Analysis on Loop Spaces

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Contents

Planning

2 January, 2022: Work on torus chapter - include material from HNote.26.November.2021

Preface

A main objective of these lectures, originally planned for Fall 2020, but 'delivered' in the difficult conditions of the Covid19 pandemic of Winter/Spring 2021, is to answer the question:- Does the Dirac-Ramond operator exist and can we work with it? I have thought about this question over a period of some years and I hope to relate here what I know. My usual joke is that the Dirac-Ramond operator is like the 'Tasmanian Tiger' () – there have been many claimed sightings but most if these turn out to be dogs. I leave it to you, gentle reader, to come to your own conclusion.

The answer to the first question is yes, although the definition is still 'weak' (in a technical sense). The answer to the second is still a little unclear but my hope is that these notes will shed a little light on that too. To define *the* Dirac-Ramond operator – which is on the loop space of a string manifold – I need to go through a rather substantial preparation which includes differential analysis (function spaces, operators), differential topology (string structures, gerbes, transgression), representation theory (loop and diffeomorphism groups), index theory (including the Witten genus) and more.

I would like to thank the hardy souls who persisted to the end of the course, in particular Robert Burklund, Ken Kim, Frank Xu and Tristan Yang; without their interest these notes would not have come to fruition.

Introduction

Initially I will give a 'colloquim style' discussion of the background leading to the definition of the (or really a) Dirac-Ramond operator. If you understand everything here you should probably be giving the lectures. By the end of the semester I hope everything will be adequately described.

The idea of a differential operator on an infinite-dimensional space arose quite early in the development of string theory as an analogue of the spin Dirac operator; in this case it is intended to describe 'spinning strings.' However there are substantial mathematical difficulties which have obstructed the precise definition of this as an operator, and some of these issues remain. I hope to convince you that some progress has been made and that there is interesting Mathematics in what might otherwise be thought of as a quixotic enterprise.

Let me start with a 'topological description of geometric structures', in particular spin structures. Consider the Whitehead (my erstwhile colleague George) tower for the group $\mathrm{O}(n)$. Here $n > 2$ and it creeps up a bit below, take $n \geq 5$ throughout if you want to be safe from low dimensional annoyances. At some point n might be even as well. So the tower in question is

27.2.2020.1 (1)
$$\begin{array}{ccccccccc}
\mathbb{Z}_2 & & & & \mathbb{Z}_2 & & K(\mathbb{Z},2) & & \\
\uparrow{\scriptstyle \det} & & & & \downarrow & & \downarrow & & \\
\mathrm{O}(n) & \longleftarrow & \mathrm{SO}(n) & \longleftarrow & \mathrm{Spin}(n) & \longleftarrow & \mathrm{String}(n) & \longleftarrow & \dots
\end{array}$$

The successive maps here 'remove' the lowest homotopy group while keeping the higher ones unchanged. In the first step the map is injective but in higher steps it is surjective. Thus $\mathrm{O}(n)$ has two components, '$\pi_0 = \mathbb{Z}_2$,' then $\pi_1 = \mathbb{Z}_2$ as well then $\pi_2 = \{\mathrm{Id}\}$, $\pi_3 = \mathbb{Z}$ and I'm not going to talk about the higher groups (look up 'fivebrane' if you want to know). All the spaces here are, or really can be taken to be, topological groups but they are actually only determined up to homotopy equivalence.

What is the relation of this to geometry? A smooth (finite-dimensional) manifold, which is really what we are interested in throughout, has a tangent bundle which, being a vector bundle, has a frame bundle – the elements of the fibre F_p at each point $p \in M$, are just the bases of the tangent space T_pM. Here F is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle with the action being change of basis

27.2.2020.2 (2)
$$\begin{array}{ccc}
\mathrm{GL}(n,\mathbb{R}) & \longrightarrow & F \\
 & & \downarrow \\
 & & M.
\end{array}$$

We can recover TM as the bundle associated to the standard representation of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$

27.2.2020.3 (3)
$$TM = FM \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n.$$

Equipping M with a Riemann metric, as we always can, reduces the structure group from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$ by taking the orthonomal frames

27.2.2020.4 (4)
$$\begin{array}{ccccc} \mathrm{O}(n) & \text{---} & F_{\mathrm{O}} & \hookrightarrow & F \\ & & \downarrow & \swarrow & \\ & & M. & & \end{array}$$

Thus we are at the bottom of the tower.

Now the first horizontal arrow in (27.2.2020.1 1) corresponds to the existence, and choice, of an orientation on the manifold. Let me spell this out explicitly in one way for later reference. An orientation is a reduction of the structure group from $\mathrm{O}(n)$ to $\mathrm{SO}(n)$, a subbundle of consistently oriented orthonormal frames

27.2.2020.5 (5)
$$\begin{array}{ccccc} \mathrm{SO}(n) & \text{---} & F_{\mathrm{SO}} = o^{-1}(1) & & \\ \downarrow & & \downarrow & & \\ \mathrm{O}(n) & \text{---} & F_{\mathrm{O}} & \xrightarrow{o} & \mathbb{Z}_2 \\ & & \downarrow & & \\ & & M. & & \end{array}$$

One way to specify an orientation is to give a continuous map o to $\mathbb{Z}_2$, as indicated, with the property that it takes both values on each fibre. An orientation exists if and only if the first Stieffel-Whitney class vanishes and then, assuming as I will, that M is connected, there are two choices.

The next step in the Whitehead tower corresponds to a spin structure on the manifold. Here $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$ and is simply connected; it is a compact Lie group. Let me pause to indicate one relevant construction of it. Namely consider the path and loop groups of $\mathrm{SO}(n)$, these are major characters in this story. For the moment we can take continuous 'pointed' paths and loops – and we might as well consider a general (connected) Lie group

27.2.2020.6 (6)
$$\begin{gathered} \dot{\mathcal{P}}(G)) = \{\chi : [0,\pi] \longrightarrow G,\ \text{continuous and with } \chi(0) = \mathrm{Id}\} \\ \dot{\mathcal{L}}(G) = \{\lambda : \mathbb{S} \longrightarrow G,\ \text{continuous and with } \lambda(0) = \mathrm{Id}\}. \end{gathered}$$

Here I am thinking of the circle as $\mathbb{R}/2\pi\mathbb{Z}$ so the loops λ are 2π-periodic maps from the line. The reason I take π to be the parameter length for paths will show up below, it is simply a normalization.

Now $\dot{\mathcal{P}}(G)$ and $\dot{\mathcal{L}}(G)$ are groups under pointwise composition and there is a short exact sequence of groups

27.2.2020.7 (7)
$$\begin{array}{ccc} \dot{\mathcal{L}}(G) & \longrightarrow & \dot{\mathcal{P}}(G) \\ & & \downarrow \\ & & G \end{array}$$

where the last map is evaluation at the endpoint π. The kernel of this homorphism is the subgroup of pointed paths with endpoint at Id; halving the parameter allows this to be identified with $\dot{\mathcal{L}}(G)$; so my choice of normalization is not for this reason!

The path space is contractible, through path shortening, and as a result (27.2.2020.7 7) is a classifiying sequence so

27.2.2020.24 (8)
$$G = B\dot{\mathcal{L}}(G) \text{ and } \pi_j(\dot{\mathcal{L}}(G)) \simeq \pi_{j+1}(G).$$

Returing to the orthogonal group, the statement that $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ therefore means that $\dot{\mathcal{L}}(\mathrm{SO}(n))$ has two components and then

27.2.2020.8 (9)
$$\mathrm{Spin}(n) = \dot{\mathcal{P}}\,\mathrm{SO}(n))/\dot{\mathcal{L}}_{\mathrm{Id}}(\mathrm{SO}(n))$$

identifies paths if they are homotopic through paths with the same endpoint.

The question of the existence of a spin structure on M is the search for an extension of the oriented frame bundle

27.2.2020.9 (10)
$$\begin{array}{ccc} \mathrm{Spin}(n) & \text{———} & F_{\mathrm{Spin}} \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) & \text{———} & F_{\mathrm{SO}} \\ & & \downarrow \\ & & M. \end{array}$$

Such a principal bundle exists if and only if the second Stieffel-Whitney class vanishes.

Since the objective is to generalize it, let me now remind you of the Spin Dirac operator – let's take the dimension to be even, $n = 2m$. The spin group has a fundamental representation of dimension 2^m coming from the identification $\mathrm{Spin}(2m) \subset \mathrm{Cl}_{\mathbb{C}}(2m) \simeq M(2^m)$ of the complexified Clifford algebra with the corresponding matrix algebra. This induces a bundle, the spinor bundle, $S = S^+ \oplus S^-$ over M with grading coming from the two irreducible parts of the spin representation.

Now the bundle F_{spin} is a double cover, so the Levi-Civita connection lifts from F_{SO} to a connection and induces a connection ∇ on S. The spin action corresponds to an action of the bundle of Clifford algebras $\mathrm{Cl}_{\mathbb{C}}(T^*M)$ on S,

27.2.2020.10 (11)
$$\mathrm{cl} : T^*M \longrightarrow \mathrm{GL}(S)$$

and combining these leads to the definition of the spin Dirac operator

27.2.2020.11 (12)
$$\eth_{\mathrm{Spin}} = \begin{pmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{pmatrix} : \mathcal{C}^\infty(M; S) \longrightarrow \mathcal{C}^\infty(M; S),\ \eth_{\mathrm{Spin}} = \mathrm{cl} \circ \nabla.$$

The spin Dirac operator is elliptic, hence Fredholm and its graded index, computed as part of the Atiyah-Singer index theorem,

27.2.2020.12 (13)
$$\mathrm{ind}(\eth_{\mathrm{Spin}}) = \dim \mathrm{Nul}(\eth^+) - \dim \mathrm{Nul}(\eth^-) = \int_M \widehat{A}$$

is the $\widehat{A}$ genus of M. This was one of the early achievements of Atiyah and Singer, explaining the integrality of the $\widehat{A}$ genus for spin manifolds (which was known previously).

So, it is this we are trying to 'emulate' at the next step up, for string structures. Before proceeding in this way, let me describe the 'trangression' of spin structures.

Consider now the free (rather than the pointed) loop and path spaces spaces now for a general manifold M :

27.2.2020.13 (14)
$$\begin{gathered}\mathcal{P}(M)=\{\chi:[0,\pi]\longrightarrow M,\ \text{continuous}\}\\ \mathcal{L}(M)=\{\lambda:\mathbb{S}\longrightarrow M,\ \text{continuous}\}.\end{gathered}$$

Each path has two endpoints and (M assumed connected) the loop space has a similar surjective map by evaluation at 1 and $-1\in\mathbb{S}$

AnLoSp.1 (15)
$$\mathcal{P}(M)\longrightarrow M^2,\ \mathcal{L}(M)\longrightarrow M^2.$$

Both are fibre bundles; these are important later.

Loop and path spaces have functorial properties arising by pull-back along the defining maps. For instance, if F is a principal G-bundle over M then there are corresponding principal bundles arising from pull-back

27.2.2020.14 (16)
$$\begin{array}{ccccccc}\mathcal{P}(G) & \longrightarrow & \mathcal{P}(F) & \quad & \mathcal{L}(G) & \longrightarrow & \mathcal{L}(F)\\ & & \downarrow & & & & \downarrow\\ & & \mathcal{P}(M) & & & & \mathcal{L}(M).\end{array}$$

Here the fibre at a path or loop consists of all the paths/loops into F which 'cover' the given map into M.

In particular for an oriented manifold, as was observed by Atiyah in the 1980s,

A spin structure on M induces an 'orientation' on $\mathcal{L}(M)$.

Note that the latter notion is not clearly defined (because whatever the tangent space to the loop space is, it is infinite-dimensional) but we find a picture very reminiscent of the finite dimensional case

27.2.2020.15 (17)
$$\begin{array}{ccccc}\mathcal{L}(\mathrm{Spin}(n)) & \text{———} & \mathcal{L}(F_{\mathrm{Spin}}) & & \\ \downarrow & & \downarrow & & \\ \mathcal{L}(\mathrm{SO}(n)) & \text{———} & \mathcal{L}(F_{\mathrm{SO}}) & \xrightarrow{o_{\mathrm{Spin}}} & \mathbb{Z}_2\\ & & \downarrow & & \\ & & \mathcal{L}(M). & & \end{array}$$

Here o_{Spin} is ±1 on a given loop in F_{SO} as it is, or is not, the image of a loop in $\mathcal{L}(F_{\mathrm{Spin}})$ – it is a continuous map which takes both values on each fibre.

Conversely, it was show by McLaughlin that if M is 2-connected, connected with $\pi_1(M)=\{0\}$, then the converse is true, but not without some such restriction. The relationship was finally clarified by Stolz and Teichner around 2005. They observed that the orientation in (27.2.2020.15 17) induced by a spin structure has an additional property corresponding to the endpoint maps (AnLoSp.1 15). To see this, form the fibre products with respect to this map

27.2.2020.17 (18)
$$\mathcal{P}^{[k]}(M)=\{(\chi_1,\dots,\chi_k)\in(\mathcal{P}(M))^k;\chi_1(0)=\cdots=\chi_k(0),\ \chi_1(\pi)=\cdots=\chi_k(\pi)\}.$$

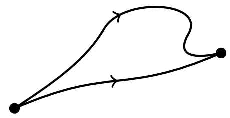


FIGURE 1. Joining paths to a loop F:Join

These form a simplicial space where the 'face' maps are just the maps forgetting one of the paths

27.2.2020.18 (19)
$$\mathcal{P}(M) \Lleftarrow \mathcal{P}^{[2]}(M) \overset{f_*}{\Lleftarrow} \mathcal{P}^{[3]}(M) \Lleftarrow \dots \qquad \mathcal{P}^{[2]}(M) \leftrightarrow \mathcal{L}(M).$$

So this consists of all the k-tuples of paths with the same initial and terminal endpoints.

The vertical bijection here is by 'joining' paths. If two paths χ_1, χ_2 have the same endpoints then traversing the first and then, in reverse and with parameter renormalized, the second gives a loop

LC.4 (20)
$$\mathcal{P}^{[2]}_{\mathcal{C}} M \ni (\chi_1, \chi_2) \longmapsto \lambda_{12} \in \mathcal{L}_{\mathcal{C}}(M)$$

and conversely a loop can be divided into 2 paths with the same endpoints by shifting and reversing the second half of a loop..

Applying this construction to $\mathcal{L}F_{\mathrm{SO}}$ gives three pull-back maps and hence the simplicial differential

LC.5 (21)
$$\mathcal{C}(\mathcal{L}F_{\mathrm{SO}}; \mathbb{Z}_2) \ni o_{\mathrm{Spin}} \longmapsto f_i^* o_{\mathrm{Spin}} \in \mathcal{C}(\mathcal{L}F_{\mathrm{Spin}}; \mathbb{Z}_2)$$
$$\delta o_{\mathrm{Spin}} = f_3^* o_{\mathrm{Spin}} (f_2^* o_{\mathrm{Spin}})^{-1} f_1^* o_{\mathrm{Spin}} = o_{\mathrm{Spin}}(\lambda_{12})(o_{\mathrm{Spin}}(\lambda_{13}))^{-1} o_{\mathrm{Spin}}(\lambda_{23}) = 1$$

(where in the case of a $\mathbb{Z}_2$-valued function inversion does nothing). Stolz and Teichner showed that

QSpin
> Spin structures on M are in 1-1 correspondence with fusion orientation structures on $\mathcal{L}(M)$, in the sense that $\delta o_{\mathrm{Spin}} = 1$ on $\mathcal{L}(M)$.

Even though this is correct, it is a little misleading as regards subsequent developments because of the discreteness of $\mathbb{Z}_2$. The important point is that 'objects' on M are often in 1-1 correspondence with 'fusive' objects on $\mathcal{L}(M)$.

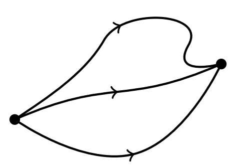


FIGURE 2. Fusion of three paths to three loops F:figure

So far I have considered continuous paths and loops but it is important to consider other regularity, with

1.3.2020.1 (22)
$$\begin{gathered}\mathcal{P}_\infty(M)=\{\lambda:[0,\pi]\longrightarrow M;\ \text{infinitely differentiable}\}\\ \mathcal{L}_\infty(M)=\{\lambda:\mathbb{S}\longrightarrow M;\ \text{infinitely differentiable}\}\end{gathered}$$

perhaps the most natural. In fact the Sobolev-regular, for $s>\frac{1}{2}$, and Lipschitz-regular paths and loops are also significant. These are well-defined as spaces of maps into a finite-dimesional manifold because the corresponding spaces of functions on $[0,\pi]$ or $\mathbb{S}$ are '$\mathcal{C}^\infty$ algebras'. Not only are they closed under the product of functions but if f_i are n of these maps into $\mathbb{R}$ collectively taking values in an open set $\Omega\subset\mathbb{R}^n$ and $F:\Omega\longrightarrow\mathbb{R}$ is a smooth function on Ω then the composite $F\circ f_*$ is in the space. This allows one to define the varous path and loop spaces with dense inclusions

1.3.2020.2 (23)
$$\begin{array}{ccccc} & & \mathcal{P}_s(M) & & \\ & \nearrow & & \searrow & \\ \mathcal{P}_\infty(M) & & & & \mathcal{P}_{\mathcal{C}}(M) \\ & \searrow & & \nearrow & \\ & & \mathcal{P}_\Lambda(M) & & \end{array}$$

Then $\mathcal{P}_{\mathcal{C}}(M)$ and $\mathcal{P}_\Lambda(M)$ are Banach manifolds, the $\mathcal{P}_s(M)$ are Hilbert manifolds and $\mathcal{P}_\infty(M)$ is a Fréchet manifold. Assuming M to be smooth they are in fact all $\mathcal{C}^\infty$ manifolds in the appropriate sense. To see why this is so, take a metric on M and, for any $\epsilon>0$ smaller than the injectivity radius, consider the covering by these open metric balls. The exponential map at each $p\in M$ identifies the ball with the ball of the same radius around the origin in T_pM and the composites $F_{pq}=\exp_p^{-1}\exp_q$ give smooth transition maps.

Now for a given element $\chi\in\mathcal{P}_{\mathcal{C}}(M)$ consider the paths

1.3.2020.3 (24)
$$\mathcal{N}(\chi)=\{\chi'\in\mathcal{P}_{\mathcal{C}}(M);d_M(\chi'(t),\chi(t))<\epsilon\ \forall\ t\in[0,\pi]\}.$$

These are open subsets and pull-back along the base curve gives a bijection

1.3.2020.4 (25)
$$\sigma'=\exp^*_{\chi(t)}\chi',\ \operatorname{Exp}_\chi:\mathcal{N}(\chi)\longleftrightarrow\{\sigma'\in\mathcal{C}([0,\pi];\chi^*TM);|\sigma'(t)|_g<\epsilon\}$$

with the continuous sections of the pull-back of the tangent bundle under the curve and similarly for the other regularities. Two of the $\mathcal{N}(\chi)$ intersect if and only if for the corresponding base curves $d(\chi_1(t),\chi_2(t))<\epsilon$ for all $t\in[0,\pi]$. Then the induced transition map

1.3.2020.5 (26)
$$\mathcal{F}_{\chi_2,\chi_1}:\operatorname{Exp}_{\chi_2}\circ\operatorname{Exp}_{\chi_1}^{-1}:\operatorname{Exp}_{\chi_1}(\mathcal{N}(\chi_1)\cap\mathcal{N}(\chi_2))\longrightarrow\operatorname{Exp}_{\chi_2}(\mathcal{N}(\chi_1)\cap\mathcal{N}(\chi_2))$$

is a diffeomorphism, i.e. it is infinitely differentiable on these open subsets of Banach spaces.

Indeed, the transition maps are non-linear bundle maps. The derivatives of the transition maps on M are symmetric $|\alpha|$-multilinear maps

LC.6 (27)
$$F^\alpha_{pq}(m):T_qM\times T_qM\times\cdots\times T_qM\longrightarrow T_pM$$

depending smoothly on m in the intersection of the coordinate balls. From (26) the corredponding (weak) derivatives exist for the transition maps on the path or loop spaces given by pointwise action:

LC.7 (28) $$\mathcal{F}^{\alpha}_{\lambda_2,\lambda_1}:\mathcal{C}^\infty(\mathbb{S};\lambda_1^*TM)\times\cdots\times\mathcal{C}^\infty(\mathbb{S};\lambda_1^*TM)\ni(\sigma'_2,\dots,\sigma'_{|\alpha|})\longmapsto F_{\lambda_1(*),\lambda_2(*)}(\sigma'_1(*),\dots,\sigma'_{|\alpha|})(*))\in\mathcal{C}^\infty(\mathbb{S};\lambda_2^*TM)$$

This corresponds to the fact that the only differentiation which arises is of the transition maps for exp on M. Note that χ^*TM is trivial as a bundle over the interval. This carries over to the other regularities and to the loop spaces, with the triviality statement for the pull-back under loops being orientability.

It is rather natural to think of these spaces being 'thickenings' of $\mathcal{P}_\infty(M)$ or $\mathcal{L}_\infty(M)$ in which they are dense. In all cases the tangent space at a path or loop is naturally the space of sections, of the corresponding regularity, of χ^*TM. In the standard approach to manifolds the cotangent space would be defined as the dual of the tangent space. Since it is rather natural to think of this as a space of sections of the cotangent space on the manifold and the most natural pairing between sections of TM and sections of T^*M pulled back is

AnLoSp.1a (29) $$\int_{\mathbb{S}}\lambda^*(v(s)\cdot w(s))ds$$

this would realize the cotangent space as the dual space of sections of λ^*T^*M, i.e. measures for $\mathcal{L}_{\mathcal{C}}(M)$ and distributions for $\mathcal{L}_\infty(M)$. Not only is this 'handist' but it is unwisely prescriptive since in fact on $\mathcal{L}_s(M)$ all the spaces

AnLoSp.2 (30) $$H^t(\mathbb{S};\lambda^*TM),\ H^t(\mathbb{S};\lambda^*T^*M)\text{ for }-s<t<s$$

make invariant sense. Thus the pointwise value of a 'vector field' of a '1-form' on $\mathcal{L}_s(M)$ can be reasonably taken to lie in any one of these spaces. From this it is already clear that there are many notions of regularity of objects over the loop spaces. Note in particular that continuity of a function on $\mathcal{L}_s(M)$ is a stronger statement than continuity on the dense subspace $\mathcal{L}_\infty(M)$.

One reason that the Lipshitz paths and loops are relevant is that (affine) arclength (re-)parameterization of a curve (so the parameter length is still 2π) is well-defined as a map

AnLoSp.3 (31) $$\mathcal{L}_1(M)\longrightarrow\mathcal{L}_\Lambda(M).$$

Let me make an apparent digression. As Mathematicians we can ask a question that perhaps the Physists do not feel bound to ask themselve. Namely, what precisely is a String? Clearly it is related to a loop, an element of the free loop space, say smooth

27.2.2020.19 (32) $$\mathcal{L}_\infty(M)=\{\lambda:\mathbb{S}\longrightarrow M,\ \mathcal{C}^\infty\}.$$

The diffeomorphism group of the circle acts on this by reparameterization and one (not quite ideal) definition of a String is that it is an element of the quotient space

27.2.2020.20 (33) $$\mathcal{L}_\infty(M)/\operatorname{Dff}^+(\mathbb{S})$$

given as the orbits under the action of the orientation-preserving diffeomorphisms of the circle. Thus a String is (perhaps) an 'unparameterized' loop. Of course taking a quotient like this is dangerous since the diffeomorphism group does not act freely, so the quotient is bound to be rather singular. Instead, as is done in many contexts, we look for $\operatorname{Dff}^+(\mathbb{S})$-invariant or -equivariant objects on $\mathcal{L}(M)$ and think of them as objects on the quotient space. So, it is important for us to understand the action of the reparameterization group.

All this suggests that we can 'transgress' things from M to $\mathcal{L}(M)$ without loss of information if we are careful. One version of this is coholomogy. There is an evaluation map

27.2.2020.21 (34)
$$\mathrm{ev} : \mathbb{S} \times \mathcal{L}(M) \ni (\theta, \lambda) \longmapsto \lambda(\theta) \in M$$

which allows cohomology to be pulled back and then integrated over the circle to define transgression from the upper left part of the diagram for each $k \geq 1$

27.2.2020.22 (35)
$$\begin{array}{ccc} H^k(\mathbb{S} \times \dot{\mathcal{L}}(M); \mathbb{Z}) & \xleftarrow{\ \mathrm{ev}^*\ } & H^k(M; \mathbb{Z}) \\ \downarrow \pi_* & \swarrow \tau & \uparrow \tau_{\mathrm{fus}} \\ H^{k-1}(\dot{\mathcal{L}}(M); \mathbb{Z}) & \xleftarrow{\ \mathrm{fg}\ } & H^{k-1}_{\mathrm{fu}}(\dot{\mathcal{L}}(M), \mathbb{Z}). \end{array}$$

However the diagonal trangression map loses information, in general it is neither injective nor surjective. For this reason Chris Kottke and I introduced 'fusive' cohomology (in Čech cohomology) by imposing fusion and a second (figure-of-eight) requirements at the chain level. This makes the corresponding cohomology spaces invariant under reparameterization and gives an isomorphism as indicated on the right, with a forgetful map to ordinary cohomology.

So a general 'principle' here is that

> Objects can be trangressed, without loss, to fusive objects on
> 27.2.2020.23 the loop space.

One particular, and fundamental, case of this is the notion of a *string structure*. This corresponds to the third step in the Whitehead tower (27.2.2020.1 1); now we are getting to the heart of the matter.

The question then is analogous to (27.2.2020.9 10). Now we ask about the existence of a lift of the spin frame bundle

28.2.2020.1 (36)
$$\begin{array}{ccc} \mathrm{String}(n) & \text{———} & F_{\mathrm{String}} \\ \downarrow & & \downarrow \\ \mathrm{Spin}(n) & \text{———} & F_{\mathrm{Spin}} \\ & & \downarrow \\ & & M \end{array}$$

to a principal bundle with structure group $\mathrm{String}(n)$.

This string group is not well-defined as a group, only up to homotopy equivalence, but the existence of a string structure in the sense of (28.2.2020.1 36) is independent of any choice and the final word here (as always there is a lot of history I am suppressing) is due to Redden:

> A string structure exists if and only $\frac{1}{2}p_1 = 0$ and then the equivalence classes are parameterized by $H^3(M; \mathbb{Z})$.

The somewhat confusingly denoted obstruction, $\frac{1}{2}p_1 \in H^4(M; \mathbb{Z})$, is the Pontryagin class of the *Spin* principal bundle F_{Spin}, it is an integral class and $2 \times \frac{1}{2}p_1$ is the 'usual' Pontryagin class of F_{SO}.

Whatever realization of $\mathrm{String}(n)$ one takes, it cannot be a finite-dimensional Lie group, since it must have trivial π_3. It is quite difficult to contemplate doing analysis direcctly on F_{String} or related spaces. However the principle in (27.2.2020.23) holds, as was understood, with some caveats, in the 1980s and we should 'trangress' to the loop space.

Since $\pi_3(\mathrm{Spin}(n)) = \mathbb{Z}$, we know from (27.2.2020.24. 8) that $\pi_2(\dot{\mathcal{L}}(\mathrm{Spin})) = \mathbb{Z}$ and that this loop group is simply-connected. It follows that there is a circle bundle with Chern class a generator of $H^2(\dot{\mathcal{L}}(\mathrm{Spin}))$. In fact this class is equivariant and the circle bundle corresponds to a central extension

28.2.2020.2 (37)
$$\mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}}(\mathrm{Spin}) \longrightarrow \mathcal{L}(\mathrm{Spin}).$$

There is a $\mathbb{Z}$ of such extensions, but all may be obtained from the 'basic' one (28.2.2020.2 37) by covering. The corresponding element of $H^2(\dot{\mathcal{L}}(\mathrm{Spin}))$ is called the *level* of the central extension.

So now we can see a corresponding lifting question over the loop space. The principal spin bundle, the top part of (27.2.2020.9 10), pulls back to a principal $\mathcal{L}(\mathrm{Spin})$ bundle over the loop space, as in (27.2.2020.14 16), and we can ask whether this has a 'lift' to a principal bundle for the basic central extension

28.2.2020.3 (38)
$$\begin{array}{ccc} \widehat{\mathcal{L}}(\mathrm{Spin}(n)) & \text{———} & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathrm{Spin}(n)) & \text{———} & \mathcal{L}(F_{\mathrm{Spin}}) \\ & & \downarrow \\ & & \dot{\mathcal{L}}(M); \end{array}$$

The projections from the top line here correspond to circle bundles.

The situation is very similar to the spin to orientation transgression already discussed. Here the existence of a string structure implies the existence of a extension. One way of seeing this is to note that such a U(1) lifting problem corresponds to the triviality of a 'lifting bundle gerbe' in the sense of Murray. Thinking more abstractly of a principal bundle with a (possibly large but topological) group with a central extension $\mathrm{U}(1) \longrightarrow \widehat{\mathcal{G}} \longrightarrow \mathcal{G}$ we can ask the same existence question – we look for a diagram

28.2.2020.4 (39)
$$\begin{array}{ccccccccccc} \widehat{\mathcal{G}} & \text{——} & \widehat{\mathcal{P}} & & L & & s : \delta L \simeq \mathrm{U}(1) & & \delta s = 1 & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{G} & \text{——} & \mathcal{P} & \longleftarrow & \mathcal{P}^{[2]} & \longleftarrow & \mathcal{P}^{[3]} & \longleftarrow & \mathcal{P}^{[4]} & \longleftarrow & \cdots \\ & & \downarrow & & & & & & & & \\ & & M & & & & & & & & \end{array}$$

Here $\mathcal{P}$ is the total space of the principal bundle and to the right are the various fibre products over M forming a simplicial space (so there are k maps from $\mathcal{P}^{[k]}$ to $\mathcal{P}^{[k-1]}$). Then L is a circle bundle defined from the central extension of $\mathcal{G}$. Namely the fibre of $\mathcal{P}^{[2]}$ over m is the set of pairs $(f_1, f_2) \in \mathcal{P}_m \times \mathcal{P}_m$ so the $\mathcal{G}$ action induces a shift map $\mathcal{P}^{[2]} \longrightarrow \mathcal{G}$ mapping $(f_1, f_2 = gf_1)$ to g. Then L is the pull-back of the circle bundle given by the central extension. This circle bundle is simplicial (in the sense of Brylinski), as indicated – the simplicial differential gives a line bundle over $\mathcal{P}^{[3]}$ as the tensor product of the three pull-backs of L. From its definition this has a section, s trivializing it. Again the simplicial differential of a section is a section of the simplicial differential of the new circle bundle, where now we are over $\mathcal{P}^{4]}$. Here the line bundle is canonically trivial and the pulled back section is given by this canonical trivialization. This is Murray's notion of a bundle gerbe.

> A bundle gerbe induces a Dixmier-Douady class $\mathrm{DD}(L) \in H^3(M;\mathbb{Z})$ and for a lifting bundle gerbe the vanishing of this class is equivalent to the existence of a lifted principal bundle $\widehat{\mathcal{P}}$ such that as a circle bundle over $\mathcal{P}$ its image under δ is L. 28.2.2020.5

Now, the Dixier-Douady class of this bundle gerbe is the trangression of the obstruction to the existence of a string structure.

28.2.2020.6 (40)
$$\mathrm{DD}(\mathcal{L}F_{\mathrm{Spin}}) = \tau(\frac{1}{2}p_1)$$

Thus again, the existence of a string structure implies the existence of a 'spin structure' in the sense of a lift of the principal bundle as in (28.2.2020.4 39). However, just as before this transgressed spin structure has additional properties, fusion, figure-of-eight and equivariance. We call these more refined structures 'loop spin structures'. To make real progress we also need to consider regularity, continuous objects do not suffice.

Ignoring these niceties, it is shown by Waldorf and in KM-equivalence [**6**] that

> String structures on M are in 1-1 correspondence with loop spin structures.

This corresponds to the fact that the Dixmier-Douady invariant for the trangressed spin bundle takes values in the fusive cohomology discussed above.

At this stage we are getting close to the setup for the spin Dirac operator. If M is a string manifold then the string structure corresponds to a principal bundle over the loop space for the basic central extension of the loop group on Spin. Of course we need to be careful and understand the regularity of these objects – I have glossed over all this. In particular I did note that we want equivariance under the reparameterization group; this is not quite what we get, rather there is an action of the basic central extension of the diffeomorphism group of the circle. This is the Bott-Virasoro group.

Now, in place of the spin representation of $\mathrm{Spin}(2n)$ the central extension of $\mathcal{L}(\mathrm{Spin}(2n))$ has a distinguished representation, also called the spin representation to avoid any confusion! This is discussed for instance in the book of Pressley and Segal [**?**]Pressley-Segal1. It is a unitary representation but for our geometric applications we want the smaller Fréchet representation rather than a Hilbert space. In either case this is a form of Fock space.

The corresponding Dirac-Ramond operator acts on sections of the bundle over the loop space of the bundle associated to the trangressed sting structure and the spin representation. We need to show that it has a connection extending the (transgression of the) Levi-Civita connection and that the cotangent bundle has an appropriate Clifford action. Then the Dirac-Ramond operator is given by the same formula (27.2.2020.11 12) plus a term ensuring equivariance.

Witten defined a 'genus' by formal analogy with the $\widehat{A}$ genus given by the index of the finite-dimensional spin Dirac operator. This is a formal (meaning non-convergent) Fourier series associated to the action of the rotation part of the reparameterization group. The defininition has been made rigorous from an algebraic perspective and shown to be a modular form – the series is the expansion around the circle at infinity.

So this is where the gaps in understanding start to appear. We can see that the Dirac-Ramond operator has 'index' – the graded space given by the difference of the null space and complement to the range – which carries an action of the Virasoro group. What I would like to do, but at this stage I cannot do (or if you want to be optimistic, have not done) is all this. Here is a serious list of what one might hope for.

(1) Show that the the index space for the Dirac-Ramond operator has finite multiplicity over the representations of the Virasoro group.
(2) Conclude that the corresponding modular form given by the formal trace is the same as that constructed by Witten and subsequently.
(3) Show that there is a geometric model for elliptic cohomolgy in these bundles over loop space (as suggested by Brylinski).

Taken together, even the first two, would give an explanation, similar to that given by Atiyah-Singer of the integrality of the Ahat genus, for the modularity of the 'Witten genus' of a string manifold.

So, it is going to be quite a trip – I hope that some of you at least will come along.

0.1. Notes for Lecture 1

One slant on what these lectures are about is that they concern the 'quantization of string structures'. The situation here is very like that concerning spin structures in the early 1960s. For a compact oriented manifold it was known how to define the $\widehat{A}$ genus. This is a rational number given as an integral over the manifold of a form generated from a Riemann metric, although the integral does not depend on the choice of metric. What was known was that if the manifold, M, is spin – has a spin structure – then this number is an integer. Is Singer, who died last week, records being asked by Michael Atiyah to think about why this was so. The result, eventually was their index theorem. In this case the 'answer' is that the $\widehat{A}$ genus is the index of the spin Dirac operator, the difference between the dimension of its null space and the codimension of its range.

15.2.2021.5 QUESTION 1. Should I give a full proof of this?

In the 1980s Witten defined a more complicated invariant associated to a spin string on a compact manifold. A string structure is the next step in the 'Whitehead tower' which in this context corresponds to

Orientation:Spin:String:(Five brane):$\cdots$

where I don't really know anything about the last of these. This sequence actually corresponds to the stable homotopy groups of $\mathrm{O}(2n)$ – for our purposes $2n > 4$ works.

The choice of a Riemann metric on M (of dimension $2n$ although there is no particular reason to take the dimension to be even for some time) means that the notion of an orthornormal basis of the tangent space, at each point, is defined. These spaces fit together smoothly to define a smooth manifold which is a fibre bundle F_{O} over M. The fibre at $m \in M$ is modelled by $\mathrm{O}(2n)$ in the sense that, it acts freely and transitively on the fibre by change of basis. Thus F_O is a principal $\mathrm{O}(2n)$-bundle. Then M is orientable if and only if there is a smooth subbundle

15.2.2021.6 (1) $$F_{\mathrm{SO}} \subset F_{\mathrm{O}}$$

which is a principal $\mathrm{SO}(2n)$ bundle. This is the case iff the first Stieffel-Whitney class vanishes. Then there are exactly two choices, assuming as I implicitly do, that M is connected.

If $2n > 2$ then $\Pi_1(\mathrm{SO}(2n) = \mathbb{Z}_2$. That is, there is a loop in $\mathrm{SO}(2n)$ which is not homotopic to a constant loop and only one such loop up to homotopy – in particular the loop obtained by traversing this one twice is homotopically trivial. The second step in the 'Whitehead tower' is given by the group $\mathrm{Spin}(2n)$. This is a double cover of $\mathrm{SO}(2n)$ which is connected and simply connected but otherwise has the same homotopy groups as $\mathrm{SO}(2n)$. We can, and will, construct it from the loop space. In fact $\mathrm{Spin}(2n)$ can be realized as the group of pointed path (those starting at the identity) up to homotopy with fixed endpoints. The map back to $\mathrm{SO}(2n)$ is just evaluation at the far end.

Now a spin structure on an oriented manifold M is a principal bundle, not a subbundle of F_{SO} but mapping to it to give a commutative diagram

15.2.2021.7 (2) $$\begin{array}{ccc} \mathrm{Spin}(2n) & \text{———} & F_{\mathrm{Spin}} \\ \downarrow & & \downarrow \\ \mathrm{SO}(2n) & \text{———} & F_{\mathrm{SO}} \\ & & \downarrow \\ & & M. \end{array}$$

Such a spin structure exists if and only if the second Stieffel-Whitney class vanishes.

So, what was the answer that Atiyah and Singer gave to the question of the integrality of $\widehat{A}$ in this case? It involves the Spin Dirac operator. This acts on sections of the spinor bundle over M which is a complex vector bundle $W \longrightarrow M$, 'associated' to F_{Spin} through the spin representation of $\mathrm{Spin}(2n)$. It has two subbundles, the positive and negative parts (it is a graded bundle). The spin Dirac operator then acts from one to the other

15.2.2021.8 (3) $$\eth_{\mathrm{Spin}} : \mathcal{C}^\infty(M; W^+) \longrightarrow \mathcal{C}^\infty(M; W^-).$$

It is elliptic, so it has finite dimensional null space and range of finite codimension and the index is the difference of these dimensions. The theorem of Atiyah and Singer in this case is

15.2.2021.9 (4) $$\mathrm{ind}(\eth_{\mathrm{Spin}}) = \widehat{A}(M).$$

15.2.2021.10 (5) $$\eth_{\mathrm{Spin}} = \mathrm{cl} \circ \nabla$$

What I will spend a large part of the semester talking about are *String structures* on manifolds. We have 'arrived' at $\mathrm{Spin}(2n)$. This is a connected compact simple semisimple Lie group and all such have $\Pi_2 = \{0\}$. So there is nothing there. However it is *always* the case that $\Pi_3 = \mathbb{Z}$, and is so for $\mathrm{Spin}(2n)$, $2n \geq 4$. You can picture this as a 'copy' of $\mathrm{SU}(2)$ being in there somewhere.

The next step in the Whitehead tower is to kill off this group, so there is an exact sequence of groups

15.2.2021.11 (6)
$$N \longrightarrow \mathrm{String}(2n) \longrightarrow \mathrm{Spin}(2n)$$

where the homotopy groups of $\mathrm{String}(2n)$ vanish in dimension 3 or less and are otherwise the same as those of $\mathrm{Spin}(2n)$. This sequence is a Serre fibration, so the conclusion is that N, the preimage of Id, must be a '$K(\mathbb{Z},2)$' – it has one non-trivial homotopy group and that is $\Pi_2 = \mathbb{Z}$ – this maps onto the $\Pi_3(\mathrm{Spin}(2n))$.

The description of the homotopy groups already warns us that $\mathrm{String}(2n)$ cannot be a finite dimensional Lie group. One possibility, which can be realized, is that one can take $N = \mathrm{PU}(H) = \mathrm{U}(H)/\,\mathrm{U}(1)$ to be the projective unitary group on a separable Hilbert space H. The precise nature of $\mathrm{String}(2n)$ – it is only defined up to homotopy – is not very important to us, although I will likely include a construction of it. The point really is that it is pretty difficult to deal with analytically.

Still we want to consider a string structure on a spin manifold which is a extension of the 'Spin frame bundle' to a 'String frame bundle'

15.2.2021.12 (7)
$$\begin{array}{ccc} \mathrm{String}(2n) & \text{---} & F_{\mathrm{String}} \\ \downarrow & & \downarrow \\ \mathrm{Spin}(2n) & \text{---} & F_{\mathrm{Spin}} \\ & & \downarrow \\ & & M \end{array}$$

Again we know exactly when this is possible, and how many possibilities there are. Such a string structure exists iff and only if $\frac{1}{2}p_1(M) = 0$ where this is a weird way of writing a Pontryagin class in $H^4(M;\mathbb{Z})$ determined by the spin structure – the point is that $2 \times \frac{1}{2}p_1(M)$ is the usual Pontryagin class from the orthonormal frame bundle.

Now, the hope is that one can use this in a way similar to that for a spin structure, to investigate the geometry of M. In the way that the spin Dirac operator is related to scalar curvature, the string strucutre is *supposed* to be related to Ricci curvature. There are no theorems that I am aware of here.

The problem I want to address then, is how to define an operator analogous to the spin Dirac operator (15.2.2021.10 5) but for a string structure. This idea is due to Ramond from the early 70s. A direct approach looks pretty hopeless – the indirect approach I will follow is hard enough!

The idea, certainly coming out of string theory, is that one should work on the loop space of the manifold – some space, $\mathcal{L}(M)$, of maps $\mathbb{T} \longrightarrow M$ (I often write $\mathbb{T}$ for the circle so $\mathbb{T}^2$ is he torus) Certainly continuous but we are going to try to do analysis so we need to consider the various classes of maps and their relationships carefully; this is where I will really start.

What we will do is 'transgress' the string structure to a 'spin structure' on the loop space – I call this a loop spin structure.

Transgression is most easily seen in a couple of cases. The first is in terms of cohomology. Evaluatiing a map at a point gives us a map

15.2.2021.13 (8)
$$\mathrm{ev} : \mathcal{L}(M) \times \mathbb{T} \longrightarrow M$$

Pull-back as a map on cohomology gives the basic transgression map

15.2.2021.14 (9) $$\mathrm{ev}^* : H^k(M) \longrightarrow H^k(\mathcal{L}(M) \times \mathbb{T}) \longrightarrow H^{k-1}(\mathcal{M}(M)).$$

The second map is 'integration over $\mathbb{T}$'. In general this map is neither injective nor surjective.

This is a problem for us, since we really want to 'move things from M to $\mathcal{L}(M)$' without loss. To do this we need to understand the structure of $\mathcal{L}(M)$ and in particular *fusion* and *parameter equivariance*.

Questions?

CHAPTER 1

Manifolds and functions

Man

I will collect here standard material, without proofs, on finite-dimensional manifolds and function spaces that will subsequently be used freely. Since we wish to consider paths, maps from an interval, and eventually 'iterated paths' we work from the beginning in the context of manifolds with corners.

1.1. Topological manifolds

A topological manifold with corners, M, is a separably metrizable topological space with a covering by open subsets ('coordinate patches') equipped with homeomorphisms to (relatively) open subsets of $[0,\infty)^n$. Usually we either suppose the space to be connected, or require n to be fixed. Note that in the topological context this is the same as a manifold with boundary, because $[0,\infty)^n$ is homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$. The distinction appears when we require more regularity.

The assumptions on the topology of M imply that it is Hausdorff and paracompact. By the assumed separability we can take the coordinate covering to be countable. In any case there is a covering

ALS.18 (1.1.1.1)
$$M=\bigcup_{a\in A}V_a,\ F_a:V_a\longrightarrow U_a\subset\mathbb{R}^n$$

where the V_a and U_a are open and F_a is a homeomorphism. This defines a homeomorphism for each pair $V_a\cap V_b\neq\emptyset$

ALS.19 (1.1.1.2)
$$F_{ab}:F_b(V_a\cap V_b)=U_{ba}\longrightarrow F_a(V_a\cap V_b)=U_{ab}$$

between the open subsets U_{ab} of $\mathbb{R}^n$ and

ALS.20 (1.1.1.3)
$$F_{ac}\circ F_{cb}\circ F_{ba}=\mathrm{Id}\ \text{ on } U_{ab}\cap U_{ac}.$$

Additional regularity on the manifold is obtained by requiring that the transition maps (ALS.19 1.1.1.2) have additional regularity.

1.2. Hilbert spaces of functions

We are basically interested in $\mathcal{C}^\infty$ manifolds. The $\mathcal{C}^\infty$ (or smooth) functions form a sheaf over $[0,\infty)^n$. Namely, for each open set $U\subset\mathbb{R}^n$ we define $\mathcal{C}^k(U)\subset\mathcal{C}(U)$ by iterative regularity. Thus $\mathcal{C}^1(U)\subset\mathcal{C}(U)$ is the subspace of elements with continuous partial derivatives in $\mathcal{C}(U)$ and then $\mathcal{C}^k(U)$ is defined inductively by the condition that the partial derivatives lie in $\mathcal{C}^{k-1}(U)$. The intersection is the space of smooth functions

ALS.2 (1.1.2.1)
$$\mathcal{C}^\infty(U)=\bigcap_k\mathcal{C}^k(U).$$

These spaces form a fine sheaf of rings, where the fineness property comes from the existence of partitions of unity – for any collection of open subsets $\{U_a\}_{a\in A}$ there exist countable many $a_i \in A$ and $\rho_i \in \mathcal{C}^\infty(U_{a_i})$ of compact support $\mathrm{supp}(\rho_i \Subset U_{a_i}$ such that only finitely many of the $\mathrm{supp}(\rho_i)$ meet any compact subset $K \subset \bigcup_a U_a = U$ and

ALS.3 (1.1.2.2)
$$\sum_i \rho_i = 1 \text{ on } U.$$

In particular of course the U_{a_i} must cover U.

If you are unfamiliar with the behaviour at corners it is important to note that the two obvious definitions of smooth functions over (relatively of course) open subsets of $[0,\infty)^n$ are the same:

ALS.4 Proposition 1. *If $u \in \mathcal{C}(U)$ for $U \subset [0,\infty)^n$ open, then the existence of an open set $\tilde U \subset \mathbb{R}^n$ such that $\tilde U \cap [0,\infty)^n = U$ on which there is an extension $\tilde u \in \mathcal{C}^\infty(\tilde U)$ with $u = \tilde u\big|_U$ is equivalent to the condition that $u' = u\big|_{\mathrm{int}\,U} \in \mathcal{C}^\infty(\mathrm{int}\,U)$, $\mathrm{int}\,U = U \cap (0,\infty)^n$ and the partial derivatives of all orders extend to $\mathcal{C}(U)$.*

Of particular interest to us are the composition properties. If $U \subset [0,\infty)^n$ and $U' \subset [0,\infty)^m$ are open and $f : U \longrightarrow U'$ has components in $\mathcal{C}^\infty(U)$ then $g \circ f \in \mathcal{C}^\infty(U)$ for all $g \in \mathcal{C}^\infty(U')$. If we write $\mathcal{C}(U;U')$ for the space of maps $U \longrightarrow U'$ with components in $\mathcal{C}^\infty(U)$ then $F \in \mathcal{C}^\infty(U;U')$ is a diffeomorphism iff and only if it has a two-sided inverse $F^{-1} \in \mathcal{C}^\infty(U';U)$; this implies $m = n$. Moreover the codimension of boundary points is preserved (this is a form of 'invariance of domains').

The diffeomorphisms between open subsets of $\mathbb{R}^n$ form a groupoid, as do the homeomorphism. A $\mathcal{C}^\infty$ structure is determined by a choice of coordinate cover such that the transition cocycle takes values in the groupoid of $\mathcal{C}^\infty$ diffeomorphisms. It does not always exist. One can define structures with respect to any groupoid of homeomorphism in place of $\mathcal{C}^\infty$, but this is not really a pressingin matter here since our interest really lies in $\mathcal{C}^\infty$ manifolds.

1.3. $\mathcal{C}^\infty$algebras of functions

We can abstract the properties of $\mathcal{C}^\infty$ functions and consider a general fine subsheaf of the contionuous functions on $[0,\infty)^n$. Then it makes sense to say this is $\mathcal{C}^\infty$ *invariant* if composition with $\mathcal{C}^\infty$ maps on the right preserves the space and a $\mathcal{C}^\infty$ *algebra* if composition on the left with $\mathcal{C}^\infty$ maps preserves the space.

Proposition 2. *Any fine, $\mathcal{C}^\infty$-invariant, subsheaf of the continuous functions on $[0,\infty)^n$ defines a corresponding sheaf of any n-manifold.*

When it comes to maps between manifolds, the second condition as well.

Definition 1. A subsheaf of the continous functions on $[0,\infty)^n$ which is $\mathcal{C}^\infty$-invariapnt and a $\mathcal{C}^\infty$-algebra defines a space of map $\mathcal{C}^\infty(Q,M)$ for $\mathcal{C}^\infty$-manifolds Q and M

Having said all this abstractly, the examples we have most in mind are the familiar ones.

ALS.5 Proposition 3. *For any $k \in \mathbb{N}_0$ and $0 < \alpha \le 1$ the Hölder spaces $C^{k,\alpha}$ form a fine $\mathcal{C}^\infty$-invariant sheaf of $\mathcal{C}^\infty$-algebras the same is true of the Sobolev spaces $s \ge n/2$.*

We will want to make some use of the negative Sobolev spaces as well, so for all $s \in \mathbb{R}$. These form a fine sheaf over $\mathbb{R}^n$ but we have to be a bit more careful about $[0,\infty)^n$ for $s < 0$. They are only sheaves of continuous functions for $s > n/2$ and we need some extension properties. I will add this later.

Repeated here to end of section?

We can abstract the properties of $\mathcal{C}^\infty$ functions and consider a general fine subsheaf of the continuous functions on $[0,\infty)^n$. Then it makes sense to say this is $\mathcal{C}^\infty$ *invariant* if composition with $\mathcal{C}^\infty$ maps on the right preserves the space and *a* $\mathcal{C}^\infty$ *algebra* if composition on the left with $\mathcal{C}^\infty$ maps preserves the space.

Proposition 4. *Any fine, $\mathcal{C}^\infty$-invariant, subsheaf of the continuous functions on $[0,\infty)^n$ defines a corresponding sheaf of any n-manifold.*

1.3.1. Global spaces recalled. At some points we will want to make some use of the negative Sobolev spaces as well, so for all $s \in \mathbb{R}$. These form a fine sheaf over $\mathbb{R}^n$ but we have to be a bit more careful about $[0,\infty)^n$ for $s < 0$. They are only sheaves of continuous functions for $s > n/2$ and we need some extension properties. I will add this later.

1.4. Fibre bundles

1.5. Simplicial spaces

Simplicial

Fibre products are examples of simplicial spaces, so let me discuss this at least briefly here. From a primitive point of view a simplicial space is a sequnce of topological spaces – manifolds for us – which we can denote $\mathcal{Q}_k$ where $k \in \mathbb{N}$. [I should warn you that my numbering convention is a little non-standard.] Then for each $k \in \mathbb{N}$ and each $1 \le j \le k$ there are 'forgetful' maps – usually called face maps –

158L3.24 (1.1.5.1) $$\sigma_j^{[k]} : \mathcal{Q}_k \longrightarrow \mathcal{Q}_{k-1}.$$

There are also 'degeneracy maps' going the other way. The maps in (158L3.24 1.1.5.1) are required to satisfy the identities

158L3.25 (1.1.5.2) $$\sigma_q^{[k-1]} \circ \sigma_p^{[k]} = \sigma_{p-1}^{[k-1]} \circ \sigma_q^{[k]} \ \forall\ q < p.$$

For fibre products you can check this easily. The maps are actually the restrictions of the projections $X^k \longrightarrow X^{k-1}$. Then (158L3.25 1.1.5.2) just comes from renumbering, on the right if you drop the qth factor first then the pth factor slips to position $p-1$.

Now, suppose that A is an abelian topological group, so we can consider continuous maps $\mathcal{Q}_k \longrightarrow A$ forming $\mathcal{C}(\mathcal{Q}_k; A)$. Writing the group composition additively, we can define the 'simplicial differential'

158L3.26 (1.1.5.3) $$\delta u = \sum_{j=1}^{k} (-1)^j \sigma_j^* u,\ \ \delta : \mathcal{C}(\mathcal{Q}_k; A) \longrightarrow \mathcal{C}(\mathcal{Q}_{k+1}; A).$$

158L3.27 Lemma 1. *For any simplicial space*

158L3.28 (1.1.5.4) $$\delta^2 = 0.$$

Proof. The definition gives for $u \in \mathcal{C}(\mathcal{Q}_k; A)$ gives

158L3.29 (1.1.5.5) $$\delta^2 = \sum_{l=1}^{k+1} (-1)^l \sum_{j=1}^{k} \sigma_l^* \sigma_j^* u = \sum_{j<l} (-1)^{l+k} \sigma_l^* \sigma_j^* u + \sum_{j \ge l} (-1)^{l+k} \sigma_j^* \sigma_{l-1}^* u = 0$$

using (158L3.25 1.1.5.2). □

Note that a simplicial space is better formalized as a functor. Consider the small (in the technical, but also the casual sense) with objects the integers, realized as the sets $J(n) = \{1, \dots, n\}$ and arrows the increasing (i.e. non-decreasing) maps $J(m) \longrightarrow J(n)$. Then the simplical space is a *contravariant* functor into the category of topological spaces and continuous maps.

The face maps $\sigma_j : \mathcal{Q}_k \longrightarrow \mathcal{Q}_{k-1}$ are the images of the maps $J(k-1) \longrightarrow J(k)$ which are stricly increasing and 'miss' j. The degeracy maps are the images of the maps $J(k) \longrightarrow J(k)$ which are stricly decreasing except for one pair j and $j+1$ which are mapped to $j \in J(k-1)$.

158L3.30 Exercise 1. Show that the fibre products form a simplicial space with the face maps discussed above and degeneracy maps $X^{[k-1]} \longrightarrow X^{[k]}$ are the restrictions of the diagonal maps $X^{k-1} \ni (x_1, \dots, x_j, \dots) \longrightarrow (x_1, \dots, x_j, x_j, \dots)$.

1.6. Tensor and vector bundles

1.7. Lie groups

1.8. Principal bundles

1.9. Riemann metrics

Let me also recall some of the basic statemnts about Riemann metrics, for the most part here I restrict attention to the boundaryless case.

1.9.1. Levi-Civita connection.

1.9.2. Exponential map. Geodesic on a Riemann manifold can be defined in several ways, most directly here as parallel curves for the Levi-Civita connection. The exponential map, defined by geodesics, is a smooth map for each p (which depends smootly on p as well)

ALS.6 (1.1.9.1)
$$\mathrm{Exp}_p : T_pM \longrightarrow M.$$

The main points we want are that Exp_p is a diffeomorphism from sufficiently small balls around $0 \in T_pM$ (in terms of the fibre metric) to their images, which are the metric balls in M around p of the same radius. One of the important properties is that for small open balls the exponential map around *any point* defines a contraction. This is really the statement that small balls are geodesically convex. In consequence

ALS.7 Proposition 5. *Small balls in a Riemann manifold (of fixed small radius if the manifold is compact) form a 'good' open cover, one in which all non-trival intersections are contractible.*

1.10. Čech cohomology

One way we want to use this is to discuss Čech cohomology. If we take an abelian topological group Z (we are really interested in the cases $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_p$) then we can consider the sheaf $\mathcal{C}(U; Z)$ for open subsets $A \subset M$ of a manifold. If U_a, $a \in A$ is an open cover then

Athough very little of what is done later regarding loop spaces carries over to the general case of mapping spaces we nevertheless start in the setting of maps $F : M \to N$ between two smooth manifolds M and N. For the most part below M

will be one-dimensional but products of 1-dimensional manifolds will also appear; N is the much more general manifold that we are really studying.

To do analysis on mapping spaces we first need to understand the smoothness of these spaces themselves before trying to come to grips with the regularity of functions and other objects defined on them. We start by recalling the familiar topic of regularity of functions of finite-dimensional manifolds. Next the 'local coordinate' patches on these spaces are considered with particular attention to the structure groupoid.

1.11. Notes ftom L2

Since we wish to consider paths, maps from an interval, and eventually 'iterated paths' as well as loops and iterated loops so we work from the beginning in the context of manifolds with corners. This is really for the domain space – the range space, which is what we are really interested in, will usually be a compact manifold without boundary.

A topological manifold with corners is a metrizable, separable, topological space with a covering by open subsets ('coordinate patches') equipped with homeomorphisms to (relatively) open subsets of $[0,\infty)^n$. Usually we either suppose the space to be connected, or require n to be fixed. Note that in the topological context this is the same as a manifold wih boundary, because $[0,\infty)^n$ is homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$. The distinction appears when we require more regularity. [In that context I actually prefer a slightly stronger definition, but it is not very relevant here.]

ALS.1 Exercise 2. Show that this is equivalent to the standard notion – unless you allow non-separability (as in the long line).

We are basically interested in $\mathcal{C}^\infty$ manifolds. The $\mathcal{C}^\infty$ (or smooth) functions form a sheaf over $[0,\infty)^n$. For each open set $U\subset[0,\infty)^n$ we define $\mathcal{C}^k(U)\subset\mathcal{C}(U)$ by iterative regularity. Namely $\mathcal{C}^1(U)\subset\mathcal{C}(U)$ is the subspace of elements with continuous partial derivatives in $\mathcal{C}(U)$ and then $\mathcal{C}^k(U)$ is defined inductively by the condition that the partial derivatives lie in $\mathcal{C}^{k-1}(U)$.

1.12. O(n), $G^{-\infty}$ and Bott periodicity

The orthogonal group O(n), which is homotopy equivalent to GL($n,\mathbb{R}$) but has the evident virtue of compactness, plays an important rôle starting from the fact that the tangent bundle of a manifold with a Riemann structure has structure group O(n), i.e. is associated to the prinicpal O(n)-bundle P_{O} of orthonormal frames.

The homotopy groups, at least their stable versions, correspond to 'structures' on a manifold – orientation, spin and string structures that we consider here, and then five-brane structures which you are welcome to study (but I'm not going to do so).

The kth stable homotopy group may be obtained by considering the inclusion

ALS.35 (1.1.12.1) $$\mathrm{O}(n)\longrightarrow\mathrm{O}(n+N),\ N\geq 0$$

where an $n\times n$ matrix is 'stabilized' as the upper block in and $n+N$ dimensional decomposition with 0 off diagonal and Id_N in the other diagonal block. Then the homotopy groups $\Pi_k(\mathrm{O}(n))\longrightarrow\Pi_k(\mathrm{O}(n+N))$ stabilize for sufficiently large N; they were computed by Bott:

ALS.36 THEOREM 1 (Bott periodicity). *The stable homotopy groups of the orthogonal groups are 8-periodic*

ALS.37 (1.1.12.2)
$$\lim_{N\to\infty}\Pi_k(\mathrm{O}(N)) = \begin{cases} \mathbb{Z}_2 & k\equiv 0 \\ \mathbb{Z}_2 & k\equiv 1 \\ \{0\} & k\equiv 2 \\ \mathbb{Z} & k\equiv 3 \\ \{0\} & k\equiv 4 \\ \{0\} & k\equiv 5 \\ \{0\} & k\equiv 6 \\ \mathbb{Z} & k\equiv 7 \end{cases} \; mod\ 8.$$

These 'stable' homotopy computations are equivalent to the computation of the homotopy groups of a group. This is normally written $\mathrm{O}(\infty)$ and defined as an inductive limit. Since we are interested in smoothness it is useful to note that this group is homotopy equivalent to one of the groups that will arise below in the discussion of the spin representation of the loop group.

Namely, consider any compact smooth manifold M (you can use $\mathbb{R}^n$ instead by replacing smooth spaces by Schwartz spaces below). Then the space of *real* smoothing operators consists of the integral operators with smooth kernels

ALS.38 (1.1.12.3)
$$\Psi^{-\infty}_{\mathbb{R}}(M) = \{\mathcal{C}^\infty(M^2;\Omega_R)\}$$

here Ω_R is the real density bundle pulled back from the right factor of M – this is so one can integrate invariantly. As operators these are defined by integration

ALS.39 (1.1.12.4)
$$A\in\mathcal{C}^\infty(M^2;\Omega_R) \text{ defines } A:\mathcal{C}^\infty(M;\mathbb{R})\longrightarrow\mathcal{C}^\infty(M;\mathbb{R}),\ Au(m)=\int_M A(m,m')u(m')$$

where for each $m\in M$ the integrand is a smooth density on M. These operators form a associative algebra which is an infinite dimensional analogue of the matrix algebra (in particular it is simple).

We can make a group by considering the smoothing perturbations of the identity

ALS.40 (1.1.12.5)
$$\mathrm{Id}+A:\mathcal{C}^\infty(M;\mathbb{R})\longrightarrow\mathcal{C}^\infty(M;\mathbb{R}),\ A\in\Psi^{-\infty}_{\mathbb{R}}(M).$$

These operators are Fredholm of vanishing index and if injective have inverses of the same form. Thus

ALS.41 (1.1.12.6)
$$G^{-\infty}_{\mathbb{R}}(M) = \{\mathrm{Id}+A \text{ injective},\ A\in\Psi^{-\infty}_{\mathbb{R}}(M)\}$$

is a group. It is a smooth realization of $\mathrm{GL}(\infty,\mathbb{R})$, the inductive limit of the general linear groups.

To get an analogous orthogonal group one only needs to consider a Riemann metric on M which determines a smooth density and an L^2 inner product

ALS.42 (1.1.12.7)
$$\langle,\rangle:\mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\ni(u,v)\longrightarrow\int_M uvdg.$$

[In fact any smooth positive density will work as well.]

Requiring $\mathrm{Id}+A$ to be orthogonal

ALS.43 (1.1.12.8)
$$\langle(\mathrm{Id}+A)u,(\mathrm{Id}_A)v\rangle = \langle u,v\rangle\ \forall\ u,v\in\mathcal{C}^\infty(M)$$

gives the orthogonal subgroup $G^{-\infty}_{\mathrm{O}}(M)$.

These groups are isomorphic independent of the manifold (even if it is not connected!) and Bott periodicity becomes the two statments that the $\Pi_k(G^{-\infty}_{\mathrm{O}})$ (or $\Pi_k(G^{-\infty})$ are given by the formula (ALS.37 1.1.12.2) and that the map

ALS.44 (1.1.12.9)
$$\Pi_k(\mathrm{O}(n)) \longrightarrow \Pi_k(G^{-\infty}_{\mathrm{O}})$$

become isomorphisms for each k for large enough n.

This means $G^{-\infty}_{\mathrm{O}}$ is a classifying group for KO-theory.

If one allows complex coefficients the situation is analogous but simpler. The group $G^{-\infty}(M)$ consists of the invertible (equivalently injective or surjective) operators of the form $\mathrm{Id} + A$ where $A \in \mathcal{C}^\infty(M^2; \Omega_R \otimes \mathbb{C})$ and now

ALS.45 (1.1.12.10)
$$\Pi_k(G^{-\infty}(M)) = \begin{cases} \{0\} & k \equiv 0 \\ \mathbb{Z} & k \equiv 1 \end{cases} \text{ modulo 2.}$$

So $G^{-\infty}$ is a classifying group for complex K-theory.

1.13. Bargmann decomposition

Bargmann

Although it is at the moment unmotivated, the 'Bargmann decomposition' of $L^2(\mathbb{R}^2)$ plays an important rôle below; for the origins see [MR157250 **4**] and [MR1178936 **2**]. The elements

8.5.2021.1 (1.1.13.1)
$$\phi_{k,j} = z^k \overline{z}^j \exp(-\frac{1}{2}|z|^2) \in \mathcal{S}(\mathbb{R}^2),\ z = x + iy,\ k, j \in \mathbb{N}_0,$$

are dense - as follows from the Hermite expansion, the eigen-series for the harmonic oscillator. Rather than this we start from the Bargmann space,

8.5.2021.2 (1.1.13.2)
$$B = \{u \in L^2(\mathbb{R}^2),\ \partial_{\overline{z}}(u \exp(\frac{1}{2}|z|^2)) = 0\}$$

which is the closure of the span of the $z^k \exp(-\frac{1}{2}|z|^2)$.

Alternatively this is the closure in $L^2(\mathbb{R}^2)$ of the null space of the 'isotropic' operator

ALS.54 (1.1.13.3)
$$\not{b}^+ = i(2\partial_{\overline{z}} + z) : \mathcal{S}(\mathbb{R}^2) \longrightarrow \text{ on } \mathcal{S}(\mathbb{R}^2)$$

which is surjective. We proceed to analyze $\not{b}^+$ and its formal adjoint

ALS.55 (1.1.13.4)
$$\not{b}^- = i(2\partial_z - \overline{z}),\ [\not{b}^+, \not{b}^-] = 4,$$

which is injective on $\mathcal{S}(\mathbb{R}^2)$, as unbounded operators on L^2.

Since

8.5.2021.4 (1.1.13.5)
$$\int_0^\infty r^{2k} \exp(-r^2) dr = \frac{1}{2} \int_0^\infty s^{k-\frac{1}{2}} \exp(-s) ds = \frac{1}{2} \Gamma(k + \frac{1}{2})$$

the functions RBM:Adjust(ed) measure so $\beta_{0,0} = 1$

8.5.2021.3 (1.1.13.6)
$$\beta_{k,0} = (\Gamma(k + \frac{1}{2}))^{-\frac{1}{2}} z^k \exp(-\frac{1}{2}|z|^2)$$

form an orthonormal basis of B with respect to the normalized Lebesgue measure

ALS.111 (1.1.13.7)
$$\nu = \frac{1}{\pi} \pi dz d\overline{z}.$$

The functions

8.5.2021.5 (1.1.13.8)
$$\beta_{0,k} = \overline{\beta_{k,0}}$$

are orthogonal to the $\beta_{j,0}$ for $j > 0$ and

8.5.2021.6 (1.1.13.9)
$$B \cap \overline{B} = \mathbb{C} = \mathbb{C}\phi_{0,0}.$$

By direct computation

8.5.2021.8 (1.1.13.10)
$$\eth^+\phi_{k,j} = (i(2\partial_{\overline{z}} + z)(z^k\overline{z}^j \exp(-\frac{1}{2}|z|^2)) = 2ij\phi_{k,j-1},$$
$$\eth^-\phi_{k,j} = (i(2\partial_z - \overline{z})(z^k\overline{z}^j \exp(-\frac{1}{2}|z|^2)) = 2ik\phi_{k-1,j} - 2i\phi_{k,j+1}.$$

For $p \ge 0$ let L_p be the closure of the span of the $\phi_{p+j,j}$ for $j \ge 0$. From (8.5.2021.8 1.1.13.10)

8.5.2021.10 (1.1.13.11)
$$\eth^-\phi_{k,j} \in L_{k-j-1},\ k > j.$$

ALS.57 LEMMA 2. *For $p \ge 0$ the*

8.5.2021.11 (1.1.13.12)
$$\beta_{p+l,l} = 2^{-l}(l!)^{-\frac{1}{2}} \left(\eth^-\right)^l \beta_{p+l,0},\ l \ge 0$$

form an orthonormal basis of L_p.

PROOF. Orthogonality of $\beta_{p+l,l}$ and $\beta_{p+l',l'}$ for $l \ne l'$ follows from the usual 'creation-annihilation' computation. Thus if $l' \ge l > 0$

ALS.58 (1.1.13.13)
$$\langle(\eth^-)^{l'}\beta_{p+l',0}, (\eth^-)^l\beta_{p+l,0}\rangle_{L^2} = \langle(\eth^-)^{l'-1}\beta_{p+l',0}, \eth^+(\eth^-)^l\beta_{p+l,0}\rangle_{L^2}$$
$$= 4l\langle(\eth^-)^{l'-1}\beta_{p+l',0}, (\eth^-)^{l-1}\beta_{p+l,0}\rangle_{L^2} = 4^l l!\langle(\eth^-)^{l'-l}\beta_{p+l',0}, \beta_{p+l,0}\rangle_{L^2}.$$

□

Setting $L_p = \overline{L_p}$,

8.5.2021.13 (1.1.13.14)
$$L^2(\mathbb{R}^2) = \bigoplus_{p\in\mathbb{Z}} L_p.$$

The 'Laplacians'

8.5.2021.14 (1.1.13.15)
$$A^+ = \eth^-\eth^+ = -(2\partial_{\overline{z}} + z)(2\partial_z - \overline{z}) \text{ and } A^- = \eth^+\eth^- = A^+ + 4$$

are diagonalized in this basis,

8.5.2021.15 (1.1.13.16)
$$A^+\beta_{k,j} = 4j\beta_{k,j},\ A^-\beta_{k,j} = (4+4j)\beta_{k,j}.$$

So all the eigenspaces have infinite multiplicity. The inverse of A^- and the generalized inverse of A^+

ALS.72 (1.1.13.17)
$$B^+\beta_{k,j} = (4j)^{-1}\beta_{k,j},\ j > 0,\ B^+\beta_{k,0} = 0,\ (A^-)^{-1}\beta_{k,j} = (4j+4)^{-1}\beta_{k,j}$$

give generalized inverses

ALS.73 (1.1.13.18)
$$Q^- = \eth^= B + +,\ Q^\beta_{k,j} = \ -$$
$$Q^+ = \eth^-(A^-)^{-1},\ \eth^- Q^- =$$

where both $Q^\pm$ are bounded and map into the domain.

The domain of both these essentially self-adjoint operators is the dense subspace

8.5.2021.16 (1.1.13.19)
$$\sum_{k,j} c_{k,j}\beta_{k,j},\ \sum_{k,j}(j+1)^2|c_{k,j}|^2 < \infty.$$

It follows that $\eth^+$ and $\eth^-$ are closed 'shift' operators, adjoints of each other, with the common domain

8.5.2021.17 (1.1.13.20)

$$\mathrm{Dom}(\eth^+) = \mathrm{Dom}(\eth^-) = \left\{ \sum_{k,j} c_{k,j}\beta_{k,j};\ \sum_{k,j}(j+1)|c_{k,j}|^2 < \infty \right\} \subset L^2(\mathbb{R}^2)$$

$$\eth^+\beta_{k,j} = 2j^{\frac12}\beta_{k-1,j-1},\ \eth^-\beta_{k,j} = 2(j+1)^{\frac12}\beta_{k+1,j+1}.$$

As operators from the Hilbert space defined by (8.5.2021.17 1.1.13.20) both operators are semi-Fredholm, the one surjective with null space B, the other injective with range the orthocomplement of B.

- Rotation action
- 'Sobolev spaces'
- Multiplication operators z and $\overline{z}$.
- Invariance

1.14. Covers and orientation

One of the underlying themes in this subject is the 'geometric' representation of integral cohomology classes. This is straightforward for dimensions 0 and 1. Namely for any (here always separable, metrizable) space, X, $H^0(X;\mathbb{Z})$ consists of the continous maps into $\mathbb{Z}$. Of course continuity can be replaced by local constancy so this is just a copy of $\mathbb{Z}$ for each connected component and we are normally dealing with connected manifolds.

For $H^1(X;\mathbb{Z})$ there are two standard approaches. We can consider 'torsors' over X, which is to say pricipal $\mathbb{Z}$ bundles over X. The other alternative is the space $\mathcal{C}(X,\mathbb{T})$ of continous maps in the circle. The connection between these two is that

ALS.33 (1.1.14.1)
$$\mathbb{Z}\mathbb{R} \longrightarrow \mathbb{R} \longrightarrow \mathbb{T}$$

is a principal $\mathbb{Z}$ bundle with contractible total space. This makes $\mathbb{T}$ into a $K(\mathbb{Z},1)$, an Eileberg-McLean space which is classifying for $H^1(X;\mathbb{Z})$. Thus

ALS.34 (1.1.14.2)
$$H^1(X;\mathbb{Z}) = \mathcal{C}(X;\mathbb{T})/\text{homotopy}.$$

1.15. Spin structures

For an oriented Riemann manifold, M, of dimension $n \geq 3$, the bundle of oriented orthonormal frames, F_{SO}, is a princpal $\mathrm{SO}(n)$ bundle and a spin structure is a double cover which is a $\mathrm{Spin}(n)$ principal bundle with compatible actions

158L5.8 (1.1.15.1)
$$\begin{array}{ccc} \mathrm{Spin}(n) & \longrightarrow & F_{\mathrm{Spin}} \\ \downarrow \pi & & \downarrow \pi \\ \mathrm{SO}(n) & \longrightarrow & F_{\mathrm{SO}} \\ & & \downarrow \\ & & M. \end{array}$$

Thus, for $s \in \mathrm{Spin}(n)$ and $f \in F_{\mathrm{Spin}}$ we require

158L5.9 (1.1.15.2)
$$\pi(sf) = \pi(s)\pi(f).$$

158L5.10 Theorem 2. *A spin structure exists if and only if the second Stiefel-Whitney class $w_2 \in H^2(M;\mathbb{Z}_2)$ vanishes and then the structures up to isomorphism are identified with $H^1(M;\mathbb{Z}_2)$.*

Let me go quickly through the proof, as much as a guide to things to come. Of course if you don't know what w_2 is this just says there is such a class. I am thinking of this result as a precursor of the 'lifting bundle gerbe' which I want to introduce soon.

Take a good open cover of M – so this can be by small geodesic balls; we can take the cover to be finite if M is compact, or in general countable but it really does not matter in the argument. Over each of these sets, B_i, the frame bundle is trivial, which is to say it has a section given by parallel transport along radial geodesics from the central point

158L5.11 (1.1.15.3)
$$\phi_i : B_i \longrightarrow F_{\mathrm{SO}},\ \pi\phi_i = \mathrm{Id}\,.$$

On the intersection of pairs of sets there is a transition map between the sections

158L5.12 (1.1.15.4)
$$\theta_{ji} : B_i \cap B_j \longrightarrow \mathrm{SO}(n),\ \phi_j(x) = \theta_{ji}\phi_i(x)\ \forall\ x \in B_i \cap B_j.$$

Everything here is continuous, but can be taken to be smooth. Again, since $B_i \cap B_j$ is contractible we can lift these maps

158L5.13 (1.1.15.5)
$$\begin{array}{ccc} B_i \cap B_j & \xrightarrow{\tilde\theta_{ji}} & \mathrm{Spin}(n) \\ & \searrow^{\theta_{ji}} & \downarrow \\ & & \mathrm{SO}(n). \end{array}$$

Over triple intersections we know, from the definition, that

158L5.14 (1.1.15.6)
$$\theta_{ik}\theta_{kj}\theta_{ji} = \mathrm{Id}$$

so for the lifts we find

158L5.15 (1.1.15.7)
$$\mu_{kji} = \tilde\theta_{ik}\tilde\theta_{kj}\tilde\theta_{ji} : B_i \cap B_j \cap B_k \longrightarrow \mathbb{Z}_2.$$

This is a Čech cochain with values in $\mathbb{Z}_2$. In fact it is a cocycle, the Čech differential vanishes. The Čech differential is

ALS.8 (1.1.15.8)
$$(\partial\mu)_{lkji} = \mu_{lkj}\mu^{-1}_{lki}\mu_{lji}\mu^{-1}_{kji} = 1.$$

This follows by inserting (158L5.15 1.1.15.7) and noting that $\mathbb{Z}_2$ lies in the centre of $\mathrm{Spin}(n)$.

At some point I will get around to writing out a brief treatment of Čech theory.

158L5.16 Exercise 3. Write out the proof.

In fact since we are working with a good cover the class $w_2 = [\mu] \in H^2(M;\mathbb{Z}_2)$ vanishes if and only if μ is exact. A primitive of μ can be used to modify the $\tilde\theta_{ji}$ so that the new μ in (158L5.15 1.1.15.7) vanishes. This means that these $\tilde\theta_{ji}$ form a cocycle and hence we can define a principal bundle by taking these as the transition map and

158L5.18 (1.1.15.9)
$$F_{\mathrm{Spin}} = \bigsqcup_i (\mathrm{Spin}(n) \times B_i)/\simeq$$

is a double cover of F_{SO} and (158L5.8 1.1.15.1) follows.

You should also check that that two different choices of primitive for μ lead to a class in $H^1(M;\mathbb{Z}_2)$ which vanishes if and only if the resulting bundles are isomorphic as principal bundles.

This completes the cursory discussion of spin structures provided we identify w_2 as the second Stiefel-Whitney class.

CHAPTER 2

Mapping spaces

Map

The finite-dimensional manifolds of a particular regularity form a category – which is a convenient way to organize some basic aspects of the subject. The objects in this category are the manifolds and then the arrows between two manifolds are the maps of the regularity under discussion – so this is a subcategory of 'Set'. The arrows from X to Y form the mapping space, which is precisely what we are interested in studying. We will work under the assumption that X is a compact manifold with corners. These mapping spaces $\mathcal{M}(X,Y)$ can then be thought of as infinite-dimensional manifolds, Banach, Hilbert of Fréchet depending on the regularity. Whilst they *are*, this rather misses special bundle-like properties that they have.

2.1. Continuous maps

For $f \in \mathcal{M}_0(M,N)$ and $\epsilon > 0$ smaller than the infimum over $f(M)$ of the injectivity radius for the chosen metric on N consider the metric ball

Tube (2.2.1.1) $$\mathcal{T}(f,\epsilon) = \{g \in \mathcal{M}_0(M,N); \sup d_N(f(m),g(m)) < \epsilon\}$$

which is a 'tube' consisting of maps uniformly close to f.

PROPOSITION 6. *The open sets $\mathcal{T}(f,\epsilon)$, where $\epsilon > 0$ is chosen sufficiently small depending on f, form a good open cover of $\mathcal{M}_0(M,N)$.*

PROOF. The main point here is that on N itself the open balls of sufficiently small radius, where the radius is bounded below on compact sets. form a good cover. For instance one can take the balls $B(p,\epsilon)$ such that for all $q \in B(p,\epsilon)$ the injectivity radius at q is at least 2ϵ. Such balls are geodesically convex so their non-empty intersections are always radially contractible along the geodesics emanating from any point. Explicitly

AnLoSp.3a (2.2.1.2) $$[0,1] \times B(q,\epsilon) \ni (t,x) \longmapsto R_t(p,x) = \exp_p(t\exp_p^{-1}(x)) \in B(q,\epsilon)$$

defines a smooth retraction to p in terms of the exponential map. This also depends smoothly on p.

The contractability of the tube $\mathcal{T}(f,\epsilon)$ where now the injectivity radius at all points of all balls $B(f(m(,\epsilon))$, for all $m \in M$, is at least 2ϵ, follows where the contraction is

$$[0,1] \times \mathcal{T}(f,\epsilon) \ni (t,g) \longmapsto g_t(p) = R_t(f(p),g(p)) \in \mathcal{T}(f,\epsilon)$$

with $R_t(p',q)$ the retraction around p'. There is a similar retraction around *any* element of $\int(f,\in)$ and hence of any non-trivial intersections. □

2.2. Tubular spaces

Map.TS

To emphasize this let us consider an appropriate categorical structure with these mapping spaces as the objects. What should the arrows be? They should certainly be maps, but the idea is to restrict these severely. Namely for two mapping spaces $\mathcal{M}(X_1, Y_1)$ and $\mathcal{M}(X_2, Y_2)$ (always of the same, if unspecified, regularity) consider as space of arrows

ALS.13 (2.2.2.1)
$$\mathcal{M}(X_1 \times Y_1, Y_2) \times \mathcal{M}(X_2, X_1).$$

The action of the second part is by pull-back, so it acts on the domain, and the action of the second is dually by post-composition. Specifically,

ALS.14 (2.2.2.2)
$$\begin{gathered} \text{If } (F, f) \in \mathcal{M}(X_1 \times Y_1, Y_2) \times \mathcal{M}(X_2, X_1) \text{ then} \\ \mathcal{M}(X_1, Y_1) \ni u \longmapsto T_F f^* u \in \mathcal{M}(X_2, Y_2), \text{ where} \\ \mathcal{M}(X_1, Y_1) \ni u \longmapsto f^* u = u \circ f \in \mathcal{M}(X_2, Y_1), \\ \mathcal{M}(X_2, Y_1) \ni v \longmapsto T_F v(x_2) = F(x_2, v(x_2, y_1)) \in \mathcal{M}(X_2, Y_2). \end{gathered}$$

If $G \in \mathcal{M}(X_2 \times Y_2, Y_3)$ and $g \in \mathcal{M}(X_3, X_2)$ then

ALS.15 (2.2.2.3)
$$T_G g^* T_F f^* = T_{G \circ g^* F}(g \circ f)^*, \ G \circ g^* F(x_3, y_1) = G(g(x_2), F(g(x_2), y_1)) \\ \Longrightarrow G \circ g^* F \in \mathcal{M}(X_3 \times Y_1, Y_3).$$

Associativity follows directly as does the existence of units.

This categorical structure actually produces an exact 2-category, where the 0-cells are the manifolds, the 1-cells are the mapping spaces and the 2-cells are the products (2.2.2.1).

ALS.16 EXERCISE 4. Check the 2-category axions if you are interested!

Restricting the morphisms in this way amounts to thinking of $\mathcal{M}(X, Y)$ as the space of sections of $X \times Y$ as a fibre bundle over X. Indeed if we use the, slighly informal, notation $Y(*)$ for the total space of a trivial fibre bundle over X and $\mathcal{M}(X, Y(*))$ for the space of section then we get a somewhat larger category with these as objects. For the arrows between two such bundles $Y_i(*)$, over bases X_i, (2.2.2.1) is then replaced by

ALS.24 (2.2.2.4)
$$\mathcal{M}(Y_1(*), Y_2(*)) \times \mathcal{M}(X_2, X_1)$$

where the first space consists of the fibre-preserving maps. The notation is marginally consistent with intrepreting $\mathcal{M}(X, Y)$ as the fibre-preserving maps from X as the trivial bundle over itself to $X \times Y$.

2.3. Open covers

Map.OC

This restricted class of 'fibre' maps (2.2.2.2) is enough to capture a great deal of the local structure of the mapping spaces. The mapping spaces, with X compact, are separable metric spaces, with topologies depending on the regularity. For the moment we will assume, as always, that X is compact but also that Y has a complete Riemann metric - although this is not essential. Then $\mathcal{M}(X, Y)$ acquires a supremum metric

ALS.27 (2.2.3.1)
$$d(u, v) = \sup_{x \in X} d_Y(u(x), v(y)).$$

The space of continuous maps is separable and complete with this metric. To see separability, take a compact exhaustion, K_n, of Y and embed Y in some Eucludean space. On compact subsets of a manifold any two Riemann metrics induce equivalent distances so the separability follows from the separability of $\mathcal{M}(X, \mathbb{R}^N) = \mathcal{C}(X)^N$.

In the standard notion of a manifold any open subset is also a manifold. Clearly this is not the case that an arbitrary open set in a mapping space. However if we take the $\mathcal{C}^0$ topology, given by metrics on X and Y then the balls are consider special open sets, in that they are *tubular domains*. By a tubular domain in $X \times Y$ we mean a trivial open fibre subbundle $Z(*)$ over X. Thus each $Z(x) \subset Y$ is open and there is a fibre-preserving diffeomorphism

ALS.25 (2.2.3.2) $$\phi(x) : Z(x) \longrightarrow Z,\ \phi : Z(*) \longrightarrow Y$$

for some manifold Z. Associated with such a tubular domain is the open subset

ALS.26 (2.2.3.3) $$\Gamma = \{f \in \mathcal{M}(X, Z(*))\} \subset \mathcal{M}(X, Y).$$

Thus Γ is a mapping space identified with the space of map from X to Y with $f(x) \in Z(x)$ for all x. Under the fibre-preserving diffeomorphism (ALS.25 2.2.3.2), Γ is identified with $\mathcal{M}(X, Z)$.

Since Γ determines $Z(*)$ we regard it as the tubular domain. Thus an open set $\Gamma \subset \mathcal{M}(X, Y)$ is a tubular domain if

ALS.21 (2.2.3.4) $$\begin{gathered} Z(x) = \Gamma \cdot x \subset Y \text{ is open for all } x \in X, \\ f \in \mathcal{M}(X, Y),\ f(x) \in \Gamma \cdot x\ \forall\ x \in X \Longrightarrow f \in \Gamma \\ \text{and } Z(*) \text{ is a trivial fibre bundle.} \end{gathered}$$

The invertibles in $\mathcal{M}(X \times Y, Y) \times \mathcal{M}(X, X)$, i.e. the pairs (F, f) with $F(x) \in \mathrm{Dff}(Y)$ and $f \in \mathrm{Dff}(X)$, transform one such tubular domain to another

ALS.22 (2.2.3.5) $$\Gamma \longmapsto T_F f^* \Gamma.$$

ALS.23 PROPOSITION 7. *If X is compact then $\mathcal{M}(X, Y)$ has a countable, good, open covering by tubular domains.*

We use the same idea to see the local structure of the mapping spaces. Namely take a Riemann metric on M and consider for $\lambda \in \mathcal{C}(N; M)$ the set

158L3.18 (2.2.3.6) $$\Gamma(\lambda, \epsilon) = \{\lambda' \in \mathcal{C}(N; M); d(\lambda(n); \lambda'(n)) < \epsilon\}$$

for $\epsilon < \epsilon_0$, the injectivity radius (or its inf over the image of $\lambda(N)$).

The exponential map on M gives an identification

158L3.19 (2.2.3.7) $$\Gamma(\lambda, \epsilon) \longrightarrow \{u \in \mathcal{C}(N; \lambda^* TM); \sup |u| < \epsilon\},\ \lambda'(n) = \mathrm{Exp}_{\lambda(n)}(u(n)).$$

158L3.20 PROPOSITION 8. *Under the identification (158L3.19 2.2.3.7) all the mapping spaces are identified with the corresponding spaces of sections of $\lambda^* TM$ and if $\epsilon_1 + \epsilon_2 < \epsilon_0$ they all have the same transition maps*

158L3.21 (2.2.3.8) $$\begin{array}{ccc} & \Gamma(\lambda_1, \epsilon_1) \cap \Gamma(\lambda_2, \epsilon_2) & \\ {}^{\mathrm{Exp}_{\lambda_1}}\swarrow & & \searrow^{\mathrm{Exp}_{\lambda_2}} \\ \mathcal{C}(N; \lambda_1^* M) & \xrightarrow[\mathrm{Exp}_{\gamma_1} \circ \mathrm{Exp}_{\lambda_1}^{-1}]{} & \mathcal{C}(N; \lambda_1^* M). \end{array}$$

Note that this is a good reason to think of all the mapping spaces as being 'the same manifold' in so far as they all have the same transition cocycle.

Proof. By picture. □

So the 'manifold structure' on say $H^s(N;M)$ is defined by the homeomorphisms 158L3.19 (2.2.3.7) from a neighbourhood $\Gamma(\lambda,\epsilon)$ of a given element λ to a neighbourhood of the zero section of $H^s(N;\lambda^*TM)$. So we do not have exactly fixed linear space as 'model'. In fact they are not even isomorphic. However the component of $H^s(N;M)$ can be identified with the homotopy classes of maps and then the bundles λ^*TM are all isomorphic over a fixed component. So this is basically just notationlly different from assuming a fixed model.

The most fundamental point is that the tangent space of a 'manifold' should actually be isomorphic to the model space. In our case we can see that the tangent space (which really needs to be defined carefully in this infinite-dimensional context) is isomorphic to $H^s(N;\lambda^*M)$. What should it be as an abstract linear space?

I have not made essential use of the compactness of M (maybe I will remvoe that restriction in the notes to this point) so the mapping spaces on the right in the definition

158L3.22 (2.2.3.9) $$TH^s(N;M)=\mathcal{M}_*(N;TM)$$

make sense in all cases (I just decided to use $\mathcal{M}_*$ for the mapping space with regularity $*$).

158L3.23 Proposition 9. *The space $\mathcal{M}_*(N;TM)$ fibres over $\mathcal{M}_*(N;M)$ with fibre at λ naturally identified with $H^s(N;\lambda^*TM)$.*

This justifies our definition of the tangent space.

There is much more to say about all this and I will come back to it. Notice that for any $s>\frac{1}{2}\dim N$ and $\lambda\in H^s(N;M)$ the spaces $H^r(N;\lambda^*TM)$ make sense fo $-s\le r\le s$. So there are 'bigger' versions of the tangent space lurking in the background. We need to come to grips with this rather seriously.

2.4. Differentiable functions

As indicated above the term 'lithe' is used here as an analogue of the term 'smooth' where the latter refers (somewhat non-specifically) to the existence of derivatives and the former to their pointwise regularity. In a finite dimensional context 'litheness' does not arise.

To define the space of s-lithe $\mathcal{C}^1$ functions on the Fréchet space $\mathcal{X}=\mathcal{C}^\infty(X;\mathbb{R})$ we will make use of an increasing sequence $P_k\subset\mathcal{X}$ of finite dimensional subspaces with dense union $P_*\subset\mathcal{X}$. In the case of main interest here, $X=\mathbb{T}$, so $\mathcal{X}=\mathcal{C}^\infty(\mathbb{T};\mathbb{R})=\mathcal{L}\mathbb{R}$ these may well be the (real-valued) Fourier polynomials. The definitions here are independent of this choice.

We first consider the standard spaces

30.4.2021.6 (2.2.4.1)
$$\begin{gathered}\mathcal{C}^0(\mathcal{X})=\{F:\mathcal{X};\ \text{continuous}\},\\ \mathcal{BC}^0(\mathcal{X})=\{F\in\mathcal{C}^0(\mathcal{X});\sup_l|F(l)|<\infty\}.\end{gathered}$$

Here continuity is with respect to the Fréchet topology on $\mathcal{X}$ so the second space is Banach as it is for any metric space.

Now consider a function $F : \mathcal{X} \longrightarrow \mathbb{C}$ which is differentiable when restricted to the affine $l + P_k$, for each $l \in \mathcal{X}$ and each k. This weak differentiability means that the derivative is defined as a linear function(al)

30.4.2021.7 (2.2.4.2)
$$dF(l) : P_* \longrightarrow \mathbb{C},\ P_* \subset T_l\mathcal{X} = \mathcal{X}.$$

The we demand the s-lithness condition that

30.4.2021.8 (2.2.4.3)
$$dF(l) \in H^s(X),\ s \in \mathbb{R}.$$

Here the implicit pairing is always the standard L^2 pairing on X – there is no coordinate-independent pairing existence is indepenent of any choices.

Then we define the space

30.4.2021.9 (2.2.4.4)
$$\begin{gathered}\mathcal{C}^{1,(s)}(\mathcal{X}) = \{F \in \mathcal{C}^0(\mathcal{X}); F \text{ is weakly differentiable with } s\text{-lithe derivative and} \\ \mathcal{X} \ni l \longmapsto dF(l) \in H^s(X) \text{ is continuous}, \\ \mathcal{BC}^{1,(s)}(\mathcal{X}) = \{F \in \mathcal{BC}^0(\mathcal{X}) \cap \mathcal{C}^{1,(s)}(\mathcal{X}) \text{ and } \sup_l \|dF(l)\|_{H^{-s}(X)} < \infty.\end{gathered}$$

Continuity here is always with respect to the Fréchet topology on $\mathcal{X}$. Again the second of these is a Banach space.

For higher derivatives we proceed similarly with an apropriate choice of tensor completion. Thus for $F \in \mathcal{C}^{N,(s)}(\mathcal{X})$ we require that F have derivatives up to order N on each affine $l + P_k$ and then consider the symmetric N-multilinear form so defined

30.4.2021.4 (2.2.4.5)
$$F^{(N)}(l) : P_* \times P_* \times \dots P_* \longrightarrow \mathbb{C}.$$

We require that this extends to a separately, hence jointly, multilinear form

30.4.2021.5 (2.2.4.6)
$$F^{(N)}(l) : H^{-s}(X) \times H^{-s}(X) \times \cdots \times H^{-s}(X) \longrightarrow \mathbb{C}$$

with contnuity of $F^{(N)}(l)$ as a function of l in terms of the implied norm

30.4.2021.12 (2.2.4.7)
$$\|F^{(N)}(l)\| = \sup_{\|u_i\|=1 \text{ in } H^{-s}(X)} |F^{(N)}(l)(u_1, \dots, u_N)|.$$

Then $\mathcal{C}^{N,(s)}(\mathcal{X})$ is defined inductively by requring $F \in \mathcal{C}^{N-1,(s)}(\mathcal{X})$ to satisfy (2.2.4.5) and (2.2.4.6). Similarly $\mathcal{BC}^{N,(s)}(\mathcal{X})$ is defined inductively as consisting of the elements of $\mathcal{BC}^{N-1,(s)}(\mathcal{X}) \cap \mathcal{C}^{N,(s)}(\mathcal{X})$ such that

30.4.2021.11 (2.2.4.8)
$$\sup_l \|F^{(N)}(l)\| < \infty.$$

As usual the spaces of infinitely differentiable functions are the intersections over N.

The product rule for differentiation certainly holds in the weak sense so the $\mathcal{C}^{1,(s)}(\mathcal{X})$ spaces are clearly multiplicative. The 'direct product' of two linear functionals u_i is a continuous bilinear functional

ALS.47 (2.2.4.9)
$$H^{-s}(X) \times H^{-s}(X) \ni (v, w) \longmapsto u_1(v)u_2(w)$$

and similarly for higher order products of multilinear functionals from which it follows that all the higher order spaces are also multiplicative.

In particular the elements $v \in H^{-s}(X)$ define linear function on $\mathcal{X}$ through the L^2 pairing on X and these are differentiable (in almost any sense, but in particular weakly) with constant s-lithe derivative v at each point, so $v \in \mathcal{C}^{\infty,(s)}(\mathcal{X})$. Similarly a continuous symmetric multilinear form

ALS.46 (2.2.4.10)
$$g : H^{-s}(X) \times H^{-s}(X) \times \cdots \times H^{-s}(X) \longrightarrow \mathbb{C}$$

defines a polynomial function on $\mathcal{X}$ and this is s-lithe infinitely differentiable with successive derivatives arising from the symmetric multlinlinear forms corresponding to evaluating some of the variables.

Combining the multiplicativity we can define s-lithe Schawartz spaces by

ALS.48 (2.2.4.11)
$$u \in \mathcal{S}^{(s)}(\mathcal{X}) = \left\{ u \in \mathcal{C}^{\infty,(s)}(\mathcal{X}); gu \in \mathcal{BC}^{\infty,(s)}(\mathcal{X}) \ \forall\ g \text{ and continuous in } g \right\}.$$

Of course the question arises as to whether these spaces are empty! As in the finite-dimensional case this is easily answered by the existence of Gaussians. More generally let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a Scwhartz function then

ALS.49 (2.2.4.12)
$$f(\|l\|^2) \in \mathcal{S}^{(s)}(\mathcal{X}) \text{ if } \|\cdot\| \text{ is a Hilbert norm on } H^{-s}(X).$$

Note that all these spaces are defined in terms of functions on $\mathcal{X} = \mathcal{C}^\infty(X)$. This is mainly to simplify the discussion. If $u : \mathcal{X} \longrightarrow \mathbb{R}$ is continuous with respect to the Fréchet norm then it extends to a continuous function on $H^{-s}(X)$ if it is locally uniformly continuous with respect to the H^{-s} norm, so uniformly continuous on small balls. This however follows if $u \in \mathcal{BC}^{1,(s)}(\mathcal{X})$. Namely the mean value theorem applies and gives a Lipschitz condintion on the restriction to the P_k and shows that

ALS.50 (2.2.4.13)
$$|u(l_1) - u(l_2)| \le \|l_1 - l_2\|_{H^s} \sup_{t_1[0,1]} \|du(tl_1 + (1-t)l_2\|_{H^{-s}(X)} \le C\|l_1 - l_2\|_{H^s}$$

with C the supremum of $\|du\|(l)$ for the $-s$ norm. It follows that u extends by continuity to $H^{-s}(X)$. Similarly arguments apply using higher derivatives.

Notice that the absence of local compactness means that for a function u on $\mathcal{X}$ with support say in a ball with respect to the H^{-s} norm the condition that $u \in \mathcal{S}^{(s)}(\mathcal{X})$ is *stronger* than the condition $u \in \mathcal{BC}^{\infty,(s)}(\mathcal{X})$.

$\mathcal{C}^\infty$ algebra

ALS.51 Proposition 10. *If U_i is an open cover of $\mathcal{C}^\infty(X)$ in the H^{-s} topology then there is an s-lithe $\mathcal{C}^\infty$ partition of unity subordinate to it.*

To transfer these spaces to $\mathcal{M}(X, M)$ we need to show 'coordinate invariance'.

2.5. Notes from L3

Let me start by recalling what I did at the end last time. If N is a compact manifold with corners we define (in principle by iterated doubling and restriction) the spaces $\mathcal{C}^{k,\alpha}(N)$ and $H^s(N)$ for $k \in \mathbb{N}_0$, $0 \le a \le 1$; we also have the fundamental space $\mathcal{C}^\infty(N)$. If $\alpha = 0$ then we just get the spaces $\mathcal{C}^k(N)$. These are all Banach/Hilbert/Fréchet spaces.

Using the properties of these spaces (for $s > \frac{1}{2}\dim N$) we defined the corresponding mapping spaces where the image is a smooth manifold M. The mapping spaces are correspondingly Banach/Hilbert/Fréchet manifolds – although much better than that. One way to see this is to using Whitney embedding $M \subset \mathbb{R}^N$ and observe (if you want, define)

158L3.16 (2.2.5.1)
$$H^s(N; M) = \{\lambda \in H^s(N; \mathbb{R}^N); \lambda(N) \subset M\}.$$

From this approach we see that these are all separable, complete metric spaces (even if M is not compact, but we will always take N to be compact).

158L3.17 Proposition 11. *For and* $s > \frac{1}{2}\dim N$, $\mathcal{C}^\infty(N;M)$ *is dense in all the spaces* $H^s(N;M)$, $\mathcal{C}^{k,\alpha}(N;M)$.

Proof. Retract onto the image of M using a normal fibration and use the density of $\mathcal{C}^\infty(N)$ in $H^s(N)$ etc. □

2.6. Path and loop spaces

Now, I want to get to one case, and I hope more cases, of transgression today or at least this week. Let's specialize to the path and loop spaces, the special cases $N = I = [0,1]$ and $N = \mathbb{T}$ the circle. Then I will use the notations $\mathcal{P}_s M = H^s(I;M)$ and $\mathcal{L}_s M = H^s(\mathbb{T};M)$ etc for the path and loop spaces.

2.7. Hardy-Bargmann decomposition

Hardy-Bargmann

Let $H^{\frac{1}{2}}(\mathbb{T}_+)$ denote the $\frac{1}{2}$ order Sobolev-Hardy space, were the notation pretends it is the Sobolev space on the 'Hardy half-circle'. As a space it is

9.5.2021.3 (2.2.7.1) $$\Pi_+ H^{\frac{1}{2}}(\mathbb{T}).$$

The constants need to be treated separately so we consider the Hilbert norm

9.5.2021.4 (2.2.7.2) $$\|u\|^2_{\frac{1}{2}} = |a_0|^2 + \sum_{k\ge 0} k|a_k|^2.$$

Although as yet unmotivated we consider Hilbert spaces obtained by completion of function spaces on the 'mean-zero' functions on the circle which may be identified with the orthocomplement of the constants in the (necessarily complex-valued) smooth Hardy space:

ALS.59 (2.2.7.3) $$\mathcal{C}^\infty_0(\mathbb{S};\mathbb{R}) = \mathcal{C}^\infty_0(\mathbb{T}_+).$$

Let $z_p = x_p + iy_p$ be the pth Fourier coefficients, representing the functions

ALS.60 (2.2.7.4) $$\frac{1}{\sqrt{2}} x_p \cos(p\theta) + i\frac{1}{\sqrt{2}} y_p \sin(p\theta)$$

We denote the rescaled version of this variable as RBM:Maybe wrong normalization

ALS.61 (2.2.7.5) $$Z_p = \sqrt{p} z_p.$$

Consider the set of maps

ALS.63 (2.2.7.6) $$\Sigma = \{\sigma : \mathbb{N} \longrightarrow \mathbb{N}^2; \operatorname{supp}(\sigma) \text{ is finite.}\}$$

where the support is the set of $p \in \mathbb{N}$ such that $\sigma(p) \neq (0,0)$. For each $\sigma \in \Sigma$ consider the function on $\mathcal{C}^\infty_0(\mathbb{T}_+)$

ALS.62 (2.2.7.7) $$\gamma_\sigma = \left(\prod_{p\in\operatorname{supp}(\sigma)} \beta_{\sigma(p)}(Z_p, \overline{Z}_p) \right) \exp(-\frac{1}{2}\sum_{k\ge 0} |Z_k|^2).$$

Here the Bargmann basis β_{ij} is defined in §1.13 (Bargmann).

Each such function is in the $\frac{1}{2}$-lithe Schwartz space and we define a Hilbert space completion by identifying the finite span as the finite sequences in $l^2(\Sigma)$. Thus formally an element of this space is identified as an infinite series

ALS.65 (2.2.7.8)
$$\mathcal{H}(H_+) = \{\sum_{\sigma\in\Sigma} c_\sigma \gamma_\sigma,\ \sum_{\sigma\in\Sigma} |c_\sigma|^2 < \infty\}.$$

For the moment at least we leave the questions of the convergence to a function on $\mathcal{C}_0^\infty(\mathbb{T}_+)$ unresolved, but by definition it has a dense subspace of such functions.

We can also consider subbspaces corresponding to weighted versions of the l^2 norms. For $s = (s_1, s_2, s_3)] \in [0,\infty)^3$ consider the weight

ALS.71 (2.2.7.9)
$$m_s(\sigma)(p) = p^{s_1}(k(\sigma(p)+1)^{s_2}(1+j(\sigma(p))^{s_3} \ge 1$$

defined by $\sigma(p) = (k(\sigma(p)), j(\sigma(p)))$. Then set

ALS.66 (2.2.7.10)
$$\mathcal{H}^s(H_+) = \{\sum_{\sigma\in\Sigma} c_\sigma \gamma_\sigma,\ \sum_{p,\sigma\in\Sigma} m_s(\sigma)^2 |c_\sigma|^2 < \infty\}.$$

Again these are Hilbert spaces in which the finite sums are dense and clearly

ALS.70 (2.2.7.11)
$$\mathcal{H}^s(H_+) \subset \mathcal{H}(H_+).$$

ALS.67 LEMMA 3. *Each of the rescaled operators*

ALS.68 (2.2.7.12)
$$\not{b}_j^+ = i(\partial_{\overline{Z}_j} + Z_j),\ \not{b}_j^- = i(\partial_{Z_j} - \overline{Z}_j)$$

extend to bounded operators

ALS.69 (2.2.7.13)
$$l^r \not{b}_l^\pm : \mathcal{H}^s(H_+) \longrightarrow \mathcal{H}^{s'}(H_+),\ s' = (s_1 - r, s_2, s_3 - \frac{1}{2}),\quad \text{if } s_1 \ge r,\ s_3 \ge \frac{1}{2}.$$

These operators are by no means closed by the domains of their closures are easily computed. We introduce these spaces for the treatment of the Dirac-Ramond operator on the torus in Chapter 7 below.

2.8. Bott-Virasoro group

9.5.2021.5 THEOREM 3. *There is an exact sequence of groups*

9.5.2021.6 (2.2.8.1)
$$G^{-\infty}_{\frac{1}{2}}(\mathbb{T}_+) \longrightarrow \mathrm{Dff}^+_{\frac{1}{2}}(\mathbb{T}_+) \longrightarrow \mathrm{Dff}^+(\mathbb{T})$$

where $G^{-\infty}_{\frac{1}{2}}(\mathbb{T}_+)$ *is the group of invertible smoothing perturbations of the identity which are unitary on* $H^{\frac{1}{2}}(\mathbb{T}_+)$ *and* $\mathrm{Dff}^+_{\frac{1}{2}}(\mathbb{T}_+)$ *is the group of unitary perturbations of* $\Pi_+\phi^*\Pi_+$ *on* $H^{\frac{1}{2}}(\mathbb{T}_+)$ *for* $\phi \in \mathrm{Dff}^+(\mathbb{T})$.

PROOF. It is convenient to work on real-valued functions on the circle. The Toeplitz projection

9.5.2021.7 (2.2.8.2)
$$\Pi_+ : \mathcal{C}_0^\infty(\mathbb{T};\mathbb{R}) \longrightarrow \mathcal{C}_0^\infty(\mathbb{T}_+),$$

from real function with mean zero to the subspace of the complex Hardy space without constant terms, is a bijection. On these subspaces the $\frac{1}{2}$ norm in (2.2.7.2) is

9.5.2021.8 (2.2.8.3)
$$\|\Pi_+ v\|^2_{\frac{1}{2}} = \frac{1}{2}\langle |D|v, v\rangle_{L^2},\ |D| = -D\Pi_- + D\Pi_+.$$

Now, consider the *standard* pull-back action of $\mathrm{Dff}^+(\mathbb{T})$ on $\mathcal{C}^\infty(\mathbb{T})$ (even though this does not preserve the mean-zero condition) $\phi^* v(\theta) = v(\phi(\theta))$. From the final part of (9.5.2021.8 2.2.8.3),

9.5.2021.9 (2.2.8.4)
$$|D|\phi* = -D\phi^*\Pi_- + D\phi^*\Pi_+ - D[\Pi_-, \phi^*] + D[\Pi_+, \phi^*] = \phi'(\phi^*|D|) + |D|W_- + |D|W+$$

where $W_\pm$ are smoothing operators. Now

9.5.2021.10 (2.2.8.5)
$$\langle \phi'(\phi^*|D|)v, \phi^* v\rangle_{L^2} = \langle |D|v, v\rangle_{L^2}.$$

It follows that there is a smoothing perturbation of ϕ^*, projected to $\mathcal{C}^\infty_0(\mathbb{T}_+)$ which is unitary with respect to the $\frac12$ norm. Since ϕ^* acts as the identity on the constants this can be extended to a smoothing perturbation of $\Pi_+\phi^*\Pi_+$ which is unitary with respect to the full norm (9.5.2021.4 2.2.7.2). The composite on left or right of ϕ^* with a smoothing operator is smoothing so these operators form a group with normal subgroup $G^{-\infty}_{\frac12}(\mathbb{T}_+)$. The quotient homorphism recovering ϕ gives the canonical trsnsformation of ϕ^* as a Fourier integral operator. □

2.9. String Gaussian

StringGaussian

It follows from Theorem 9.5.2021.5 3 that the 'String form' Uses Hardy extension of $\mathcal{L}\,\mathrm{SO}$

ALS.81 (2.2.9.1)
$$\sum_{k\ge 0} k|z_k|^2$$

is 'almost invariant' under the standard action of $\mathrm{Dff}^+(\mathbb{T})$ by pull-back on $\mathcal{C}^\infty(\mathbb{T};\mathbb{R})$.

We define a related function on the smooth loop space of the orthonormal frame bundle, $\mathcal{L}F_{\mathrm{SO}}$, of a Riemann manifold M with value at e given in terms of the tangent vector field $\tau(\theta)$ to the projection $l = l(e) \in \mathcal{L}M$ onto the underlying loop in M. Expressing $\tau \in \mathcal{C}^\infty(\mathbb{T}; l^*TM)$ in terms of the orthonormal basis

ALS.82 (2.2.9.2)
$$\begin{gathered}\tau(\theta) = \sum_{j=1}^{2n} T_i(\theta)e_i(\theta),\ T \in \mathcal{C}^\infty(\mathbb{T};\mathbb{R}^{2n})\\ S(e) = \sum_{k>0} k^{-1}|\zeta_k(T)|^2,\ T = \sum_k \zeta_k(T)e^{ik\theta},\ \zeta_k(T) \in \mathbb{C}^{2n}\end{gathered}$$

being the Fourier expansion of T. This is an $H^{-\frac12}$ seminorm on $T \in \mathcal{C}^\infty(\mathbb{T};\mathbb{R}^{2n})$ but, as τ is the tangent vector to $l(e)$, has $H^{\frac12}$ continuity on $\mathcal{L}F_{\mathrm{SO}}$. It is $(s-1)$-lithe on H^s loops.

It is of particular significance that the leading part of S effectively descends to a function on $\mathcal{L}M$. This is a direct consequence of the properties of Toeplitz operators as pseudodifferential operators, but we give a direct proof.

ALS.83 Lemma 4. *Under the action of* $\mathrm{Dff}^+ \ltimes \mathcal{L}\,\mathrm{SO}(2n) \ni (\phi, \gamma)$ *on* $\mathcal{L}F_{\mathrm{SO}}$

ALS.84 (2.2.9.3)
$$\begin{gathered}S(\gamma e) = S(e) + E_\gamma,\ |E_\gamma| \le C_\gamma\|\tau\|_{H^{-1}},\\ S(\phi^* e) = S(e) + E_\phi,\ |E_\phi| \le C_{\phi,N}\|\tau\|_{H^{-N}},\ \forall\ N.\end{gathered}$$

Proof. Let Π_1 be the orthogonal projection onto the strictly postive Fourier coefficients. Then

ALS.87 (2.2.9.4)
$$S(e) = \frac{1}{2\pi}\int_{\mathbb{T}} \langle D^{-1}\Pi_1\tau, \Pi_1\tau\rangle d\theta$$

in terms of the sesquilinear inner product on $\mathbb{C}^{2n}$.

The action by $\gamma \in \mathcal{L}\,\mathrm{SO}(2n)$ produces

ALS.88 (2.2.9.5)

$$S(\gamma e) = \frac{1}{2\pi}\int_{\mathbb{T}} \langle D^{-1}\Pi_1\gamma\tau, \Pi_1\gamma\tau\rangle d\theta = \frac{1}{2\pi}\int_{\mathbb{T}} \langle D^{-1}\Pi_1 U_\gamma\tau, \Pi_1 U\gamma\tau\rangle d\theta + E'_\gamma(e)$$

where $U_\gamma = \Pi_1(\gamma + \alpha_\gamma)\Pi_1$ is unitary with α a smoothing operator and the error $E'_\gamma(e)$ also smoothing, i.e. bounded in all Sobolev norms on τ. This gives the first part of (2.2.9.3) with

ALS.89 (2.2.9.6)
$$E_\gamma = E'_\gamma + E''_\gamma,\ E''_\gamma(e) = \frac{1}{2\pi}\int_{\mathbb{T}} \langle [U_\gamma, D^{-1}]\Pi_1\tau, \Pi_1 U\gamma\tau\rangle d\theta$$

The commutator can be written, as an operator on the range of Π_1

ALS.90 (2.2.9.7)
$$[U_\gamma, D^{-1}] = D^{-1}[D, U_\gamma]D^{-1}$$

with the differential commutator bounded on all Sobolev spaces. The bound on E_γ follows.

The argument for the action of the reparametrization group is similar. The chain rule for functions on the circle may be written

ALS.94 (2.2.9.8)
$$D\phi^* = \phi'\phi^* D.$$

Compressing to the Hardy space and composing with the inverse of D there

ALS.95 (2.2.9.9)
$$\Pi_1 D^{-1}\phi'\phi^*\Pi_1 = \Pi_1\phi^* D^{-1}\Pi_1.$$

Under the reparametrization action

ALS.91 (2.2.9.10)
$$S(\phi^* e) = \frac{1}{2\pi}\int_{\mathbb{T}} \langle D^{-1}\Pi_1\phi'\phi^* T, \Pi_1\phi'\phi^* T\rangle d\theta.$$

The operator $\Pi_1\phi'\phi^*(\mathrm{Id} - \Pi_1)$ is smoothing so, up to a smoothing error

ALS.93 (2.2.9.11)
$$S(\phi^* e) \equiv \frac{1}{2\pi}\int_{\mathbb{T}} \langle \Pi_1\phi^* D^{-1}\tau, \Pi_1\phi'\phi^*\tau\rangle d\theta.$$

Then, since the commutators $[\Pi_1, \phi^*]$ and $[\Pi_1, \phi']$ are smoothing, up to such an error

ALS.96 (2.2.9.12)
$$S(\phi^* e) \equiv \frac{1}{2\pi}\int_{\mathbb{T}} \phi'\phi^* \left(\langle \Pi_1 D^{-1}\tau, \Pi_1\tau\rangle\right) d\theta = S(e).$$

This completes the proof of (2.2.9.3). □

It follows that the Gaussian, on $\mathcal{L}F_{\mathrm{SO}}$

ALS.85 (2.2.9.13)
$$\exp(-\frac{1}{2}S)$$

descends to a function on $\mathcal{L}M$ up to a positive factor which is considerably smoother. On the space of finite energy loops this factor is H^1-lithe, so can effectively be localized to bounded tubes using an H^1-lithe partition of unity; see §§Find.

2.10. Fusion and simplicial structure

I mentioned the (iterative) doubling of manifolds with corners to manifolds without boundary above – without going into details. One of the fundamental properties of the path space concerns the case of an interval $I = [0, 1]$ which doubles to a circle. Thus if

158L3.7 (2.2.10.1) $$\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$$

then we can define two smooth maps

158L3.8 (2.2.10.2) $$\iota_\pm : I \ni t \longmapsto \exp(\pm\pi i t) \in \mathbb{T}$$

which cover the circle with just the boundary points 0 and 1 mapped to the same point.

Pull-back under these two maps defines two maps from the (continuous for the moment) loop space to the path space

158L3.9 (2.2.10.3) $$\iota_\pm^* : \mathcal{L}_{\mathcal{C}} M \longmapsto \mathcal{P}_{\mathcal{C}} M,\ \lambda \longmapsto \lambda \circ \iota_\pm.$$

For any M these maps are separately surjective. The range is easily determined.

Namely, for the path space we can consider the endpoint map

158L3.10 (2.2.10.4) $$\mathcal{P}_{\mathcal{C}} M \longrightarrow M^2,\ p \longmapsto (p(0, p(1)).$$

Then

158L3.11 (2.2.10.5) $$e(\iota_+^* \lambda) = e(\iota_-^* \lambda)\ \forall\ \lambda \in \mathcal{L}_{\mathcal{C}} M.$$

Conversely this is the only constraint on the image.

We can formalize this in terms of fibre products. Namely with respect to any continuous map $\pi : X \longrightarrow Y$ the fibre products of X are defined to be

158L3.12 (2.2.10.6) $$X^{[k]} = \{(x_1, \dots, x_k) \in X^k; \pi(x_1) = \dots = \pi(x_k)\}.$$

Thus $X^{[k]}$ is the inverse image of the diagonal space $Y \subset Y^k$ with respect to the product map $X^k \longrightarrow Y^k$ so it is at least a closed subspace.

158L3.13 Proposition 12. *The restriction maps* (158L3.9 2.2.10.3) *define a homeomorphism (and more ...)*

158L3.14 (2.2.10.7) $$\mathcal{L}_{\mathcal{C}} M \equiv \mathcal{P}_{\mathcal{C}}^{[2]}$$

to the fibre product with respect to the end-point map to M^2.

Proof. The only constraint on two continuous maps $p_\pm : I \longrightarrow M$ that their 'join'

158L3.15 (2.2.10.8) $$\lambda(\exp(i\pi t) = \begin{cases} p_+(t) & 0 \le t \le 1 \\ p_-(-t) & -\pi \le t \le 0t \end{cases}$$

to define a continuous map is that $e(P_+) = e(p_-)$. This shows surjectivity of the map in (158L3.14 2.2.10.7) and injectivity is clear. That the map and its inverse are continuous is left as an exercise! □

2.11. Simplicial spaces

Last time we saw that the loop spaces are fibre products of the path space with respect to the end-point map $\mathcal{P}M \longrightarrow M^2$ – for $\mathcal{C}^{0,\alpha}$ regularity with $0 \le \alpha \le 1$ and H^s regularity for $\frac{1}{2} < s < 3/2$ –

158L4.1 (2.2.11.1) $$\mathcal{L}_{\mathcal{C}}M \equiv \mathcal{P}_{\mathcal{C}}^{[2]}M$$

where the 'join' map is involved – the first path is mapped to the first half of a loop and the second path, reversed, is mapped to the second half of the loop.

2.12. Spin group

If $n \ge 3$ then $\Pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$; in fact for later reference, $\Pi_2(\mathrm{SO}(n)) = \{0\}$ and $\Pi_3(\mathrm{SO}(n) = \mathbb{Z}$. So as discussed earlier, the universal cover of $\mathrm{SO}(n)$ is a double cover, the spin group which is a $\mathbb{Z}_2$ central extension

158L5.4 (2.2.12.1) $$\mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n).$$

We can define the spin group as the quotient

158L5.5 (2.2.12.2) $$\mathrm{Spin}(n) = \dot{\mathcal{P}}\,\mathrm{SO}(n)/\simeq$$

where two pointed paths with the same endpoint are equivalent if the loop formed by their join is contractible.

From the properties of the universal cover it follows that the projection map

158L5.6 (2.2.12.3) $$\dot{\mathcal{L}}\,\mathrm{Spin}(n) \longrightarrow \dot{\mathcal{L}}\,\mathrm{SO}(n)$$

is an isomorphism onto the connected component of the identity. Indeed, a path $p \in \dot{\mathcal{P}}\,\mathrm{SO}(n)$ has a unique lift to a path $\tilde{p} \in \dot{\mathcal{P}}\,\mathrm{Spin}(n)$, from which injectivity of (158L5.6 2.2.12.3) follows. If $\tilde{P}$ is closed for $p \in \dot{\mathcal{L}}\,\mathrm{SO}(n)$, i.e. $\tilde{p}(1) = \mathrm{Id}$, then p is homotopic through pointed loops to the constant loop.

Thus in fact

158L5.7 (2.2.12.4) $$\mathcal{L}\,\mathrm{Spin}(n) = \mathrm{Spin}(n) \ltimes (\dot{\mathcal{L}}\,\mathrm{SO}(n))_{\mathrm{Id}}.$$

CHAPTER 3

Transgression and regression

As already emphasized, the idea of doing analysis on the loop space is to study objects on the manifold itself which are rather hard to get at directly – here I mean string structures. It is natural then to see what, and how, objects on M can be equivalently, in an appropiate sense, studied on the loop space. This transfer is called here *trangression* altough that is applied to other similar operations. The reverse process of passage from (objects on) the path space to the manifold we call *regression.*

3.1. Maps to the circle

The first example of transgression starts from continuous functions with values in the circe – so this is really a mapping space in itself $\mathcal{C}(M;\mathbb{T})$ but we really think of it as a 'function space'. There is a 'direct' notion of trangression here given by pull-back:

ALS.28 (3.3.1.1) $$\tau : \mathcal{C}(M;\mathbb{T}) \longrightarrow \mathcal{C}(\mathcal{L}M;\mathcal{C}(\mathbb{T};\mathbb{T})),\ ((\tau(u))(l))(\theta) = u(l(\theta)) \in \mathbb{T}.$$

Here the pull-back for a fixed loop, l, defines a map

ALS.29 (3.3.1.2) $$\mathbb{T} \ni \theta \longrightarrow u(l(\theta)) \in \mathbb{T}$$

and then as l varies this defines a map (ALS.28 3.3.1.1).

Now, there is a natural map

ALS.30 (3.3.1.3) $$\mathrm{wn} : \mathcal{C}(\mathbb{T};\mathbb{T}) \longrightarrow \mathbb{Z},$$

the *winding number*. The winding number of $v \in \mathcal{C}(\mathbb{T};\mathbb{T})$ can be defined for instance by choosing a continuous branch of $\frac{1}{2\pi i}\log v$ along the path $[0, 2\pi]$ and taking the difference of the final and initial values. It is of course just a direct representation of the fact that

ALS.31 (3.3.1.4) $$\Pi_1(\mathbb{T}) = \mathbb{Z}$$

and also of the fact that $H^1(\mathbb{T}) = \mathbb{Z}$. The winding number is locally constant under variation of g.

Combining the two maps gives a courser version of trangression

ALS.32 (3.3.1.5) $$\widetilde{\tau} : \mathrm{wn} \circ \widetilde{\tau} : \mathcal{C}(M;\mathbb{T}) \longrightarrow \mathcal{C}(\mathcal{L}M;\mathbb{Z}).$$

Written in terms of the evaluation map 158L3.2 3.3.4.1 we need to 'push-forward' to $\mathcal{L}(M)$. Our 'coarsened' trangression map is Repetition here

158L3.4 (3.3.1.6) $$\tau : \mathcal{L}M \ni f \longmapsto w(\gamma^* f) \in \mathcal{C}(\mathcal{L}M;\mathbb{Z}).$$

Clearly the transgressed function is constant on components of $\mathcal{L}(M)$; as noted above these are identified with $\Pi_1(M)$.

Thus $\mathcal{C}(M;\mathbb{T})$ trangresses into $\mathcal{C}(\mathcal{L}M;\mathbb{Z})$ but it is most important to note that this map is not surjective (in general) but has an additional 'fusion' property. Namely in terms of the simplicial structure $\mathcal{L}_{\mathcal{C}}M = \mathcal{P}_{\mathcal{C}}M^{[2]}$,

158L3.5 (3.3.1.7)
$$\delta\tau(f) = 0.$$

158L3.6 Proposition 13. *For M connected, the transgression map (3.3.4.3) is a bijection from $H^1(M;\mathbb{Z})$ to the continuous maps $\phi : \mathcal{L}M \longrightarrow \mathbb{Z}$ satisfying the fusion condition $\delta\phi = 0$.*

Let me quickly go through the proof of this again. First let's think about the *regression* map, going back from $\phi : \mathcal{L}M \longrightarrow \mathbb{Z}$ to M. The idea is to construct a covering of M^2. We do this by taking the product with the endpoint map back

158L4.2 (3.3.1.8)
$$\mathcal{P}M \times \mathbb{Z} \longrightarrow M^2.$$

Now, define an equivalance relation on this product by

158L4.3 (3.3.1.9)
$$(p_1, k_1) \simeq (P_2, k_2) \text{ iff } (p_1, p_2) \in \mathcal{P}^{[2]}M \text{ and } k_2 - k_1 = \phi(p_1, p_2).$$

We need to check some properties of ϕ which follow from $\delta\phi = 0$. First ϕ must vanish on the component of $\mathcal{L}M$ consisting of the contractible paths. It is certainly constant on components and this component contains the trivial loop fixing some point $m \in M$. However, this loop is the join of the constant path at m with itself and the fusion of three such paths gives the same loop three times. So $2\phi = \phi$ on this loop and hence $\phi = 0$ on the whole contractible component.

This in particular implies that $(p, k) \equiv (p, k)$ for the relation (3.3.1.9) since the loop involved is goes out p and back 'the same way' so this is contractible by 'shortening' p.

Similarly if we reverse a loop, l reversed being written l_-, then ϕ must change sign. To see this we can think of a loop as being a path with the far endpoint the same as the initial point. Then we can join the loop with its reverse and we get a contractible loop – where the contraction moves what is now the midpoint back to the initial point by path shortening. Adding a constant middle path and using the fusion condition shows that $\phi(l) + \phi(l_-) = 0$. This is symmetry for the relation (3.3.1.9).

Finally we need to check transitivity of the relation. However, this is precisely the fusion condition.

So we can 'take the quotient' by this relation and this defines a space

158L4.4 (3.3.1.10)
$$M_\phi = \mathcal{P}M \times \mathbb{X}/ \simeq \longrightarrow M^2.$$

The endpoint map to M^2 survives since it is constant on equivalence classes and it is actually surjective.

What can we say about M_ϕ? It has an action by $\mathbb{Z}$ which is free and locally trivial. Indeed, $\mathbb{Z}$ acts on the $\mathbb{Z}$ factor of the product $\mathcal{P}M \times \mathbb{Z}$ and is preserved by the equivalence relation. Moreover the quotient by the $\mathbb{Z}$ action is $\mathcal{P}M$ before the equivalence quotient and hence M^2 afterwards. So

158L4.5 (3.3.1.11)
$$M_\phi \longrightarrow M^2 \text{ is a } \mathbb{Z} \text{ principal bundle.}$$

It is also smooth. Namely we can see that the endpoint map $\mathcal{P}M \longrightarrow M^2$ has local smooth sections.

Now, such a principal $\mathbb{Z}$ bundle (also called a 'torsor' in some circles) admits a connection, just like any principal bundle. In this case this means we can find a smooth (continuous is fine) function $l : M_\phi \longrightarrow \mathbb{R}$ which shifts with the $\mathbb{Z}$ action

158L4.6 (3.3.1.12)
$$l(k \cdot q) = l(q) + k,\ q \in M_\phi.$$

This means that we can define a map

158L4.7 (3.3.1.13)
$$f_\phi = \exp(2\pi i l) : M^2 \longrightarrow \mathbb{T}.$$

How come we finished up on M^2 instead of M. We need to understand this properly but for the moment just obverse that if we find say the second point in M^2 we get an element of $\mathcal{C}(M;\mathbb{T})$ which lies in a fixed homotopy class independent of the point chosen (remember M is connected). So our claim is that ϕ is the regression of this function (unless I made a sign mistake) and similarly that if we started with some f and passed to $\phi = \tau f$ then f_ϕ is homotopic to f.

I have not written this out.

Rigidity is deceptive.

3.2. Stolz-Teichner theorem

Now, we have the ingredients for the theorem. Recall that

158L5.19 (3.3.2.1)
$$\begin{array}{ccccc} \mathcal{L}\,\mathrm{SO}(n) & \text{---} & \mathcal{L}F_{\mathrm{SO}} & \xrightarrow{o} & \mathbb{Z}_2 \\ & & \downarrow & & \\ & & \mathcal{L}M & & \end{array}$$

is a principal bundle and a continuous map o is an orientation if it takes both values on each fibre. It is a *fusion (or loop) orientation* if it satisfies the fusion condition

158L5.20 (3.3.2.2)
$$\delta o = 0$$

in terms of the simplicial space formed from $\mathcal{P}F_{\mathrm{SO}}$ – so δo is a $\mathbb{Z}_2$-valued function on $\mathcal{L}^{[3]}F_{\mathrm{SO}}$.

158L5.21 Theorem 4. Stolz-Teichner2005 **[12]** *For a connected manifold of dimension greater than* 3, *Fusion orientations are in 1-1 correspondence with spin structures.*

Proof. The transgression part was observed by Atiyah, atiyah_loop_orientation [1] in the 80s, without the fusion condition. Namely if $F \longrightarrow F_{\mathrm{SO}}$ is a spin bundle over M then

158L5.22 (3.3.2.3)
$$\begin{array}{ccc} \mathcal{L}\,\mathrm{Spin}(n) & \text{---} & \mathcal{L}F \\ & & \downarrow \\ & & \mathcal{L}M \end{array}$$

is a principal bundle which maps to the looped frame bundle

158L5.23 (3.3.2.4)
$$\begin{array}{ccc} \mathcal{L}\,\mathrm{Spin}(n) & \text{---} & \mathcal{L}F \\ \downarrow & & \downarrow \\ \mathcal{L}\,\mathrm{SO}(n) & \text{---} & \mathcal{L}F_{\mathrm{SO}} \\ & & \downarrow \\ & & \mathcal{L}M. \end{array}$$

As we saw above, $\mathcal{L}\,\mathrm{Spin} = \mathrm{Spin} \ltimes (\dot{\mathcal{L}}\,\mathrm{SO})_{\mathrm{Id}}$ maps onto the identity component of $\mathcal{L}\S 0 = \mathrm{SO} \ltimes \dot{\mathcal{L}}\,\mathrm{SO}$. Thus the image of $\mathcal{L}F$ in $\mathcal{L}F_{\mathrm{SO}}$ consists of one of the components of each fibre over $\mathcal{L}M$. This defines an orientation o.

We proceed to show that this orientation satisfies the fusion condition. Consider a fusion triple of paths $p_i \in \mathcal{P}F_{\mathrm{SO}}$, three paths with the same initial and end points. Choosing a lift of the intial point to F_{Spin} determines a unique lift, $\tilde{p}_i$ of each path. The end points of the $\tilde{p}_i$ project to the same point in F_{SO}. For each pair of paths $p-i$, p_j, the loop $j(p_i, p_j)$ in F_{SO} formed by their join lifts to a loop in F_{Spin} if and only if the end points are the same so

158L5.24 (3.3.2.5)
$$o(j(p_i, p_j)) = z \in \mathbb{Z}_2 \text{ iff } z\tilde{p}_j(1) = \tilde{p}_j(1).$$

This implies the fusion condition on o (written additiviely) that

158L5.25 (3.3.2.6)
$$p(j(p_1, p_3)) = o(j(p_1, p_2)) + o(j(p_2, p_3)).$$

Thus a spin structure determiens a fusion orientation. Conversely given such a function o we choose a point $q \in F_{\mathrm{SO}}$ and define a spin bundle by descent from $\dot{\mathcal{P}}F_{\mathrm{SO}} \times \mathbb{Z}_2$. That is, we consider the relation

158L5.26 (3.3.2.7)
$$(p_1, z_1) \equiv (p_2, z_2) \Longleftrightarrow p_1(1) = p_2(1) \in F_{\mathrm{SO}},\ z_2 = o(j(p_1, p_2))z_1.$$

As in the earlier discussion of transgression for functions with values in the circle this is an equivalence relation and the quotient is a spin refinement of F_{SO}.

It remains to show that these two constructions give right and left inverses of each other. Starting from a spin structure the corresponding loop orientation is equal to the identity on the loops on F_{SO} which have lifts as loops to F_{Spin}. Fixing a point in F_{Spin} the whole bundle is reconstructed from the endpoint map for paths starting at that point. Similarly in the other direction.

Finally, the product of two loop orientations is a map $\mathcal{L}M \longrightarrow \mathbb{Z}_2$ which is fusive, i.e. δ-closed. This defines an element of $H^1_{\mathrm{fus}}(\mathcal{L}M; \mathbb{Z}_2)$ and conversely. [1] □

This is our model for the trangression of string structures to loop spin structures.

158L5.27 Remark 1. In fact I see that it would almost certainly have been clearer to start with the trangression of real line bundles. Over a manifold M these correspond, by Čech theory, to elements of $H^1(M; \mathbb{Z}_2)$ and we can see following the discussion of the transgression of $H^1(M; \mathbb{Z})$ that these trasgress to fusive maps $\mathcal{L}M \longrightarrow \mathbb{Z}_2$. Applying this to the real line bundle defined by F_{Spin} over F_{SO} gives the loop orientation.

3.3. Transgression of Spin structure

I want to do a second example of transgression, this time a 'real one'.

158L4.8 Theorem 5 (Stolz-Teichner). *A spin structure on an oriented manifold transgresses to a unique fusion orientation on $\mathcal{L}M$.*

So, let's examine all the terms in the statement! First an orientation on a (connected) manifold M. The standard definition is that this is a conistent, i.e. continous, orientation on the tangent spaces T_mM. This in turn can be thought of as a continuous choice of positive half of $\Lambda^m M_m$, where $n = \dim M$, the maximal exterior power of the cotangent bundle – so it is induced by a non-vaniishing section

[1] I need to go through transgression in Čech theory to clarify this.

of $\Lambda^n M$. I prefer to think of an orientation as a continuous choice of component of $F_O M$, the orthonormal frame bundle for a choice of metric on M. This we can think of more explicitly as a continuous map

158L4.9 (3.3.3.1)
$$F_O M \longrightarrow \mathbb{Z}$$

3.4. Maps to the circle

The first example of transgression I want to discuss starts from continuous functions with values in the circe – so this is really a mapping space in itself $\mathcal{C}(M;\mathbb{T})$ but we really think of it as a 'function space'.

Transgression is defined as usual, with the first step being pull-back to

158L3.2 (3.3.4.1)
$$\mathrm{ev} : \mathbb{T} \times \mathcal{L}(M) \longrightarrow M.$$

Thus $\mathrm{ev}^* f \in \mathcal{C}(\mathbb{T} \times \mathcal{L}(M))$ for $f \in \mathcal{C}(M;\mathbb{T})$. Then we need to 'push-forward' to $\mathcal{L}(M)$. Fixing $\gamma \in \mathcal{L}(M)$ makes $\mathrm{ev}^* f = \gamma^* f$ into an element of $\mathcal{C}(\mathbb{T},\mathbb{T})$ and we take the push-forward to be the winding number.

Thus if $g \in \mathcal{C}(\mathbb{T};\mathbb{T})$ then there exists a real-valued function $\lambda \in \mathcal{C}(\mathbb{R})$ such that $g(\theta) = \exp(2\pi i\lambda(\theta))$. This is defined as a continuous branch of $\log g/2\pi i$ and so is determined by to addition of an integral constant. The winding number is then well-defined as

158L3.3 (3.3.4.2)
$$w(g) = \lambda(2\pi) - \lambda(0).$$

It is locally constant under variation of g.

So, our trangression map is

158L3.4 (3.3.4.3)
$$\tau : \mathcal{L}M \ni f \longmapsto w(\gamma^* f) \in \mathcal{C}(\mathcal{L}M;\mathbb{Z}).$$

Clearly the transgressed function is constant on components of $\mathcal{L}(M)$; as noted above these are identified with $\Pi_1(M)$.

It is most important that $\tau(f)$ has an additional 'fusion' property. Namely in terms of the simplicial structure $\mathcal{L}_{\mathcal{C}} M = \mathcal{P}_{\mathcal{C}} M^{[2]}$,

158L3.5 (3.3.4.4)
$$\delta\tau(f) = 0.$$

158L3.6 Proposition 14. *The transgression map* (158L3.4 3.3.4.3) *is a bijection from* $H^1(M;\mathbb{Z})$ *to the continuous maps* $\phi : \mathcal{L}M \longrightarrow \mathbb{Z}$ *satisfying the fusion condition* $\delta\phi = 0$.

As I have already mention a couple of times, the discrete objects so far are a bit misleading. With this in mind, I want to go through two further cases of trangression/regression which are not discrete in this sense.

3.5. Complex line and circle bundles

Connections
Holonomy
Transgression

Spin

3.6. Spin structures

Next I will go through the theorem of Stolz and Teichner relating spin structures and loop orientations.

3.7. Loop and pointed loop groups

As we have already seen, if G is a topological group (we are mostly interested in Lie groups) then $\mathcal{P}G$ and $\mathcal{L}G$, with any of the regularities we are considering, are also groups with the pointwise product – indeed this is true for the mapping space $\mathcal{M}(X;G)$ which is often called a gauge group. Thus the product is

158L5.1 (3.3.7.1)
$$(\gamma_1\gamma_2)(x) = \gamma_1(x)\gamma_2(x) \ \forall \ x \in X.$$

If X is *pointed*, so some $\bar{x} \in X$ is distinguished, then the pointed mapping group

158L5.2 (3.3.7.2)
$$\dot{\mathcal{M}}(X;G) = \{\gamma \in \mathcal{M}(X;G); \gamma(\bar{x}) = \mathrm{Id}\}$$

is a normal subgroup and

158L5.3 (3.3.7.3)
$$\mathcal{M}(X;G) = G \ltimes \dot{\mathcal{M}}(X;)$$

is a semidirect product where G is identified as the constant maps.

3.8. $\mathrm{Spin}_{\mathbb{C}}$ structures

Central extensions.
Lifting principal bundles
The lifting bundle gerbe
General bundle gerbes
Dixmier-Douady class
Trivial gerbes

CHAPTER 4

Bundle gerbes

Ger

Gerbes come in several flavours but their origin lies in more-or-less geometric representations of integral cohomology three classes. I will mostly consider *bundle gerbes*, due to Murray [**8**] (Murray1), and eventually bigerbes, from [**?**] due to Chris Kottke and myself. For today just bundle gerbes.

By back-construction, line bundles or principal circle bundles, are regarded as '0-gerbes' and $\mathbb{Z}$-equivariant covers ($\mathbb{Z}$-torsors) as (-1)-gerbes. They represent respectively integral 2- and 1-classes in the cohomology of the base.

Notice that there is a close relationship between (always principal) circle bundles and $\mathbb{Z}$-torsors. Take a circle bundle

L7.1 (4.4.0.1)
$$\begin{array}{ccc} \mathbb{T} & \text{---} & Y \\ & & \downarrow \\ & & M \end{array}$$

which in the case of compact manifolds is the same as a free action of $\mathbb{T}$ on Y – local triviality is automatic. Now we can associate to this the fibre product

L7.2 (4.4.0.2)
$$\begin{array}{ccc} Y^{[2]} & \xrightarrow{\sigma} & \mathbb{T} \\ \downarrow & & \\ M & & \end{array}$$

Thus the fibre $Y^{[2]}_m$ of $Y^{[2]}$ above $m \in M$ is $Y_m \times Y_m$, where Y_m is the fibre of Y. A point $(p, q) \in Y^{[2]}$ is therefore necessarily of the form (p, zp) where $z \in \mathbb{T}$ is the principal circle action on Y_m, so transitive and free. This defines the 'fibre shift map' σ in (4.4.0.2) (L7.2), namely

$$\sigma(p, q) = z, \ q = zp.$$

This works for any principal G-bundle

L7.8 (4.4.0.3)
$$\begin{array}{ccccccc} G & \text{---} & P & \leftleftarrows & P^{[2]} & \xrightarrow{\sigma} & G \\ & & \downarrow & \swarrow & & & \\ & & M & & & & \end{array}$$

Returning to the case of a circle bundle, we can construct a $\mathbb{Z}$-cover, a torsor, from the map σ by considering all the possible values of $\frac{1}{2\pi i} \log \sigma$ at each point. I leave you to check the details.

The second thing to recall, is that a circle bundle defines an integral 2-class, its Chern class $c_1(Y) \in H^2(M; \mathbb{Z})$. This can be seen readily in terms of Čech cohomology. Take a good open cover, U_i, of M, so over each U_i the circle bundle is trivial – it has smooth (or just continuous) section

L7.9 (4.4.0.4) $$\mu_i : U_i \longrightarrow Y,\ \pi \circ \mu_i = \mathrm{Id}\,.$$

Over the overlaps this gives

L7.10 (4.4.0.5) $$\gamma_{ij} I_i \cap U_j \longrightarrow \mathbb{T},\ \gamma_{ij}(m) = \sigma(\mu_i(m), \mu_j(m)).$$

In patricular $\gamma_{ji} = (\gamma_{ij})^{-1}$. This is a Čech cocycle, since over triple intersections

L7.11 (4.4.0.6) $$(\partial\gamma)_{ijk} = \gamma_{ij}\gamma_{jk}\gamma_{ki} = 1$$

since it maps $\mu_i(m)$ successively to $\mu_k(m)$, to $\mu_j(m)$ and back to $\mu_i(m)$.

Thus we have defined the class

L7.12 (4.4.0.7) $$c_1(Y) = [\gamma] \in H^1(M; \mathbb{T}).$$

First check that it is independent of the choice of the sections μ_i. Indeed, new sections are related by maps into the circle, $\mu_i' = Z_i \mu_i$ with $Z_i : U_i \longrightarrow \mathbb{T}$. This shifts the new Čech class to

$$\gamma_{ij}' = \sigma(\mu_i'(m), \mu_j'(m)) = Z_i^{-1} Z_j \gamma_{ij}$$

which changes it by a Čech boundary. In principle we should also check what happens under change of the covering.

Note that the Bockstein homomorphism, corresponding to change of coefficients, is an isomorphism in this case

L7.15 (4.4.0.8) $$\beta : H^k(M; \mathbb{T}) \longrightarrow H^{k+1}(M; \mathbb{Z}).$$

So we have the more conventional view that $c_1(Y) \in H^2(M; \mathbb{Z})$. The homomorphism arises by taking smooth choices for $\lambda_{ij} \frac{1}{2\pi i} \log \gamma_{ij}$. The Čech boundary of these real-valued functions in then intervalued on triple overlaps precisely because of (L7.11 4.4.0.6). This defines $c_1(Y) \in H^2(M; \mathbb{Z})$.

The triviality of c_1 means (because we have a good cover) that $\gamma_{ij} = \tau_i^{-1}\tau_j$ is exact, where $\tau_i : U_i \longrightarrow \mathbb{T}$ are smooth. These can be used to 'correct' the original sections to $\tau_i \mu_i$ which now combine to give a global section of Y over M. Thus the vanishing of c_1 implies that Y is trivial.

L7.16 EXERCISE 5. Show that the Chern class in $H^2(M; \mathbb{Z})$ is additive under tensor product and hence that two circle bundles with the same Chern class are isomorphic.

I have gone through this, presumably familiar, discussion of the Chern character for circle bundles to motivate the discussion of the corresponding invariant for bundle gerbes.

Before passing on, note another naturality property of the Chern class. If $\phi : N \longrightarrow M$ is a smooth map (just continuous works) and Y is a circle bundle over M then the pull-back

L7.17 (4.4.0.9) $$(\phi^* Y)_n = Y_{\phi(n)}$$

is a circle bundle over N and

L7.18 (4.4.0.10) $$c_1(\phi^* Y) = \phi^* c_1(Y) \in H^2(N; \mathbb{Z}).$$

This is straightforward to check.

Let's apply this to $\pi : Y \longrightarrow M$ as a map. So Y pulls back to a circle bundle over its own total space. Note that the total space of $\pi^* Y$ is naturally identified with $Y^{[2]}$ with the projection to Y being the corresponding face map $Y^{[2]} \longrightarrow Y$. However

π^*Y is clearly a trivial circle bundle since it has the global section given by the 'diagonal' in $Y^{[2]}$ in which $p \in Y$ is mapped to $(p, p) \in Y^{[2]}$ (this is the degeneracy map of the simplicial structure). From this we learn what can be checked readily in other ways, that

L7.19 (4.4.0.11) $$\pi^* c_1(Y) = 0 \text{ in } H^2(Y; \mathbb{Z}).$$

Now, to bundle gerbes. The basic idea is that we want to 'add another level' to a circle bundle, really by analogy with the circle bundle and a principal $\mathbb{Z}$-bundle. Anyway, we start by considering a space mapping to M :

L7.20 (4.4.0.12) $$\begin{array}{c} X \\ \downarrow{\scriptstyle \beta} \\ M. \end{array}$$

I will assume that this is a fibre bundle – let me not be quite specific at the moment but this will cover all the cases we want. In any case we suppose that there are local continuous sections of β over sufficiently small contractible sets in M.

The idea, although you will not see this for a while, is that the 3-class in $H^3(M; \mathbb{Z})$ that we are trying to 'represent' should vanish when pulled back to X. This is a small problem compare to 2-classes since, whilst it is always possible as we shall see, it may not be possible with X a finite-dimensional bundle over M.

Anyway, the idea is that we consider again the simplicial space defined by the fibre products of X with respect to β :

L7.21 (4.4.0.13) $$\begin{array}{ccccccc} X & \Leftarrow & X^{[2]} & \Lleftarrow & X^{[3]} & \Lleftarrow & X^{[4]} \\ \downarrow & \swarrow & & \swarrow & & \swarrow & \\ M & & & & & & \end{array}$$

Over $X^{[2]}$ we suppose there is a given circle bundle

L7.22 (4.4.0.14) $$\begin{array}{c} Y \\ \downarrow \\ X^{[2]} \\ \downarrow \\ M \end{array}$$

and we suppose that this is *simplicial* in the following sense. First we can define a 'simplicial differential' on circle bundles, modelled on that on functions with values in $\mathbb{T}$. Namely we combine the three face maps from $X^{[3]}$ to $X^{[2]}$ and define

L7.23 (4.4.0.15) $$\delta Y = f_1^* Y \otimes f_2^* Y^{-1} \otimes f_3^* Y$$

where $f_i : (p_1, p_2, p_3)$ 'forgets' p_i. The inverse means the dual circle bundle. We can do the same for all levels of the simplicial. For instance if $Z \longrightarrow X$ is a circle bundle then

$$\delta Z = f_1^* Z \otimes f_2 Z^{-1} \longrightarrow X^{[2]}.$$

Recalling that if $f_i^{[k]}$ are the face maps from $X^{[k]}$ to $X^{[k-1]}$ then

L7.25 (4.4.0.16) $$f_j^{[k-1]} f_l^{[k]} = f_l^{[k-1]} f_j^{[k]} \text{ if } j < l, \;\; f_j^{[k-1]} f_l^{[k]} = f_l^{[k-1]} f_{j-1}^{[k]} \text{ if } j \le l$$

we can see that $\delta^2 Z$ is naturally trivial – has a canonical section defined by choosing any point in the fibre of Z. This actually holds at any level. So *if* $Y = \delta Z$ then Y is trivial. What we shall assume is that

L7.26 (4.4.0.17) $$\delta Y \longrightarrow X^{[3]} \text{ is trivial with section } s.$$

CHAPTER 5

Central extensions

Cen

Much of the material for today's lecture can be found in Pressley-Segal [9, Chapter 4] (Pressley-Segal1), see also [3] (Baez) and [7] (meinrenken_basic).

For the most part I will work with a compact, connected, simple, simply-connected Lie group G – for instance $\mathrm{Spin}(n)$, $n \geq 3$. For such a group $\Pi_1(G) = \{0\}$ by assumption, $\Pi_2(G) = \{0\}$ as well and $\Pi_3(G) = \mathbb{Z}$. By Hurewicz theorem $H^2(G, \mathbb{Z}) = \{0\}$, $H^3(G, \mathbb{Z}) = \mathbb{Z}$. So there is a gerbe, except that I did not prove existence last time..

We are interested in central extensions – why? Ultimately it is closely related to the realization of string structures over the loop space of a spin manifold.

In the case of the group itself, there are no non-trivial central extensions

15.3.2021.1 (5.5.0.1)
$$\mathbb{T} \longrightarrow \widehat{G} \longrightarrow G.$$

To see that,:pass to the Lie algebra which similarly has a central extension

15.3.2021.3 (5.5.0.2)
$$\mathbb{R} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g}, \ \longrightarrow \widehat{\mathfrak{g}} = \mathbb{R} \oplus \mathfrak{g}$$

as a vector space.

Since the multiples of the line are central, the Lie bracket on $\widehat{\mathfrak{g}}$ must be of the form $[(s, \xi), (t, \eth)] = (\omega(\xi, \eta), [\xi, \eta])$ where $\omega : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ must be an antisymmetric bilinear form. Moreover if we allow the central extension $\widehat{G}$ to act on $\widehat{\mathfrak{g}}$ by conjugation then $x(s, \xi)x^{-1} = (s, g\xi g^{-1})$ where $x \in \widehat{G}$ projects to $g \in G$. Thus in fact

L8.1 (5.5.0.3)
$$\omega(g\xi g^{-1}, g\eta g^{-1}) = \omega(\xi, \eta)$$

must be conjugation-invariant. However, there are no such bilinear forms. Recall that there is a conjugation-invariant positive-definite symmetric form $\langle , \rangle$ on $\mathfrak{g}$ for any compact Lie group, obtained by averaging any chosen positive-definite symmetric form. If G is simple then this is unique up to a positive multiple. This follows from the fact that there can be no non-trivial invariant subspaces for the conjugation action on $\mathfrak{g}$. Indeed, such a subspace would be an ideal in the Lie algebra, trivial by assumption. Any other invariant, positive-definite symmetric form can be diagonalized with respect to $\langle , \rangle$ by the spectral theorem. All eigenvalues must be equal by the same argument using the assumtion that $\mathfrak{g}$ is simple.

Similarly, any antisymmetric blinear form on $\mathfrak{g}$ must vanish, since again it could be diagonalized with respect to $\langle , \rangle$ and would then have both positive and negative eigenspaces which would be invariant.

In the case of $\mathrm{Spin}(n)$ the Lie algebra is $\mathfrak{so}(n)$ and then

L8.2 (5.5.0.4)
$$\left\{ \begin{matrix} \langle \xi, \eta \\ = -\operatorname{Tr}(\xi\eta) \end{matrix} \right\}$$

is an invariant bilinear form.

Clearly the durth of central extensions has to do with the fact that $H^2(G) = 0$. On the other hand $H^3(G,\mathbb{Z}) = \mathbb{Z}$. If we look for a generator of the image of $H^3(G;\mathbb{Z}) \longrightarrow H^3(G,\mathbb{R})$ it is represented by a deRham 3-form δ. Again we can average to make this left-invariant.

15.3.2021.4 (5.5.0.5) $$\int_G g^*\delta dg = \delta + d\gamma.$$

This allows us to average the form over the left action of G and show that each class is represented by a left-invariant form – in fact $H^3(G;\mathbb{Z}) = H^3_G(G,\mathbb{Z})$ and all classes are invariant under the conjugation action. At the identity we can consider

15.3.2021.2 (5.5.0.6) $$\operatorname{tr}(\xi[\eta,\zeta])$$

and then extend it to be left-invariant form. This is non-zero on a copy SU(2) given by the root-space decomposition. I stated last time but did not prove that there is a gerbe on the pointed path space of G representing $\delta \in H^3(G;\mathbb{Z})$. I want to construct this. . . The path sequence

15.3.2021.5 (5.5.0.7) $$\dot{\mathcal{L}}G \longrightarrow \dot{\mathcal{P}}G \longrightarrow G,$$

say for finite-energy paths and loops, is a Serre fibration. It follows that

15.3.2021.6 (5.5.0.8) $$\Pi_{k-1}(\dot{\mathcal{L}}G) = \Pi_k G,\ k \geq 1.$$

So there is a circle bundle over $\dot{\mathcal{L}}G$, this is the central extension. Again look for the Lie algebra cocycle as above. Now the Lie algebra is $\dot{\mathcal{L}}\mathfrak{g}$ and we look for a conjugation-invariant bilinear form

15.3.2021.8 (5.5.0.9) $$\omega : \dot{\mathcal{L}}\mathfrak{g} \times \dot{\mathcal{L}}\mathfrak{g} \longrightarrow \mathbb{R}.$$

The 'obvious' possibility is

15.3.2021.7 (5.5.0.10) $$\omega(f,g) = \int_{\mathbb{T}} \operatorname{tr}(f(\theta)\frac{dg}{d\theta}(\theta))d\theta.$$

I will outline the double-loop space construction. Next I will set up the Hardy space construction.

Recall that we are working under the assumption that the Lie group G is compact, connected and simply connected. In practice I will take $G = \operatorname{Spin}(n)$ for $n \geq 3$. Even for a non-simply connected group such as U(n), which is definitely of interest, or even GL($n,\mathbb{C}$)) we can get a central extension of the identity component of the loop group this way.

Any map $\lambda \in L^\infty(\mathbb{T}, \operatorname{GL}(n,\mathbb{R}))$ acts as a multiplication operator on $L^(\mathbb{R};\mathbb{R}^n)$

L9.3 (5.5.0.11) $$(\lambda u)(\theta) = \lambda(\theta)u(\theta),\ u \in L^2(\mathbb{R};\mathbb{R}^n).$$

For $\lambda \in \mathcal{L}\operatorname{Spin}(n) = \mathcal{L}_{\mathrm{Id}}(\mathbb{T};\operatorname{SO}(n))$ this action is orthogonal action with respect to the inner product

L9.2 (5.5.0.12) $$\int_{\mathbb{T}} \langle v(\theta), w(\theta)\rangle d\theta$$

in terms of the Euclidean inner product on $\mathbb{R}^n$.

As we know the oriented diffeomorphism group of the circle acts through pull-back on $\mathcal{L}M$ for any manifold M,

L9.4 (5.5.0.13) $$f^*l(\theta) = f(l(\theta))$$

and this action leads to the semi-direct product of groups

L9.5 (5.5.0.14) $\mathcal{L}G \rtimes \mathrm{Dff}^+(\mathbb{T})$ with product $(\lambda_1, f_1) \cdot (\lambda_2, f_2) = (\lambda_1 \cdot f_2^* \lambda_2, f_1 \circ f_2)$.

Then the larger group acts on $L^2(\mathbb{T}; \mathbb{R}^n)$,

L9.6 (5.5.0.15)
$$(\lambda, f)u = (f^*\lambda)u = f^*(\lambda((f^{-1})^*u),\ (\lambda_1, f_1)(\lambda_2, f_2)u = (f_1^*\lambda_1 \dot{f}_1^{-1} * f_2^*\lambda_2)u.$$

L9.1 (5.5.0.16) $\omega(f, g) - \mathrm{tr}_H(\pi f(\mathrm{Id} - \pi)g\pi - \pi g(\mathrm{Id} - \pi)f\pi - \pi[f, g]\pi),\ f, g \in \mathcal{C}^\infty(\mathbb{T}; \mathfrak{g})$

Last time I described a central extension of $\mathcal{L}\,\mathrm{Spin}(n)$ in terms of Toeplitz operators. I will go into a little more detail about this since it is rather important to what follows. In fact, initially at least, consider the bigger group $\mathrm{GL}(n, \mathbb{C})$.

As for $\mathcal{L}\,\mathrm{Spin}(n)$, an element of the loop group $\mathcal{L}\,\mathrm{GL}(n, \mathbb{C})$ acts on the loop space

L10.1 (5.5.0.17)
$$\mathcal{L}\mathbb{C}^n = \mathcal{C}^\infty(\mathbb{T}; \mathbb{C}^n)$$

as a multiplication operator. Then we consider the Hardy subspace which is the $\mathcal{C}^\infty$ closure of

L10.2 (5.5.0.18)
$$H = \mathrm{sp}(e_i e^{ik\theta}; k \ge 0\}$$

where e_i is the standard basis for $\mathbb{C}^n$. We also consider the closure of H in $L^2(\mathbb{T}; \mathbb{C}^n)$ and especially the orthogonal projection Π onto H. Thus $\mathrm{Id} - \Pi$ is the projection onto the span of the Fourier series with vanishing coefficients of $e^{ik\theta}$ for $k \ge 0$.

Now consider the Toeplitz operator corresponding to $\lambda \in \mathcal{L}\,\mathrm{GL}(n, \mathbb{C})$,

L10.3 (5.5.0.19)
$$T_\lambda = \Pi\lambda\Pi : H \longrightarrow H.$$

Last time I showed that

L10.4 (5.5.0.20)
$$T_\lambda T_\gamma - T_{\lambda\gamma} = \Pi\kappa(\lambda, \gamma)\Pi,\ \kappa \in \Psi^{-\infty}(\mathbb{C}; \mathbb{C}^n)$$

is a smoothing operator – so an operator with Schwartz kernel in $\mathcal{C}^\infty(\mathbb{T}\times\mathbb{T}; M(n; \mathbb{C})$.

Then we can see that T_λ is a Fredholm operator and that

L10.5 (5.5.0.21)
$$\mathrm{ind}_H(T_\lambda) = -\,\mathrm{wn}(\det \lambda).$$

This result is due to Toeplitz. It can be proved readily by using the stability of the index but let me give another proof which, while needing more machinery, is in the spirit of the discussion below.

We can see that T_λ is Fredhom using (5.5.0.20) since $T_{\lambda^{-1}}$ is a 2-sided parameterix, i.e.

L10.6 (5.5.0.22)
$$T_{\lambda^{-1}} \circ T_\lambda = \mathrm{Id} - \Pi\kappa(\lambda^{-1}, \lambda)\Pi,\ T_{\lambda \circ T_\lambda^{-1}} = \mathrm{Id} - \Pi\kappa(\lambda, \lambda^{-1})\Pi$$

is an inverse up to errors in the smoothing ideal – these form an ideal in the Toeplitz algebra but not an ideal in the bounded operators, however they are contained in the trace ideal in the bounded operators

It follows even from the compactness of the errors in (5.5.0.22) that T_λ is Fredholm and so has a unique generalized inverse Q, a bounded operator on the Hilbert completion of H satisfying

L10.7 (5.5.0.23)
$$QT_\lambda = \mathrm{Id} - P_{\mathrm{Nul}},\ T_\lambda Q = \mathrm{Id} - P_{\mathrm{Ran}}^\perp.$$

The errors here are finite rank projections, so trace class and

L10.8 (5.5.0.24) $\mathrm{ind}(T_\lambda) = \dim \mathrm{Nul} - \mathrm{codim}\,\mathrm{Ran} = \mathrm{Tr}(P_{\mathrm{Nul}}) - \mathrm{Tr}(P_{\mathrm{Ran}}^\perp) = \mathrm{Tr}([P, Q]).$

The individual terms, PQ and QP in the commutator are not trace class – since they are Fredholm – but the commutator is in the trace ideal.

Now, an observation of Calderón is that

L10.9 $$Q \text{ a parmaetrix modulo trace class errors } \Longrightarrow \operatorname{ind}(P) = \operatorname{Tr}([P,Q]). \tag{5.5.0.25}$$

This follows immediately from the properties of the trace functional.

Indeed, the space of parameterices as in (5.5.0.25) is convex, namely the interval between two, $Q_t = tQ_1 + (1-t)Q_0$, consists of such parametrices since

L10.10 $$Q_tP = \operatorname{Id} - (tA_1 + (1-t)A_0),\ PQ_t = \operatorname{Id} - t_1 + (1-t)B_0). \tag{5.5.0.26}$$

Moreover $dQ_t/dt = Q_1 - Q_0$ is of trace class. It follows that

L10.11 $$\frac{d}{dt}\operatorname{Tr}([P,Q_t]) = \operatorname{Tr}([P,Q_1 - Q_0] = 0 \tag{5.5.0.27}$$

as the trace of the commutator of a bounded and a trace class operator. Thus $\operatorname{Tr}([P,Q_t])$ is constant and the general case (5.5.0.25) follows from (5.5.0.24).

Since $T_{\lambda^{-1}}$ is a parametrix for T_λ we proceed to compute

L10.12 $$\operatorname{Tr}_H(T_{\lambda^{-1}}T_\lambda - T_\lambda T_{\lambda^{-1}}). \tag{5.5.0.28}$$

In fact since the pointwise matrices $\lambda(\theta$ and $\lambda^{-1}(\theta)$ commute we can proceed more generally to compute

L10.17 $$\operatorname{Tr}([T_\gamma, T_\lambda] - T_{[\gamma,\lambda]}) \tag{5.5.0.29}$$

for any two smooth maps γ, $\lambda \in \mathcal{C}^\infty(\mathbb{T}; \operatorname{GL}(n,\mathbb{C}))$. Then

L10.19 $$[T_\gamma, T_\lambda] - T_{[\gamma,\lambda]} = -\Pi\gamma(\operatorname{Id} - \Pi)\lambda\Pi + \Pi\lambda(\operatorname{Id} - \Pi)\gamma\Pi \tag{5.5.0.30}$$

The Fourier basis of H allows an opeator on it to be represented as an infinite matrix

L10.13 $$A(e_ie^{ik\theta}) = \sum_{j\ge 0, 1\le l\le n} A_{jk,li}e_le^{ij\theta},\ k \ge 0 \tag{5.5.0.31}$$

and then if it is in the trace ideal the trace is

L10.18 $$\operatorname{Tr}_H(A) = \sum_{k,i} A_{kk,ii}. \tag{5.5.0.32}$$

For a Toeplitz operator T_λ the matrix representation is

L10.14 $$A_{jk,li} = \frac{1}{2\pi}\lambda_{j-k,li},\ \lambda_{li}(\theta) = \frac{1}{2\pi}\sum_{p\in\mathbb{Z}}\lambda_{p,li}e^{ip\theta}. \tag{5.5.0.33}$$

It follows that the operator in (5.5.0.30) has the matrix representation and trace

L10.20 (5.5.0.34)

$$\begin{aligned}
B_{jk,il} &= \frac{1}{4\pi^2}\sum_{p<0,\ 1\le q\le n}(\lambda_{j-p,iq}\gamma_{p-l,ql} - \gamma_{j-p,iq}\lambda_{p-l,ql}) \\
\operatorname{Tr}(B) = \sum_{j\ge 0,i} B_{jj,ii} &= \frac{1}{4\pi^2}\sum_{j\ge 0,\ p<0,\ 1\le q,i\le n}(\lambda_{j-p,iq}\gamma_{p-j,qi} - \lambda_{p-j,iq}\gamma_{j-p,qi}) \\
&= \frac{1}{4\pi^2}\sum_{k,\ 1\le q,i\le n} k\lambda_{k,iq}\gamma_{-k,qi}
\end{aligned}$$

Here the last line is obtained by noting that the term $\lambda_{k,iq}\gamma_{-k,qi}$ occurs when $j - p = k$ in the first sum, so only for $k > 0$ and then k times for $j = 0, \dots, k-1$

and $p = -k, \dots, -1$; similarly in the second sum $\lambda_{k,iq}\gamma_{-k,qi}$ occurs only for $k < 0$ and then $-k$ times. Then in terms of the maps themeselves

L10.21 (5.5.0.35)

$$\operatorname{Tr}([T_\gamma, T_\lambda] - T_{[\gamma,\lambda]}) = \frac{1}{2\pi}\int_{\mathbb{S}} \operatorname{tr}(\frac{1}{i}\partial_\theta\lambda(\theta)\cdot\gamma(\theta)) = -\frac{1}{2\pi}\int_{\mathbb{S}} \operatorname{tr}(\lambda(\theta)\cdot\frac{1}{i}\partial_\theta\gamma(\theta))d\theta.$$

in terms of the trace on matrices.

Applying this to (L10.12 5.5.0.28) gives

L10.22 (5.5.0.36) $\operatorname{ind}(T_\lambda)$

$$= \operatorname{Tr}_H(T_{\lambda^{-1}}T_\lambda - T_\lambda T_{\lambda^{-1}}) = -\int_{\mathbb{S}} \operatorname{tr}(\lambda^{-1}\frac{1}{i}\partial_\theta\lambda) = -\frac{1}{2\pi i}\int_0^{2\pi} \partial_\theta \log\det(\lambda)d\theta$$
$$= -\operatorname{wn}\lambda$$

proving (L10.5 5.5.0.21).

There is a more concetual version of this computational proof of the Toeplitz index formula

The product, $\lambda_{j-k,li}$, of two such Toeplitz operators is therefore repreented by the matrix based on the diagonal operator $A = \frac{1}{i}\partial_+ 1$ which acts as $k+1$ on $e_i\exp(ik\theta)$. The complex powers A^{-z} are well-defined, unbounded normal operators on H with eigenvalues $(k+1)^{-z}$. Then

L10.24 Lemma 5. *If for any $\lambda \in \mathcal{C}^\infty(\mathbb{T};\mathbb{C}^n)$ the composite $A^{-z}T_\lambda$ is in the trace ideal for $\operatorname{Re} z > 1$ and for any smoothing operator κ on H*

L10.25 (5.5.0.37)
$$D(z) = \operatorname{Tr}(A^{-z}(T_\lambda + \kappa))$$

is holomorphic in $\operatorname{Re} z < -1$ with a meromorphic extension to the complex plan which is regular at $z = 0$.

We define the *regularized trace* on Toeplitz operators to be

L10.23 (5.5.0.38)
$$\overline{\operatorname{Tr}}(T_\lambda + \kappa) = D(0) = -\frac{1}{2}\int \operatorname{tr}\lambda + \operatorname{Tr}(\kappa).$$

the value of the analytic continuation.

Proof. Since the composite $A^{-z}\kappa$ is in the trace class ideal for $z \in \mathbb{C}$ and has entire trace it follows that

L10.26 (5.5.0.39)
$$\overline{\operatorname{Tr}}(T_\lambda + \kappa) = \overline{\operatorname{Tr}}(T_\lambda) + \operatorname{Tr}(\kappa)$$

From (L10.14 5.5.0.33) it follows that the matrix entries of the regularized operator are

L10.27 (5.5.0.40)
$$(A^{-z}T_\lambda)_{kj,li} = (k+1)^{-z}\frac{1}{2\pi}\lambda_{k-j,li}.$$

Thus the diagonal entries are multiples of $\lambda_{0,li}$ and

L10.28 (5.5.0.41)
$$(A^{-z}T_\lambda)_{kj,li} = \left(\sum_{k\ge 0}(k+1)^{-z}\right)\frac{1}{2\pi}\operatorname{tr}(\lambda_0) = \zeta(z)\operatorname{tr}(\lambda_0)$$

is given by a multiple of the Riemann zeta function. Since $\zeta(0) = -\frac{1}{2}$, (L10.23 5.5.0.38) follows. □

The previous computation now gives us

L10.30 LEMMA 6 (Trace-defect formula). *The regularized trace on Toeplitz operators satisfies*

L10.29 (5.5.0.42)
$$\overline{\mathrm{Tr}}([T_\lambda+\kappa, T_\gamma+\mu]) = -\frac{1}{2\pi}\int_{\mathbb{S}} \mathrm{tr}(\lambda(\theta)\cdot\frac{1}{i}\partial_\theta\gamma(\theta))d\theta$$

for $\lambda,\ \gamma\in\mathcal{C}^\infty(\mathbb{T};\mathrm{GL}(n,\mathbb{C})$ *and any smoothing operators* κ, μ *on* H.

PROOF. This follows by combining (5.5.0.38), (5.5.0.35) and the fact that

L10.31 (5.5.0.43)
$$\overline{\mathrm{Tr}}(T_{[\lambda,\gamma]}) = 0.$$

□

Now recall that the central extension discussed above, of the component $\mathcal{L}_0G$ of the identity for $G=\mathrm{U}(n)$ or $\mathrm{SO}(n)$, is obtained by considering the invertible perturbations $\mathcal{G}=\{T_\lambda+\kappa;\ \text{invertible}\}$ where κ is a smoothing operator (projected onto H) and then taking the quotient $\widehat{\mathcal{L}}_0G=\mathcal{G}/G_1^{-\infty}$, by the normal subgroup $G_1^{-\infty}=\{\mathrm{Id}+\kappa\in G^{-\infty};\det(\mathrm{Id}+\kappa)=1\}$. The Mourier-Cartan form $g^{-1}dg$ is well-defined on the tangent space to $\mathcal{G}$ and we can define the complex-valued 1-form

L10.32 (5.5.0.44)
$$\alpha=\overline{\mathrm{Tr}}(g^{-1}dg)\text{ in }\mathcal{G}.$$

In fact it descends to a 1-form on $\widehat{\mathcal{L}}_0G$. Namely if $h\in G_1^{-\infty}$ then

L10.33 (5.5.0.45)
$$(hg)^{-1}d(hg)=g^{-1}dg+g^{-1}(h^{-1}dh)g\Longrightarrow$$
$$\alpha_{hg}=\overline{\mathrm{Tr}}((hg)^{-1}d(hg))=\overline{\mathrm{Tr}}(g^{-1}dg)+\mathrm{Tr}(g^{-1}(h^{-1}dh)g)$$

since $h^{-1}dh=(\mathrm{Id}+\kappa)^{-1}d\kappa$ is a smoothing operator. Moreover $\det h=1$ on $G_1^{-\infty}$ so

$$\mathrm{Tr}(g^{-1}(h^{-1}dh)g)=\mathrm{Tr}(h^{-1}dh)=d\log\det h=0\Longrightarrow\alpha_{hg}=\alpha_g.$$

Thus α is constant on the orbits of $G_1^{-\infty}$ so descends to $\widehat{\mathcal{L}}_0G$.

L10.35 LEMMA 7. *The 1-form* $\alpha=\overline{\mathrm{Tr}}(g^{-1}dg)$ *is a connection form on* $\widehat{\mathcal{L}}_0G$ *as a circle bundle over* $\mathcal{L}_0G$.

PROOF. The circle action on $\widehat{\mathcal{L}}_0G$ is the quotient of the multiplicative action of $G^{-\infty}$. Thus, taking a map $\mathbb{T}\longrightarrow h_t\in G^{-\infty}$ such that $\det(h_t)(e^{it}=\exp(it)$ – for instance complex rotation around any basis vector – the same formula (5.5.0.45) shows that

L10.36 (5.5.0.46)
$$\alpha_g(\partial_t)=\alpha_g$$

□

We can also compute the lie algebra cocyle corresponding to the central extension. Recall that we choose a decomposition over the identity

L10.37 (5.5.0.47)
$$T_{\mathrm{Id}}\widehat{\mathcal{L}}_0G=\mathbb{R}+T_{\mathrm{Id}}\mathcal{L}_0\text{ by setting the image of }T_{\mathrm{Id}}\mathcal{L}_0=\mathrm{Nul}(\alpha)$$
$$\Longrightarrow\mathcal{L}_0G\ni\lambda\longmapsto T_\lambda+\kappa\in T_{\mathrm{Id}}\widehat{\mathcal{L}}_0G,\ \overline{\mathrm{Tr}}(T_\lambda+\kappa)=0.$$

Then the 2-form on $\mathcal{L}_0G$ corresponding (and determining) the central extension is

L10.38 (5.5.0.48)
$$\omega(\lambda,\gamma)=\overline{\mathrm{Tr}}([T_\lambda+\kappa,T_\gamma+\mu])=\overline{\mathrm{Tr}}([T_\lambda,T_\gamma])=-\frac{1}{2\pi}\int_{\mathbb{T}}\mathrm{Tr}(\lambda\frac{d\gamma}{d\theta})d\theta.$$

Thus, we are back at the 2-form we 'guessed' earlier in (??).

We can see that this is just the differential of α at Id, i.e. the curvature 2-form of the central extensio, since

L10.39 (5.5.0.49)
$$d\alpha = -\overline{\mathrm{Tr}}(g^{-1}dg \wedge g^{-1}dg) \Longrightarrow$$
$$(d\alpha)_{\mathrm{Id}}(\lambda, \gamma) = -\overline{\mathrm{Tr}}([T_\lambda, T_\gamma]) = -\frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{Tr}(\lambda \frac{d}{d\theta}\gamma)d\theta.$$

Let us make sure we understand the simplicial nature of this central extension (as it turns out, all central extensions of loop groups at least in the simply connected case). We can already see this in the behaviour of the curvature – the form on $\mathcal{L}_0 G$

L10.40 (5.5.0.50)
$$d\alpha = -\frac{1}{2\pi i}\int_{\mathbb{T}} g^{-1}dg \wedge \partial_\theta(g^{-1}dg)d\theta.$$

Recalling that the loop group is the fibre product $\mathcal{P}^{[2]}G$ we notice that in fact the curvature is the simplicial derivative of a 2-form on $\mathcal{P}G$, namely

L10.41 (5.5.0.51)
$$B = -\frac{1}{2\pi i}\int_0^1 g^{-1}dg \wedge \partial_\theta(g^{-1}dg)d\theta.$$

This is called a 'B-field' in mathematical physics, and is part of the connection on a bundle gerbe.

L10.42 LEMMA 8. *For the Hardy central extension dB (restricted to the pointed path space) is the pull-back of the left-invariant 3-form on G which at the identity is*

L10.43 (5.5.0.52)
$$c\,\mathrm{tr}(\lambda_1, [\lambda_2, \lambda_3]).$$

Despite a couple of efforts I have not succeeded in nailing down the constant in (5.5.0.52) – and we need to know this. The deRham 3-form determined by (5.5.0.52) is necessarily integral, the question is whether it corresponds to a generator of $H^3(G; \mathbb{Z})$.

PROOF. □

5.1. Bott-Virasoro group

Next we turn to the group $\mathrm{Dff}^+(\mathbb{T})$. Similar arguments to those above show that this also has a central extension, the Bott-Virasoro group (or groups). Recall that an element $F \in \mathrm{Dff}^+(\mathbb{T})$ acts on complex-valued $\mathcal{C}^\infty(\mathbb{T})$ (or one can take functions valued in $\mathbb{C}^n$) by pull-back. The half-density part of the action is to ensure orthogonality on $L^2(\mathbb{T})$. So the action is

L10.44 (5.5.1.1)
$$F^{\#}u = J_F^{\frac{1}{2}} F^* u.$$

Again we 'compress' this to the Hardy space so

L10.45 (5.5.1.2)
$$T_F = \Pi F^{\#} \Pi : H \longrightarrow H$$

where it preserves smoothness. Moreover as for the actions of loops, this operator is Fredholm of index zero. So we can consider the big group $\mathcal{F}$ of invertible (unitary) perturbations by smoothing operators and its quotient

L10.46 (5.5.1.3)
$$\widehat{\mathrm{Dff}}^+(\mathbb{T}) = \mathcal{F}/G_1^{-\infty}.$$

This is again a central extension so naturally we want to compute the extension 2-form on the Lie algebra.

The Lie algebra of $\mathrm{Dff}^+(\mathbb{T})$ consists of the smooth vector fields on the circle, this is a version of the 'Witt algebra'. Since ∂_θ is a non-vanishing vector field (generating the rotations) the whole Lie algebra is

L10.47 (5.5.1.4) $$\mathcal{C}^\infty(\mathbb{T})\partial_\theta.$$

These should be real-valued, but we can also consider the complexification – this is not in fact the Lie algebra of a group as for the loop groups, rather we do this to simplify the computation.

As for the loop group it is conveneint to use the regularized trace to fix a decomposition of the Lie algebra

L10.51 (5.5.1.5) $$T_{\mathrm{Id}}\widehat{\mathrm{Dff}}^+(\mathbb{T}) = \mathbb{R} + \tau\mathcal{C}^\infty(\mathbb{T})$$

where the central $\mathbb{R}$ is comes from smoothing operators and

L10.52 (5.5.1.6) $$\tau(V) = \Pi(L_V + \kappa)\Pi,\ \overline{\mathrm{Tr}} \circ \tau = 0.$$

Here $\overline{\mathrm{Tr}}$ is the regularized but extended a little further to be the value at $z = 0$ of the analytic continuation of

L10.53 (5.5.1.7) $$\mathrm{Tr}_H\left((\frac{1}{i}\partial_\theta + 1)^{-z})\Pi(V\partial_\theta + \frac{1}{2}V' + \kappa)\Pi\right)$$

so only the first term is new.

So the extension cocylce on the Witt algebra is the Bott form

L10.50a (5.5.1.8) $$\omega_B(V, W) = \mathrm{Tr}_H([\Pi(L_V)\Pi, \Pi(L_W)\Pi] - \Pi L_{[V,W]}\Pi).$$

As indicated above, since ω_B is a bilinear form we can complexify the Lie algebra and look at the spanning vector fields

L10.51a (5.5.1.9) $$V_p = e^{ip\theta}\partial_\theta,\ L_P = e^{ip\theta}\partial_\theta + \frac{1}{2}ipe^{ip\theta},\ p \in \mathbb{Z}.$$

Compressed to the Hardy space

L10.52a (5.5.1.10) $$\Pi L_p \Pi e^{ik\theta} = e^{i(k+p)\theta} \times \begin{cases} i(k + \frac{1}{2}p) & \text{iff } k + p \geq 0 \\ 0 & \text{iff } k + p < 0, \end{cases},\ k \geq 0.$$

So for any p, $q \in \mathbb{Z}$,

L10.53a (5.5.1.11) $$[\Pi(L_p)\Pi, \Pi(L_q)\Pi]e^{ik\theta} =$$
$$e^{i(k+p+q)\theta} \times \begin{cases} \frac{1}{2}(p + q + 2k)(p - q) & \text{if } p + k \geq 0,\ q + k \geq 0,\ p + q + k \geq 0 \\ -(k + p + \frac{1}{2}q)(k + \frac{1}{2}p) & \text{if } q + k < 0,\ p + q + k \geq 0 \\ (k + q + \frac{1}{2}p)(k + \frac{1}{2}q) & \text{if } p + k < 0,\ p + q + k \geq 0 \\ 0 & \text{if } p + k < 0,\ q + k < 0. \end{cases}$$

Since $[V_p, V_q] = i(q - p)V_{p+q}$

L10.54a (5.5.1.12) $$\Pi L_{[V_p,V_q]}\Pi e^{ik\theta} = e^{i(k+p+q)\theta} \times \begin{cases} (p - q)(k + \frac{1}{2}p + \frac{1}{2}q) & \text{if } p + q + k \geq 0 \\ 0 & \text{if } p + q + k < 0 \end{cases}$$

The trace is therefore

L10.55a (5.5.1.13) $$\omega_B(V_p, V_q) = \begin{cases} 0 & \text{if } p + q \neq 0 \\ & \text{if } p + q = 0 \end{cases}$$

CHAPTER 6

The spin representation

The spin representation, as a representation of the basic central extension of $\mathcal{L}\,\mathrm{Spin}(2n)$ is constructed here using Hardy space techniques. The main reference is Pressley-Segal [**9**, Chapter 12], but see also the construction by Bortheick [**5**] and the work of Valiveti.

6.1. Complex structures on $\mathbb{R}^{2n}$

We start with a treatment in the finite-dimensional setting. This is quite a good guide to the infinite-dimensional case although there are some significant conceptual differences; it is however rather a complicated derivation of the spin representation, as a projective representation of $\mathrm{SO}(2n)$ and a true representation of $\mathrm{Spin}(2n)$ or $\mathrm{Spin}_{\mathbb{C}}(2n)$.

Take $\mathbb{R}^{2n}$ with it standard Euclidean inner product, we could take an abstract real even-dimensional Euclidean vector space but I shall stick with the model case. So we have the standard basis e_i, $i = 1, \dots, 2n$. A complex structure on $\mathbb{R}^{2n}$ is a real $2n \times 2n$ matrix J with $J^2 = -\operatorname{Id}$. The 'standard' complex structure on $\mathbb{R}^{2n}$ is

L11.1 (6.6.1.1) $$J_0 e_i = e_{i+n},\ J_0 e_{i+n} = -e_i,\ 1 \le i \le n.$$

This is norm-preserving, i.e. an orthogonal matrix.

So consider the space of all such *orthogonal* complex structures,

L11.2 (6.6.1.2) $$\mathcal{J} = \{J \in M(2n, \mathbb{R}); J^2 = -\operatorname{Id},\ J^t = -J\}.$$

These form a smooth complex, indeed algebraic, manifold as we shall see by writing it as a homogenous space.

If we pass from $\mathbb{R}^{2n}$ to its *complexification* – just $\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes \mathbb{C}$ – then of course J still acts, now as a complex-linear map. As the complexification of a real space, $\mathbb{C}^{2n}$ has a complex-conjugation map, $z \longrightarrow \overline{z}$ and an element of $M(2n\mathbb{C})$ is real if and only if it commutes with complex conjugation.

The spectral theorem shows that J, being anti-Hermitian, has an eigenbasis and this can only correspond to eigenvalues $\pm i$. Both eigenspaces are complex subspaces of $\mathbb{C}^{2n}$. In the case of J_0 we can see exactly what these are. Namely we set

ALS.9 (6.6.1.3) $$w_i = \frac{1}{\sqrt{2}}(e_i - ie_{i+n}),\ \overline{w}_i = \frac{1}{\sqrt{2}}(e_i + ie_{i+n})$$

and then $J_0 w_i = iw_i$, $J_0\overline{w}_i = -i\overline{w}_i$, for $1 \le i \le n$. Thus the w_i span $\mathbb{C}^n = W_0$ and we can write

L11.3 (6.6.1.4) $$\mathbb{C}^{2n} = W_0 \oplus \overline{W}_0 = \mathbb{C}^n \oplus \overline{\mathbb{C}}^n,\ J = i\Pi - i(\operatorname{Id} - \Pi)$$

where W_0 is the i-eigenspace, which is thought of as $\mathbb{R}^{2n}$ with the complex structure determined by J_0 and $\overline{\mathbb{C}}^n$ is $\mathbb{R}^{2n}$ with the opposite complex structure (determined by $-J_0$). It is also important that both eigenspaces are 'isotropic'. The inner product on $\mathbb{R}^{2n}$ has a unique complex-linear (not hermitian) extension to $\mathbb{C}^{2n}$ which we denote by B. Then

L11.4 (6.6.1.5) $$B = 0 \text{ on } W_0 = \mathbb{C}^n \text{ and on } \overline{W}_0 = \overline{\mathbb{C}}^n.$$

On the other hand the form

L11.14 (6.6.1.6) $$\langle w', w\rangle_W = B(w', \overline{w})$$

is a positive-definite Hermitian form. This also means we could identify $\overline{W}$ as the dual of W with complex conjugaton then being the usual identification with the 'opposite' Hilbert space. We will not make this identification notationally as it would be a bit confusing.

The behaviour of a general orthogonal complex structure is 'the same'. Namely J has $\pm i$ eigenspaces in $\mathbb{C}^{2n}$ which are exchanged by complex conjugation – since J is a real matrix. Writing the i-eigenspace as W_J we see that

L11.6 (6.6.1.7) $$\mathbb{C}^{2n} = W_J \oplus \overline{W}_J,\ W_J \text{ is isotropic for } B.$$

Again the form $\langle w', w\rangle = B(w', \overline{w})$ is a positive-definite Hermitian form on W_J identifying $\overline{W}_J$ as its dual although we shall not make this identification implicit.

Moreover every such complex subspace W, which isotopic and has

L11.15 (6.6.1.8) $$\mathbb{C}^{2n} = W \oplus \overline{W}$$

arises as W_J for a unique $J \in \mathcal{J}$. If Π_W is the projection onto W implied by the decomposition (6.6.1.7) – so it has null space $\overline{W}$ and range W – then

L11.7 (6.6.1.9) $$J = i\Pi_W - i(\operatorname{Id} - \Pi_W)$$

is a real and orthogonal matrix with $J^2 = -\operatorname{Id}$.

ALS.10 Lemma 9. *Any two orthogonal complex structures on $\mathbb{R}^{2n}$ are conjugate by an element of* $\mathrm{O}(2n)$.

Proof. It suffices to show that any J is conjugate to J_0. It follows from (6.6.1.8) that the i eigenspace, W, of J has complex dimension n and hence there exists a unitary isomorphism $a : W_0 \longrightarrow W$ indetifying the sequilinear inner products. Then $\bar{a} : \overline{W}_0 \longrightarrow$ is also a unitary isomorphism and $O = a \oplus \bar{a}$ is real and orthogonal. □

Thus the orthogonal group $\mathrm{O}(2n)$ acts transitively on $\mathcal{J}$ and it follows that it is a homogeneous space

ALS.12 (6.6.1.10) $$\mathcal{J} = \mathrm{O}(2n)/\,\mathrm{U}(n)$$

since the isotropy group of J_0 consists of the orthogonal matrices which are diagonal with respect to the splitting and these are of the form $U \oplus \overline{U}$ with $U \in \mathrm{U}(n)$. Since $\mathrm{O}(2n)$ has two components while $\mathrm{U}(n)$ is cnnected, $\mathcal{J}$ has two components.

To see that $\mathcal{J}$ is a complex manifold, pass from $\mathrm{O}(2n)$ to the larger Lie group $\mathrm{O}_{\mathbb{C}}(2n)$ consisting of the complex orthogonal transformations on $\mathbb{C}^{2n}$ with respect to B. So the inverse of $O \in \mathrm{O}_{\mathbb{C}}(2n)$ is O^t and $\det O = \pm 1$. In fact $\mathrm{O}_{\mathbb{C}}(2n)$ has two components. Every orthogonal complex structure is of the form

L11.8 (6.6.1.11) $$J = OJ_0O^t,\ O \in \mathrm{O}_{\mathbb{C}}(2n).$$

Indeed $J^2 = -\operatorname{Id}$ and $J \in \mathrm{O}_{\mathbb{C}}(2n)$. The $\pm i$ eigenspaces of J must be $W = OW_0$ and $O\overline{W}_0$ where $W_0 = \mathbb{C}^n$ is the $+i$ eigenspace of J_0. It follows that W is isotropic, that $\overline{W} = O^t\overline{W}_0$ and that J is real. Conversely if we take an orthonormal basis $f_i \in W$ with respect to the Hermitian form and define

L11.16 (6.6.1.12) $$O_J(\frac{1}{\sqrt{2}}(e_i - ie_{i+n}) = f_i,\ O_J(\frac{1}{\sqrt{2}}(e_i + ie_{i+n}) = \overline{f_i}$$

and extend by complex linearity then (6.6.1.11) follows and $O \in \mathrm{O}_{\mathbb{C}}(2n)$.

This identifies

L11.9 (6.6.1.13) $$\mathcal{J} = \mathrm{O}_{\mathbb{C}}(2n)/\,\mathrm{U}(n)$$

initially as the quotient of $\mathrm{O}_{\mathbb{C}}(2n)$ by the subgroup which commutes with J_0. Writing out $O \in \mathrm{O}_{\mathbb{C}}(2n)$ as a 2×2 matrix with respect to the 'base' decomposition (6.6.1.4) it must be diagonal

L11.10 (6.6.1.14) $$O \in \mathrm{O}_{\mathbb{C}}(2n),\ J_0O = OJ_0 \Longrightarrow O = \begin{pmatrix} U & 0 \\ 0 & U^t \end{pmatrix}.$$

where $U \in \mathrm{U}(n)$. Since $\mathrm{U}(n)$ is connected we see that $\mathcal{J}$ has two components.

From (6.6.1.13) it follows that $\mathcal{J}$ is a complex manifold and there is a 'determinant bundle' over $\mathcal{J}$ with fibre at J the quotient of the unitary group on W by the subgroup of determinant one. In fact it follows that $\mathrm{SO}(2n)$ acts holomorphically on the corresponding complex line bundle. This is where the finite-dimensional and infinite-dimensional pictures start to diverge significantly. However let us proceed a little further.

We can consider the special 'decomposed' complex structures, with respect to the basis e_i. We label these J_S where $S \subset \{1, \dots, n\}$. The $+i$ eigenspace of J_S is W_S which is spanned by $e_i - ie_{i+n}$, $i \notin S$, and $e_i + ie_{i+n}$ for $i \in S$. This is clearly isotropic and

L11.11 (6.6.1.15) $$\mathbb{C}^{2n} = W_S \oplus \overline{W}_S,\ \overline{W}_S = W_{S'},\ S' = \{1, \dots, n\} \setminus S.$$

There is an clear choice of complex orthogonal transformation conjugating J_0 to J_S, satisfying

L11.17 (6.6.1.16)
$$O_S(e_i) = e_i,\ O_S(e_{i+n}) = e_{i+n},\ i \notin S,\ O_S(e_i) = e_i,\ O_S(e_{i+n}) = -e_{i+n},\ i \in S.$$

Notice that $\det(O_S) = (-1)^{\#S}$ is the parity of S.

For any J and corresponding W consider subspaces

L11.12 (6.6.1.17) $$A_J = \{J' \in \mathcal{J}; \Pi_J : W_{J'} \longrightarrow W_J \text{ is invertible.}\}.$$

These are certainly open. For $J = J_S$ we set $A_J = A_S$.

L11.13 Proposition 15. *The A_S cover $\mathcal{J}$ with $A_{S_1} \cap A_{S_2} \neq \emptyset$ if and only if S_1 and S_2 have the same parity.*

Proof. If $J = OJ_0O^t$ then $W_J = OW_0 = OO_SW_S$. The projection onto W_S is therefore invertible if and only if

L11.21 (6.6.1.18) $$\det{}_{W_S}(\Pi_S OO_S \Pi_S) \neq 0.$$

□

To introduce coordinates in A_S consider first A_0, the affine around J_0. Then for $J \in A_0$, $\Pi_0 W_J \longrightarrow W_0$ is, by assumption, invertible and hence each $w \in W_J$ can be written uniquely in terms of a linear map

L11.22 (6.6.1.19) $$W_0 \ni w = v + \beta_J(v) \in W_J,\ v \in W_0,\ \beta_J : W_0 \longrightarrow \overline{W}_0.$$

The fact that W_J is isotropic reduces to the antisymmetry of β_J :

L11.23 (6.6.1.20)
$$B(v+\beta_J(v),v'+\beta_J(v'))=0\ \forall\ v,v'\in W_0\Longleftrightarrow B(\beta_J(v),v')=-B(\beta_J(v'),v).$$

The same argument shows that in general

L11.20 (6.6.1.21)
$$A_S=\{\beta:W_S\longrightarrow\overline{W}_S;B\text{-antisymmetric}\}.$$

If $J\in A_0$ then

L11.24 (6.6.1.22)
$$W_J=(\operatorname{Id}+\beta)W_0.$$

L11.18 Proposition 16. *An element of* $\operatorname{Det}(\mathcal{J})$ *determines a holomorphic section of* $\operatorname{Det}'(\mathcal{J})$.

Proof. Take $P\in\operatorname{Det}(\mathcal{J})$, this is determined by a choice of $O_P\in\mathrm{O}_{\mathbb{C}}(2n)$ where P is the point $[O_P]\in\mathrm{O}_{\mathbb{C}}(2n)/\operatorname{SU}(n)$ above $J=O_PJ_0O_P^t$. Now consider

L11.19 (6.6.1.23)
$$\mathrm{O}_{\mathbb{C}}(2n)\ni O\longmapsto\det_{W_0}(\Pi_0O_P^tO\Pi_0).$$

This descends to a complex-valued function on $\operatorname{Det}=\mathrm{O}_{\mathbb{C}}(2n)/\operatorname{SU}(n)$ where $\mathrm{U}(n)$ is the stabilizer of J_0 as in (L11.10 6.6.1.14). Similarly the function only depends on P. Under the action of the whole of $\mathrm{U}(n)$ on O it trasforms by $\det(U)$ so does indeed define a fibre-linear function on Det and hence a section of Det', the dual line bundle. □

If we write an orthogonal matrix in terms of the base complex structure

L11.25 (6.6.1.24)
$$O=\begin{pmatrix}a&c\\b&d\end{pmatrix}\text{ on }W_0\oplus\overline{W}_0$$

then for $J=AJ_0O^t$

L11.26 (6.6.1.25)
$$W_J=(a+b)W_0$$

and $J\in A_0$ if and only if a is invertible. Then

L11.27 (6.6.1.26)
$$W_J=(\operatorname{Id}+ba^{-1})W_0\Longrightarrow\beta_J=ba^{-1}$$

in terms of the coordinate on A_0. The section of γ_0 of Det' is

L11.28 (6.6.1.27)
$$\gamma_0([O])=\det(a)\neq0.$$

If we consider another decomposed complex structure J_S then the corresponding section of Det' is

L11.29 (6.6.1.28)
$$\alpha_S([O])=\det_{W_0}(O_SO)=\det(\Pi_0\Pi_Sa+(\operatorname{Id}-\Pi_S)b)=\det_S(ba^{-1})\det(a)$$

where $\det_S$ is the determinant of the $S\times S$ submatrix. This shows that

L11.30 (6.6.1.29)
$$\alpha_S=\det_S(\beta_J)\alpha_0\text{ on }A_0.$$

This in turn determines the transition cocycle for Det' (and for Det).

Now, and we are getting near the crux of the story, for antisymetric $2k\times2k$ matrices ($\#S$ must be even if A_S is to meet A_0) the determinant, as a polynomial in the matrix entries has a square-root, the Pfaffian polynomial,

L11.31 (6.6.1.30)
$$\operatorname{Pf}^2=\det.$$

Namely for a $2k\times2k$ antisymmetric matrix

L11.34 (6.6.1.31)
$$\operatorname{Pf}(\beta)=\frac{1}{k!2^k}\sum_i\operatorname{sgn}(i)\beta_{i_1i_2}\dots\beta_{i_{2k-1}i_{2k}}$$

with the sum over all permutations, i, of $\{1,\dots,2k\}$. It is certainly a polynomial of degree k and if β is invertible the derivative of $\mathrm{Pf}(\beta+t\eta)$ at $t=0$ is given by the sum as in (L11.34 6.6.1.31) with each term replaced by the sum over the terms obtained by replacing one factor with η.

The matrices $(\eta^{ij})_{pq}=\delta_{pi}\delta_{qj}-\delta_{qi}\delta_{pj}$ for $i<j$ form a basis of the antisymmetric matrices and

L11.36 (6.6.1.32)
$$\frac{d}{dt}\Big|_{t=0}=\frac{1}{k!2^k}\sum_{1\le s\le k\ i,}\ \sum_{i_{2s-1}=i,i_{2s}=j}\operatorname{sgn}(i)\beta_{i_1i_2}\dots\dots\widehat{\beta_{ij}}\beta_{i_{2k-1}i_{2k}}.$$

Using the inversion formula satisfied by antisymmetric matrices

L11.37 (6.6.1.33)
$$\sum_{i<j}\beta_{ij}(\beta^{-1})_{jp}=\frac{1}{2}\,\mathrm{Id}_{ip}$$

it follows that, when β is invertible

L11.38 (6.6.1.34)
$$\frac{d}{dt}\,\mathrm{Pf}(\beta)=\frac{1}{2}\,\mathrm{Pf}(\beta)\,\mathrm{Tr}(\beta^{-1}\dot\beta)\Longrightarrow\mathrm{Pf}(\beta)^2=\det(\beta)\text{ everywhere.}$$

L11.32 LEMMA 10. *The determinant bundle has a holomorphic square-root defined by replacing the determinant by the Pfaffian in the transition cocycle, a holomorphic section of* Pf' *over* $\mathcal{J}_0$ *(the component containing the base complex structure) is then a holomorphic function* f *on* A_0 *such that* $f/\det(\beta_S)$*, defined on* $A_0\cap A_S$*, extends to be holomorphic on* A_S *for all* S *of even parity.*

And similarly for the other component.

L11.33 LEMMA 11. *The space of holomorphic sections of* Pf' *is isomorphic to* Λ^*W_0 *and the projective representation of* $\mathrm{SO}(2n)$ *defined on it is the spin representation of* $\mathrm{Spin}_{\mathbb{C}}(2n)$.

This follows by checking that

L11.40 (6.6.1.35)
$$\beta\wedge\beta\dots k\text{ factors}\wedge\beta=k!2^k\,\mathrm{Pf}(\beta)\mathrm{vol}$$

and

L11.41 (6.6.1.36)
$$\exp\frac{1}{2}\beta=\sum_S\mathrm{Pf}(\beta_S)(dw)_S$$

6.2. Complex structutes on $\mathcal{L}(\mathbb{T};\mathbb{R}^{2n})$

Now we pass to the infinite dimensional setting and consider complex structures on the Euclidean space

L12.1 (6.6.2.1)
$$\mathcal{C}^\infty(\mathbb{T};\mathbb{R}^{2n})\subset L^2(\mathbb{T};\mathbb{R}^{2n}).\ \|u\|^2=\int_{\mathbb{T}}|u(\theta)|^2d\theta.$$

We mostly work with the complexification

L12.2 (6.6.2.2)
$$\mathcal{C}^\infty(\mathbb{T};\mathbb{C}^{2n})\supset W_0$$

where W_0 is the 'Hardy space' for our base complex structure

L12.3 (6.6.2.3)
$$W_0=\mathrm{sp}\{\mathbb{C}^n,\ e^{ik\theta}\mathbb{C}^{2n},\ k>0\}$$

consisting of the base complex subspace $\mathbb{R}^{2n}\otimes\mathbb{C}=\mathbb{C}^n\oplus\overline{\mathbb{C}}^n$ of the constants, that I talked about last time, and the positive Fourier terms.

This corresponds to base projections

L12.4 (6.6.2.4) $$\Pi_0 : \mathcal{C}^\infty(\mathbb{T};\mathbb{C}^{2n}) \longrightarrow W_0,\ \overline{\Pi_0} = \operatorname{Id} - \Pi_0$$

in terms of the complex conjugation on $\mathcal{C}^\infty(\mathbb{T};\mathbb{C}^{2n})$. The complex structure itself, which restricts to a map on $\mathcal{C}^\infty(\mathbb{T};\mathbb{R}^{2n})$ is $J_0 = i\Pi_0 - i(\operatorname{Id} - \Pi_0)$. The index in (6.6.2.3) will be denoted

L12.33 (6.6.2.5) $$\mathcal{I} = \{1, 2, \dots, n, (1,1), \dots (1, 2n), \dots\}$$

labelling the basis

L12.34 (6.6.2.6) $$e_1, \dots, e_n, e^{i\theta}e_1, \dots, e^{i\theta}e_{2n}, \dots .$$

The complex structures we want to consider should be orthogonal and 'comparable' to J_0

L12.5 (6.6.2.7)
$$\mathcal{J}^\infty = \{J : \mathcal{C}^\infty(\mathbb{T};\mathbb{R}^{2n}) \longrightarrow \mathcal{C}^\infty(\mathbb{T};\mathbb{R}^{2n}),\ J^2 = -\operatorname{Id},\ J^t = -J,\ J - J_0 \text{ smoothing}.$$

Since it is an anti-selfadjoint operator on the Hilbert space $L^2(]bbT;\mathbb{C}^{2n})$ the spectral theorem applies to $J \in \mathcal{J}^\infty$, as in the finite-dimensional case and the $\pm i$-eigenspace give a decomposition

L12.7 (6.6.2.8) $$\mathcal{C}^\infty(\mathbb{T};\mathbb{C}^{2n}) = W \oplus \overline{W}.$$

As usual I am not distinguishing between W as a subspace of $\mathcal{C}^\infty(\mathbb{T};\mathbb{C}^{2n})$ and its closure in L^2 which is the actual $+i$-eigenspace. That the smooth part is dense follows from the last part of (6.6.2.7) since for instance the projection onto W given by (6.6.2.8) is

L12.8 (6.6.2.9) $$\Pi_W = \frac{1}{2i}(J + i\operatorname{Id}) = \frac{1}{2i}(J_0 + i\operatorname{Id}) + E = \Pi_0 + E,\ E \in \Psi^{-\infty}(\mathbb{T};\mathbb{C}^{2n}).$$

In particular the map

L12.9 (6.6.2.10) $$\Pi_o\Pi_W : W_0 \longrightarrow W_0$$

differs from the identity by a smoothing operator projected to W_0. This is a Fredholm operator (completing to Hilbert spaces as you think necessary ...) of index zero. This can be pertturbed by a finite rank smoothing operator to be invertible. With a little more effort we find

L12.10 LEMMA 12. *Each element $J \in \mathcal{J}^\infty$ is conjugate to J_0, $J = OJ_0O^t$ where*

L12.11 (6.6.2.11) $$O \in G_O^\infty = \{A \in \operatorname{Id} + \Psi^{-\infty}(\mathbb{T};\mathbb{C}^{2n});\ \overline{A} = A.\ A^tA = \operatorname{Id} = AA^t\}.$$

PROOF. □

Thus we again see that $\mathcal{J}^\infty$ is a homogeneous space with G_O^∞ replacing the finite-dimensional orthogonal group $O(\mathbb{R}^{2n})$. Again we pass from this orthogonal group to the corresponding complex orthogonal group

L12.12 (6.6.2.12) $$G_{\mathbb{C}O}^\infty = \{A \in \operatorname{Id} + \Psi^{-\infty}(\mathbb{T};\mathbb{C}^{2n});\ A^tA = \operatorname{Id} = AA^t\}$$

to see that

L12.13 (6.6.2.13) $$\mathcal{J}^\infty = G_{\mathbb{C}O}^\infty / G^\infty(W_0).$$

Here the isotropy group is the subgroup of the unitary elements of the invertible operators of the form $\Pi_0 + \Pi_0 E \Pi_0$, E smoothing, on W_0.

This is all directly analogous to the finite-dimensional case. The first departure is the action of the group $\mathcal{L}\,\mathrm{Spin}(2n) = \mathcal{L}_0\,\mathrm{SO}(2n)$ we are interested in. In the finite-dimensional case $\mathrm{SO}(2n)$ is a subgroup of the (complex) orthogonal group so acts in the obvious way. Here this is not the case, but nevertheless the conjugation action by loops

L12.14 (6.6.2.14) $$J \longmapsto lJl^{-1} \text{ defines an action of } \mathcal{L}\,\mathrm{Spin}(2n) \text{ on } \mathcal{J}^\infty.$$

Certainly lJl^{-1} is an orthogonal complex structure, the issue is to show that it is in $\mathcal{J}^\infty$.

For this it is enough to consider J_0. Then is can use our earlier result on the Hardy space – which we replace by W_0 without any significant change. Thus we know that $\Pi_0 l \Pi_0$ has a smoothing perturbation $\lambda = \Pi_0(l+E)\Pi_0$, where E is projected to W_0, which is unitary on W_0. Taking the orthogonal operator which is λ on W_0 and $\overline{\lambda}$ on $\overline{W}_0$ gives an operator $\tilde{\lambda}$ which is orthogonal, differs from l by a smoothing operator and commutes with J_0. Then

L12.15 (6.6.2.15) $$l\lambda^{-1}J_0\lambda l^{-1} = lJ_0l^{-1} \text{ and } \lambda^{-1}l \in G^\infty_{\mathbb{C}O}.$$

This proves (6.6.2.14) since

L12.16 (6.6.2.16) $$lJl^{-1} = lOJ_0O^tl^{-1} = lO\lambda^{-1}J_0\lambda O^tl^{-1},\ lO\lambda^{-1}J_0 \in G^\infty_{\mathbb{C}O}.$$

Then we proceed to analyze the determinant bundle over $\mathcal{J}^\infty$. This is defined as in the finite-dimensional case as the homogeneous bundle given by the finer quotient

L12.17 (6.6.2.17) $$\mathrm{Det} = G^\infty_{\mathbb{C}O}/\{A \in G^\infty_{\mathrm{U}}(W); \det(A) = 1\}.$$

Here the determinant is the Fredholm determinant, well-defined on operators of the form $\mathrm{Id}+E$ on a Hilbert space if E is of trace class – so in particular if it is smoothing in our case.

The group $\mathcal{L}\,\mathrm{Spin}(2n)$ does not act on Det as a circle bundle over $\mathcal{J}^\infty$ because there is no canonical way to choose λ in (6.6.2.15). Different choices are related by an element of the unitary group $G^\infty_{\mathrm{U}}(W_0)$ on W_0 so what we recover is the determinant central extension of $\mathcal{L}\,\mathrm{Spin}(2n)$ now acting on Det.

We need to define the Pfaffian bundle over $\mathcal{J}^\infty$. As in the finite dimensional case we can cover $\mathcal{J}^\infty$ by affine coordinate patches. Now

L12.18 (6.6.2.18) $$S \subset \mathcal{I} = \{1,\dots,n\} \cup \bigcup_{k\in\mathbb{N}} \{1,\dots,2n\} \text{ finite}$$

corresponds to a 'decomposed' complex structure. The countable set on the right labels to the Fourier basis of W_0 and then $W_S \subset \mathcal{C}^\infty(\mathbb{T};\mathbb{C}^{2n})$ is defined by replaced the basis elemetns w_s with $s \in S$ by $\overline{w_s}$. This defines an element $J_S \in \mathcal{J}^\infty$. Then the affines are the

L12.19 (6.6.2.19) $$A_S = \{J \in \mathcal{J}^\infty; \Pi_S : W_J \longrightarrow W_S \text{ is an isomorphism.}\}$$

Since the map here is Fredholm of index zero this is equivalent to its being injective or surjective, as in the finite-dimensional case.

Just as in the finite-dimensional case we can associate a section of the dual bundle, Det', to each point in Det. Namely the latter is determined by an element of $G^\infty_{\mathbb{C}O}$. Fixing one element, A, the map

L12.21 (6.6.2.20) $$\alpha_A : G^\infty_{\mathbb{C}O} \ni O \longrightarrow \det_{W_0}(A^tO) \in \mathbb{C}$$

descends to a function on Det depending only on the image of A in Det and has the correct behaviour under the circle action on Det to define a holomorphic section of Det$'$. We are particularly interested in the sections corresponding to the decomposed elements, taking $A = O_S$.

If we decompose O as a 2×2 matrix with respect to $W_0 \oplus \overline{W}_0$,

L12.22 (6.6.2.21) $$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \alpha_0 = \det(a).$$

In particular, α_0 is non-zero over A_0 so trivializes Det and Det$'$ there.

So we concentrate on A_0 since $A_S = O_S \cdot A_0$ where O_s is the orthogonal operator as in the finite-dimensional case. An element of A_0 can be identified with the graph of W_J over $W_0 \oplus \overline{W}_0$ and hence with

L12.20 (6.6.2.22) $$\beta = \overline{\Pi}_0(\Pi_0\Pi_J)^{-1}{}_0 \longrightarrow \overline{W}_0,$$
$$A_0 \longrightarrow \{\beta : W_0 \longrightarrow \overline{W}_0;\ \text{smoothing and antisymmetric}\}.$$

In terms of (L12.22 6.6.2.21) this means that $\beta = ba^{-1}$. Now, just as in the finite-dimensional case

L12.24 (6.6.2.23) $$\alpha_S(O) = \det(\beta)\alpha_0(0),\ [O] \in A_0.$$

Restricted to A_0 we can express α_S in terms of β. Indeed using (L12.22 6.6.2.21)

L12.29 (6.6.2.24) $$\alpha_S(O) = \det_{W_0}(O_S O) = \det(\beta_S)\det(a).$$

So we are reduced to essentially the same proposition, that a holomoprhic section of Det$'$ is equivalent to a holomorphic function on A_0, f such that

L12.25 (6.6.2.25) $$f/\det(\beta_S) \text{ extends holomophically from } A_0 \cap A_S \text{ to } A_S.$$

I have ignored to issue of the precise meaning of holomorphy on such an infinite-dimensional space. All this means for a continuous function is weak holomorphy, meaning holomorphy on any finite-dimensional complex submanifold – or in this case subspace.

Thus the transition cocycle for Det$'$ is determined by the holomorphic functions $\det(\beta_S)$. These are finite-dimensional determinants of antisymmetric matrices of even rank. So, just as before we can define a new holomophic line bundle, for the moment only over the component of $\mathcal{J}^\infty$ containing J_0, the Pfaffian bundle, by replacing the transition cocycle by the

L12.26 (6.6.2.26) $$\mathrm{Pf}(\beta_S) \text{ on } A_0 \cap A_S.$$

This amounts to replacing the trivialized determinant bundle over A_0 by a square root – formally taking a square root of $\det(a)$ – and then a global holomorphic section of Pf$'$ is a holomorphic function $g : A_0 \longrightarrow \mathbb{C}$ such that

L12.27 (6.6.2.27) $$g/\mathrm{Pf}(\beta_S) \text{ extends to be holomorphic on } A_S.$$

Since this function is already defined on a dense subset of A_S this is actually equivalent to boundedness on A_S.

L12.28 PROPOSITION 17. *A holomorphic section of* Pf$'$ *which vanishes on each of the lines above the* J_S *vanishes identically.*

We proceed to show the existence of holomorphic sections associated to each point of A_0.

Now if $O \in G^{\infty}_{\mathbb{C}O}$ is decomposed as in (L12.22 6.6.2.21) then

$$O^t = \begin{pmatrix} d^t & c^t \\ b^t & a^t \end{pmatrix}$$

L12.31 Definition 2. Fock space is the exterior algebra of W_0, as a Hilbert space identified with the complex sequences in $l^2(\mathcal{I})$ written as sums

L12.32 (6.6.2.28) $$\sum_{S\in\mathcal{I}} a_S dw_S,\ dw_S = dw_{(k_1,i_1)} \wedge \ldots dw_{(k_l,i_l},\ S = \{(k_1,i_1),(k_l,i_l)\},\ l = |S|.$$

We will also need a Schwartz version.

L12.30 Theorem 6. *The spin representation is a unitary representation of* $\widehat{\mathcal{L}}\,\mathrm{Spin}(2n)$ *on Fock space.*

CHAPTER 7

The torus

Torus

The Dirac-Ramond operator on the $2n$-torus can be analysed rather directly. This is essentially an extension of the work of Spera and Wurzbacher, MR1957676 [**11**], who computed an L^2 version of the U(1)-equivariant index of the Dirac-Ramond operator on Euclidean space.

7.1. Loops in the torus

The torus has universal covering space $\mathbb{R}^n$

ALS.97 (7.7.1.1)
$$\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n.$$

As a Lie group the tangent bundle is trivial and can be identified with $\mathbb{T}^n \times \mathbb{R}^n$. Each (smooth) loop $l \in \mathcal{L}(\mathbb{T}^n)$, therefore has a multi-winding number

ALS.98 (7.7.1.2)
$$\omega(l) = \frac{1}{2\pi}\int_0^{2\pi} \frac{dl}{d\theta} \in \mathbb{Z}^n = \pi_1(\mathbb{T}^n)$$

which labels the components of the loop space.

ALS.102 PROPOSITION 18. *The components of the loop space of the torus*

ALS.103 (7.7.1.3)
$$\mathcal{L}(\mathbb{T}^n) = \bigsqcup_{\omega\in\mathbb{Z}^n} \mathcal{L}_\omega(\mathbb{T}^n)$$

fibre via the mean value $\mu : \mathcal{L}_\omega(\mathbb{T}^n) \longrightarrow \mathbb{T}^n$ *with fibre* $\mathcal{L}_0(\mathbb{R}^n)$*, the loops in* $\mathbb{R}^n$ *of mean value* 0.

PROOF. Each loop in $\mathbb{T}^n$ has a $\mathbb{Z}^n$ of lifts to a smooth map $[0, 2\pi] \longrightarrow \mathbb{R}^n$ so the space

ALS.99 (7.7.1.4)
$$\{\tilde{l} : [0, 2\pi] \longrightarrow \mathbb{R}^n; \tilde{l}(2\pi) - \tilde{l}(0) = 2\pi\omega \in \mathbb{Z}^n\}$$

is a $\mathbb{Z}^n$ cover of $\mathcal{L}_\omega(\mathbb{T}^n)$. Then each such $\tilde{l}$ has a unique extension to a smooth map $\hat{l} : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $\hat{l}(t + 2\pi) - \hat{l}(t) = 2\pi\omega(l)$.

For the component of winding number zero, consisting of the contractible loops in $\mathbb{T}^n$, the lifts are of the form $\hat{l}(t) + 2\pi\eta$, $\eta \in \mathbb{Z}^n$ and $\hat{l} \in \mathcal{L}(\mathbb{R}^n)$. A loop in $\mathbb{R}^n$ has a mean value

$$\mu(\hat{l}) = \frac{1}{2\pi}\int_0^{2\pi} \hat{l} \in \mathbb{R}^n.$$

We conclude that

ALS.100 (7.7.1.5)
$$\mathcal{L}_{\omega=0}(\mathbb{T}^n) = \mathbb{T}^n \oplus \mathcal{L}_0(\mathbb{R}^n),\ \mathcal{L}_0(\mathbb{R}^n) = \{\hat{l} \in \mathcal{L}(\mathbb{R}^n); \int_0^{2\pi} \hat{l}(\theta)d\theta = 0\},$$

being the Euclidean loops of mean value 0.

For non-zero winding number we choose the base loop which is the projection of $\theta \longmapsto \theta\omega$. Then a general a loop with this winding number has lifts of the form

ALS.101 (7.7.1.6) $$\hat{l}(\theta) = \theta\omega t + \check{l},\ \check{l} \in \mathcal{L}(\mathbb{R}^n).$$

Even though the mean value of the base path is not in general in the torus this does allow each component to be identified as

ALS.104 (7.7.1.7) $$\mathbb{T}^n \oplus \mathcal{L}_0(\mathbb{R}^n).$$

□

If we take the flat metric on $\mathbb{T}^n$ corresponding to the standard Euclidean metric on $\mathbb{R}^n$ then the orthormal frame bundle is naturally trivial

ALS.105 (7.7.1.8) $$\mathcal{F}(\mathbb{T}^n) = \mathbb{T}^n \times \mathrm{SO}(n)$$

and hence the loop frame bundle is a product

ALS.106 (7.7.1.9) $$\mathcal{L}(\mathcal{F}(\mathbb{T}^n)) = \mathcal{L}(\mathbb{T}^n) \times \mathcal{L}\,\mathrm{SO}(n)$$

and there is a product spin structure $\mathcal{F}_{\mathrm{Spin}}(\mathbb{T}^n) = \mathbb{T}^n \times \mathrm{Spin}(n)$.

The loop orientation corresponding to this product spin structure is therefore

ALS.109 (7.7.1.10) $$\mathcal{L}(\mathcal{F}_{\mathrm{Spin}}(\mathbb{T}^n)) = \mathcal{L}(\mathbb{T}^n) \times \mathcal{L}\,\mathrm{Spin}(n)$$

and there is always (for $n \geq 3$) a flat loop spin structure identified as

ALS.107 (7.7.1.11) $$\mathcal{L}(\mathbb{T}^n) \times \widehat{\mathcal{L}}\,\mathrm{Spin}(n)$$

for the basic central extension of $\mathcal{L}\,\mathrm{Spin}(n)$.

The associated loop spinor bundle is also naturally trivial for the Euclidean metric on $\mathbb{T}^{2n}$.

7.2. Complex structure on $\mathcal{L}\mathbb{R}^{2n}$

The tangent bundle to

$$\mathcal{L}\mathbb{T}^{2n} \simeq \bigsqcup_{\omega \in \mathbb{Z}^{2n}} \mathbb{T}^{2n} \times \mathcal{L}_0\mathbb{R}^{2n} \cdot \tilde{\omega}$$

is trivial with the fibre identified as

ALS.114 (7.7.2.1) $$T_l\mathcal{L}\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathcal{L}_0\mathbb{R}^{2n},\ \forall\ l \in \mathcal{L}\mathbb{R}^{2n}.$$

It follows that $\mathcal{L}\mathbb{T}^{2n}$ is a complex manifold with the assigment of the complex structure $\mathbb{R}^{2n} = \mathbb{C}^n$ and in terms of the Fourier expansions

ALS.115 (7.7.2.2) $$\mathcal{L}_0\mathbb{R}^{2n} \ni \tau = \sum_{j \in \mathbb{Z}\setminus\{0\}} \sum_{p=1,\dots,2n} c_{j,p} e^{ij\theta} e_p$$
$$J\tau = \sum_{j \in \mathbb{Z}\setminus\{0\}} \sum_{p=1,\dots,2n} \operatorname{sgn} j c_{-j,p} e^{ij\theta} e_p$$

where e_p is the standard basis of $\mathbb{R}^{2n}$.

Thus $z_k = x_k + i_{k+n}$ on $\mathbb{R}^{2n}$ and $\zeta_{j,p} = c_{j,p} + ic_{-j,p}$ are complex coordinates on $\mathcal{L}\mathbb{T}^{2n}$. Fock space, in the sense of the spin representation of $\mathcal{L}\,\mathrm{Spin}(2n)$, is the infinite tensor product of the four dimesnial spaces

ALS.116 (7.7.2.3) $$\Lambda\mathbb{C}_k \text{ and } \Lambda\mathbb{C}_{j,p},\ k = 1, \dots, n,\ p = 1, \dots, 2n, j \in \mathbb{N}.$$

Following Spera and Wurzbacher, the Dirac-Ramond operator decomposes as a (formal) infinite dirct sum of operators acting on the tensor factors in (ALS.116 7.7.2.3). Namely each of the factors $\mathbb{C}_{j,p}$

7.3. Sections of Fock spaces

In the even dimensional case where $\mathbb{T}^{2n}$ inherits the complex structure

ALS.110 (7.7.3.1)
$$\mathbb{T}^{2n} = \mathbb{C}^n / (\mathbb{Z}^n + i\mathbb{Z}^n)$$

Denote the complex coordinates as $z_j = x_j + iy_j$, $j = 1, \ldots, n$ and the real basis of the cotangent space as e_l, $l = 1, \ldots, 2n$.

ALS.108 (7.7.3.2)
$$\mathcal{L}(\mathbb{T}^{2n}) \times \mathrm{Fo}$$

whatever version of the fock space forming the spin representation Fo, is taken.

We consider Hilbert spaces obtained by completion of spaces of maps into the loop spin representation. The orthonormal basis of maps

ALS.74 (7.7.3.3)
$$\mathcal{C}^\infty(\mathbb{T}_+) \longrightarrow \mathcal{F}$$

is labelled by $\Sigma \times \mathcal{S}_0$ where, as in (ALS.63 2.2.7.6), Σ is the set of maps $\mathbb{N} \longrightarrow \mathbb{N}_0 \times \mathbb{N}_0$ and $\mathcal{S}$ is the collection of finite subsets of $\mathbb{N} \times \{1, \ldots, 2n\}$. Specifically

ALS.75 (7.7.3.4)
$$\Phi_{\sigma,\alpha} = \gamma_\sigma dz^{\alpha'} e^{\alpha''}$$

where γ_σ is given by (ALS.62 2.2.7.7).

The basic Hilbert spaces corresponds to l^2 of the set of labels

ALS.76 (7.7.3.5)
$$\mathcal{HF} = \left\{ \sum_{(\sigma,\alpha)} d_{\sigma,\alpha}\Phi_{\sigma,\alpha}; \sum_{(\sigma,\alpha)} |d_{\sigma,\alpha}|^2 < \infty \right\}.$$

We also consider Hilbert spaces given by weights

ALS.77 (7.7.3.6)
$$\mathcal{H}^s\mathcal{F}^N = \left\{ \sum_{(\sigma,\alpha)} d_{\sigma,\alpha}\Phi_{\sigma,\alpha}; \sum_{(\sigma,\alpha)} w(s,\sigma)^2 l(\alpha')^{2N} |d_{\sigma,\alpha}|^2 < \infty \right\}.$$

Questions:Dff$^+$-equivariance?
The $H^{\frac{1}{2}}$ (semi)norm and projections
The approach of Spera and Wurzbacher
Dirac-Ramond vs 'plain' Dirac
Null space

CHAPTER 8

Loop spin structures

The next objective is the construction of the (loop) spin bundle, and the corresponding spinor bundle, over the loop space of a *string* manifold. I will approach this from the loop end, starting with a compact spin manifold M of even dimension at least 4. What we know already is that over the loop space we have the looped spin frame bundle giving a principal bundle

L13.1 (8.8.0.1)
$$\begin{array}{ccc} \mathcal{L}\,\mathrm{Spin}(2n) & \longrightarrow & \mathcal{L}P_{\mathrm{Spin}} \\ & & \downarrow \\ & & \mathcal{L}M. \end{array}$$

For the moment think of the continuous loop spaces, but regularity is one of the things we need to sort out.

Last week I talk about the spin representation, which is a projective representation of $\mathcal{L}\,\mathrm{Spin}(2n)$ in the sense that it is a representation of the basic central extension $\widehat{\mathcal{L}}\,\mathrm{Spin}(2n)$. We know how to construct a vector bundle from a principal bundle and a representation of the structure group (and conversely) so what we need is a covering principal bundle giving a commutative diagram

L13.2 (8.8.0.2)
$$\begin{array}{ccc} \widehat{\mathcal{L}}\,\mathrm{Spin}(2n) & \longrightarrow & \mathcal{L}P_{\widehat{\mathrm{Spin}}} \\ \downarrow & & \downarrow \\ \mathcal{L}\,\mathrm{Spin}(2n) & \longrightarrow & \mathcal{L}P_{\mathrm{Spin}} \\ & & \downarrow \\ & & \mathcal{L}M. \end{array}$$

8.1. Brylinski-McLaughlin bigerbe

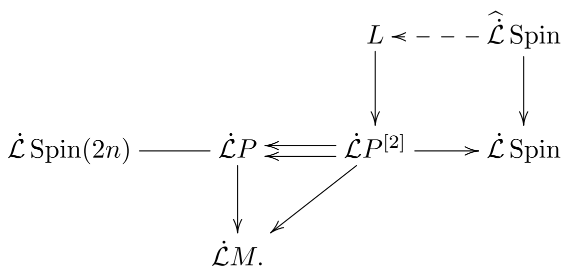
We also know that, at least at a topological level, the existence of such a 'loop-spin frame bundle' is equivalent to the triviality of the corresponding lifting bundle gerbe

L13.3 (8.8.1.1)
$$\begin{array}{ccccccc} & & & & L & \leftarrow - - - & \widehat{\mathcal{L}}\,\mathrm{Spin} \\ & & & & \downarrow & & \downarrow \\ \mathcal{L}\,\mathrm{Spin}(2n) & \longrightarrow & \mathcal{L}P_{\mathrm{Spin}} & \leftleftarrows & \mathcal{L}P^{[2]}_{\mathrm{Spin}} & \longrightarrow & \mathcal{L}\,\mathrm{Spin} \\ & & \downarrow & \swarrow & & & \\ & & \mathcal{L}M. & & & & \end{array}$$

This is the Brylinski-McLaughlin gerbe. Furthermore we know that this triviality is equivalent to the vanishing of the Dixmier-Douady invariant in $H^3(\mathcal{L}M;\mathbb{Z})$.

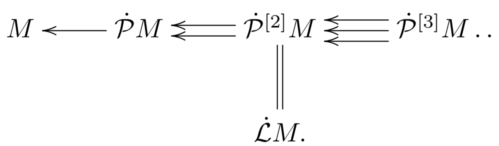
However, we do want this loop-spin bundle to 'come from M' and, as I have emphasized, that this should correspond to appropriate 'fusive' conditions. The first of these conditions is fusion itself. To simplify the initial discussion here I will drop down to pointed spaces, although ultimately we need the full spaces. So choose a base point in P_{Spin} and hence a base point in M given by its image. Then we consider pointed loops all round (with the Spin suffix dropped as well):

L13.4 (8.8.1.2)



The fusion condition should correspond to writing the loop space on M in the simplicial space formed from the fibre products of the path space with respect to the end-point map

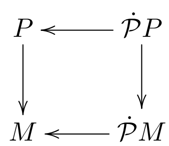
L13.5 (8.8.1.3)



It is rather clear that this should arise from fusion in $\dot{\mathcal{P}}P$.

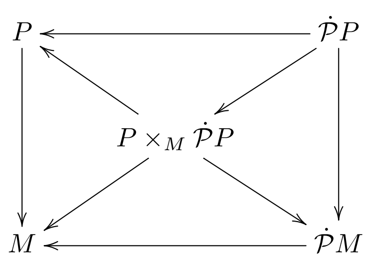
So consider the relation between the path spaces forming the commutative square

L13.6 (8.8.1.4)



Here all the maps are fibrations, the horizontal maps being to the endpoints. We can consider the fibre product of P with $\dot{\mathcal{P}}M$ over the map to M which gives the larger diagram

L13.7 (8.8.1.5)



In fact *all* these maps are fibrations where only for the map from $\dot{\mathcal{P}}P$ to $P\times_M\dot{\mathcal{P}}P$ is this not immediate – but is clear enough.

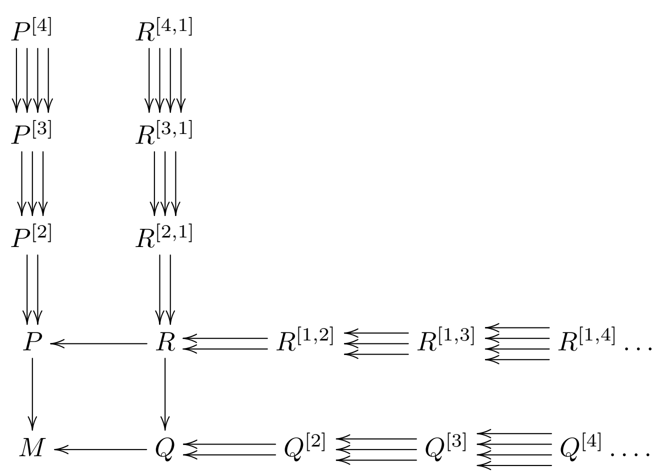
In [?] Chris Kottke under a more general condition than this call (8.8.1.4) a split square. Here I will say (8.8.1.4) is a fibred square if the maps in (8.8.1.5) are all fibrations – this is a stronger condition than just that the maps in (8.8.1.4) are fibrations.

To simultaneously simplify the notation and generalize, let's suppose we start with a fibred square

L13.8 (8.8.1.6)
$$\begin{array}{ccc} P & \longleftarrow & R \\ \downarrow & & \downarrow \\ M & \longleftarrow & Q. \end{array}$$

Using these four fibrations we can erect simplicial spaces

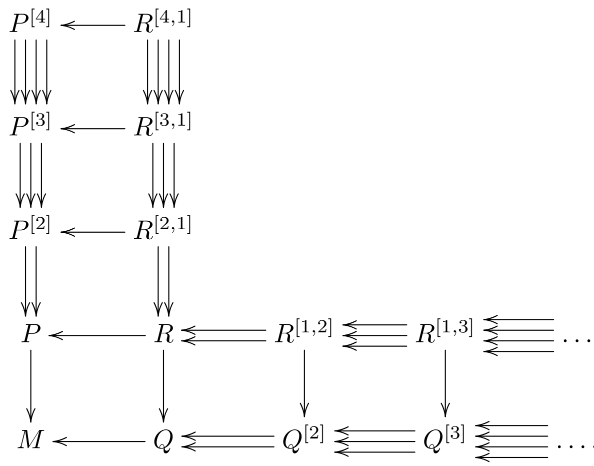
L13.9 (8.8.1.7)



Here the notation $R^{[k,1]}$ just indicates that the fibre produt is taken with respect to the 'first' map, being the map from R to Q and similarly $Q^{[1,k]}$ means the fibre products with respect to the map to P. This is justified as follows. A point in $R^{[k,1]}$ consists of a point in R^k, which is to say k points in R which must all map to a fixed point in Q (that is the '1'). It follows that they all map to the same point in M. This in turn means that the image in P^k of this point in R^k lies in $P^{[k]}$ – the k-fold fibre product of P over M. Thus we have well-defined maps from $R^{[k,1]}$ to $P^{[k]}$ for each k. These are all fibrations, as follows from (8.8.1.5). The same argument applies to the $R^{[1,k]}$ showing that they fibre over the $Q^{[k]}$.

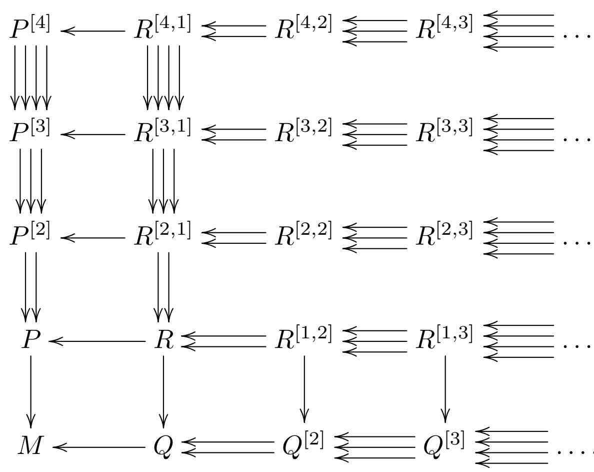
So now we have a diagram

L13.10 (8.8.1.8)



We can extend this to a planar diagram by expanding the rows, say, to simiplicial spaces.

L13.11 (8.8.1.9)



Here the space denoted $R^{[k,l]}$ is defined as follows. A point in $R^{[k,l]}$ is by definition a point in the l-fold fibre product of $R^{[k,1]}$ over $P^{[k]}$. Thus it consists of l points in $R^{[k,1]}$ mapping to the same point in $P^{[k]}$. Each of the points in $R^{[k,1]}$ consists of k points in R mapping to the same point in Q. Thus a point in $R^{[k,l]}$ can be thought of as a $k \times l$ array of points in R where each row is mapped to a particular point of P and each column is mapped to a particular point of Q, these two points being over the same point of M. Thus the space is essentially symmetric with respect to rows and colums and the erection of the vertical simplicial spaces in (8.8.1.8) constructs canonically isomorphic spaces.

So in fact we arrive at a full 'bisimplicial complex' which I shall prune a little

L13.12 (8.8.1.10)

$$
\begin{array}{ccccccccc}
P^{[4]} & \leftarrow & R^{[4,1]} & \leftleftarrows & R^{[4,2]} & & & & \\
\downarrow\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\downarrow & & & & \\
P^{[3]} & \leftarrow & R^{[3,1]} & \leftleftarrows & R^{[3,2]} & \Lleftarrow & R^{[3,3]} & & \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \\
P^{[2]} & \leftarrow & R^{[2,1]} & \leftleftarrows & R^{[2,2]} & \Lleftarrow & R^{[2,3]} & \Lleftarrow\!\!\leftarrow & R^{[2,4]} \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
P & \longleftarrow & R & \leftleftarrows & R^{[1,2]} & \Lleftarrow & R^{[1,3]} & \Lleftarrow\!\!\leftarrow & R^{[1,4]} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & \longleftarrow & Q & \leftleftarrows & Q^{[2]} & \Lleftarrow & Q^{[3]} & \Lleftarrow\!\!\leftarrow & Q^{[4]}
\end{array}
$$

Here the face maps – dropping a particular row and column – commute in the squares.

As discussed earlier there are simplicial differentials which act on circle bundles, sections of circle bundles or functions and these again commute to give a double complex. We write the horizontal simplicial differentials as δ_h and the vertical ones as δ_v.

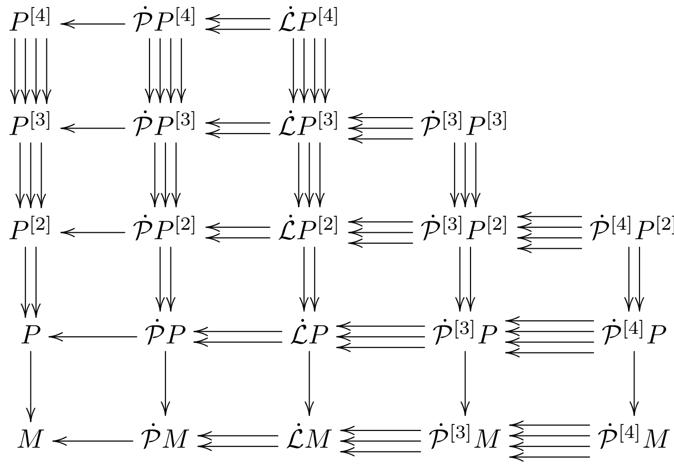
L13.13 Definition 3. A bigerbe for a fibred square (L13.8 8.8.1.6) is a doubly simplicial circle, or complex line, bundle, L, over $R^{[2,2]}$ in the sense that $\delta_h L$ has a global section s_h over $R^{[2,3]}$, $\delta_v L$ has a global section s_v over $R^{[3,2]}$ and these are compatible in the sense that $\delta_v s_h = \delta_h s_v$ over $R^{[3,3]}$ and simplicial in the sense that $\delta_h s_h$ and $\delta_v s_v$ are the canonical trivializations of $\delta_h^2 L$ and $\delta_v^2 L$ over $R^{[2,4]}$ and $R^{[4,2]}$.

The main result about bigerbes is that they 'represent' integral 4-cohomology classes over M. You can think about this as follows. The row starting at $P^{[2]}$, with the circle bundle over $R^{[2,2]}$ is a bundle gerbe, using the horizontal trivialization, so represents a Dixmier-Douady 3-class on $P^{[2]}$. The simplicial differential extends to cohomology and the compatible vertical trivialization of the line bundle means that the vertical simplicial differential of this 3-class vanishes on $P^{[3]}$, more correctly it is a simplicial class. This descends to a 4-class on M. It is important that there is a degree of symmetry here, the column through $R^{[2m2]}$ also reprsents a gerbe, fixing a three class on $Q^{[2]}$ which dedcends to the same 4-class on M with a change if sign. So there is quite a lot to show here.

We are primarily interested in the triviality of a bigerbe, the consequences of the vanishing of the 4-class. As for a bundle gerbe the vanishing of the obstruction class means that there are line bundle over $R^{[1,2]}$ and $R^{[2,1]}$ with simplicial images the given bundle over $R^{[2,2]}$ but now these bundles are simplicial in the other direction.

Before doing any of this, and we need more, I will proceed to show that this applies in the loop space setting. Consider the bisimplicial diagram (L13.12 8.8.1.10) for the fibred square (L13.6 8.8.1.4) underlying the Brylinski-McLaughlin gerbe (L13.3 8.8.1.1)

L13.14 (8.8.1.11)



L13.15 PROPOSITION 19. *The line bundle over $\dot{\mathcal{L}}P^{[2]}$ induced by the central extension of $\mathcal{L}\,\mathrm{Spin}(2n)$ by the shift map $\dot{\mathcal{L}}P^{[2]} \longrightarrow \mathcal{L}\,\mathrm{Spin}(2n)$ defines a bigerbe for the fibred square (L13.6 8.8.1.4).*

This is the Brylinski-McLaughlin bigerbe.

PROOF. (Incomplete) The stucture in the vertical direction is the lifting bundle gerbe, it is the horizontal behaviour we need to investigate. Elements of $\mathcal{P}^{[3]}P^{[2]}$ are pairs, corresponding to the last superscript, of triples of paths in P with the same endpoints, where the two triples project to the same triple of paths in M. So there are three maps to $\mathcal{L}\,\mathrm{Spin}(2n)$ arising from the shift maps from the three pairs of loops constructed by joining the three paths. Each of these pairs of paths lie over a path in M so define an element

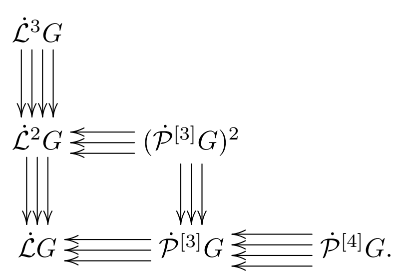
L13.16 (8.8.1.12)
$$\sigma \in \dot{\mathcal{P}}^{[3]}\,\mathrm{Spin}(2n)$$

where the fibre product is over the end-point map, i.e. the three pointed paths have the same end-point.

Thus the circle from $\delta_h L$ at this point in $\dot{\mathcal{P}}^{[3]}P^2$ is the alternating tensor product of the circles defined by $\widehat{\mathcal{L}}\,\mathrm{Spin}(2n)$ over the three elements in $\mathcal{L}\,\mathrm{Spin}(2n)$ formed from (L13.16 8.8.1.12).

The additional property of the central extension $\widehat{\mathcal{L}}$ of $\dot{\mathcal{L}}\,\mathrm{Spin}(2n) = \dot{\mathcal{P}}^{[2]}G$, $G = \mathrm{Spin}(2n)$, is that as a circle bundle it is bisimplicial with respect to the diagram

L13.17 (8.8.1.13)



I will discuss this below in the non-pointed case but it is a natural consequence of the fact that the Chern class, the curvature, is simplicial since it is the transgression of a three class on G. □

8.2. Loop spin bundle

I see that I have not gone into Čech theory far enough - I will fill in the whole in the notes and maybe talk about it later.

Recall from the end of last lecture the 'two-ended' version of the Brylinski-McLaughlin bigerbe for a compact spin manifold of dimension $2n \geq 4$:

L14.1 (8.8.2.1)

$$\begin{array}{ccccccccc}
(P^{[4]})^2 & \longleftarrow & \mathcal{P}P^{[4]} & \Longleftarrow & \mathcal{L}P^{[4]} & & & & \\
\downarrow\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow\downarrow & & & & \\
(P^{[3]})^2 & \longleftarrow & \mathcal{P}P^{[3]} & \Longleftarrow & \mathcal{L}P^{[3]} & \Lleftarrow & \mathcal{P}^{[3]}P^{[3]} & & \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \\
(P^{[2]})^2 & \longleftarrow & \mathcal{P}P^{[2]} & \Longleftarrow & \mathcal{L}P^{[2]} & \Lleftarrow & \mathcal{P}^{[3]}P^{[2]} & \Lleftarrow & \mathcal{P}^{[4]}P^{[2]} \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
P^2 & \longleftarrow & \mathcal{P}P & \Longleftarrow & \mathcal{L}P & \Lleftarrow & \mathcal{P}^{[3]}P & \Lleftarrow & \mathcal{P}^{[4]}P \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M^2 & \longleftarrow & \mathcal{P}M & \Longleftarrow & \mathcal{L}M & \Lleftarrow & \mathcal{P}^{[3]}M & \Lleftarrow & \mathcal{P}^{[4]}M.
\end{array}$$

The circle bundle defining the bigerbe is given by the pull-back of the central extension of $G = \mathrm{Spin}$ under the shift map

L14.2 (8.8.2.2)

$$\begin{array}{ccc}
L = \sigma^*\widehat{\mathcal{L}} & & \widehat{\mathcal{L}}G \\
\downarrow & & \downarrow \\
\mathcal{L}P^{[2]} & \xrightarrow{\sigma} & \mathcal{L}G
\end{array}$$

What we aim to construct is a circle bundle over

L14.3 (8.8.2.3)

$$\begin{array}{c}
S \\
\downarrow \\
\mathcal{L}P,
\end{array}$$

given that the obstruction we find vanishes. I will proceed to do this 'by hand' in an effort to get as much regularity as we can.

The path and loop spaces will all be taken to be finite energy, i.e. with H^1 regularity.

One thing we need to do is choose sections of various fibre bundles. Let me illustrate this with the end-point map $\mathcal{P}M \longrightarrow M^2$. As discussed earlier we choose to cover M by all the balls of some fixed small radius since this gives a good open cover. For each open set $B(m_1, \epsilon) \times B(m_2, \epsilon)$ of the product cover we choose a section

L14.15 (8.8.2.4)

$$B(m_1, \epsilon) \times B(m_2, \epsilon) \longrightarrow \mathcal{P}M.$$

It is helpful to make this section smooth. We proceed by choosing a smooth path $p \in \mathcal{P}M$ with endpoints (m_1, m_2). The pull back to p of TM is trivialized by parallel transport along p, identifying $\mathcal{C}^\infty(I; p^T M)$ with $\mathcal{C}^\infty(I; T_{p(0)})$, The the balls $B(m_i, \epsilon)$ are identified, using the exponential map with the ball around the origin in $B_1 \subset T_{m_1}M$ of radius ϵ. Thus $(a, b) \in B(m_1, \epsilon) \times B(m_2, \epsilon)$ is identified with $(v, w) \in B_1 \times B_1$. A smooth section (L14.15 8.8.2.4) is defined by the exponential map from points on p

L14.16 (8.8.2.5)
$$(a, b) \longmapsto q(t) = \exp_{p(t)}(u(t))$$

where $u(t) \in T_{p(t)}$ is the parallel translate of $1-t)v + tw)$ along p. This gives a curve with each pair of endpoints and hence a section

L14.17 (8.8.2.6)
$$q : B(m_1, \epsilon) \times B(m_2, \epsilon) \longrightarrow \Gamma(p, \epsilon).$$

We start at the path space in P fibering over $\mathcal{P}M$. By assumption M is a compact spin manifold, so in particular has a Riemannian metric. For each element of $\mathcal{P}M$ we can consider the tubular domain of some small radius ϵ which will be suppressed in the notation:

L14.4 (8.8.2.7)
$$\Gamma(\lambda) = \{\lambda' \in \mathcal{P}M; \sup_{t \in I} d_M(\lambda(t), \lambda'(t)) < \epsilon\}.$$

These open sets (balls in the supremum distance) certainly cover $\mathcal{P}M$ and they form a good open cover. We could easily drop down to a countable subcover but that would just complicate the notation.

Since $\mathcal{P}P \longrightarrow \mathcal{P}M$ is a fibre bundle we can consider the preimages of these open sets which form a covering (not a 'good' one) of $\mathcal{P}P$,

L14.5 (8.8.2.8)
$$\pi^{-1}(\Gamma(\lambda)) = \{l \in \mathcal{P}P; \pi l \in \Gamma(\lambda)\}.$$

Next we consider sections of $\mathcal{P}P$ over the $\Gamma(\lambda)$. We can do this smoothly. To do so we use the natural metric on P. Since P is the finite cover of P_{SO} it inherits the Levi-Civita connectiion. So at each point of P there is a natural horizonatal tangent space projecting onto the tangent space to M given by the connection. The fibre of P is a principal G space and this has a natural bi-invariant metric, unique if we fix the volume to be 1. This defines a fibre metric on P and this extends to a metric given as the sum of the vertical metric extended to vanish on the horizontal tangent space plus the metric on the base pulled back to P. This makes π into a Riemannian submersion. Geodesics on P which start horizontal stay horizontal and these project onto geodesics on M.

This bundle P is trivial over small balls and the metric determines smooth sections of over each of these once we choose an image $\phi_m(m) \in P_m$ of the central point. Namley each point in $B(m; \epsilon)$ is the end-point of a geodesic starting at m and then

L14.6 (8.8.2.9)
$$\phi_m : B(m; \epsilon) \longrightarrow P, \ \pi\phi_a = \mathrm{Id} \text{ on } B(m, \epsilon)$$

is defined by lifting these geodsics to horizontal geodesics starting at the chosen $\phi_m(m)$ and mapping to the end-points.

Now, for each path $\lambda \in \mathcal{P}M$ we choose a lift $\tilde{\lambda} \in \mathcal{P}P$ such that

L14.7 (8.8.2.10)
$$\tilde{\lambda}(0) = \phi_{\lambda(0)}, \ \tilde{\lambda}(1) = \phi_{\lambda(1)}$$

There does not appear to be an natural way to do this, so make an appropriately smooth choice.

We then extend the choice of $\tilde{\lambda}$ to a lifting map

L14.8 (8.8.2.11)
$$\begin{gathered}\Phi_\lambda : \Gamma(\lambda) \longrightarrow \Gamma(\tilde{\lambda}),\\ \Phi_\lambda(\lambda')(0) = \phi_{\lambda(0)}(\lambda'(0)),\ \Phi_\lambda(\lambda')(1) = \phi_{\lambda(0)}(\lambda'(1)).\end{gathered}$$

This lift of λ' is defined by recalling that in M

L14.9 (8.8.2.12)
$$\lambda'(t) = \exp_{\lambda(t)}(v(t)),\ v \in H^1(I; \lambda^* TM),\ |v(t)| < \epsilon..$$

Now the lift of λ' is defined from $\tilde{\lambda}$ using the same formula but with the exponential map on P and the horizontal lift of $v(t)$:

L14.10 (8.8.2.13)
$$\Phi_\lambda(\lambda')(t) = \exp_{\tilde{\lambda}(t)}(v_h(t)).$$

By construction this path in P lies in
$Gamma(\tilde{\lambda})$, projects to λ' and has end-points as required in (8.8.2.11). This fixes the section Φ_λ of $\mathcal{P}P$ over $\Gamma(\lambda)$ for each $\lambda \in \mathcal{P}M$.

Now, we use this section to define a further section

L14.11 (8.8.2.14)
$$\mu_\gamma : \mathcal{P}P \supset \pi^{-1}(\gamma(\lambda)) \longrightarrow \mathcal{P}P^{[2]},\ \mu_\gamma(\kappa)(t) = (\kappa(t); \Phi_\lambda(\pi\kappa)(t)).$$

for each $\lambda \in \mathcal{P}M$. This does map into curves in the fibre product since $\Phi_\lambda(\pi\kappa)$ covers $\pi\kappa$. Clearly μ_γ followed by the projection onto P in the first factor induces the identity map on $\pi^{-1}(\Gamma(\lambda))$.

Now we can use these maps to define a 'Čech' circle bundle over $\mathcal{L}P$. That is, we define a circle bundle over each of the sets $\pi^{-1}(\Gamma(l)) \subset \mathcal{L}P$, $l \in \mathcal{L}M$. To do so identify $\mathcal{L}P$ with $\mathcal{P}^{[2]}P$, by dividing each element it into two paths with the same endpoints. For $l \in \mathcal{L}P$ let me use the notation l_i $i = 1, 2$ for the component paths so $l = j(l_1, l_2)$. Thus $\pi^{-1}(\Gamma(l))$ is the join of $\pi^{-1}(\Gamma(l_1)) \subset \mathcal{P}P$ and $\pi^{-1}(\Gamma(l_2)) \subset \mathcal{P}P$ over the end-point maps – every element of $\pi^{-1}(\Gamma(l))$ is a unique joint of such paths. Using the maps Φ constructed above this gives a map

L14.12 (8.8.2.15)
$$\Psi_l : \pi^{-1}(\Gamma(l)) \longrightarrow \mathcal{L}P^{[2]},\ \Psi(l') = j(\Phi_{l_1}(l'_1), \phi_{l_2}(l'_2))$$

where the end-points match by construction.

In this way we obtain a circle bundle

L14.13 (8.8.2.16)
$$S_l = \Psi_l^* \widehat{\mathcal{L}} \text{ over } \pi^{-1}\Gamma(l) \subset \mathcal{L}P\ \forall\ l \in \mathcal{L}M.$$

The idea is to assemble these into a global circle bundle over $\mathcal{L}P$ but this is clearly obstructed. To glue two of the local bundles together we need a section of the homomorphism bundle which is

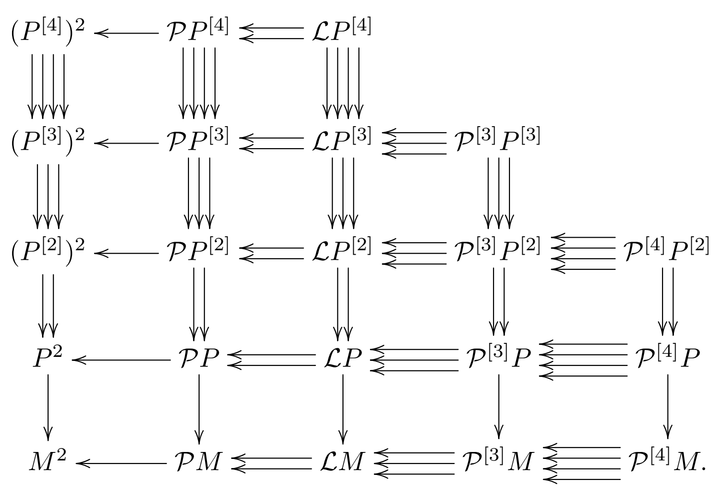
L14.14 (8.8.2.17)
$$S_{l_1} \otimes S_{l_2}^{-1} \text{ over } \pi^{-1}\Gamma(l_1) \cap \pi^{-1}\Gamma(l_2).$$

These two circle bundles are the pull-backs of the central extension under the maps σ_{l_1} and $\sigma_{l_2}^{-1}$. The group property of the central extension identifies this with the pull-back under the product $\sigma_{l_1}\sigma_{l_2}^{-1}$. In principal this is a map from $\pi^{-1}\Gamma(l_1) \cap \pi^{-1}\Gamma(l_2)$ to $\mathcal{L}G$ but in fact it factors through the projection to $\Gamma(l_1) \cap \Gamma(l_2)$ since it is just the shift map between the two sections of $\mathcal{L}P$ over these two tubular domains.

Thus the transition bundles (8.8.2.17) are actually pulled back from bundles over each of the intersections$\Gamma(l_1) \cap \Gamma(l_2)$ of the cover of $\mathcal{L}M$ where they are given as the pull-back of the central extension over $\mathcal{L}G$ under the transition map between the sections of $\mathcal{P}$ over each of the tubes $\Gamma(l_i)$. These sections are defined by parallel transport from an initial choice of the lift of the base loop l in terms of the exponential map on M and the metric on P and this section is simplicial in the sense that it comes from sections of the tubes in $\mathcal{P}M$. Each such intersection of tubes in $\mathcal{P}M$ maps to intersecting balls in M over the end-points. So for all pairs of intersecting balls we choose a point in the intersection. Then for each pair of intersecting tubes chooce a smooth path $i(p,p') \in \Gamma(p) \cap \Gamma(p')$, whenver this is non-empty, joining the chosen points in the images of the end-point map. The the exponential retraction in M retracts $\Gamma(p) \cap \Gamma(p')$ to $i(p,p')$. The shift map between the images of the sections of P chosen above over these tubes in the path space is deformed under this retraction to the shift map between the core paths p and p'. Now, for a loops in M the join construction, gives a similar retraction over intersections $\Gamma(l) \cap \Gamma(l')$ since the chosen central elements of the two tubes in the path space have the same end-points so join to a core loop onto which the intersection of tubes is retracted. The shift map over the intesection of tubes is therby retracted onto its value at the core loop. This gives a retraction of the shift map into $\mathcal{L}G$. Using the connection on $\widehat{\mathcal{L}G}$ as a circle bundle over $\mathcal{L}G$ this gives an equivriant retraction of the circle bundle restricted to $\Gamma(l) \cap \Gamma(l')$ to its fibre over the core loop.

8.3. Partitions of unity and litheness

L15.0 (8.8.3.1)



L15.1 **Proposition 20.** *In a (separable) metric space any open cover has a partition of unity subordinate to it.*

Proof. (M. Rudin MR236876 [**10**]) □

We want as smooth a partition of unity as we can get on the path and loop spaces.

Weak differentiability.

Derivatives of $|u|^2_{H^1}$.

Partitions of unity.

Čech cohomology

Smoothness of loop spin structures.

8.4. Local Dirac-Ramond

We are now in a position to give a 'local coordinate' form of the Dirac-Ramond operator. So, I will work at a given loop $l \in \mathcal{L}M$, assuming it to be smooth and M to be 'string' in the sense of the obstruction to the existence of the loop-spin structure vanishing. However, since this is local it does not really matter.

Over an appropriately small tubular neighbourhood $\Gamma(l)$ of l we 'constructed' the loop-spin bundle from the fibre product $(\mathcal{L}P)^{[2]}$ by choosing a section on the right. The identification of the tubular neighbourhood of l with a neighbourhood of the zero section in $H^1(\mathbb{T}; l^*TM)$ is made by parallel transport. The choice of a lift of l to $\lambda \in \mathcal{L}P$ reduces $\pi^{-1}\Gamma(l)$ to $\Gamma(l) \times \mathcal{L}G$ again by parallel transport, now in P. These sections are flat, over l, with respect to the Levi-Civita connection. Having fixed a lift of l to λ we can replace $\mathcal{L}G$ by $\widehat{\mathcal{L}G}$.

The spinor bundle is now given by the Fock space at each point above $\Gamma(l)$, where the section of P reduces $\mathcal{C}^\infty(\mathbb{T}, l^*TM)$ to $\mathcal{C}^\infty(\mathbb{T}; \mathbb{R}^{2n})$ and we take the base Hardy space W_0 at each point. Thus the Fock space is some space (I have yet to talk about smoothness, i.e. convergence) with elements

L16.11 (8.8.4.1)
$$u = \sum_{I,\alpha} a_{I,\alpha} e^I \otimes (dz)^\alpha.$$

The coefficients here are functions on $\Gamma(l)$.

In trying to define $\eth$ we can start with finite rank sections in (8.8.4.1). Note that the even α correspond to $\mathcal{F}^+$ and the odd to $\mathcal{F}^-$. The 'plain' Dirac-Ramond operator is supposed to be the composite of the action of the connection and Clifford multiplication. I have not yet discussed the connection, but it is supposed to be just a 1-D extension of the Levi-Civita connection. This means that the basis in (8.8.4.1) is flat at the base point. So in trying to define $\eth u(l)$ for any one term we see that

L16.12 (8.8.4.2)
$$\nabla u(l) = da_{I,\alpha} \otimes e^I \otimes (dz)^\alpha + a_{I,\alpha} \mu e^I \otimes (dz)^\alpha$$

is 'diagonal' since the second, connection, term only comes from the central part of the group and so must commute with the group action and hence be scalar.

So the 'plain' Dirac-Ramond operator reduces to the Clifford action on Fock space at the base point – both $da_{I,\alpha}$ and μ are 1-forms. So they are sections of some space of sections, let's say $\gamma \in H^1(\mathbb{T} : \mathbb{R}^{2n}$ at l since everything has been trivialized. The Clifford action is supposed to be

L16.13 (8.8.4.3)
$$(\Pi\gamma) \wedge \pm\iota((\mathrm{Id} - \Pi)\gamma), \ \gamma \in H^1(\mathbb{T} : \mathbb{R}^{2n})$$

in terms of the Hardy projection. Thus the first term is a sum $\sum_j c_j^+ e_l dz_j$ and the second part the conjugate of it.

This makes sense formally if we start off with finite sums and finish up with infinite ones. Remaining issues include convergence – what sorts of Fock spaces we are talking about – corresponding to regularity of the coefficients. The fact that this local operator is independent of choices is not so hard now but we still need to understand reparameterization.

The actual Dirac-Ramond operator is an equivariant version obtained by adding a non-differential term

ALS.52 (8.8.4.4)
$$\eth = \eth_\# + i \,\mathrm{cl}(\tau)$$

where τ is the tangent vector to l.

L16.14 CONJECTURE 1. *The (Spin) Dirac-Ramond operator is well defined on a compact string manifold and has null space which is a representation of the Bott-Virasoro group with finite-dimensional multiplicities.*

8.5. Reparametrization

A question I mentioned earlier is the following: What is a string? This is a serious question only in so far as it asks what we should think about. One standard response is 'A string is a loop up to reparametrization'. We know, more or less, that a loop is a map from the circle to the space of interest. So one intetpretation of a string is as an element of

L16.1 (8.8.5.1)
$$\mathcal{L}M/\operatorname{Dff}^+\mathbb{T}$$

and indeed this is a motivation for looking at$\operatorname{Dff}^+\mathbb{T}$-equivariance, the quotient (8.8.5.1) being fairly nasty (as is not unusual for group quotients).

If you think about the loops with non-trivial isotropy groups for the action of $\operatorname{Dff}^+\mathbb{T}$ the most obvious ones are the constant loops which are the fixed points. There are other problematic loops, such as those with a constant segment. This suggests considering the non-stationary loops. These form an open dense subset of the smooth loops; it might be appropriate to think of these, perhaps along with the constant loops as being 'really smooth'.

Let's concentrate first on paths rather than loops. The oriented diffeomorphism group of $\mathbb{T}$ or I consists of strictly increasing maps. The closure in an appropriate topology consists of the 'reparameterization semigroup' $\operatorname{Rep}^+ I \subset \operatorname{Dff}^+ I$ consisting of weakly increasing maps. These still act on $\mathcal{P}M$ by pull-back and the orbit space, the quotient, is not so horrible.

L16.2 PROPOSITION 21. *The quotient $\mathcal{P}_1M/\operatorname{Rep}_1^+ I$ of the finite-energy paths by the finite-energy reparameterization semigroup is identified (by a choice or Riemann metric on M) with the subspace of paths*

L16.3 (8.8.5.2)
$$\Sigma_1 M = \{p \in \mathcal{P}_1 M; |p'(t)| = c \ a.e. \ \int_I |p'(t)| df = L(p)\}.$$

So the constant paths are there, with $c = L(p) = 0$ and otherwise these paths are parameterized by a multiple of arclenght, so $c = 1/L(p)$.

PROOF. This is just the standard theory of arclength. A finite-energy curve is rectifiable with arclength defined by

L16.4 (8.8.5.3)
$$\frac{d\alpha}{dt} = |p'(t)|, \alpha(0) = 0.$$

Clearly the length of the tangent vector defines a function in $L^2(I)$ so the arclength is in $H^1(I)$ with $\alpha(1) = L(p)$. Since $L(\gamma = 0$ if and only if p is constant, after affine rescaling the non-constant paths are identified as the pull-backs

L16.5 (8.8.5.4)
$$p = \alpha^* q, \ q(s/L(p)) = p(t) \text{ if } s = \alpha(t).$$

Note that $q \in \mathcal{P}_1 M$ is a well-defined finite-energy path, in fact it is Lipschitz, satisfying (8.8.5.4) and hence

L16.6 (8.8.5.5)
$$q'(\alpha(t)/L(p)) = \frac{1}{L(p)} \frac{p'(\alpha(t'))}{|p'(\alpha(t')|}.$$

□

Of course even for smooth paths the affine-arclength-parameterized curve q may not be smooth; it is smooth if p is non-stationary. Thus $\Sigma_1 M$ is in a reasonable sense the space of unparameterized paths in M. In fact it is reasonably smooth away from the constant paths which represent a large 'conic' singularity.

L16.7 Proposition 22. *The space $\Sigma_1 M \subset \mathcal{P}_1 M$ is smooth away from the constant paths with tangent space at q the subspace*

L16.8
$$\{v \in H^1(I; q^*TM); v(t)\cdot q'(t) = 0 \ a.e.\}. \tag{8.8.5.6}$$

Proof. If $q \in \Sigma_1 M$ then the tubular neighbourhood of q in $\mathcal{L}_1 M$ consists of the paths

L16.9
$$t \longmapsto \exp_{p(t)}(w(t)),\ v \in H^1(I; p^*TM),\ |v(t)| < \epsilon. \tag{8.8.5.7}$$

The tangent vector field is therefore

L16.10
$$(\exp_{p(t)})_* \tag{8.8.5.8}$$

□

Recall that we are working under the assumption that G is a compact, simply connected and simple Lie group; the last condition is easily dropped and the former too but with more trouble.

We have seen that the central extensions of the loop group $\mathcal{L}G$ are classified by a bilinear form on the Lie algebra which is

L17.1
$$\int_{\mathbb{T}} \langle u', v\rangle ds,\ u,\ v \in \mathcal{L}\mathfrak{g}. \tag{8.8.5.9}$$

Here $\langle , \rangle$ is a binvariant Euclidean inner product on $\mathfrak{g}$. For a simple group there is only one of these up to a scalar multiple and then there is a minimal inner product for which (8.8.5.9) corresponds to a central extension of the loop group (as opposed to its Lie algebra) and the others are integer multiples.

L17.2 Proposition 23. *Any central extension*

L17.3
$$\mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}G} \longrightarrow \mathcal{L}G \tag{8.8.5.10}$$

of the loop group of a complact, simple, simply-connected Lie group is fusion in the sense that $\delta\widehat{\mathcal{L}G}$ has a natural section over $\mathcal{P}^{[3]}G$ with image the canonical section over $\mathcal{P}^{[4]}G$.

Proof. This is not just the multiplicativity of $\widehat{\mathcal{L}G}$. It is more related to the fact that (say with H^1 regularity) we have a natural inclusion

L17.4
$$\dot{\mathcal{L}}G \longrightarrow \mathcal{L}G,\ l \longmapsto \tilde{l} \tag{8.8.5.11}$$

where a pointed loop $l \in \dot{\mathcal{L}}G$ is mapped to

L17.5
$$\tilde{l}(t) = \begin{cases} l(2t) & 0 \le t \le \pi \\ \mathrm{Id} & \pi \le t \le 2\pi. \end{cases} \tag{8.8.5.12}$$

Thus we are joining the constant loop at Id to the loop.

Clearly this is an injective group homomorphism and the image is a normal subgroup. In fact the restriction of $\widehat{\mathcal{L}G}$ to this subgroup is 'the same' central extension since the bilinear form in (8.8.5.9) is scale-invariant.

What we note is that we can similarly glue another copy of $\dot{\mathcal{L}}G$ on the 'return' pathusing the join map

$$J : \dot{\mathcal{L}}G \times \dot{\mathcal{L}}G \ni (l_1, l_2) \longrightarrow j(l_1, l_2) \in \dot{\mathcal{L}}G \subset \operatorname{cl} G$$

L17.6 (8.8.5.13)
$$j(l_1, l_2)(t) = \begin{cases} l_1(2t) & 0 \le t \le \pi \\ l_2(2(2\pi - t)) & \pi \le t \le 2\pi \end{cases}$$

As usual for the join construction the second path is reversed and we see from (8.8.5.9) that this reverses the sign so as circle bundles

L17.7 (8.8.5.14)
$$R^* \widehat{\mathcal{L}}G = (\widehat{\mathcal{L}}G)^{-1},\ Rl(t) = t(2\pi - t).$$

It follows that the pull-back of the central extension under (8.8.5.13) is

L17.8 (8.8.5.15)
$$J^* \widehat{\mathcal{L}}G = \widehat{\mathcal{L}}G \boxtimes (\widehat{\mathcal{L}}G)^{-1}.$$

This is the 'external tensor product' of the circle bundle so is not trivial.

Now, finally we can consider fusion. From the multiplicativity of $\widehat{\mathcal{L}}G$ it follows that the simplicial differential

L17.11 (8.8.5.16)
$$\delta \widehat{\mathcal{L}}G \longrightarrow \mathcal{P}^{[3]}G$$

is a central extension.

The pointed loop group acts on each facto giving a principl bundle

L17.9 (8.8.5.17)
$$(\dot{\mathcal{L}}G)^3 \longrightarrow \mathcal{P}^{[3]}G \longrightarrow G^2.$$

Restricted to each fibre over G^2 the action of the simplicial differential on circle bundles is the tensor product of the three pull-backs in (8.8.5.13) corresponding to the three joins. Thus over each fibre the pull-back of the central extension is

L17.10 (8.8.5.18)
$$\widehat{\mathcal{L}}G_1 \boxtimes (\widehat{\mathcal{L}}G_2)^{-1} \boxtimes \widehat{\mathcal{L}}G_2 \boxtimes (\widehat{\mathcal{L}}G_3)^{-1} \boxtimes \widehat{\mathcal{L}}G_3 \boxtimes (\widehat{\mathcal{L}}G_1)^{-1} \equiv \mathrm{U}(1)$$

by the commuatativity of the tensor product.

It follows that $\delta \widehat{\mathcal{L}}G$ as a circle bundle over $\mathcal{P}^{[3]}G$ descends to G^2 where it is again a central extension since (8.8.5.17) is a normal subgroup on which the extension is trivial. As a simply connected Lie group G^2 any central extension is equivariantly trivial. This is the fusion trivialization of $\delta \widehat{\mathcal{L}}G$ over $\mathcal{P}^{[3]}G$. The naturality of the induced trivialization of the trivial central extension $\delta^2 \widehat{\mathcal{L}}G$ over $\mathcal{P}^{[4]}G$ follows directly. □

We can see from either the direct or the Toeplitz construction of the central extesion of $\mathcal{L}G$ that these circle bundles extend to central extensions of the semi-direct product

L17.12 (8.8.5.19)
$$\mathcal{L}G \rtimes \mathrm{Dff}^+(\mathbb{T})$$

where the diffeomorphism group acts by pull-back.

One of the problems we have with the construction of loop-spin structures is that this full group is not naturally fusion. The restriction to paths corresponds to the smaller group

L17.13 (8.8.5.20)
$$\mathcal{L}P \rtimes \mathrm{Dff}^+(I)$$

which (necessarily) keeps the end-points fixed. This means that the most fundamental invariance, the rotation action, does not survive fusion – there is tension between the simultaneous desire for both, but that is what we want.

How can we overcome this? Consider first the simple case corresponding to the transgression of circle bundles that I discussed earlier. If we start from a smooth circle bundle over M then the holonomy of a connection is a map

L17.14 (8.8.5.21)
$$\eta : \mathcal{L}M \longrightarrow \mathrm{U}(1)$$

If $l \in \mathcal{L}M$ and $\tilde{l} \in \mathcal{L}S$, where $S \longrightarrow M$ is the total space of the circle bundle, then

L17.15 (8.8.5.22)
$$\eta = \int_{\mathbb{T}} \tilde{l}^* \alpha.$$

It follows directly that η is fusive in the sense that it satisfies the fusion condition for functions and is $\mathrm{Dff}^+(\mathbb{T})$-invariant

L17.16 (8.8.5.23)
$$\delta\eta = 0 \text{ on } \mathcal{P}^{[3]}M,\ \phi^*\eta = \eta,\ \phi \in \mathrm{Dff}^+(\mathbb{T}).$$

We can immediately see from (8.8.5.22) that η is weakly smooth with first derivative

L17.17 (8.8.5.24)
$$d\eta = \int_{\mathbb{T}} d\alpha(v, \tau_l) \text{ at } l \in \mathcal{L}M,\ v \in H^1(\mathbb{T}; l^*TM)$$

where $\tau_l \in L^2(\mathbb{T}; l^*TM)$ is the tangent vector to l. Since α is smooth this shows litheness of η in the sense that (8.8.5.24) extends to a linear functional on $L^2(\mathbb{T}; TM)$ and if l has Sobolev regularity $s > 1$ then $d\eta \in H^{s-1}(\mathbb{T}; l^*TM)$.

Suppose conversely we have a function

L17.18 (8.8.5.25)
$$\mu : \mathcal{L}M \longrightarrow \mathrm{U}(1)$$

which is continuous (say in the H^1 topology0 and fusion, so

L17.19 (8.8.5.26)
$$\delta\mu = 1 \text{ on } \mathcal{P}^{[3]}M.$$

Then we can construct a circle bundle from it, but over M^2 by using μ as descent data from the trivial circle bundle $\mathcal{P} \times \mathrm{U}\,1$ over $\mathcal{P}M$. That is, we define the relation

L17.20 (8.8.5.27)
$$(p_1, z_1) \simeq (p_2, z_2) \text{ iff } p_1(0) = p_2(0),\ p_1(1) = p_2(1),\ z_2 = \mu(j(p_1, p_2) z_1.$$

This identifies all the circles over paths with the same end-points in M^2 and the fusion condiition is precisely transitivity making this an equivalence relation.

In general this circle bundle is not δS for a circle bundle over M but we know that the figure-of-eight condtion ensures this.

Suppose instead that μ is invariant under the action of $\mathrm{Dff}^+(I) \times \mathrm{Dff}^+(I)$, the subgroup leaving the end and mid-points of a loop fixed. This we might expect to get by a construction via the path space.

L17.21 LEMMA 13. *If μ is weakly continuously differentiable with derivative lithe in the sense that it lies in $H^t(\mathbb{T}; l^*TM \otimes \Omega)$ for some $t > 0$ at each point of $\mathcal{L}_E M$ and is invariant under $\mathrm{Dff}^+(I) \times \mathrm{Dff}^+(I)$ then it is invariant under $\mathrm{Dff}^+(\mathbb{T})$.*

PROOF. If ϕ_t is a $\mathcal{C}^1$ path in $\mathrm{Dff}^+(\mathbb{T})$ starting t the identity then $\phi_t^*\mu$ is differentiable in t and its derivative at $t = 0$ is

L17.22 (8.8.5.28)
$$d\mu(v\tau_l),\ v \in \mathcal{C}^\infty(\mathbb{T}) \text{ at } l \in \mathcal{L}M.\ \frac{d}{dt}\phi_t\big|_{t=0} = v\frac{d}{d\theta}$$

with $\tau_l \in L^2(\mathbb{T}; l^*TM)$ the tangent vector field along l. The subspace of smooth functions which vanish near any finite number of points is dense in any L^p topology for any $p < \infty$ and $H^t \subset L^{2+q}$ for appropriately small $q > 0$. Thus For the initial and mid-points the subspace vector fields generate elements of the subgroup $\mathrm{Dff}^+(I) \times \mathrm{Dff}^+(I)$ and so $d\mu(v\tau_l) = 0$ on the subspace and hence the closure. □

We need to do something like this to ensure that the circle bundle representing the loop-spin structure over $\mathcal{L}M$ has appropriate equivariance.

Next week I want to – finally – start to put all the pieces together to define the Dirac-Ramond operator. The idea is just to follow the usual prescription for the spin, or spin-$\mathbb{C}$, Dirac operator.

8.6. Lecture 22

At this stage I hope I have convinced you of the possibility of defining the Dirac-Ramond operator on a string manifold. This part can be generalized to other string bundles, I hope this will make it into the notes. My misgivings about the case of the torus last time were the result of a mild confusion (but see below) and I do believe that the results in that case are explicit.

8.7. Questions, questions, questions!

- How do the Bargmann-Fock spaces I introduced last time for the torus transform under $\mathrm{Dff}^+(\mathbb{T})$? They are clearly invariant under the rotation action. This is closely related to multiplication by $\mathcal{C}^\infty(\mathbb{T})$.
- How does the space $H^{\frac{1}{2}}(\mathbb{T}; M)$ behave? It is possible to define $H^s(\mathbb{T}; M)$ for $s \geq 0$ but for $s \leq \frac{1}{2}$ these contain non-continuous, and more significantly 'unbounded' loops. It seems (I have not tracked this down precisely) that $\mathcal{C}^\infty(\mathbb{T}; M)$ is not dense in $H^{\frac{1}{2}}(\mathbb{T}; M)$ unless $\pi_1(M)$ is trivial.
- As in the case of the torus the identity component of $\mathcal{L}(M)$ can be identified with $\mathcal{L}(\widetilde{M})/\pi_1(M)$ - this is where $H^{\frac{1}{2}}$ maps really are unbounded. Does this help with the analysis? Can we understand in a reasonable way the behaviour at infinity?
- Even in the simply-connected case the $H^{\frac{1}{2}}$ balls are hard to understand. Does the 1-dimensional version of Moser-Trudinger inequality play a role here - it means that the set where an $H^{\frac{1}{2}}$ function is large is exponentially small.
- Would the existence of a $-\frac{1}{2}$-lithe partition of unity help.
- All this is heading towards problem of transferring the Bargmann-Fock spaces to $\mathcal{L}(M)$ to act as domains for the Dirac-Ramond operator.
- Of course then one would like to use the analysis on the torus to prove semi-Fredholm properties.
- Even with the weak definition of Dirac-Ramond, can one show that the generator of the circle action is bounded below on the null space?
- Have I missed something relating to fusion and the Dirac-Ramond operator? What is the fusion proprty of the loop-spinor bundle corresponding to fusion for the loop-spin principal bundle?
- Can one come to grips analytically with the modular properties of the (formal) trace of the positive energy representations of Bott-Virasoro group?

CHAPTER 9

Musings

9.1. The sphere

Are things simpler on $\mathbb{S}^{2n}$? It is simply connected and covered by two coordinate patches. So, suppose we think of a measurable map $l : \mathbb{T} \longrightarrow \mathbb{S}^{2n}$ as a pair of measurable maps into the unit ball

(9.9.1.1) $$l_\pm : D_\pm \longrightarrow \mathbb{B}^{2n},\ \mathbb{T} = D_+ \sqcup D_- ALS.78$$

by restriction. Both will be in L^p for all p but not in general in L^∞. Set

ALS.79 (9.9.1.2) $$\delta(x,y) = d_{\mathbb{S}}(x,y) \text{ on } \mathbb{B}^{2n} \sqcup \mathbb{B}^{2n}.$$

The condition for l to be $H^{\frac{1}{2}}$

ALS.80 (9.9.1.3) $$\int_{\mathbb{T}\times\mathbb{T}} \frac{d(l(\theta_1), l(\theta_2))^2}{d(\theta_1,\theta_2)^2} < \infty$$

can be expressed in terms of the restrictions

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