18.199 Lecture on Čech Cohomology

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1 Introduction

In this lecture, we define the Čech Cohomology of a topological space X, and if time permitting, the relationship between Čech and with other types of cohomology. The content of this lecture is drawn from Chapter 2 of Bott and Lu [BL].

2 Presheaf on a topological space

Let X be a topological space.

Definition 2.1. A presheaf $\mathcal{F}$ on a topological space X is a function that assigns to each open set U of X an abelian group $\mathcal{F}(U)$ and to each inclusion of open subsets $i_{V\to U} : V \to U$ a group homomorphism

$$\mathcal{F}\left(i_{V\to U}\right) : \mathcal{F}(U) \to \mathcal{F}(V),$$

called a restriction, such that

1. $\mathcal{F}\left(i_{V\to V}\right)$ is the identity on $\mathcal{F}(V)$, and
2. $\mathcal{F}\left(i_{W\to V}\right)\mathcal{F}\left(i_{V\to U}\right) = \mathcal{F}\left(i_{W\to U}\right)$ if we have the inclusions $W \to V \to U$.

Example 2.1. Let $U \subset X$ be open, and denote by $\Omega^*(U)$ the space of differential forms on U. If for each inclusion of open sets $i_{V\to U} : V \to U$ we associate the usual restriction of differential forms $|_V = \Omega^*(i_{V\to U})$, then Ω^* is a presheaf on X.

Definition 2.2. $\mathcal{F}$ is a constant presheaf with group G if for each open set U, $\mathcal{F}(U) = G$ and for each restriction of open sets $i_{V\to U}$, $\mathcal{F}\left(i_{V\to U}\right)$ is the identity map on G.

Given two presheafs $\mathcal{F}$ and $\mathcal{G}$, a homomorphism between them is defined as:

Definition 2.3. A homomorphism $f : \mathcal{F} \to \mathcal{G}$ between two presheafs $\mathcal{F}$ and $\mathcal{G}$ on a topological space X is a collection of maps f_U, for $U \subset X$ open, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ {\scriptstyle \mathcal{F}(i_{V\to U})}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(i_{V\to U})} \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

3 Čech cohomology with values in a presheaf

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of the topological space X and $\mathcal{F}$ a presheaf on X.

For $q \geq 0$, define the following direct product of abelian groups

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0<\alpha_1<\cdots<\alpha_q} \mathcal{F}(U_{\alpha_0\cdots\alpha_q}) \tag{1}$$

where $\alpha_j \in I$ and $U_{\alpha_0\cdots\alpha_q} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$ is a non-empty intersection. Notice that $C^q(\mathcal{U}, \mathcal{F})$ is an abelian group with component-wise group operations.

Definition 3.1. A q-cochain is an element of $C^q(\mathcal{U}, \mathcal{F})$ (1).

Let $\omega \in C^q(\mathcal{U}, \mathcal{F})$ be a q-cochain. Below, we introduce a few notations to make the discussion going forward easy to follow.

1. We use $\omega_{\alpha_0\cdots\alpha_q}$ to represent the component of ω corresponding to the intersection $U_{\alpha_0\cdots\alpha_q}$, that is $\omega_{\alpha_0\cdots\alpha_q} \in \mathcal{F}(U_{\alpha_0\cdots\alpha_q})$
2. $\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_q$ means that the index α_j is omitted.
3. $\omega_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_q}|_j = \mathcal{F}\left(i_{U_{\alpha_0\cdots\alpha_q}\to U_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_q}}\right)\left(\omega_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_q}\right)$.

We define $\delta : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ which for $\omega \in C^q(\mathcal{U}, \mathcal{F})$ is given by

$$(\delta\omega)_{\alpha_0\cdots\alpha_{q+1}} = \sum_{i=0}^{q+1} (-1)^j \omega_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_{q+1}}|_j \tag{2}$$

Proposition 3.1. *The cochain*

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots$$

is a cochain complex, that is $\delta^2 = 0$.

Proof. Let $\omega \in C^q(\mathcal{U}, \mathcal{F})$. Then, we have that

$$\begin{aligned}(\delta^2\omega)_{\alpha_0\cdots\alpha_{q+2}} &= \sum_{j=0}^{q+2}(-1)^j(\delta\omega)_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_{q+2}}|_j \\ &= \sum_{j=0}^{q+2}\left(\sum_{l<j}(-1)^{j+l}\left(\omega_{\alpha_0\cdots\hat{\alpha}_l\cdots\hat{\alpha}_j\cdots\alpha_{q+2}}|_l\right)|_j\right. \\ &\left. + \sum_{j<l}(-1)^{j+l-1}\left(\omega_{\alpha_0\cdots\hat{\alpha}_j\cdots\hat{\alpha}_l\cdots\alpha_{q+2}}|_l\right)|_j\right) \\ &= 0\end{aligned}$$

The last equality is true because

$$\left(\omega_{\alpha_0\cdots\hat{\alpha}_j\cdots\hat{\alpha}_l\cdots\alpha_{q+2}}|_l\right)\Big|_j = \left(\omega_{\alpha_0\cdots\hat{\alpha}_l\cdots\hat{\alpha}_j\cdots\alpha_{q+2}}|_l\right)\Big|_j.$$

□

Define for $q \geq 1$ the quotient group

$$\check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F}) = \ker\left(\delta : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})\right)/\mathrm{Im}\left(\delta : C^{q-1}(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{F})\right) \tag{3}$$

where ker and Im refer to the kernel and image respectively.

Definition 3.2. $\check{\mathrm{H}}^*(\mathcal{U}, \mathcal{F})$, with $\check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F})$ given by (3), is called the Čech cohomology of the cover $\mathcal{U}$ with values in $\mathcal{F}$.

Eventually, we want to get to a definition of the Čech cohomology that is independent of the open cover. This leads us to discussing refinements.

Definition 3.3. An open cover $\mathcal{B} = \{V_\beta\}_{\beta\in J}$ is a refinement of $\mathcal{U}$ if there is a map

$$\phi : J \to I$$

such that $\forall\beta \in J$, $V_\beta \subset U_{\phi(\beta)}$

The refinement map ϕ induces a map

$$\phi^\# : C^q(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{B}, \mathcal{F})$$

which maps $\omega \in C^q(\mathcal{U}, \mathcal{F})$ to

$$(\phi^\#\omega)_{\beta_0\cdots\beta_q} = \mathcal{F}\left(i_{V_{\beta_0\cdots\beta_q}\to U_{\phi(\beta_0)\cdots\phi(\beta_q)}}\right)\left(\omega_{\phi(\beta_0)\cdots\phi(\beta_q)}\right) \tag{4}$$

Lemma 3.1. *The map $\phi^\#$ (4) is a chain map, that is it commutes with δ.*

Proof. The proof follows from the computation below. Let $\omega \in C^q(\mathcal{U}, \mathcal{F})$. By definition,

$$
\begin{aligned}
(\delta\phi^{\#}\omega)_{\beta_0\cdots\beta_{q+1}} &= \sum_{j=0}^{q+1}(-1)^j \; (\phi^{\#}\omega)_{\beta_0\cdots\hat{\beta}_j\cdots\beta_{q+1}}\Big|_j \\
&= \sum_{j=0}^{q+1}(-1)^j \mathcal{F}\left(i_{V_{\beta_0\cdots\beta_{q+1}}\to U_{\phi(\beta_0)\cdots\widehat{\phi(\beta_j)}\cdots\phi(\beta_q)}}\right)\left(\omega_{\phi(\beta_0)\cdots\widehat{\phi(\beta_j)}\cdots\phi(\beta_q)}\right)
\end{aligned}
$$

where the last equality follows from the definition of $\phi^{\#}$ (4) and the composition formula for restrictions $\mathcal{F}(i_{V\to U})$. Similarly, we have

$$
\begin{aligned}
(\phi^{\#}\delta\omega)_{\beta_0\cdots\beta_{q+1}} &= \mathcal{F}\left(i_{V_{\beta_0\cdots\beta_{q+1}}\to U_{\phi(\beta_0)\cdots\phi(\beta_{q+1})}}\right)\left((\delta\omega)_{\phi(\beta_0)\cdots\phi(\beta_{q+1})}\right) \\
&= \sum_{j=0}^{q+1}(-1)^j \mathcal{F}\left(i_{V_{\beta_0\cdots\beta_{q+1}}\to U_{\phi(\beta_0)\cdots\widehat{\phi(\beta_j)}\cdots\phi(\beta_q)}}\right)\left(\omega_{\phi(\beta_0)\cdots\widehat{\phi(\beta_j)}\cdots\phi(\beta_q)}\right)
\end{aligned}
$$

where the last equality follows from the definition of δ (2) and the composition of the restriction maps $\mathcal{F}(iV \to U)$. □

Lemma 3.1 tells us that

$$\phi^{\#} : \check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^q(\mathcal{B}, \mathcal{F})$$

is a map on the cohomology. The next lemma says that if we have another refinement map $\psi : J \to I$, then the induced maps $\phi^{\#}$ and $\psi^{\#}$ are the same map on the cohomology chain.

Lemma 3.2. *Suppose that $\mathcal{B} = \{V_\beta\}_{\beta\in J}$ is a refinement of the cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ with refinement maps $\phi, \psi : J \to I$. Then there exists a homotopy operator $K : C^q(\mathcal{U}, \mathcal{F}) \to C^{q-1}(\mathcal{B}, \mathcal{F})$, $q \geq 1$, between $\phi^{\#}$ and $\psi^{\#}$, that is*

$$\phi^{\#} - \psi^{\#} = \delta K + K\delta$$

Proof. Let $\omega \in C^q(\mathcal{U}, \mathcal{F})$ and define

$$
(K\omega)_{\beta_0\cdots\beta_{q-1}} = \sum_{j=0}^{q-1}(-1)^j \mathcal{F}\left(i_{V_{\beta_0\cdots\beta_{q-1}}\to U_{\phi(\beta_0)\cdots\phi(\beta_j)\psi(\beta_j)\cdots\psi(\beta_{q-1})}}\right)\left(\omega_{\phi(\beta_0)\cdots\phi(\beta_j)\psi(\beta_j)\cdots\psi(\beta_{q-1})}\right)
$$

A computation shows that $\delta K + K\delta = \psi^{\#} - \phi^{\#}$. □

Definition 3.4. A directed set A is a non-empty set with a reflexive and transitive binary operation $\leq$ such that for any two pairs $a, b \in A$, there exist $c \in A$ such that $a \leq c$ and $b \leq c$.

Notice that the set of open covers $\mathcal{U}$ is a directed set with respect to the binary operation of refinement, that is, $\mathcal{U} \leq \mathcal{B}$ means $\mathcal{B}$ is a refinement of $\mathcal{U}$.

Definition 3.5. A directed system of group is a collection of group $\{G_i\}_I$ indexed by a directed set such that for any pair $a \leq b$ in I there is a group homomorphism

$$f_b^a : G_a \to G_b$$

satisfying

1. f_a^a is the identity on G_a, and
2. $f_c^a = f_c^b \circ f_b^a$ for any $a \leq b \leq c$

Lemma 3.1 and 3.2 imply that the collection of groups $\{\check{\mathrm{H}}^*(\mathcal{U}, \mathcal{F})\}_{\mathcal{U}}$ is a directed system of groups under binary operation of refinement. The group homomorphisms f_b^a in the definition (3.5) are given by $\phi^{\#}$ (4), which is induced by a refinement map ϕ.

For a directed system of groups $\{G_i\}_{i \in I}$, we can define what is called a direct limit of a direct system. Let $\amalg G_i$ be the disjoint union of the G_i. Define an equivalence $\sim$ where for $g_a, g_b \in \amalg G_i$

$$g_a \sim g_b \iff f_c^a(g_a) = f_c^b(g_b) \tag{5}$$

for some upper bound c of a and b. With this equivalence relation, we define

Definition 3.6. The direct limit of a directed system of groups $\{G_i\}_{i \in I}$ is

$$\lim_{i \in I} G_i = \amalg G_i / \sim \tag{6}$$

where $\sim$ is defined is the equivalence relation defined by (5)

We make the direct limit $\lim_{i \in I} G_i$ a group by defining

$$[g_a] + [g_b] = [f_c^a(g_a) + f_c^b(g_b)] \tag{7}$$

where $[g_a]$ is the equivalence class of g_a and c some upper bound of a and b.

Now we are ready to define the Čech cohomology that is independent of the cover $\mathcal{U}$.

Definition 3.7. The Čech cohomology of a topological space X with values in a presheaf $\mathcal{F}$ is defined as

$$\check{\mathrm{H}}^*(X, \mathcal{F}) = \lim_{\mathcal{U}} \check{\mathrm{H}}^*(\mathcal{U}, \mathcal{F}) \tag{8}$$

where $\check{\mathrm{H}}^*(\mathcal{U}, \mathcal{F})$ is defined in 3.2.

4 Čech cohomology with coefficients in an abelian group

We start by defining a sheaf. Let X be a topological space. Suppose $U \subset X$ is open and $\{U_\alpha\}_{\alpha \in I}$ is an open cover for U with $U_\alpha \subset U$ for all α.

Definition 4.1. A sheaf $\mathcal{F}$ is a presheaf on X that safisfies two additional properties:

1. Locality: For $s, t \in \mathcal{F}(U)$,

$$\mathcal{F}(i_{U_\alpha \to U})(s) = \mathcal{F}(i_{U_\alpha \to U})(t) \;\; \forall \alpha \implies s = t$$

2. Gluing: If $\{s_\alpha \in \mathcal{F}(U_\alpha)\}_\alpha$ is such that $\mathcal{F}(i_{U_{\alpha_0\alpha_1} \to U_{\alpha_0}})(s_{\alpha_0}) = \mathcal{F}(i_{U_{\alpha_0\alpha_1} \to U_{\alpha_1}})(s_{\alpha_1})$ on the intersections $U_{\alpha_0\alpha_1} = U_{\alpha_0} \cap U_{\alpha_1}$, then there exist $s \in \mathcal{F}(U)$ such that $\mathcal{F}(i_{U_\alpha \to U})(s) = s_\alpha$.

Given an abelian group A, we set $\mathcal{F}_A$ to be the constant sheaf on X associated to A, that is, the sheaf of locally constant functions on X with values in A. Recall that a function f is locally constant at $x \in X$ if there exist a neighborhood U of x such that f is constant on U. f is locally constant on X if f if locally constant at every point $x \in X$.

Definition 4.2. $\check{\mathrm{H}}^*(X; A) = \check{\mathrm{H}}^*(X, \mathcal{F}_A)$ is the Čech Cohomology of X with coefficients in an abelian group A. $\mathcal{F}_A$ is the constant sheaf on X associated to A and $\check{\mathrm{H}}^*(X, \mathcal{F}_A)$ is specified in definition 3.7.

References

[BL] Raoul Bott, Loring W. Tu, *Differential forms in algebraic topology*, Springer (1982)