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Since no one seems very keen to speak, I thought I would try to move things along and at the same time address for his loopers's simplified question: 'Is there any analysis here?' He was more polite than Ted, but it is a fair point.

Claim: The group of orientation-preserving diffeomorphisms of the circle has a natural central extension giving an exact sequence of groups:

$$S \hookrightarrow \text{Diff}^+(S) \xrightarrow{\tau} \text{Diff}^+_1(S).$$

This is of course a theorem, I just have not defined the terms.
We identify the circle as the quotient
\[ S = \mathbb{R} / 2\pi \mathbb{Z}. \]
Then there is a bijection,
\[ \Theta : [0, 2\pi) \to S \]
Any continuous map \( f : S \to S \) has a
left inverse the covering map \( S \leftarrow \mathbb{R} \) as
so lifts uniquely to a continuous map
\[ \tilde{f} : \mathbb{R} \to \mathbb{R}, \quad \tilde{f}(0) \in [0, 2\pi). \]
Using the coordinate \( \Theta \) the lift
then lifts to a unique continuous map
\[ \tilde{f} : \mathbb{R} \to \mathbb{R}, \quad \tilde{f}(0) \in [0, 2\pi), \]
\[ \tilde{f}(\Theta + 2\pi) = \tilde{f}(\Theta) + 2\pi \mathbb{Z} \]
where \( \mathbb{Z} \) is the winding number of \( f \).
The \( f \) is smooth if \( \tilde{f} \) is smooth.
If \( f \in \text{Diff}(S) \) is a diffeomorphism
then $\tilde{f}' = 0$ and $\tilde{f} > 1$ so extra $f > 0$ as $w = 1$ a $f < 0$ as $w = -1$.

Thus the group of orientation-preserving diffeomorphisms can be written as

\[(A) \quad Diff^+(\mathbb{S}) = \mathbb{S} \times \{g \in C^\infty(\mathbb{S}), \int g = 2\pi \}
\]

where the maps take $\tilde{f}$ to $(f|_0), f'$.\]

\textbf{Lemma 1} $Diff^+(\mathbb{S})$ is smoothly contractible to the circle $\mathbb{S}$.

\textbf{Proof} Take the map on the second factor $w(A)$,

\[g_t = (1-t)g + t2\pi, \quad t \in [0, 1],
\]

which contracts not to be constant at $t = 0$.\]

We are interested in the action
of $\text{Diff}^+(S)$ in functions, by pull-back

$$f^n = n \circ f, \quad n \in C^0(S).$$

This extends to a bounded quote on $L^2(S)$ but observe that

$$\int |n \circ f| \, d\theta = \int |n| \, f \, d\theta,$$

then $f = d\theta / d\theta'$ with $\theta' = f(\theta)$. So

this is not a matching factor. To correct this we can instead consider the action

on half-densities

\((T)\)

$$T \, n = (f')^{1/2} \, n \circ f$$

which is non-forcing. It is an action by the chain rule. The

$$T_f : L^2(S) \to L^2(S), \quad \|T_f n\| = f.$$
Although there are bounded families on $L^2(S)$ the map
\[ \text{diff}^+ (S) \to \bigcup L^2(S) \]
is not norm continuous. It is easily seen to be strongly continuous. The topology I am telling you is the topology coming from (A), it is $C^0$ topology on the second factor— but this has no relationship with the lack of norm continuity.

Exercise 5: Show that (B) is not norm continuous— even on the Abelian subgroup $\mathcal{S} \subset \text{diff}^+ (S)$.

Next we think about the Haag
span (S). One approach is to recall the basic properties of Fourier series. If \( u \in L^2(S) \) then

\[
(c) \quad u(\theta) = \sum_{k \in \mathbb{Z}} v_k e^{i k \theta}, \quad \sum_{k \in \mathbb{Z}} |v_k|^2 < \infty
\]

with convergence in \( L^2(S) \) (e.g., pointwise a.e. if you want a harder result...)

The Sobolev spaces on \( S \) are defined as

\[
1^\text{st} \text{ order Sobolev: } H^1(S) = \left\{ u \in L^2(S) \mid \sum_{k \in \mathbb{Z}} (1 + k^2) |v_k|^2 < \infty \right\}
\]

In fact, for \( s > 0 \) boundary is in the \( s \)-th order Sobolev space \( C^{s-1} \), \( C^s(S) \).

In terms of (c) it is Hardy projection on the unit circle

\[
(d) \quad H u = \sum_{k \geq 0} v_k e^{i k \theta}
\]
There is always an issue of what to do with \( h \to 0 \) but we will keep it. A fundamental fact about \( H \) is that it 'almost commutes' with multiplication by smooth functions. This is due to the fundamental fact that the kernel of \( A \) & the two 'infinite'. Anyway

**Proposition.** If \( v \in C^0(S) \) acts on \( L^2(S) \) as a multiplication operator

then

\[
[H, v] = Hv - vH
\]

is a smoothing operator a \( L^2(S) \), an \( \epsilon \) for

\[
u \to \int A(\theta, \theta') u(\theta') d\theta', \quad \epsilon \in C^0(S^2)
\]
This is true, I think, to Toeplitz.

Proof. The expansion in Fourier series (C), identify \( L^2(\mathbb{S}) \) with \( l^2(\mathbb{Z}) \).

This

\[ (\omega \cdot u) = \sum_{k} \frac{1}{k} \langle u, e^{-i k \theta} \rangle. \]

The Fourier coefficients are given by

\[ u_k = \frac{1}{2\pi} \int_{\mathbb{S}} u(\theta) e^{-i k \theta} \, d\theta \]

and applying Plancherel to (\(\star\))

\[ (\omega \cdot u)_j = \sum_{k} \frac{1}{k} V_{jk} u_k \]

\[ V_{jk} = \frac{1}{2\pi} \int_{\mathbb{S}} u(\theta) e^{i (k-j) \theta} \, d\theta \]

\[ = \langle u, e^{i (k-j) \theta} \rangle. \]

Thus \( \omega \) acts as the Fourier series.
in (14 superposing) in the form of equation
\[
(\mathbf{v}_k) \cdot \mathbf{u} = \sum_{i,j} \beta_{ijk} \mathbf{u}_i \cdot \mathbf{v}_{j-k} \mathbf{u}_k.
\]
From this we can exactly compute the form of the commutator. Namely,
\[
(\mathbf{H} \mathbf{v} - \mathbf{v} \mathbf{H}) \mathbf{u} = \sum_{i,j} \beta_{ijk} \mathbf{u}_i \cdot \mathbf{v}_{j-k} \mathbf{u}_k.
\]
= \left( \mathbf{H} (\mathbf{v} \mathbf{u}) \right)_j - \left( \mathbf{v} \mathbf{H} \mathbf{u} \right)_j
= \sum_{i,j} \left( \mathbf{v}_{j-k} \mathbf{u}_i \right) - \sum_{i,j} \mathbf{v}_{j-k} \mathbf{u}_i \cdot \mathbf{v}_{j-k} \mathbf{u}_i.
= \left\{ \begin{array}{ll}
\sum_{i,j=0}^{2\pi} \mathbf{v}_{j-k} \mathbf{u}_i & \text{if } j \geq 0 \\
-\sum_{i,j=0}^{2\pi} \mathbf{v}_{j-k} \mathbf{u}_i & \text{if } j < 0
\end{array} \right.
\]
\[ \beta_{jk} = \begin{cases} -1 & j < 0, k \geq 0 \\ 1 & j > 0, k \leq 0 \\ 0 & \text{otherwise} \end{cases} \]

Le focus here is that the entry \( j - k \) is negative in the second case and positive in the first case. Since \( \psi \in C^0(\mathbb{S}) \),

\[ |\beta_{jk}| \leq C \psi (1 + |j| + k)^{-1/2} \]

This is sufficiently decreasing.

Thus its effect \([H, \psi]\) is an operator in \( L^2(\mathbb{S}) \) the kernel is
$$A(0, 0) = \frac{1}{2\pi i} \sum_{j,k} \frac{1}{j+k} \log \left( e^{i\theta - i\phi} e^{i\phi - i\theta} \right)$$

Notice that this is a function with conjugate axis, since we can just reduce this to a finite basis.

So this is for multiplication operators, but a single would help for
an pull-back operator $T_f$ in (1).

We only need to think about the $f^*_{\#}$

At $x_0$

$$[H, T_f] = [H, (f')^{1/2}] f^* f^* + (f')^{1/2} [H, T_f].$$

We look at this in terms of the torus $SL_2$.
\[ S_f = H T_f H \]
as an operator on the Hardy space \( H \)
\[ H = H L^2(S) \subset L^2(S). \]

**Proposition** The Toeplitz operator \( S_f \) is a Fredholm operator on \( H \) of index \( 0 \), with \( \sigma(S_f) \) compact.

**Proof.** We go back and compute the form of \( [H, f^*] \) as before, and conclude it is a Fredholm operator.
\[ f^\prime e^{i\theta} = \exp(\frac{\imath}{\hbar} H f(\theta)). \]

\[ \begin{align*}
[\hat{H}, \mathcal{F}_k] e^{i\theta} & = (\hat{H} e^{i\theta} f_0 - f_0 \hat{H} e^{i\theta}) \\
& = \frac{1}{\hbar} [\hat{H} e^{i\theta} f_0 - e^{i\theta} \hat{H} f_0] \\
& = \begin{cases} \hbar e^{i\theta} f_0 & k > 0 \\ -e^{i\theta} f_0 & k < 0. \end{cases}
\end{align*} \]

The Fourier components are therefore
\[ \mathcal{F}_k e^{i\theta} = \begin{cases} \hbar e^{i\theta} f_0 & k > 0 \\ -e^{i\theta} f_0 & k < 0. \end{cases} \]

\[ = \begin{cases} -\left( e^{-2\pi i jk} \right) & j < 0, k > 0 \\ \left( e^{-2\pi i jk} \right) & j > 0, k < 0. \end{cases} \]
at zero energy. So if \( f(\theta) \) is smooth,

\[
\left( e^{iH\tau} - \delta(\theta) \right)_j = \frac{1}{i\pi} \int e^{iH(\tau) - i\theta} \, d\theta
\]
is rapidly decreasing as \( \tau \to 0 \) if \( u < 0, k > 0, \tau > 0 \). This follows by considering the path \( \sin \)

\[
\frac{d}{d\theta} \left( Hf(\theta) - i\theta \right)
\]

\[
= Hf'(\theta) - i
\]

in the first region, i.e., the region i m.
Thus, as before, $[L_H, f^*]_f$ is a similarity operator. Consider the composite

$S_{f^{-1}} S_f = H T_{f^{-1}} H T_f$

$= H T_{f^{-1}} T H - H T_f L_{f^*} H T_f$

$= H - R_1$

The compact $R_1 = H T_{f^{-1}} [L_{f^*}, H] T_f$ is also a similarity operator (like $M_1$, $M_2$, etc., but not $G_1$, $G_2$).

It follows that

(F)

$S_{f^{-1}} S_f = T L_{f^{-1}} - R_1$

$S_f S_{f^{-1}} = T L_{f} - R_1$
So $f_S$ is handed to Fred and
puts an $H$

It remains to see that its
index is $g_0$. Certainly if $f$ is
an $H$-type the $g_0$ comes with it
as $S_f = fH$ is invertible.

The general case follows from these.

The given map a $1$-type continues
can $fH$ continue $f$ to a $H$-type.
Although this is only stability
continuing the shortness of the $R_{1,1}$ of $R_{2,1}$ defined by $(F)$ as actually
these contins - even such $x = t$. 
It follows that the index, given by
\[ \text{id}(\Sigma_t) = \text{Tr}(P_{t,t}) - \text{Tr}(P_{t+1}) \]
are I believe by Candelas) show that the index is constant, hence vanishes.

So it follows that if the first domain will space and
first dimension complete to the
ray, of the same domain. Then
are spawned by smooth subsets of
\[ L^2(\Sigma) \subseteq C^\infty(\Sigma) \] or for then
exists a (finite rank) structure
over B - given by a kernel \( B+C^\infty(\Sigma) \)
Subtend
(K) \( \Sigma_f + \mathcal{B} \text{ is addable.} \)

Define \( \delta_g(Y) \) to be the closure of \( \delta_f(\mathcal{B}) \) in \( \mathcal{B} \), which is a \( \mathcal{B} \)-module and a \( \mathcal{B} \)-submodule.

Clearly \( Y \) is a group. If \( Y \) is a \( \mathcal{B} \)-module, then \( \delta_f(\mathcal{B}) \) is a \( \mathcal{B} \)-submodule and \( \delta_f(\mathcal{B}) - \delta_f(\mathcal{B}) \) is a direct summand of \( \mathcal{B} \).

Proposition. There is a \( \mathcal{B} \)-subspace of \( \mathcal{B} \).

(ES) \( \mathcal{G} \to \mathcal{G} \to \mathcal{G} \cong \mathcal{H} \)
where $\mathcal{G}$ is the graph of $\mathbf{m}$ and

\[ \mathbf{m}(L) \text{ defines a } \mathbf{G} \text{ of the form } \mathbf{I} + \mathbf{B}, \mathbf{B} \text{ square}. \]

**Proof.** The only noise here is the definition of the 'symbol map' $\mathcal{S}$, as a procedure that we can mean to for $\mathcal{S} + \mathbf{B}$. This is again an asymptotic statement. Consider it separate $\mathcal{H}(\mathbf{O})$

\[ \mathcal{S} = (f')' e (\mathcal{S} + \mathbf{B}) e i k \theta \quad (6) \]

The claim is $\mathcal{S} \to 1 \text{ uniformly as } k \to \infty$. For a smoothing function $B$, $B e i k \theta \to 0$ in
$C_0(S)$ is only the set of functions that tend to zero as $t \to \pm\infty$. Hence for $f \in C_0(S)$,

$$f \in i\theta \Rightarrow H f \in i\theta$$

$$= \mathcal{T}_f e^{i\theta} - [T_f, H] e^{i\theta}$$

The second term is a Shurley operator. So conclude

$$(S_f + B) e^{i\theta} \to f$$

refines to $C_0(S)$.

Thus, the second map is well-defined and multiplies the right hand side, so the result follows. 2
Finitely we are about there. The subgroup \( G^0 \subset G \) is normal since

\[(S_f + B)^{-1} (\text{Id} + B)(S_f + B) \in G^0.\]

Hence we have a non-degenerate Fredholm determinant \((\bullet \circ \circ ?)\)

\[\det(\text{Id} + B) \in C^\ast\]

\[\text{for } (\text{Id} + B) \text{ similar. In fact,}\]

\[\text{in } G^0 \text{ consists of finite limits}\]

\[\text{with } \det \text{ finite values } \in B. \text{ Then}\]

\[\hat{G} = \left\{ \text{Id} + B : \|G^0\| \neq 0 \text{ and } \det(\text{Id} + B) = 1 \right\}\]
is again a novel subgroup \( \mathfrak{g} \) of \( \mathfrak{g} \) sin. It is multiplicative.

\[ \text{Theorem. The quotient group} \]
\[ (BV) \quad \hat{\mathfrak{g}}^+(S) = \mathfrak{g}^* / \mathfrak{g} \]

is a central extension \( A \to \text{Diff}^+(S) \)
\[ \wedge + \]
\[ S \to \text{Diff}^+(S) \to \text{Diff}^+(S) \]

the the subgroup
\[ S = \mathfrak{g}^* / \mathfrak{g} \to S \]

is central:
\[ \triangleright \]

This is a realization of the B"ottcher-Vinogradov group.

\textit{Next: Complete the above agenda...}