chapter

AN ANALYTIC DIRAC-RAMOND OPERATOR

RICHARD MELROSE

Abstract. This is a written, and expanded, version of a colloquium at the Simons Center, Stony Brook, on February 27th, 2020.

I would like to take this opportunity to discuss the background leading to the definition of the (or really a) Dirac-Ramond operator. This idea, of a differential operator on an infinite-dimensional space, arose quite early in the development of string theory as an analogue of the spin Dirac operator, in this case intended to describe 'spinning strings.' However there are substantial mathematical difficulties which have obstructed the precise definition of this as an operator, and some of these issues remain. Today I hope to convince you that some progress has been made and that there is interesting Mathematics in what might otherwise be thought of as a quixotic enterprise.

Let me start with a 'topological description of geometric structures', in particular spin structures. Consider the Whitehead (my erstwhile colleague George) tower for the group $\mathrm{O}(n)$. Here $n > 2$ and it creeps up a bit below, take $n \geq 5$ throughout if you want to be safe from low dimensional annoyances. At some point n might be even as well. So the tower in question is

(1)
$$\begin{array}{ccccccccc}
\mathbb{Z}_2 & & & & \mathbb{Z}_2 & & K(\mathbb{Z},2) & & \\
\uparrow{\scriptstyle \det} & & & & \downarrow & & \downarrow & & \\
\mathrm{O}(n) & \longleftarrow & \mathrm{SO}(n) & \longleftarrow & \mathrm{Spin}(n) & \longleftarrow & \mathrm{String}(n) & & \dots
\end{array}$$

The successive maps here 'remove' the lowest homotopy group while keeping the higher ones unchanged. In the first step the map is injective but in higher steps it is surjective. Thus $\mathrm{O}(n)$ has two components, '$\pi_0 = \mathbb{Z}_2$,' then $\pi_1 = \mathbb{Z}_2$ as well then $\pi_2 = \{\mathrm{Id}\}$, $\pi_3 = \mathbb{Z}$ and I'm not going to talk about the higher groups (see 'fivebrane' if you want to know). All the spaces here are, or really can be taken to be, topological groups but they are actually only determined up to homotopy equivalence.

What is the relation of this to geometry? A smooth (finite-dimensional) manifold, which is really what I am interested in, has a tangent bundle which, being a vector bundle, has a frame bundle – the elements F_p at each point $p \in M$, are just the bases of the tangent space T_pM. Here F is a principal $\mathrm{GL}(n,\mathbb{R})$-bundle with the action being change of basis

(2)
$$\begin{array}{ccc}
\mathrm{GL}(n,\mathbb{R}) & \text{———} & F \\
& & \downarrow \\
& & M.
\end{array}$$

We can recover TM as the bundle associated to the standard representation of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$

(3)
$$TM = FM \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n.$$

We can equip M with a Riemann metric and this reduces the structure group from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$ by taking the orthonomal frames

(4)
$$\begin{array}{ccc}
\mathrm{O}(n) & \text{———} & F_{\mathrm{O}} \\
& & \downarrow \\
& & M.
\end{array}$$

Here I am thinking of the circle as $\mathbb{R}/2\pi\mathbb{Z}$ so the λ are 2π-periodic functions on the line. The reason I take π to be the parameter length of the curves will show up below.

Now $\dot{\mathcal{P}}(G)$ and $\dot{\mathcal{L}}(G)$ are groups under pointwise composition and there is a short exact sequence of groups

$$\begin{array}{ccc} \dot{\mathcal{L}}(G) & \longrightarrow & \dot{\mathcal{P}}(G) \\ & & \downarrow \\ & & G \end{array} \tag{7}$$

where the last map is evaluation at the endpoint π. The kernel of this homorphism is the subgroup of pointed paths with endpoint at Id; halving the parameter allows this to be identified with $\dot{\mathcal{L}}(G)$.

The path space is contractible, through path shortening, and as a result (7) is a classifiying sequence so

$$G = B\dot{\mathcal{L}}(G) \text{ and } \pi_j(\dot{\mathcal{L}}(G)) \simeq \pi_{j+1}(G). \tag{8}$$

Returing to the orthogonal group, the statement that $\pi_1(\mathrm{SO}(n) = \mathbb{Z}_2$ just means that $\dot{\mathcal{L}}(\mathrm{SO}(n)$ has two components and then

$$\mathrm{Spin}(n) = \dot{\mathcal{P}}\,\mathrm{SO}(n))/\dot{\mathcal{L}}_{\mathrm{Id}}(\mathrm{SO}(n)) \tag{9}$$

identifies paths if they are homotopic through paths with the same endpoint.

The question of the existence of a spin structure on M is the search for an extension of the oriented frame bundle

$$\begin{array}{ccc} \mathrm{Spin}(n) & \text{———} & F_{\mathrm{Spin}} \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) & \text{———} & F_{\mathrm{SO}} \\ & & \downarrow \\ & & M. \end{array} \tag{10}$$

Such a principal bundle exists if and only if the second Stieffel-Whitney class vanishes.

Since the objective is generalize it, let me now remind you of the Spin Dirac operator – let's take the dimension to be even, $n = 2m$. The spin group has a fundamental representation of dimension 2^m coming from the identification $\mathrm{Spin}(2m) \subset \mathrm{Cl}_{\mathbb{C}}(2m) \simeq M(2^m)$ of the complexified Clifford algebra with the corresponding matrix algebra. This induces a bundle, the spinor bundle $S = S^+ \oplus S^-$ over M with grading coming from the two irreducible parts of the spin representation.

Now the bundle F_{spin} is a double cover, so the Levi-Civita connection lifts from F_{SO} to a connection and and induces a connection ∇ on S. The spin action corresponds to an action of the bundle of Clifford algebras $\mathrm{Cl}_{\mathbb{C}}(T^*M)$ on S,

$$\mathrm{cl} : T^*M \longrightarrow \mathrm{GL}(S) \tag{11}$$

and combined these define the spin Dirac operator

$$\eth_{\mathrm{Spin}} = \begin{pmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{pmatrix} : \mathcal{C}^\infty(M; S) \longrightarrow \mathcal{C}^\infty(M; S),\ \eth_{\mathrm{Spin}} = \mathrm{cl} \circ \nabla. \tag{12}$$

The spin Dirac operator is elliptic, hence Fredholm and its graded index, computed as part of the Atiyah-Singer index theorem,

$$\operatorname{ind}(\eth_{\mathrm{Spin}}) = \dim \operatorname{Nul}(\eth^+) - \dim \operatorname{Nul}(\eth^-) = \int_M \widehat{A} \tag{13}$$

is the $\widehat{A}$ genus of M. This was one of the early achievements of Atiyah and Singer, explaining the integrality of the $\widehat{A}$ genus for spin manifolds (which was known previously).

So, it is this we are trying to 'emulate' at the next step up, for string structures. Before proceeding in this way, let me describe the 'trangression' of spin structures.

Let me now consider the free, rather than the pointed loop and path spaces spaces

$$\begin{gathered}\mathcal{P}(M) = \{\chi : [0,\pi] \longrightarrow M,\ \text{continuous}\}\\ \mathcal{L}(M) = \{\lambda : \mathbb{S} \longrightarrow M,\ \text{continuous}\}.\end{gathered} \tag{14}$$

Now there are two endpoints of paths and (M being connected)

$$\mathcal{P}(M) \longrightarrow M^2 \tag{15}$$

is a fibre bundle.

Loop spaces have functorial properties. For instance, if F is a principal G-bundle over M then there are corresponding principal bundle structures

$$\begin{array}{ccccccc} \mathcal{P}(G) & \longrightarrow & \mathcal{P}(F) & & \mathcal{L}(G) & \longrightarrow & \mathcal{L}(F) \\ & & \downarrow & & & & \downarrow \\ & & \mathcal{P}(M) & & & & \mathcal{L}(M). \end{array} \tag{16}$$

In particular for an oriented manifold, as was observed by Atiyah in the 1980s, that a spin structure on M induces an 'orientation' on the loops in the frame bundle

$$\begin{array}{ccccc} \mathcal{L}(\mathrm{Spin}(n)) & \text{———} & \mathcal{L}(F_{\mathrm{Spin}}) & & \\ \downarrow & & \downarrow & & \\ \mathcal{L}(\mathrm{SO}(n)) & \text{———} & \mathcal{L}(F_{\mathrm{SO}}) & \xrightarrow{o_{\mathrm{Spin}}} & \mathbb{Z}_2 \\ & & \downarrow & & \\ & & \mathcal{L}(M); & & \end{array} \tag{17}$$

with o_{Spin} being ± 1 as the loop is, or is not, the image of a loop in $\mathcal{L}(F_{\mathrm{Spin}})$ – it is a continuous map which takes both values on each fibre.

A spin structure on M induces an 'orientation' on $\mathcal{L}(M)$.

Note that the usual notion of orientation does not really make sense on an infinite dimensional manifold like $\mathcal{L}(M)$.

Conversely, it was show by McLaughlin that if M is 2-connected, then the converse is true, but not without some such restriction. The relationship was finally clarified by Stolz and Teichner around 2005. They observed that the orientation in (17) induced by a spin structure has an additional property corresponding to the endpoint map (29). To see this, form the fibre products with respect to this map

$$\mathcal{P}^{[k]}(M) = \{(\chi_1, \ldots, \chi_k) \in (\mathcal{P}(M))^k; \chi_1(0) = \cdots = \chi_k(0),\ \chi_1(\pi) = \cdots = \chi_k(\pi)\}. \tag{18}$$

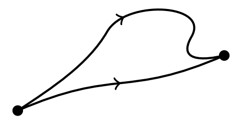

Figure 1. Joining paths to a loop

These form a simplicial space where the 'face' maps are just the maps forgetting one of the paths

$$\begin{array}{ccccccc} \mathcal{P}(M) & \Longleftarrow & \mathcal{P}^{[2]}(M) & \overset{f_*}{\Lleftarrow} & \mathcal{P}^{[3]}(M) & \Lleftarrow & \dots \\ & & \updownarrow & & & & \\ & & \mathcal{L}(M) & & & & \end{array} \tag{19}$$

The vertical bijection here is by 'joining' paths. If two paths χ_1, χ_2 have the same endpoints then traversing the first and then, in reverse, the second gives a loop

$$\mathcal{P}^{[2]}_{\mathcal{C}} M \ni (\chi_1, \chi_2) \longmapsto \lambda_{12} \in \mathcal{L}_{\mathcal{C}}(M) \tag{20}$$

and conversely a loop can be divided into 2 paths with the same endpoints.

Applying this construction to $\mathcal{L}F_{\mathrm{SO}}$ gives three pull-back maps and hence the simplicial differential

$$\begin{gathered} \mathcal{C}(\mathcal{L}F_{\mathrm{SO}}; \mathbb{Z}_2) \ni o_{\mathrm{Spin}} \longmapsto f_i^* o_{\mathrm{Spin}} \in \mathcal{C}(\mathcal{L}F_{\mathrm{Spin}}; \mathbb{Z}_2) \\ \delta o_{\mathrm{Spin}} = f_3^* o_{\mathrm{Spin}} (f_2^* o_{\mathrm{Spin}})^{-1} f_1^* o_{\mathrm{Spin}} = o_{\mathrm{Spin}}(\lambda_{12}) (o_{\mathrm{Spin}}(\lambda_{13}))^{-1} o_{\mathrm{Spin}}(\lambda_{23}) = 1 \end{gathered} \tag{21}$$

(where in the case of a $\mathbb{Z}_2$-valued function inversion does nothing). Stolz and Teichner showed that

> Spin structures on M are in 1-1 correspondence with fusion orientation structures on $\mathcal{L}(M)$, in the sense that $\delta o_{\mathrm{Spin}} = 1$ on $\mathcal{L}(M)$.

Even though this is correct, it is a little misleading because of the discreteness of $\mathbb{Z}_2$.

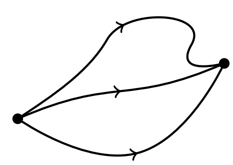

Figure 2. Fusion of three paths to three loops

So far I have considered continuous paths and loops but it is important to consider other regularity, with

$$\begin{gathered} \mathcal{P}_\infty(M) = \{\lambda : [0, \pi] \longrightarrow M;\ \text{infinitely differentiable}\} \\ \mathcal{L}_\infty(M) = \{\lambda : \mathbb{S} \longrightarrow M;\ \text{infinitely differentiable}\} \end{gathered} \tag{22}$$

perhaps the most natural. In fact the Sobolev-regular, for $s > \frac{1}{2}$, and Lipschitz-regular paths and loops (denoted by a 'Λ') are also significant. These are well-defined as spaces of maps into a finite-dimesional manifold because the corresponding spaces of functions on $[0, \pi]$ or $\mathbb{S}$ are '$\mathcal{C}^\infty$ algebras'. Not only are they closed under the product of functions but if f_i are n of these maps into $\mathbb{R}$ collectively taking values in an open set $\Omega \subset \mathbb{R}^n$ and $F : \Omega \longrightarrow \mathbb{R}$ is a smooth function on Ω then the composite $F \circ f_*$ is in the space. This allows one to define the varous path and loop space with inclusions

$$\begin{array}{ccccc} & & \mathcal{P}_s(M) & & \\ & \nearrow & & \searrow & \\ \mathcal{P}_\infty(M) & & & & \mathcal{P}_{\mathcal{C}}(M) \\ & \searrow & & \nearrow & \\ & & \mathcal{P}_\Lambda(M) & & \end{array} \tag{23}$$

Then $\mathcal{P}_{\mathcal{C}}(M)$ and $\mathcal{P}_\Lambda(M)$ are Banach manifolds, the $\mathcal{P}_s(M)$ are Hilbert manifolds and $\mathcal{P}_\infty(M)$ is a Fréchet manifold. Assuming M to be smooth they are in fact all $\mathcal{C}^\infty$ manifolds in the appropriate sense. To see why this is so, take a metric on M and, for any $\epsilon > 0$ smaller than the injectivity radius, consider the covering by these open metric balls. The exponential map at each $p \in M$ identifies the ball with the ball of the same radius around the origin in T_pM and the composites $F_{pq} = \exp_p^{-1} \exp_q$ give smooth transition maps.

Now for a given element $\chi \in \mathcal{P}_{\mathcal{C}}(M)$ consider the paths

$$\mathcal{N}(\chi) = \{\chi' \in \mathcal{P}_{\mathcal{C}}(M); d_M(\chi'(t), \chi(t)) < \epsilon \ \forall \ t \in [0, \pi]\}. \tag{24}$$

These are open subsets and pull-back along the base curve gives a bijection

$$\sigma' = \exp^*_{\chi(t)} \chi', \ \operatorname{Exp}_\chi : \mathcal{N}(\chi) \longleftrightarrow \{\sigma' \in \mathcal{C}([0, \pi]; \chi^*TM); |\sigma'(t)|_g < \epsilon\} \tag{25}$$

with the continuous sections of the pull-back of the tangent bundle under the curve and similarly for the other regularities. Two of the $\mathcal{N}(\chi)$ intersect if and only if for the corresponding base curves $d(\chi_1(t), \chi_2(t)) < \epsilon$ for all $t \in [0, \pi]$. Then the induced transition map

$$\mathcal{F}_{\chi_2,\chi_1} : \operatorname{Exp}_{\chi_2} \circ \operatorname{Exp}_{\chi_1}^{-1} : \operatorname{Exp}_{\chi_1}(\mathcal{N}(\chi_1) \cap \mathcal{N}(\chi_2)) \longrightarrow \operatorname{Exp}_{\chi_2}(\mathcal{N}(\chi_1) \cap \mathcal{N}(\chi_2)) \tag{26}$$

is a diffeomorphism, i.e. it is infinitely differentiable on these open subsets of Banach spaces.

Indeed, the transition maps are non-linear bundle maps. The derivatives of the transition maps on M are symmetric $|\alpha|$-multilinear maps

$$F^\alpha_{pq}(m) : T_qM \times T_qM \times \cdots \times T_qM \longrightarrow T_pM \tag{27}$$

depending smoothly on m in the intersection of the coordinate balls. From (26) the corredponding (weak) derivatives exist for the transition maps on the path or loop spaces given by pointwise action:

$$\begin{aligned} \mathcal{F}^\alpha_{\lambda_2,\lambda_1} : \mathcal{C}^\infty(\mathbb{S}; \lambda_1^*TM) \times \cdots \times \mathcal{C}^\infty(\mathbb{S}; \lambda_1^*TM) \ni (\sigma'_2, \ldots, \sigma'_{|\alpha|}) \\ \longmapsto F_{\lambda_1(*),\lambda_2(*)}(\sigma'_1(*), \ldots, \sigma'_{|\alpha|})(*)) \in \mathcal{C}^\infty(\mathbb{S}; \lambda_2^*TM) \end{aligned} \tag{28}$$

This corresponds to the fact that the only differentiation which arises is of the transition maps for exp on M. Note that χ^*TM is trivial as a bundle over the interval. This carries over to the other regularities and to the loop spaces, with the triviality statement for tge pull-back under loops being orientability.

It is rather natural to think of these spaces being 'thickenings' of $\mathcal{P}_\infty(M)$ or $\mathcal{L}_\infty(M)$ in which it is dense. In all cases the tangent space at a path or loop is naturally the space of sections, of the corresponding regularity, of χ^*TM. In the standard approach to manifolds the cotangent space would be defined as the dual of the tangent space. Since it is rather natural to think of this as a space of sections of the cotangent space on the manifold and the most natural pairing between sections of TM and sections of T^*M pulled back is

$$\int_{\mathbb{S}} \lambda^*(v(s)\cdot w(s))ds \tag{29}$$

this would realize the cotangent space as the dual space of sections of λ^*T^*M, i.e. measures for $\mathcal{L}_{\mathcal{C}}(M)$ and distributions for $\mathcal{L}_\infty(M)$. Not only is this 'handist' but it is unwisely prescriptive since in fact on $\mathcal{L}_s(M)$ all the spaces

$$H^t(\mathbb{S};\lambda^*TM),\ H^t(\mathbb{S};\lambda^*T^*M) \text{ for } -s<t<s \tag{30}$$

make invariant sense. Thus the pointwise value of 'vector field' of a '1-form' on $\mathcal{L}_s(M)$ can be reasonably taken to lie in any one of these spaces. There are then many notions of regularity of objects over the loop spaces. Note in particular that continuity of a function on $\mathcal{L}_s(M)$ is a stronger statement than continuity on the dense subspace $\mathcal{L}_\infty(M)$.

One reason that the Lipshitz paths and loops are relevant is that (affine) arclength (re-)parameterization of a curve (so the parameter length is still 2π) is well-defined as a map

$$\mathcal{L}_1(M)\longrightarrow\mathcal{L}_\Lambda(M). \tag{31}$$

This is another possible definition of a string.

Let me make an apparent digression. As Mathematicians we can ask a question that perhaps the Physists do not feel bound to ask themselve. Namely, what precisely is a String? Clearly it is related to a loop, an element of the free loop space, say smooth

$$\mathcal{L}_\infty(M)=\{\lambda:\mathbb{S}\longrightarrow M\ \mathcal{C}^\infty\}. \tag{32}$$

The diffeomorphism group of the circle acts on this by reparameterization and one (still not quite ideal) definition of a String is that it is an element of the quotient space

$$\mathcal{L}_\infty(M)/\operatorname{Dff}^+(\mathbb{S}) \tag{33}$$

given as the orbits under the action of the orientation-preserving diffeomorphisms of the circle. Thus a String is (perhaps) an 'unparameterized' loop. Of course taking a quotient like this is dangerous since the diffeomorphism group does not act freely, so the quotient is bound to be rather singular. Instead we might, and do, look for $\operatorname{Dff}^+(\mathbb{S})$ invariant or equivariant objects on $\mathcal{L}(M)$ and think of them as objects on the quotient space.

It suggest that we can 'transgress' things from M to $\dot{\mathcal{L}}(M)$ without loss of information if we are careful. One version of this is coholomogy threre is an evaluation map

$$\text{ev} : \mathbb{S} \times \dot{\mathcal{L}}(M) \ni (\theta, \lambda) \longmapsto (\lambda(\theta) \in M \tag{34}$$

and this allows cohomology to be pulled back and then integrated over the circle to define transgression from the upper left part of the diagram for each $k \geq 1$

$$\begin{array}{ccc} H^k(\mathbb{S} \times \dot{\mathcal{L}}(M); \mathbb{Z}) & \xleftarrow[\text{ev}^*]{} & H^k(M; \mathbb{Z}) \\ \Big\downarrow \pi_* & \swarrow \tau & \Big\updownarrow \tau_{\text{fus}} \\ H^{k-1}(\dot{\mathcal{L}}(M); \mathbb{Z}) & \xleftarrow{\text{fg}} & H^{k-1}_{\text{fu}}(\dot{\mathcal{L}}(M), \mathbb{Z}) \end{array} \tag{35}$$

Here Chris Kottke and I introduce 'fusive' cohomology in Čech cohomology by introducing fusion and a second figure-of-eight condition at chain level. This makes the corresponding cohomology spaces invariant under reparameterization and gives an isomorphism as indicated on the right, with a forgetful map to ordinary cohomology.

So a general 'principle' here is that

> Objects can be trangressed, without loss, to fusive objects on the loop space.

One particular case of this is the notion of a *string structure*. This corresponds to the third step in the Whitehead tower (1).

The question then is analogous to (10) where we ask about the existence of a lift of the spin frame bundle

$$\begin{array}{ccc} \text{String}(n) & \text{———} & F_{\text{String}} \\ \big\downarrow & & \big\downarrow \\ \text{Spin}(n) & \text{———} & F_{\text{Spin}} \\ & & \big\downarrow \\ & & M. \end{array} \tag{36}$$

The string group is not really well-defined but the existence of the is independent of choice and the final word here is due to Redden:

> A string structure exists if and only $\frac{1}{2}p_1 = 0$ and then the equivalence classes are parameterized by $H^3(M; \mathbb{Z})$.

Here $\frac{1}{2}p_1 \in H^4(M; \mathbb{Z})$ is the Pontryagin class of the *Spin* principal bundle, it is an integral class and $2 \times \frac{1}{2}p_1$ is the 'usual' Pontryagin class of S_{SO}. It is difficult to contemplate doing analysis on F_{String} or related spaces. However the principle about holds once we understand the appropriate structure over the loop space.

Before talking about this, let me addess the obvious question:- "What is a String?". The amswer is a bit murky but one interpreation is that it is a loop up to reparamerization. The diffeomorphism group of the circle, $\text{Dff}(\mathbb{S})$ acts on the loop space

Since $\pi_3(\text{Spin}(n)) = \mathbb{Z}$, we know from (8) that $\pi_2(\dot{\mathcal{L}}(\text{Spin})) = \mathbb{Z}$ and that this loop space is simply-connected. It follows that there is a circle bundle with Chern class a generator of $H^2(\dot{\mathcal{L}}(\text{Spin}))$. In fact this class is equivariant and the circle bundle corresponds to a central extension

$$\mathrm{U}(1) \longrightarrow \widehat{\dot{\mathcal{L}}}(\mathrm{Spin}) \longrightarrow \dot{\mathcal{L}}(\mathrm{Spin}). \tag{37}$$

There is a $\mathbb{Z}$ of such extensions, but all obtained from the 'basic' one (37) by covering. The corresponding element of $H^2(\dot{\mathcal{L}}(\mathrm{Spin}))$ is called the *level* of the central extension.

So now we see a corresponding lifting question over the loop space; whether there is a lift of the top principal bundle in (10) corresponding to the central extension

$$\begin{array}{ccc} \widehat{\dot{\mathcal{L}}}(\mathrm{Spin}(n)) & \text{———} & \mathcal{F} \\ \downarrow & & \downarrow \\ \dot{\mathcal{L}}(\mathrm{Spin}(n)) & \text{———} & \dot{\mathcal{L}}(F_{\mathrm{Spin}}) \\ & & \downarrow \\ & & \dot{\mathcal{L}}(M); \end{array} \tag{38}$$

The projections from the top line here correspond to circle bundles.

The situation is very similar to the spin to orientation transgression above. Here the existence of a string structure imples the existence of a extension. One way of seeing this is to note that such a U(1) lifting problem corresponds to the triviality of a 'lifting bundle gerbe' in the sense of Murray. If we think much more abstractly of a principal bundle with a (possibly large but topoligical) group with a central extension $\mathrm{U}(1) \longrightarrow \widehat{\mathcal{G}} \longrightarrow \mathcal{G}$

$$\begin{array}{ccccccccccc} \widehat{\mathcal{G}} & \text{——} & \widehat{\mathcal{P}} & & L & & s : \delta L \simeq \mathrm{U}(1) & & \delta s = 1 & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{G} & \text{——} & \mathcal{P} & \longleftarrow & \mathcal{P}^{[2]} & \longleftarrow & \mathcal{P}^{[3]} & \longleftarrow & \mathcal{P}^{[4]} & \longleftarrow & \cdots \\ & & \downarrow & & & & & & & & \\ & & M & & & & & & & & \end{array} \tag{39}$$

Here $\mathcal{P}$ is the total space of the principal bundle and to the right are the various fibre products ove M forming a simplicial space (so there are k mmps from $\mathcal{P}^{[k]}$ to $\mathcal{P}^{[k-1]}$. Then L is a circle bundle defined from the central extension of $\mathcal{G}$. Namely the fibre of $\mathcal{P}^{[2]}$ over m is the set of pairs $(f_1, f_2) \in \mathcal{P}_m \times \mathcal{P}_m$ so the $\mathcal{G}$ action induces a shift map $\mathcal{P}^{[2]} \longrightarrow \mathcal{G}$ mapping $(f_1, f_2 = gf_1)$ to g. Then L is the pull-back of the circle bundle given by the central extension. This circle bundle is simplicial (in the sense of Brylinski), as indicated – the simplicial differential gives a line bundle over $\mathcal{P}^{[3]}$ as the tensor product of the three pull-backs of L. From its definition this has a section, s trivializing it. This comes from the product on $\widehat{\mathcal{G}}$. Again the simplicial differential of a section is a section of the simplicial differential of the new circle bundle, where now we are over $\mathcal{P}^{4]}$. Here the line bundle is canonically trivial and the pulled back section is given by this canonical trivialization. So this is a bundle gerbe and Murray's theorem is

> A bundle gerbe induces a Dixmier-Douady class $\mathbb{D}(L) \in H^3(M; \mathbb{Z})$ and for a lifting bundle gerbe the vanishing of this class is equivalent

to the existence of a lifted principal bundle $\widehat{\mathcal{P}}$ such that as a circle bundle over $\mathcal{P}$ its image under δ is L.

Now, the Dixier-Doaady class of this bundle gerbe is the trangression of the obstruction to the existence of a string structure.

$$\mathbb{D}(\dot{\mathcal{L}}F_{\text{Spin}} = \tau(\frac{1}{2}p_1) \tag{40}$$

Thus again, the existence of a string structure implies the existence of a 'spin structure' in the sense of a lift of the principal bundle as in (39). However, just as before this transgressed spin structure has additional properties, fusion, figure-of-eight and equivariance; we also need to consider regularity.

Returning to the lifting question (38) we first work in the setting of continuous objects. A lifted principal bundle as in (38) is said to be fusive if the circle bundle given by $\mathcal{F}$ over $\dot{\mathcal{L}}(F_{\text{Spin}})$ is simplicial, i.e. defines a bundle gerbe, and also has an action of $\widehat{\text{Dff}}^+(\mathbb{S})$ equivariant with respect to the reparameterization action on $\dot{\mathcal{L}}(F_{\text{Spin}})$. We call such a lifted principal bundle a 'loop spin structure'. It is shown by Waldor and in [**?**] that

String structures on M are in 1-1 correspondence with loop spin structures.

Of course the automorphism of these objects need to be taken into account too.

In fact as follows from the construction in [**?**] the loop spin structure can be constructed over $\mathcal{L}_E(F_{\text{Spin}})$ (where E corresponds to $s = 1$)

(1) Positive energy representations, smoothness. M.-Valiveti.
(2) Loop spin bundle, connection, Clifford action
(3) Dirac-Ramond operator.
(4) Null spaces. Conjecture
(5) Index – Witten, Landweber.
(6) Modular forms and Bott+Virasoro.

(1) Introduction and the circle
    (a) Functions and distributions on the circle
    (b) Toeplitz operators
(2) Loop and path spaces, coordinate covers, localization
(3) Holonomy, fusion and simplicial spaces
(4) Reparameterization
(5) Transgression and regression
(6) Gerbes, loop groups, extensions
(7) Positive energy representations
(8) The Bott-Virasoro group
(9) Spin, string and loop-spin structures
(10) Dirac-Ramond operators
(11) Index and genera

References

Department of Mathematics, Massachusetts Institute of Technology
Email address: rbm@math.mit.edu