4. $G^{-\infty}_{1,\mathrm{res}}(\mathcal{H})$ is a (variant of) a loop group

G. Segal

$$\Rightarrow \pi_j\left(G^{-\infty}_{1,\mathrm{res}}(\mathcal{H})\right) = \begin{cases} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd.} \end{cases}$$

In particular not connected.

0 Chern class — degree — index — winding number

$$c_0(P) = \deg(P) = \mathrm{ind}(P) = \mathrm{wind.}\#(P)$$

$$= \frac{1}{2\pi i} \mathrm{Tr}_{\mathcal{I}}\left(P^{-1} D_t P\right)$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}_{(\mathbb{R}^{2n})}\left(P^{-1}(t,\cdot)\, D_t P(t,\cdot)\right) dt.$$

$\in \mathbb{Z}$ labels components.

$$\text{Winding } \# = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{d}{dt} \log \underline{\det P(t,\cdot)}$$

Fredholm