18.199 Talk 1 : A Crash Course on Lie Groups

Dain Kim

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1 Lie groups

Definition 1.1. A set G is a *Lie group* if it is a group and a smooth manifold such that multiplication and inversion maps are smooth.

Example 1.2.
- $(\mathbb{R}^n, +)$, $(\mathbb{R}^\times, \times)$, $(S^1, \times)$,
- (Classical Lie groups) $\mathrm{GL}(n, \mathbb{K})$, $\mathrm{SL}(n, \mathbb{K})$, $\mathrm{O}(n, \mathbb{K})$, $\mathrm{SO}(n, \mathbb{K})$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(2n, \mathbb{K})$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Note that a priori it is not straightforward that the classical Lie groups are Lie groups as we defined. Let us begin with $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{SU}(2)$. The group $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ consists of $n \times n$ matrices with nonzero determinant. Hence, $\mathrm{GL}(n, \mathbb{R})$ has a smooth structure induced from $\mathbb{R}^{n^2}$ and is of dimension n^2. Also, it is well known that $\mathrm{SU}(2) = \{A \in \mathrm{GL}(2, \mathbb{C}) | AA^* = 1, \det A = 1\}$ can be rewritten as

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

It tells you that $\mathrm{SU}(2)$ is diffeomorphic to S^3, so $\mathrm{SU}(2)$ has a smooth structure as well. So $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{SU}(2)$ are Lie groups.

Definition 1.3. If G and H are Lie groups, a map $f \colon G \to H$ is a *homomorphism of Lie groups* if it is smooth and a group homomorphism.

Later we will see that for $n \geq 3$, $\mathrm{Spin}(n)$ is a universal cover of $\mathrm{SO}(n)$. Then the following theorem guarantees that $\mathrm{Spin}(n)$ also has a Lie group structure.

Theorem 1.4. Let G be a connected Lie group. Then its universal cover $\widetilde{G}$ inherits a canonical Lie group structure induced by G such that the covering map $p \colon \widetilde{G} \to G$ is a homomorphism of Lie groups and $\ker p = \pi_1(G)$ as a group. Moreover, $\ker p$ is a discrete central subgroup in $\widetilde{G}$.

2 Exponential map and Logarithmic map

Denote by $\mathfrak{gl}(n, \mathbb{K})$ the set of $n \times n$ matrices over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. To motivate the notation, we make a note here that it will be the tangent space of $\mathrm{GL}(n, \mathbb{K})$ at $1 \in \mathrm{GL}(n, \mathbb{K})$. However, we have not justified anything yet.

The exponential map for matrices is defined as

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for each $x \in \mathfrak{gl}(n, \mathbb{K})$. This power series converges well and defines an analytic map $\mathfrak{gl}(n, \mathbb{K}) \to \mathfrak{gl}(n, \mathbb{K})$. The logarithmic map for matrices is defined as

$$\log(X) = -\sum_{k=1}^{\infty} \frac{(1-X)^k}{k}$$

for each $X \in \mathfrak{gl}(n, \mathbb{K})$ close to 1. To be precise, this power series converges when the eigenvalues of $1 - X$ have modulus less than 1. Hence, the logarithmic map defines an analytic map in a neighborhood of $1 \in \mathfrak{gl}(n, \mathbb{K})$.

Theorem 2.1.
1. $\log(\exp(x)) = x$ and $\exp(\log(X)) = X$ whenever they are defined.
2. $\exp(0) = 1$ and $d\exp(0) = \mathrm{id}$.
3. If $xy = yx$, then $\exp(x + y) = \exp(x)\exp(y)$. If $XY = YX$ and both are sufficiently close to 1, then $\log(XY) = \log(X) + \log(Y)$.
4. The exponent map is conjugation-invariant: for any $A \in \mathfrak{gl}(n, \mathbb{K})$, $\exp(AxA^{-1}) = A\exp(x)A^{-1}$.
5. The exponent map is transpose-invariant: $\exp(x^t) = \exp(x)^t$.

In particular, since $\exp(x)\exp(-x) = 1$, $\exp(x) \in \mathrm{GL}(n, \mathbb{K})$ for any $x \in \mathfrak{gl}(n, \mathbb{K})$. Therefore, we may think of the exponential map as a map $\mathfrak{gl}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$.

Note that $\mathfrak{gl}(n, \mathbb{K})$ is a vector space.

Theorem 2.2. For each classical group $G \subset \mathrm{GL}(n, \mathbb{K})$, there exists a vector space $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{K})$ such that for some neighborhoods U of $1 \in \mathrm{GL}(n, \mathbb{K})$ and $\mathfrak{u}$ of $0 \in \mathfrak{gl}(n, \mathbb{K})$, the logarithmic map and exponential map on $U \cap G$ and $\mathfrak{u} \cap \mathfrak{g}$ are mutually inverse.

Corollary 2.3. Each classical Lie group $G \subset \mathrm{GL}(n, \mathbb{K})$ is a Lie group, its tangent space at 1 is $T_1 G = \mathfrak{g}$, and $\dim G = \dim \mathfrak{g}$.

We first prove Corollary 2.3 assuming Theorem 2.2.

proof of Corollary 2.3. We first prove that G has a smooth structure. Theorem 2.2 gives an identification between a neighborhood $U \cap G$ of $1 \in G$ and an open set of a vector space $\mathfrak{g}$. Therefore, it gives a smooth structure on the neighborhood of $1 \in G$. Then for any $g \in G$, since $(g \cdot U) \cap G = g \cdot (U \cap G)$, it follows that G is smooth on a neighborhood of g. Therefore, G is smooth.

The exponential map $\exp\colon \mathfrak{g} \to G$ induces $\exp_*\colon T_0\mathfrak{g} \to T_1 G$. Since $\mathfrak{g}$ is a vector space, $T_0\mathfrak{g}$ is identified to $\mathfrak{g}$. On the other hand, since $\exp(x) = 1 + x + \cdots$, its derivative is identity, so $\mathfrak{g} = T_0\mathfrak{g} = T_1 G$. □

Now we prove Theorem 2.2 for each classical Lie group.

proof of Theorem 2.2. We proceed based on casework.

- $\mathrm{GL}(n, \mathbb{K})$: Immediately follows from Theorem 2.1. We have $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$ in this case.
- $\mathrm{SL}(n, \mathbb{K})$: For each $X \in \mathrm{SL}(n, \mathbb{K})$ close to 1, $X = \exp(x)$ for some $x \in \mathfrak{gl}(n, \mathbb{K})$. Then we have
$$1 = \det X = \det \exp(x) = \exp(\mathrm{tr}(x)).$$
where the last equality can be seen by noting that exp is conjugation-invariant and so we may write x as an upper triangular matrix. Therefore, $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{K}) : \mathrm{tr}(x) = 0\}$.
- $\mathrm{O}(n, \mathbb{K})$: For each $X \in \mathrm{O}(n, \mathbb{K})$ close to 1, $X = \exp(x)$ and $X^t = \exp(x^t)$ for some $x \in \mathfrak{gl}(n, \mathbb{K})$. Since $XX^t = I$, X and X^t are commute. In particular, $x = \log(X)$ and $x^t = \log(X^t)$ commute as well. Then $1 = \exp(x)\exp(x^t) = \exp(x + x^t)$, so $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{K}) : x + x^t = 0\}$.
- $\mathrm{SO}(n, \mathbb{K})$: Note that $\mathrm{SO}(n, \mathbb{K})$ is a connected component of $\mathrm{O}(n, \mathbb{K})$ containing 1. Therefore, $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{K}) : x + x^t = 0\}$.
- $\mathrm{U}(n)$: A similar argument gives that $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{C}) : x + x^* = 0\}$.
- $\mathrm{SU}(n)$: Since $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$, we have $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{C}) : x + x^* = 0,\ \mathrm{tr}(x) = 0\}$.
- $\mathrm{Sp}(2n, \mathbb{K})$: A similar argument gives that $\mathfrak{g} = \{x \in \mathfrak{gl}(n, \mathbb{K}) : x + Jx^tJ^{-1} = 0\}$ where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

□

Definition 2.4. The tangent space $\mathfrak{g} = T_1 G$ is called *Lie algebra*.

Although we do not justify here, $\mathfrak{g}$ has a canonical skew-symmetric bilinear map $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying an identity called Jacobi identity: for any $x, y, z \in \mathfrak{g}$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

3 Topology of classical Lie groups

3.1 π_0 and π_1 of classical Lie groups

Here, we collect π_0 and π_1 of classical Lie groups without justification. The π_0 and π_1 of classical Lie groups are as in Table 1.

G	$\pi_0(G)$	$\pi_1(G)$
$\mathrm{GL}(n,\mathbb{R})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $(n \geq 3)$
$\mathrm{SL}(n,\mathbb{R})$	$\{1\}$	$\mathbb{Z}_2$ $(n \geq 3)$
$\mathrm{O}(n,\mathbb{R})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $(n \geq 3)$
$\mathrm{SO}(n,\mathbb{R})$	$\{1\}$	$\mathbb{Z}_2$ $(n \geq 3)$
$\mathrm{U}(n)$	$\{1\}$	$\mathbb{Z}$
$\mathrm{SU}(n)$	$\{1\}$	$\{1\}$
$\mathrm{Sp}(2n,\mathbb{R})$	$\{1\}$	$\mathbb{Z}$

G	$\pi_0(G)$	$\pi_1(G)$
$\mathrm{GL}(n,\mathbb{C})$	$\{1\}$	$\mathbb{Z}$
$\mathrm{SL}(n,\mathbb{C})$	$\{1\}$	$\{1\}$
$\mathrm{O}(n,\mathbb{C})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathrm{SO}(n,\mathbb{C})$	$\{1\}$	$\mathbb{Z}_2$

Table 1: π_0 and π_1 of classical Lie groups

One important fact is that $\pi_1(\mathrm{SO}(n,\mathbb{R})) = \mathbb{Z}_2$ because this implies that spin group $\mathrm{Spin}(n)$ is a twofold cover. We roughly provide an outline of its proof.

Proposition 3.1. $\mathrm{SO}(3,\mathbb{R}) \cong \mathbb{RP}^3$ and $\pi_1(\mathrm{SO}(3,\mathbb{R})) = \mathbb{Z}_2$.

proof sketch. Let V be a vector space

$$V = \mathfrak{su}(2) = \{x \in \mathfrak{gl}(2,\mathbb{C}) : x + x^*,\ \mathrm{tr}(x) = 0\} = \mathrm{Span}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right\}$$

with a bilinar form $(\ ,\ )\colon V \times V \to \mathbb{R}$ given by $(A,B) = \mathrm{tr}(AB)$. Then $\mathrm{SU}(2)$ acts on V by conjugation, and the bilinear form $(\ ,\ )$ is invariant under the action. This gives a homomorphism $\phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ which is a covering map since $SO(3)$ and $SU(2)$ are connected Lie groups. Moreover, since $SU(2) \cong S^3$ is simply connected, ϕ is a universal covering map. By analyzing ϕ, we get $\ker(\phi) = \{\pm 1\}$ and ± 1 identify the antipodals. Therefore, $\mathrm{SO}(3,\mathbb{R}) \cong \mathbb{RP}^3$ and $\pi_1(SO(3)) = \mathbb{Z}_2$. □

The group $\mathrm{SO}(n,\mathbb{R})$ acts on the sphere $S^{n-1} \subset \mathbb{R}^n$. So we have a fiber bundle

$$\begin{array}{ccc} \mathrm{SO}(n-1,\mathbb{R}) & \longrightarrow & \mathrm{SO}(n,\mathbb{R}) \\ & & \downarrow \\ & & S^{n-1} \end{array}$$

Then we get an exact sequence

$$\pi_2(S^{n-1}) \to \pi_1(\mathrm{SO}(n-1,\mathbb{R})) \to \pi_1(\mathrm{SO}(n,\mathbb{R})) \to \pi_1(S^{n-1}) \to \{1\}.$$

In particular, we get $\pi_1(\mathrm{SO}(n-1,\mathbb{R})) \cong \pi_1(\mathrm{SO}(n,\mathbb{R}))$ for $n \geq 4$, so $\pi_1(\mathrm{SO}(n,\mathbb{R})) = \mathbb{Z}_2$ for all $n \geq 3$.

Proposition 3.2. $L\,\mathrm{Spin}(n)$ is the identity component of $L\,\mathrm{SO}(n)$.

Proof. Any loop in $\mathrm{Spin}(n)$ is uniquely projected down to $\mathrm{SO}(n)$ and any loop in $\mathrm{SO}(n)$ can be lifted to a path in $\mathrm{Spin}(n)$. The path is loop if and only if the loop in $\mathrm{SO}(n)$ is contractible. □

3.2 π_2 and π_3 of Lie groups

Here, we closely follow [Mil63].

Theorem 3.3. If G is a Lie group, then $\pi_2(G) = \{1\}$ and $\pi_3(G) = \mathbb{Z}$.

It is known that every connected Lie group deformation retracts onto its maximal compact subgroup. Therefore, we may assume that G is compact.

Definition 3.4. Let G be a Lie group and let e be the identity of G. A *piecewise smooth path based at* e is a map $p\colon [0,1] \to G$ such that $p(0) = e$ and there exists $0 = t_0 < t_1 < \cdots < t_k = 1$ for some k so that $p|_{[t_{i-1},t_i]}$ is smooth for all $1 \leq i \leq k$. The *pointed path space of* G, denoted by PG, is the space of all piecewise smooth paths based at e. If l is a piecewise smooth path based at e such that $l(1) = e$, then l is called a *piecewise smooth loop based at* e. The *pointed loop space of* G, denoted by ΩG, is the space of all piecewise smooth loops based at e.

Finally, define $\Omega G(x_0) := \{p \in PG : p(1) = x_0\}$ for $x_0 \in G$. Note that $\Omega G(x_0)$ and ΩG are homotopically equivalent.

Then $\Omega G \to PG \to G$ where the first arrow is the natural inclusion and the second arrow is $p \mapsto p(1)$ is a fibration. Since PG is contractible, it follows that $\pi_2(G) \simeq \pi_1(\Omega G)$.

Now we introduce three natural actions on Lie groups.

Definition 3.5. Let G be a Lie group. For $g \in G$, its Left action, Right action, and Adjoint action are defined as follows: given $g \in G$,

- Left action $L_g\colon G \to G$, $h \mapsto gh$
- Right action $R_g\colon G \to G$, $h \mapsto hg$
- Adjoint action $Ad_g\colon G \to G$, $h \mapsto ghg^{-1}$

Proposition 3.6. If G is a compact Lie group, then G admits a bi-invariant metric.

Proof. Let μ_e be a nonzero n-form at $\mathfrak{g}$. Then by defining $\mu_g := R^*_{g^{-1}}\mu_e$, we get a right Haar measure. Let $(\bullet,\bullet)$ be an inner product of $\mathfrak{g}$. Then we get another inner product by averaging out: for $u, v \in T_{g_0}G$,

$$\langle u, v\rangle = \int_G (Ad_g(u), Ad_g(v))\mu(dg).$$

This is Ad-invariant. Since this is also right-invariant, it follows that it is left-invariant as well. □

Proposition 3.7. Let G be a Lie group with a bi-invariant metric. If X, Y, Z, W are left invariant vector fields on G, then

(1) $\nabla_X X = 0$

(2) $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$

(3) $R(X, Y)Z = \frac{1}{4}[[X, Y], Z]$

(4) $\langle R(X, Y)Z, W\rangle = \frac{1}{4}\langle [X, Y], [Z, W]\rangle$

Proof. (1) Since the geodesics are precisely the one-parameter subgroups of G and integral curves of X are left translates of one-parameter group, the integral curve is geodesic. Therefore, $\nabla_X X = 0$.

(2) Applying (1) to $X + Y$, we get

$$0 = \nabla_{X+Y}(X + Y) = \nabla_X X + \nabla_X Y + \nabla_Y X + \nabla_Y Y = \nabla_X Y + \nabla_Y X.$$

On the other hand, the torsion freeness gives

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Combining these two yields $2\nabla_X Y = [X, Y]$. By metric compatibility, we also have

$$0 = Y\langle X, Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle.$$

Since Lie bracket is skew-symmetric, we get $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$.

(3) The definition of Riemann curvature gives

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z = -\frac{1}{4}[X, [Y, Z]] + \frac{1}{4}[Y, [X, Z]] + \frac{1}{2}[[X, Y], Z].$$

Using Jacobi identity, we get

$$R(X, Y)Z = \frac{1}{4}[[X, Y], Z].$$

(4) It follows from (2) and (3). □

Corollary 3.8. If G is a compact Lie group, then the sectional curvature is positive semi-definite.

Proof. By definition and the computations above, $\langle R(X, Y)X, Y\rangle = \frac{1}{4}\langle [X, Y], [X, Y]\rangle \geq 0$. □

Definition 3.9. Let M be a smooth manifold and γ a geodesic on M. A vector field J along γ is called a *Jacobi field* if it satisfies

$$\frac{D^2 J}{dt^2} + R(\gamma', J)\gamma' = 0.$$

Two points $x, y \in M$ are *conjugate* if there exists a geodesic $\gamma\colon [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ and a nonzero Jacobi field that vanishes at x and y.

Theorem 3.10 (Fundamental Theorem of Morse Theory)**.** Let G be a Lie group, and let $x_0 \in G$ be a point not conjugate to e along any geodesic. Then $\Omega G(x_0)$ has the homotopy type of a countable CW-complex which contains a λ-cell for each geodesic from e to x_0 of index λ.

Theorem 3.11 (Morse Index Theorem)**.** The index of a geodesic γ is finite and equal to the number of conjugate points $\gamma(t)$, $0 < t < 1$, counted with its multiplicity.

For $u \in \mathfrak{g}$, define a linear transformation $K_u \colon \mathfrak{g} \to \mathfrak{g}$ by $K_u(v) = R(u, v)u$.

Theorem 3.12. Let $\gamma \colon \mathbb{R} \to G$, $\gamma(0) = e$, be a geodesic. Then the conjugate points of e along γ are the points $\gamma(\pi k/\sqrt{e_i})$ where k is any nonzero integer and e_i is any positive eigenvalue of $K_{\gamma'}$.

Proof. Note that K_u is symmetric: for any $v, w \in \mathfrak{g}$,

$$\langle K_u(v), w\rangle = \langle v, K_u(w)\rangle$$

because this is equivalent to the symmetry relation

$$\langle R(u, v)u, w\rangle = \langle R(u, w)u, v\rangle.$$

Therefore, there exists an orthonormal basis $v_1, \ldots, v_n$ for $\mathfrak{g}$ so that $K_{\gamma'}(v_i) = \sigma_i v_i$ where σ_i are eigenvalues. Extend u and v_i to vector fields U and V_i along γ by parallel transport. Then since $R(U, V_i)U$ are parallel as well, $R(V, U_i)V = \sigma_i U_i$ holds on γ. Therefore, any vector field W along γ is uniquely written as

$$W(t) = w_1(t)U_1(t) + \cdots + w_n(t)U_n(t)$$

and the Jacobi equation $\sum_i \dfrac{D^2 w_i}{dt^2} U_i + \sum_i \sigma_i w_i U_i = 0$ is equivalent to

$$\frac{d^2 w_i}{dt^2} + \sigma_i w_i = 0$$

for all i. We are looking for solutions that vanish at $t = 0$. Therefore, if $\sigma_i > 0$, then the solution is $w_i(t) = c_i \sin(\sqrt{e_i} t)$ for some constant c_i. If $\sigma_i = 0$, then $w_i(t) = c_i t$, and if $\sigma_i < 0$, then $w_i(t) = c_i \sinh(\sqrt{-e_i} t)$. Therefore, the desired result follows. □

proof of $\pi_2(G) = \{1\}$. Let $x_0 \in G$ be a point not conjugate to e along any geodesics. By fundamental theorem of Morse Theory (Theorem 3.10), it is enough to show that each geodesic has even index. By Morse Index Theorem (Theorem 3.11), we only need to show that the geodesics have even number of conjugate points. Define adjoint homomorphism $\operatorname{ad} u \colon \mathfrak{g} \mapsto \mathfrak{g}$ for $u \in \mathfrak{g}$ by $v \mapsto [u, v]$. The conjugate points of e on a geodesic γ are determined by eigenvalues of the linear transformation $K_{\gamma'} \colon \mathfrak{g} \to \mathfrak{g}$ defined by

$$K_{\gamma'}(v) = R(\gamma', v)\gamma' = \frac{1}{4}[[\gamma', v], \gamma'] = -\frac{1}{4}(\operatorname{ad} \gamma') \circ (\operatorname{ad} \gamma')$$

Note that $\operatorname{ad}\gamma'$ is skew-symmetric: for any $v, w \in \mathfrak{g}$, $\langle(\operatorname{ad}\gamma').v, w\rangle = \langle v, (\operatorname{ad}\gamma').w\rangle$ (This can be seen by parallel transport and metric compatibility). Therefore, there exists an orthogonal basis such that $\operatorname{ad}\gamma'$ takes the form

$$\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Then

$$(\operatorname{ad}\gamma') \circ (\operatorname{ad}\gamma') = \begin{pmatrix} -\lambda_1^2 & & & & \\ & -\lambda_1^2 & & & \\ & & -\lambda_2^2 & & \\ & & & -\lambda_2^2 & \\ & & & & \ddots \end{pmatrix}.$$

In particular, nonzero eigenvalues of $K_{\gamma'}$ are positive and come in pairs. Therefore, index λ of any geodesic from e to x_0 is even. □

In addition, we get that $H_3(G) \equiv \mathbb{Z}^m$ for some m. Then Hurewicz theorem gives $\pi_3(G) \equiv \mathbb{Z}^m$. In fact, we need to work more to deduce that $\pi_3(G) \equiv \mathbb{Z}$. We refer the interested reader to [Bot56].

References

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