10. Also before $\eta(PQ) = \eta(P) + \eta(Q)$

For ϑ

$$\eta^{APS}(\vartheta) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \, \mathrm{Tr}\big(\vartheta \, \underbrace{e^{-t\vartheta^2}}_{\text{heat kernel}}\big) \, dt$$

Atiyah-Patodi-Singer. Converges at $t=\infty$ and at $t=0$ by 'local index theorem'.

Manipulation:

$$\eta^{APS}(\vartheta) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \mathrm{Tr}\big(\vartheta \, e^{-t\vartheta^2 - t\lambda^2}\big) \, dt \, d\lambda$$

$$\int_{-\infty}^\infty e^{-t\lambda^2} d\lambda = \pi^{1/2} t^{-1/2}$$

$$\text{`}=\text{'} \; \frac{1}{\pi} \int_{-\infty}^\infty \mathrm{Tr}\big((\vartheta + i\lambda)(\vartheta^2 + \lambda^2)^{-1}\big) \, d\lambda$$

$$\int_0^\infty e^{-tA} dt = A^{-1} \quad \text{odd/even}$$

$$\text{`}=\text{'} \; \frac{1}{\pi i} \int_{-\infty}^\infty \mathrm{Tr}\Big((\vartheta + i\lambda)^{-1} \frac{d}{d\lambda}(\vartheta + i\lambda)\Big) \, d\lambda$$

I lost a $\frac{1}{2}$.

$$= \eta(\vartheta + i\lambda)$$