 22.4 (2015), pp. 1177–1192. ISSN: 1073-2780. DOI: 10.4310/MRL.2015.v22.n4.a11. URL: http://dx.doi.org/10.4310/MRL.2015.v22.n4.a11.

[3] Serge Lang. *Introduction to differentiable manifolds*. Second. Universitext. Springer-Verlag, New York, 2002, pp. xii+250. ISBN: 0-387-95477-5.

[4] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[5] M.K. Murray. "Bundle gerbes". In: *J. London Math. Soc.* 54 (1996), pp. 403–416.

[6] Mauro Spera and Tilmann Wurzbacher. "The Dirac-Ramond operator on loops in flat space". In: *J. Funct. Anal.* 197.1 (2003), pp. 110–139. ISSN: 0022-1236,1096-0783. DOI: 10.1016/S0022-1236(02)00178-7. URL: https://doi.org/10.1016/S0022-1236(02)00178-7.

[7] Stephan Stolz and Peter Teichner. "The Spinor Bundle on Loop Spaces". Preprint.

[8] Yongchang Zhu. "Modular invariance of characters of vertex operator algebras". In: *J. Amer. Math. Soc.* 9.1 (1996), pp. 237–302. ISSN: 0894-0347,1088-6834. DOI: 10.1090/S0894-0347-96-00182-8. URL: https://doi.org/10.1090/S0894-0347-96-00182-8.