Hardy, Toeplitz and Bott–Virasoro

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Since no one seems very keen to speak I thought I would try to move things along and at the same time address Sasha Logunov's implied question: 'Is there any analysis here?' He was more polite than that, but it is a fair point.

<u>Claim</u> The group of orientation-preserving diffeomorphisms of the circle has a natural central extension giving a short exact sequence of groups

$$\mathbb{S} \hookrightarrow \widehat{\mathrm{Diff}}^{+}(S) \twoheadrightarrow \mathrm{Diff}^{+}(S)$$

This is of course a Theorem, I just have not defined the terms.

We identify the circle as the quotient

$$\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}.$$

Then there is a bijection,

$$\theta : [0, 2\pi) \longrightarrow \mathbb{S}$$

Any continuous map $f : \mathbb{S} \to \mathbb{S}$ has a lift under the covering map $\mathbb{S} \leftarrow \mathbb{R}$ and so lifts uniquely to a continuous map

$$f : \mathbb{S} \longrightarrow \mathbb{R}, \quad f(0) \in [0, 2\pi).$$

Using the coordinate θ this in turn lifts to a unique continuous map

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}, \quad \tilde{f}(0) \in [0, 2\pi],$$

$$\tilde{f}(\theta + 2\pi) = \tilde{f}(\theta) + 2\pi w$$

where $w \in \mathbb{Z}$ is the winding number of f.

The f is smooth iff $\tilde{f}$ is smooth.

If $f \in \mathrm{Diff}(\mathbb{S})$ is a diffeomorphism then $\tilde{f}' \neq 0$ and $\tilde{f}$ is 1-1 so either $f' > 0$ at $w = 1$ or $f' < 0$ at $w = -1$.

Thus the group of orientation-preserving diffeomorphisms can be written as

$$\text{(A)} \quad \mathrm{Diff}^+(\mathbb{S}) = \mathbb{S} \times \left\{ g \in C^\infty(\mathbb{S}),\ g > 0,\ \int_{\mathbb{S}} g = 2\pi \right\}$$

where the map takes $\tilde{f}$ to $(\tilde{f}(0), \tilde{f}')$.

<u>Lemma 1</u> $\mathrm{Diff}^+(\mathbb{S})$ is smoothly contractible to the circle $\mathbb{S}$.

<u>Proof</u> Take the map on the second factor in (A),

$$g_t = (1-t)g + t\,2\pi, \quad t \in [0,1]$$

which contracts it to be constant at $t = 1$.

We are interested in the action

of $\mathrm{Diff}^+(S)$ on functions, by pull-back

$$f^* u = u \circ f, \quad u \in C^\infty(S).$$

This extends to a bounded operator on $L^2(S)$ but observe that

$$\int |u \circ f|^2 d\theta = \int |u|^2 \delta \, d\theta'$$

where $\delta = d\theta / d\theta'$, where $\theta' = f(\theta)$. So this is not a unitary operator. To 'correct' this we can instead consider the action 'on half-densities'

$$(T) \quad T_f u = (f')^{1/2} u \circ f$$

which is norm-preserving. It is an action by the chain rule. Then

$$T_f : L^2(S) \longrightarrow L^2(S), \quad \| T_f \| = 1.$$

Although there are bounded functions on $L^2(S^1)$ the map

(B) $\quad \mathrm{Diff}^+(S^1) \longrightarrow U(L^2(S^1))$

is not norm continuous. It is easily seen to be strongly continuous. The topology I am taking here is that topology coming from (A), the C^∞ topology on the second factor — but this has nothing much to do with the lack of norm continuity

Exercise Show that (B) is not norm continuous — even on the rotation subgroup $S^1 \subset \mathrm{Diff}^+(S^1)$.

Next we think about the Hardy

space (S). One approach is to recall the basic properties of Fourier series. If $u \in L^2(\mathbb{S})$ then

$$(C) \qquad u(\theta) = \sum_{k \in \mathbb{Z}} u_k e^{ik\theta}, \quad \sum |u_k|^2 < \infty$$

with convergence in $L^2(\mathbb{S})$ (and pointwise a.e. if you want a harder result(...))
The Sobolev spaces on $\mathbb{S}$ are identified with the sequences

$$u \in H^s(\mathbb{S}) \Leftrightarrow \sum_k (1+k^2)^{s/2} |u_k|^2 < \infty$$

where for $s < 0$ convergence is in the space of distributions, $C^{-\infty}(\mathbb{S})$.

In terms of (C) the Hardy projection is the map

$$(D) \qquad Hu = \sum_{k \geq 0} u_k e^{ik\theta}.$$

There is always an issue of what to do with $k=0$ but here we keep it. A fundamental fact about H is that it 'almost commutes' with multiplication by smooth functions. This is due to the anomalous fact that the cotangent space to S has two 'infinities'. Anyway

<u>Proposition</u>. If $v \in C^\infty(S)$ acts on $L^2(S)$ as a multiplication operator then

$$[H, v] = Hv - vH$$

is a <u>smoothing operator</u> on $L^2(S)$, an integral operator of the form

$$u \longmapsto \int_S A(\theta,\theta')u(\theta')\,d\theta', \quad A \in C^\infty(S^2).$$

This is due, I think, to Toeplitz.

<u>Proof</u> The expansion in Fourier series, (C), identifies $L^2(S)$ with $\ell^2(\mathbb{Z})$.

Thus

$$(*) \quad v.u = \sum_k u_k (v e^{ik\theta}).$$

The Fourier coefficients are given by

$$u_k = \frac{1}{2\pi} \int u(\theta) e^{-ik\theta} d\theta$$

and applying this to $(*)$

$$(vu)_j = \sum_k V_{jk} u_k$$

$$V_{jk} = \frac{1}{2\pi} \int_S v(\theta) e^{i(k-j)\theta} d\theta$$

$$= v_{-k+j}.$$

Thus the action on the Fourier series

is (not surprisingly) in the form of convolution:

$$(vu)_j = \sum_k v_{-k+j} \, u_k .$$

From this we can easily compute the form of the commutator. Namely,

$$\big([Hv - vH] u \big)_j = \sum_k \beta_{jk} u_k$$

$$= \big(H(vu) \big)_j - (vHu)_j$$

$$= \sum_k \left(v_{j-k} u_k \right) - \sum_{k \geq 0} v_{j-k} u_k \qquad \cancel{j \geq 0}$$

$$= \begin{cases} \displaystyle\sum_{k<0} v_{j-k} u_k & j \geq 0 \\ \displaystyle -\sum_{k \geq 0} v_{j-k} u_k & j < 0 \end{cases}$$

So

$$\beta_{jk} = \begin{cases} v_{j-k} & j \geq 0, k < 0 \\ -v_{j-k} & j < 0, k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The point here is that the index $j-k$ is negative in the second case and positive in the first case. Since $v \in C^\infty(\mathbb{S})$,

$$|\beta_{jk}| \leq C_N (1+|j|+|k|)^{-N}$$

is rapidly decreasing.

Turning the operator $[H, V]$ back into an operator on $L^2(\mathbb{S})$ the kernel is

$$A(\theta, \theta') = \frac{1}{2\pi} \sum_{j,k} \beta_{jk}\, e^{ij\theta - ik\theta'} \in C^\infty(S \times S).$$

Note that there is no problem with convergence here, since we are just evaluating on the Fourier basis.

So this is for multiplication operators, but a similar result holds for our pull-back operator T_f in (T). We only need to think about the f^* part since

$$[H, T_f] = [H, (f')^{1/2}]\, f^* + (f')^{1/2} [H, f^*].$$

We look at this in terms of the

Toeplitz algebra

(S) $S_f = H T_f H$

as an operator on the Hardy space

$$\mathcal{H} = H L^2(\mathbb{S}) \subset L^2(\mathbb{S}).$$

Proposition The Toeplitz operator (S) is a Fredholm operator on $\mathcal{H}$ of index zero, with parametrix $S_{f^{-1}}$ up to smoothing operators (Toeplitz).

Proof. We go back and compute the form of $[H, f^*]$ as before, in particular its action on $e^{ik\theta}$.

Check

$$f^{*} e^{ik\theta} = \exp(ik\, f(\theta)).$$

So

$$[H, f^{*}]\, e^{ik\theta}$$

$$= (H f^{*}) e^{ik\theta} - f^{*} H e^{ik\theta}$$

$$= \begin{cases} H e^{ik f(\theta)} - e^{ik f(\theta)} & k \geq 0 \\ H e^{ik f(\theta)} & k < 0. \end{cases}$$

The Fourier components are therefore

$$\left([H, f^{*}]\, e^{ik\theta}\right)_j$$

$$= \begin{cases} -\left(e^{ik f(\theta)}\right)_j & j < 0,\ k \geq 0 \\ \left(e^{ik f(\theta)}\right)_j & j \geq 0\ \ k < 0 \end{cases}$$

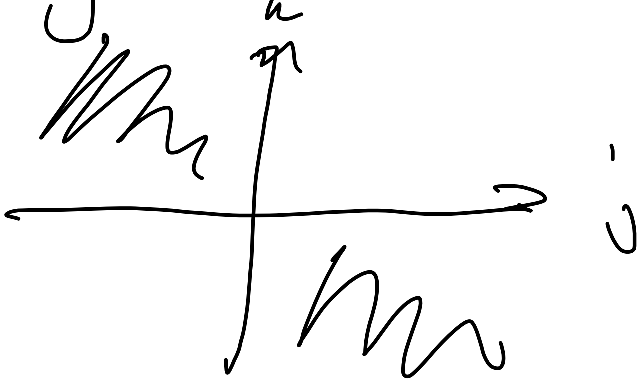
and zero otherwise. Since $f(\theta)$ is smooth

$$\left(e^{ihf(\theta)}\right)_j = \frac{1}{2\pi}\int e^{i(hf(\theta)-j\theta)}\, d\theta$$

is rapidly decreasing as $j, h \to \infty$ if $v<0, h\geq v,\ j\geq 0, h<0$. This follows by integrating by parts since

$$\frac{d}{d\theta}\left(hf(\theta)-j\theta\right) = hf'(\theta)-j$$

has fixed sign in the region in question



Thus, as before, $[H, f^*]$ is a smoothing operator. Consider the composite

$$S_{f^{-1}} S_f = H T_{f^{-1}} H T_f H$$

$$= H T_{f^{-1}} T_f H - H T_{f^{-1}} [T_f, H] H$$

$$= H - R_1.$$

The composite $R_1 = H T_{f^{-1}} [T_f, H] H$ is also a smoothing operator (the main thing for us is that it is compact).

It follows that

$$(F) \qquad \begin{aligned} S_{f^{-1}} S_f &= \mathrm{Id}_{\mathcal{H}} - R_1 \\ S_f S_{f^{-1}} &= \mathrm{Id}_{\mathcal{K}} - R_2 \end{aligned}$$

So ~~S~~ S_f is indeed a Fredholm operator on $\mathcal{H}$.

It remains to see that its index is zero. Certainly if f is a rotation then it commutes with H and so $S_f = fH$ is invertible.

The general case follows from Lemma 3.

This gives us a ~~strongly~~ continuous curve f_t connecting f to a rotation.

Although this is only strongly continuous, the shortcut operators $R_{1,t}$ and $R_{2,t}$ defined by (F) are actually norm continuous — even smooth in t.

It follows that the index, given by

$$\mathrm{ind}(S_{f_t}) = \mathrm{Tr}(R_{1,t}) - \mathrm{Tr}(R_{2,t})$$

(due I believe to Calderón) shows that the index is constant, hence vanishes.

$\square$

So it follows that S_f has finite dimensional null space and finite-dimensional complement to its range, of the same dimension. These are spanned by smooth elements of $L^2(S) \supset C^\infty(S)$ so there exists a (finite rank) smoothing operator B – given by a kernel $B \in C^\infty(S \times S)$ such that

(K) $\quad S_f + B$ is invertible.

<u>Defn</u> Let ~~X~~ $\mathcal{G}$ be the [illegible] $\mathcal{H}$ which consists of operators on $\mathcal{H}$ which are <u>unitary</u> and of the form (K) with B smoothing.

Clearly $\mathcal{G}$ is a group. The composites $S_f B$ and $B' S_f$ are smoothing and so is $S_f S_{f^{-1}} - S_{Id}$.

<u>Proposition</u>. There is a short exact sequence of groups

$$\text{(ES)} \quad \mathcal{G}^0 \hookrightarrow \mathcal{G} \xrightarrow{\sigma} \mathrm{Diff}^+(S)$$

where $\mathcal{G}^0$ is the group of unitaries, unitary operators on L^2 of the form $\mathrm{Id} + B$, B smoothing.

<u>Proof</u>. The only issue here is the definition of the 'symbol map' σ – and in particular that we can 'recover' T_f from $S_f + B$. This is again an asymptotic statement. Consider the sequence

$$S_k = (f')^{-1/2} e^{-ikf(\theta)} (S_f + B) e^{ik\theta} \qquad (\sigma)$$

The claim is that $S_k \longrightarrow I$ uniformly on $\mathbb{S}$ as $k \to \infty$. For a smoothing operator B, $Be^{ik\theta} \to 0$ in

$C^\infty(\mathbb{S})$ rapidly decreasing so only the term from S_f survives in (8). Again the commutator properties show for $k > 0$,

$$S_f e^{ik\theta} = H T_f e^{ik\theta}$$

$$= T_f e^{ik\theta} - [T_f, H] e^{ik\theta}$$

where again the second term is a smoothing operator. So indeed

$$(f')^{-1} e^{-ikf(\theta)} (S_f + B) e^{ik\theta} \longrightarrow 1$$

rapidly in $C^\infty(\mathbb{S})$.

Thus the second map in (E.S) is well-defined and multiplicative so the result follows. 2

Finally we are almost there. The subgroup $\mathcal{G}^0 \subset \mathcal{G}$ is normal

since

$$(S_f+B)^{-1}(Id+B')(S_f+B) \in \mathcal{G}^0.$$

Moreover we have at our disposal the Fredholm determinant (do we?)

$$\det(Id+B) \in \mathbb{C}^*$$

for $(Id+B)$ invertible. In fact since $\mathcal{G}^0$ consists of unitary operators the det takes values in S. The kernel

$$\tilde{\mathcal{G}} = \{ Id+B \in \mathcal{G}^0 ; \det(Id+B) = 1 \}$$

is again a normal subgroup of $\mathcal{G}$ since det is multiplicative.

<u>Theorem</u>. The quotient group

$$(BV) \quad \widehat{\mathrm{Diff}}^{+}(S) = \mathcal{G} / \tilde{\mathcal{G}}$$

is a central extension of $\mathrm{Diff}^{+}(S)$

$$\mathbb{S} \hookrightarrow \widehat{\mathrm{Diff}}^{+}(S) \to \mathrm{Diff}^{+}(S)$$

where the subgroup

$$\mathbb{S} = \mathcal{G}^{0} / \tilde{\mathcal{G}} \xrightarrow{\det} \mathbb{S}$$

is central.

This is a realization of the Bott–Virasoro group.

Next: Compute the extension cocycle.