18.199 Lecture for April 2
The Spin Dirac Operator

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1 Spinors and Spin Connections

The main goal of today's lecture is to define the spin Dirac operator. In order to do this, we first need to tie up some loose ends from the previous lectures.

1.1 Spinors

Before defining the spin Dirac operator, we review spin structures and discuss some important properties of the spin connection.

The vector space of complex n-spinors is $\Delta_n = \mathbb{C}^{2\lfloor \frac{n}{2} \rfloor}$.

Definition 1.1.1 (Spin manifold). *A Riemannian manifold M^n is* spin *if its tangent bundle TM has a spin structure.*

If M^n is spin with associated principal Spin(n)-bundle $P_{\mathrm{Spin}(n)}$, then we have the spin representation $\rho_n : \mathrm{Spin}(n) \to \mathrm{End}(\Delta_n)$.

Definition 1.1.2 ($\mathbb{C}$-spinor bundle). *The* complex spinor bundle *$S \to M$ is the associated vector bundle*

$$S = P_{\mathrm{Spin}} \times_\rho \Delta_n = (P_{\mathrm{Spin}} \times \Delta_n)/\sim,$$

where we say that $(p \cdot g, s) \sim (p, \rho(g)s)$ for $g \in P_{\mathrm{Spin}}$.

As an example, if $P_{SO(n)}$ denotes the oriented orthonormal frames of M and $\rho : SO(n) \to SO(n) \subseteq Gl(\mathbb{R}^n)$ is the standard representation, then $TM \cong P_{SO(n)} \times_\rho \mathbb{R}^n$.

Definition 1.1.3 (Clifford algebra bundle). *The* Clifford algebra bundle *is*

$$\mathrm{Cl}(M) = \bigcup_{x \in M} \mathrm{Cl}(T_x^* M, g_x).$$

It is important to note that this bundle is associated to $P_{SO(n)}$, and each Clifford algebra here is a real Clifford algebra.

Since each Clifford algebra embeds into $\mathrm{End}(\Delta_n)$, fiberwise Clifford multiplication gives a map

$$\mathcal{C}^\infty(T^*M \otimes S) \to \mathcal{C}^\infty(S),\ v \otimes s \mapsto c(v)s = v \cdot s.$$

More precisely, each fiber of the Clifford algebra bundle $\mathrm{Cl}(M)$ is formed by "products" of cotangent vectors at x, subject to the relation

$$u \cdot v + v \cdot u = -2g(u, v) \cdot \mathrm{Id}$$

for all $u, v \in T_x M$.

Definition 1.1.4 (Spinor). *Sections of the spinor bundle S are called* spinors *on M.*

1.2 Spin connections

Let (M^n, g) be an oriented Riemannian manifold with Levi-Civita connection ∇. Define the one-forms $\tilde{\omega}_j^k = \Gamma_{ij}^k dx^i$ so that $\nabla \partial_{x^j} = \tilde{\omega}_j^k \partial_{x^k}$. In a local orthonormal basis e_i of TM, $\nabla e_i = \tilde{\omega}_{ij} e_j$ (that is, $\nabla_X e_i = \tilde{\omega}_{ij}(X) e_j$). The metric compatibility of the connection implies $\tilde{\omega}_{ij}$ is skew-symmetric. From this, we obtain a skew-symmetric matrix $\tilde{\omega} = \tilde{\omega}_{ij} \in \mathfrak{so}(n)$. Although this depends on the choice of orthonormal frame, one obtains a globally defined 1-form by passing to $P_{SO(n)}$.

Definition 1.2.1. *A* spin connection *is an $SO(n)$-valued 1-form ω on $P_{SO(n)}$ that satisfies the following properties.*

1. *If X_P is the vector field on $P_{SO(n)}$ induced by the right action of $\exp(tX) \in SO(n)$, where $X \in \mathfrak{so}(n)$, then $\omega(X_P) = X$*
2. *$g^*\omega = \mathrm{Ad}_{g^{-1}}(\omega)$ for all $g \in SO(n)$.*

Proposition 1.2.2. *If M is spin, any connection on $P_{SO(n)}$ naturally induces a connection on $P_{\mathrm{Spin}(n)}$, which in turn gives a covariant derivative on the spinor bundle*

$$\nabla^S : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(T^*M \otimes S).$$

In a local orthonormal frame $e_i = dx^i$ and local coordinates x,

$$\nabla^S_{\partial_{x^k}} \psi = \partial_{x^k} \psi + \frac{1}{4} \tilde{\omega}_{ij}(\partial_{x^k}) e_i \cdot e_j \cdot \psi.$$

The induced connection is also compatible with the Clifford multiplication:

$$\nabla^S_X(v \cdot \psi) = \nabla_X v \cdot \psi + v \cdot \nabla^S_X \psi.$$

The point of all this is that the Levi-Civita connection induces a connection ∇^S on the spinor bundle.

2 Spin Dirac Operators

2.1 Definition

Let us first recall the principal symbol of a differential operator.

Definition 2.1.1. *If E and F are vector bundles, $P : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(F)$ is a differential operator of order m if, in local coordinates,*

$$P = \sum_{|\alpha| \leqslant m} c_\alpha(x) \partial_x^\alpha,$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, and $c_\alpha(x) : E_x \to F_x$ is a linear map.

Definition 2.1.2. *The* principal symbol *of a differential operator P is defined by*

$$\sigma_\xi(P) = i^m \sum_{|\alpha| = m} c_\alpha(x) \xi^\alpha : E_x \to F_x,$$

where $\xi = \xi_k dx^k \in T_x^*M$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Let (M^n, g) be a spin manifold with spinor bundle S.

Definition 2.1.3. *The* Dirac operator $D : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S)$ *is defined by* $D = c \circ \nabla^S$*, where* $\nabla^S : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(T^*M \otimes S)$ *is the spin connection and* $c : \mathcal{C}^\infty(T^*M \otimes S) \to \mathcal{C}^\infty(S)$ *is Clifford multiplication.*

In a local basis e_i, the Dirac operator is given by $D = e_i \cdot \nabla^S_{e_i}$, where we identify e_i^* with e_i via g. In local coordinates x,

$$D = dx^i \cdot \left(\partial_i + \frac{1}{4}\tilde{\omega}_{ijk} e_j \cdot e_k \cdot\right).$$

From this, we see that D is a first-order differential operator and $\sigma_\xi(D) = i\xi\cdot$. In particular, D is elliptic since $|\xi|^2 = 0$ if and only if $\xi = 0$.

Lemma 2.1.4. *There exists a Hermitian inner product* $(,)$ *on* Δ_n *such that for any unit vector* $e \in \mathbb{R}^n$ *and* $v, w \in \Delta_n$*,*

$$(e \cdot v, e \cdot w) = (v, w). \tag{2.1}$$

Proof. Let e_i denote the standard orthonormal basis of $\mathbb{R}^n$, and let $(,)_0$ be any Hermitian inner product on Δ_n. We will construct a Hermitian inner product $(,)$ such that

$$(e_i \cdot v, e_i \cdot w) = (v, w) \tag{2.2}$$

for all $i = 1, ..., n$ and $v, w \in \Delta_n$. We claim that this gives us the desired Hermitian inner product. Indeed, since $e \cdot e = -|e|^2 = -1$ for any unit vector e, we see that (2.1) holds if and only if

$$(e \cdot v, w) = -(e \cdot v, e \cdot (e \cdot w)) = -(v, e \cdot w). \tag{2.3}$$

Therefore, if $e = \sum_{i \leqslant n} \alpha_i e_i$ is any unit vector in $\mathbb{R}^n$ and $(e_i \cdot v, e_i \cdot w) = (v, w)$ for each i, then

$$(e \cdot v, w) = \sum_{i \leqslant n} \alpha_i (e_i \cdot v, w) = -\sum_{i \leqslant n} \alpha_i (v, e_i \cdot w) = -(v, e \cdot w),$$

as desired. To construct $(,)$, we use a standard averaging process. Consider the finite subgroup $\Gamma = \{\pm 1, \pm e_1, ..., \pm e_n, ..., \pm e_{i_1} \cdots e_{i_k} (i_1 < \cdots < i_k), ..., \pm e_1 \cdots e_n\}$ and define $(,)$ by

$$(v, w) := \sum_{g \in \Gamma} (g \cdot v, g \cdot w)_0.$$

Then for all $g \in \Gamma$,

$$(g \cdot v, g \cdot w) = \sum_{h \in \Gamma} (h \cdot g \cdot v, h \cdot g \cdot w)_0 = \sum_{g' \in \Gamma} (g' \cdot v, g' \cdot w)_0 = (v, w)$$

since $h\Gamma = \Gamma$. Since each e_i is in Γ, (2.2) follows. □

Proposition 2.1.5. *The Hermitian inner product on* Δ_n *as above induces a Hermitian metric* $(,)$ *on the spinor bundle* S *such that for any vector field* X*,*

$$(X \cdot \psi, \psi') = -(\psi, X \cdot \psi').$$

It is also compatible with the connection in the sense that

$$X(\psi, \psi') = (\nabla^S_X \psi, \psi') + (\psi, \nabla^S_X \psi').$$

The point of this is that the Riemannian metric induces a Hermitian metric on the spinor bundle.

2.2 Basic properties

Proposition 2.2.1 (Spin Dirac operator is self-adjoint)**.** *With respect to the metric* $(,)$ *given by Proposition 2.1.5, the spin Dirac operator* D *is formally self-adjoint, i.e.*

$$\int_M (D\psi, \psi')dV_g = \int_M (\psi, D\psi')dV_g$$

for all $\psi, \psi' \in \mathcal{C}_0^\infty(S)$.

Proof. Fix $p \in M$ and work with a local orthonormal frame e_i such that $\nabla e_i = 0$ at p. Then at p,

$$\begin{aligned}
(D\psi, \psi') &= (e_i \cdot \nabla^S_{e_i}\psi, \psi') \\
&= -(\nabla^S_{e_i}\psi, e_i \cdot \psi') && \text{(by (2.3))} \\
&= -e_i(\psi, e_i \cdot \psi') + (\psi, \nabla^S_{e_i}(e_i \cdot \psi')) && \text{(metric compatibility)} \\
&= -e_i(\psi, e_i \cdot \psi') + (\psi, e_i \cdot \nabla^S_{e_i}\psi') && (\nabla e_i(p) = 0) \\
&= (\psi, D\psi') - e_i(\psi, e_i \cdot \psi'). && (2.4)
\end{aligned}$$

The second term of (2.4) is ostensibly a divergence term, and we verify that this is indeed the case. Define a vector field X on M by

$$\langle X, Y\rangle = -(\psi, Y \cdot \psi')$$

for any vector field Y. Then at p,

$$\operatorname{div}(X) = \langle \nabla_{e_i} X, e_i\rangle = e_i\langle X, e_i\rangle = -e_i(\psi, e_i \cdot \psi').$$

Plugging this into (2.4), we obtain

$$(D\psi, \psi') = (\psi, D\psi') + \operatorname{div}(X),$$

so it follows from the divergence theorem that

$$\int_M (D\psi, \psi')dV_g = \int_M (\psi, D\psi')dV_g + \int_M \operatorname{div}(X)dV_g = \int_M (\psi, D\psi')dV_g.$$

□

We note that if M has a nonempty boundary ∂M, then we instead obtain

$$\int_M (D\psi, \psi')dV_g = \int_M (\psi, D\psi')dV_g + \int_{\partial M} (\nu \cdot \psi, \psi')dV_g,$$

where ν denotes the outward unit normal of ∂M.

We devote the rest of this lecture to the Lichnerowicz formula for the square of the Dirac operator and an important consequence of it.

Definition 2.2.2 (Connection Laplacian)**.** *Let* $E \to M$ *be a vector bundle with connection* ∇^E*. The* connection Laplacian $\nabla^*\nabla : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ *is defined by*

$$\nabla^*\nabla s = -\sum_{i \leqslant n} (\nabla^E_{e_i}\nabla^E_{e_i} - \nabla^E_{\nabla_{e_i} e_i})s,$$

where e_i *is a local orthonormal frame for* TM.

Proposition 2.2.3 (Lichnerowicz)**.** *We have*

$$D^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{R},$$

where R *is the scalar curvature of* (M^n, g)*.*

The main application of the Lichnerowicz formula is the following Bochner-type result.

Corollary 2.2.4. *If* M *is closed and* $\mathrm{R} > 0$*, then* $\ker(D) = 0$*. Therefore, if* g *has positive scalar curvature, there are no harmonic spinors on* (M^n, g)*.*

Proof. If ψ is a harmonic spinor, $D\psi = 0$, so

$$0 = D^2\psi = \nabla^*\nabla\psi + \frac{1}{4}\mathrm{R}\psi,$$

hence

$$|\nabla^S\psi|^2 + \frac{1}{4}\mathrm{R}|\psi|^2 = \left\langle \nabla^*\nabla\psi + \frac{1}{4}\mathrm{R}\psi, \psi \right\rangle = 0.$$

Integrating over M, we obtain

$$0 = \int_M |\nabla^S\psi|^2 dV_g + \frac{1}{4}\int_M \mathrm{R}|\psi|^2 dV_g.$$

Since $\mathrm{R} > 0$, it follows that $\psi \equiv 0$. □