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RICHARD MELROSE

ABSTRACT. The calculus of Heisenberg operators on a contact manifold is recalled and described in terms of parabolic compactification and blow up. The Hochschild homology is computed for its ideals and those of the related extended Heisenberg calculus. These computations are used to reduce the index formula for these algebras to a combination of the standard Atiyah-Singer theorem and the index theorem of Boutet de Monvel for Töplitz operators.

1. COMPACTIFICATIONS

Let V be a vector space, of finite dimension over $\mathbb{R}$ and let $L \subset V$ be a subspace. Choosing a complement W to L, $V = L \oplus W$, we can consider the homotheity

$$\gamma_t : V \to V\,, \gamma_t(l, \omega) = (t^2 l, t\omega)\,, t > 0\,, \tag{1}$$

in which the variables in L are 'parabolic'. The space of orbits in $V \setminus \{0\}$ is a sphere denoted ${}^L\mathbb{S}V$. The disjoint union, the *L-parabolic compactification of V,*

$${}^L\bar{V} = V \sqcup {}^L\mathbb{S}V \tag{2}$$

has a natural structure as a compact $\mathcal{C}^\infty$ manifold with boundary (a ball) where the $\mathcal{C}^\infty$ structure near the boundary (infinity) is generated by the smooth functions on $V \setminus \{0\}$ which are homogeneous of non-positive integral orders $(0, -1, -2$ suffice) under γ.

Lemma 1. *The Lie group of linear transformations of V preserving L lifts to act as $\mathcal{C}^\infty$ diffeomorphisms of ${}^L\bar{V}$ preserving the two submanifolds of the boundary*

$$\partial\bar{L} = {}^L\bar{L} \cap {}^L\mathbb{S}V \text{ and } Z = \bar{W} \cap {}^L\mathbb{S}V\,.$$

Here ${}^L\bar{L}$ and $\bar{W}$ are the closures of L and W in ${}^L\bar{V}$. As a result of this lemma ${}^L\bar{V}$ with its $\mathcal{C}^\infty$ structure is independent of the choice of W (as is Z — see Figure 1 for another complement W').

Example 1. The *radial compactification* of V corresponds to $L = \{0\}$; we write ${}^{\{0\}}\bar{V} = \bar{V}$.

Example 2. The case $L = V$ is

$${}^V\bar{V} = [\bar{V}; \partial\bar{V}\,, N^*\partial\bar{V}]$$

in terms of parabolic blow up of the boundary described in [4]. With these conventions, $\partial^V\bar{V} \equiv \partial\bar{V}$ and the notation above, ${}^L\bar{L}$ and $\bar{W} = {}^{\{0\}}\bar{W}$ is consistent.

We are most interested in the case $\dim L = 1$. Then the complement of $Z = \bar{W} \cap {}^{\mathrm{L}}\mathbb{S}V$ in the boundary, ${}^{\mathrm{L}}\mathbb{S}V$, is

$${}^{\mathrm{L}}\mathbb{S}V \setminus Z = U_+ \sqcup U_-$$

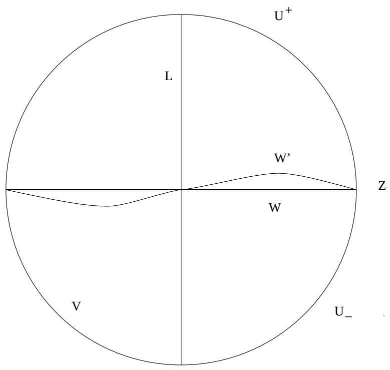

FIGURE 1. The L-parabolic compactification.

where $\pm$ correspond to the orientations of L.

Lemma 2. *In case* $\dim L = 1$, $U_\pm$ *have natural structures as vector spaces and there is a group of natural isomorphisms to* V/L,

$$\begin{gathered}\Phi_l : U_{\mathrm{sgn}(l)} \longrightarrow V/L\,,\ l \in L\setminus\{0\}\,,\\ \Phi_{l'}\circ\Phi_l^{-1} = t^{1/2}\cdot\mathrm{Id}\,,\ l' = tl\,,\ t>0\,.\end{gathered}$$

We may identify U_+ and U_-, and hence regard then as the same vector space U, using the isomorphism

$$M = (-\,\mathrm{Id})\circ\Phi_{-l}\circ\Phi_l^{-1} : U_+ \longleftrightarrow U_-$$

for any $l>0$. Then M is a diffeomorphism of ${}^{\mathrm{L}}\mathbb{S}V$ which fixes Z pointwise. The closure of U_+ in ${}^{\mathrm{L}}\mathbb{S}V$ is ${}^{\mathrm{L}}\mathbb{S}V\setminus U_-$ and this is canonically identified with its U_+-parabolic compactification. Thus ${}^{\mathrm{L}}\mathbb{S}V$ may be viewed as the sphere made from two copies of the U-parabolic compactification of U with the unique $\mathcal{C}^\infty$ structure making the identification map of the two copies into a diffeomorphism fixing the common boundary.

We also consider a larger compactification of V, the *extended L-parabolic compactification.* This is a manifold with corners and is obtained by the (radial) blow up of Z.

$${}^{\mathrm{eL}}\bar V = \left[{}^{L}\bar V; Z\right]\,.$$

Lemma 3. *There is a canonical isomorphism*

$${}^{\mathrm{eL}}\bar V = \left[{}^{L}\bar V; Z\right] \longleftrightarrow \left[\bar V; \partial\bar L\,, N^*\partial\bar V\right]$$

with the space obtained from $\bar V$ *by parabolic blow up, with respect to the boundary conormal direction, of* $\partial\bar L$. *The group of linear transformations of* V *preserving* L *lifts to a smooth action on* ${}^{\mathrm{eL}}\bar V$.

The smooth lifting of these linear transformations means that these notions can be extended directly to the case of a vector bundle V with a sub-bundle L. We use the same notation for the fibre-by-fibre compactifications; $\bar V$, ${}^{L}\bar V$ and ${}^{\mathrm{eL}}\bar V$. Ther

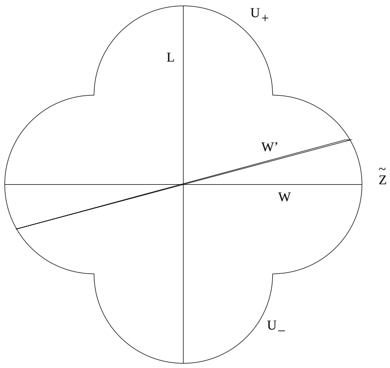

FIGURE 2. Extended L-parabolic compactification.

are natural smooth blow-down giving a commutative diagramme

$$\begin{tikzcd}
 & {}^{eL}\bar{V} \arrow[dl] \arrow[dr] & \\
{}^{L}\bar{V} & & \bar{V} \\
 & V \arrow[ul, hook'] \arrow[ur, hook] \arrow[uu, hook] &
\end{tikzcd}$$

consistent with the identifications over the interior.

2. Riemann-Weyl quantization

Let X be a compact Riemann manifold with metric g. Let $U \subset X^2$ be a small geodesically convex neighborhood of the diagonal in X^2; so if $(x, y) \in U$ there is a unique short geodesic from y to x lying in U. Denote the midpoint of the geodesic by $m(x, y)$ and by $\tau(x, y)$ the tangent vector at the midpoint with respect to arclength parameterization. Shrinking U if necessary,

$$M : U \ni (x, y) \longmapsto (m(x, y), \tau(x, y)) \in TX \tag{3}$$

is a diffeomorphism onto a neighborhood of the zero section.

Proposition 1. *If g_s is a smooth family of metrics on X and M_s are the corresponding diffeomorphisms (3) then the vector field V_s on TX, near the zero section, generating the diffeomorphisms $M_s \cdot M_0^{-1} = \exp(V_s)$ by integration is of the form*

$$V_s = \sum \tau_i \tau_j W_{ij}\,, \; \frac{d}{ds} M_s^* \varphi = M_s^*(V_s \varphi)\,,$$

where τ_i are (linear) functions vanishing at 0 in TX and the W_{ij} are smooth vector fields which are tangent to the zero section.

This proposition represents the well-known 'high invariance' properties of the Riemann-Weyl fibration of W given by (3).

The standard pseudodifferential operators on X can then be represented directly globally by choosing $\varphi \in \mathcal{C}^\infty(U)$ with $\varphi \equiv 1$ near the diagonal and considering the kernels obtained by 'Riemann-Weyl quantization'

$$\rho^{-k}\mathcal{C}^\infty(\bar{T}^*X) \ni a \longmapsto \\ A = (2\pi)^{-n}\varphi \int_{T^*_{m(x,y)}X} e^{i\tau(x,y)\cdot\xi} a(m(x,y),\xi)\, d\xi \cdot dg_R\,. \tag{4}$$

Here $d\xi$ is the Riemann density on the fibres and dg_R is the density on the right factor in X^2. The integral is best interpreted as a push-forward from the pull-back to U of T^*X.

Replacing the radial compactification by the two parabolic compactifications ${}^L\bar{T}^*X$ and ${}^{\mathrm{eL}}\bar{T}^*X$ gives the two classes of operators of interest here.

Definition 1. The spaces $\Psi^k_L(X)$ and $\Psi^{k,l}_{\mathrm{eL}}(X)$ of L-parabolic pseudodifferential operators and extended L-parabolic pseudodifferential operators are defined respectively as the sums of $\mathcal{C}^\infty(X^2;\Omega_R)$ and the ranges of the Riemann-Weyl quantization map (4) acting on $\rho^{-k}\mathcal{C}^\infty({}^L\bar{T}^*X)$ and $\rho_L^{-k}\rho_S^{-l}\mathcal{C}^\infty({}^{\mathrm{eL}}\bar{T}^*X)$.

Notice here that ${}^L\bar{T}^*X$ is a compact manifold with boundary, $\rho \in \mathcal{C}^\infty({}^L\bar{T}^*X)$ being a defining function for the boundary and ${}^{\mathrm{eL}}\bar{T}^*X$ is a manifold with corners, $\rho_L \in \mathcal{C}^\infty({}^{\mathrm{eL}}\bar{T}^*X)$ and $\rho_S \in \mathcal{C}^\infty({}^{eL}\bar{T}^*X)$ being defining functions for the two boundary hypersurfaces, the first 'arising from L' and the second, S arising from the second blow up in (1).

It is implicit in this definition that the class of kernels obtained is independent of the metric used in the definition. This follows for Proposition 1 and a deformation argument.

The space $\Psi^k_L(X)$ is the same as that considered by Taylor [6] and by Beals and Greiner in [1] (except that they take L to be a line bundle); they prove coordinate-invariance and composition properties; see also the earlier work of Dynin. Similar results hold for the extended spaces.

Proposition 2. *The spaces $\Psi^k_L(X)$ and $\Psi^{k,l}_{\mathrm{eL}}(X)$ form graded algebras of operators on $\mathcal{C}^\infty(X)$*

$$\begin{gathered}\Psi^k_L(X)\circ\Psi^{k'}_L(X) \subset \Psi^{k+k'}_L(X)\\ \Psi^{k,l}_{\mathrm{eL}}(X)\circ\Psi^{k',l'}_{\mathrm{eL}}(X) \subset \Psi^{k+k',l+l'}_{\mathrm{eL}}(X)\end{gathered} \tag{5}$$

and

$$\Psi^k_L(X) \subset \Psi^{k,k}_{\mathrm{eL}}(X)\,,\ \ \Psi^k(X) \subset \Psi^{2k,k}_{\mathrm{eL}}(X)\,. \tag{6}$$

It is straightforward to extend these definitions and properties to operators acting from sections of one vector bundle to another, simply by taking tensor products with bundle homomorphisms.

We are especially interested in the case that X is a contact manifold with contact line bundle L. Then we call $\Psi^{\mathbb{Z}}_L(X)$ and $\Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{eL}}(X)$ the Heisenberg and extended Heisenberg algebras respectively and denote them $\Psi^{\mathbb{Z}}_{\mathrm{H}}(X)$ and $\Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{eH}}(X)$. From this point on we restrict attention to these contact cases.

Taylor in [6] defines two spaces similar to $\Psi_{\mathrm{eH}}(X)$. These are respectively somewhat smaller and somewhat larger that the space considered here. Taylor's smaller space is an asymptotic completion of the sums of products of the Heisenberg and standard spaces. His larger space has non-polyhomogeneous symbols and is therefore less precise.

Example 3. One of the results shown by Taylor in [6] is that in the case of strictly pseudoconvex domains the operator $\Box_b^+$ on the boundary has inverse in the smaller of his calculi. See also the comments in section 4 below.

3. Symbols

For a manifold with corners, X, let $\dot{\mathcal{C}}^\infty(X) \subset \mathcal{C}^\infty(X)$ be the space of functions vanishing to infinite order at the boundary. The quotient

$$\mathcal{T}(X) = \mathcal{C}^\infty(X)/\dot{\mathcal{C}}^\infty(X)$$

is the algebra of Taylor series at the boundary. More generally if ρ_i, for $i = 1, \dots, N$, are defining functions for the boundary hypersurfaces then

$$\rho_1^{k_1} \cdots \rho_N^{k_N} \mathcal{T}(X) = \rho_1^{k_1} \cdots \rho_N^{k_N} \mathcal{C}^\infty(X)/\dot{\mathcal{C}}^\infty(X)$$

is the space of Laurent series with singularities of given order. We denote the union over orders by $\rho_1^{-\mathbb{Z}} \cdots \rho_N^{-\mathbb{Z}} \mathcal{T}(X)$.

Riemann-Weyl quantization (4) defines *full symbol maps*

$$\begin{aligned} \mathrm{FS} : \mathcal{A}^{\mathbb{Z}}(X) &= \Psi^{\mathbb{Z}}(X)/\Psi^{-\infty}(X) \longleftrightarrow \rho^{-\mathbb{Z}}\mathcal{T}(\bar{T}^X) \\ \mathrm{HFS} : \mathcal{A}^{\mathbb{Z}}_{\mathrm{H}}(X) &= \Psi^{\mathbb{Z}}_{\mathrm{H}}(X)/\Psi^{-\infty}(X) \longleftrightarrow \rho^{-\mathbb{Z}}\mathcal{T}({}^L\bar{T}^X) \\ \mathrm{eHFS} : \mathcal{A}^{\mathbb{Z}}_{\mathrm{eH}}(X) &= \Psi^{\mathbb{Z}}_{\mathrm{eH}}(X)/\Psi^{-\infty}(X) \longleftrightarrow \rho_{\mathrm{H}}^{-\mathbb{Z}}\rho_S^{-\mathbb{Z}}\mathcal{T}({}^{\mathrm{eL}}\bar{T}^X). \end{aligned} \tag{7}$$

Since $\Psi^{-\infty}(X)$ is an ideal in each algebra these quotients are algebras and Riemann-Weyl quantization, with inverse the appropriate full symbol map in (7), induces associative products on the Laurent series spaces.

In the standard case this is the '$\star$-product'

$$a \star b = \sum_{l=0}^{\infty} P_l(a,b) \tag{8}$$

where P_l is a bilinear differential operator on $\mathcal{T}(\bar{T}^*X)$ of the special form

$$P_l(a,b) = \rho^l \tilde{P}_l(a,b)$$

where $\tilde{P}_l$ is generated by vector fields tangent to to the boundary of $\bar{T}^*X$. The first two terms are

$$P_0(a,b) = ab,\ P_1(a,b) = \frac{1}{2i}\{a,b\} \tag{9}$$

in terms of the Poisson bracket. Thus the principal symbol, which is independent of the choice of metric,

$$\sigma_k : \Psi^k(X)/\Psi^{k-1}(X) \longleftrightarrow \mathcal{C}^\infty(\mathbb{S}^*X; R^{-k})$$

is multiplicative. Here R is the conormal bundle to the boundary $\mathbb{S}^*X$ of $\bar{T}^*X$.

There are similar principal symbol maps, independent of the choice of metric, for the Heisenberg algebras

$$\begin{aligned} {}_{\mathrm{H}}\sigma_k &: \Psi^k_{\mathrm{H}}(X)/\Psi^{k-1}_{\mathrm{H}}(X) \longleftrightarrow \mathcal{C}^\infty({}^{\mathrm{L}}\mathbb{S}^*X; R_{\mathrm{H}}^{-k}) \\ {}_{\mathrm{eH}}\sigma_k &: \Psi^k_{\mathrm{eH}}(X)/\Psi^{k-1,l-1}_{\mathrm{eH}}(X) \longleftrightarrow \mathcal{C}^\infty({}^{\mathrm{eL}}\mathbb{S}^*X; R_{\mathrm{eH}}^{-k,-l}). \end{aligned} \tag{10}$$

In the first case R_{H} is the conormal bundle to ${}^{\mathrm{L}}\mathbb{S}^*X$ as the boundary of ${}^L\bar{T}^*X$. In the second, ${}^{\mathrm{eL}}\mathbb{S}^*X$ is *not* a manifold. Rather it is the union of the two boundary hypersurfaces of ${}^{\mathrm{eH}}\bar{T}^*X$ with R_{eH} being the exterior product of the two conormal bundles; the notation $\mathcal{C}^\infty({}^{\mathrm{eL}}\mathbb{S}^*X; R_{\mathrm{eH}}^{-k,-l})$ includes the necessary compatibility condition at the common boundary hypersurface of the two components.

Let X_L be the double cover of X given by the orientation sheaf of L. Then ${}^{\mathrm{L}}\mathbb{S}^*X \setminus Z$ is a vector bundle, U, over X_L. The choice of a non-vanishing (local) section, α, of L reduces the bundle locally to T^*X/L over X_L and the differential of α induces a symplectic structure on the fibres of U; this structure is independent of the choice of α. Thus U is a symplectic vector bundle over X_L.

On any symplectic vector space U with symplectic form ω, there is a natural associative product on Schwartz' space $\mathcal{S}(U)$:

$$\begin{gathered}\#: \mathcal{S}(U) \times \mathcal{S}(U) \longrightarrow \mathcal{S}(U),\\ u\#v(\zeta) = \int e^{i\omega(\xi,\eta)} u(\zeta+\xi)v(\zeta+\eta)d\xi d\eta.\end{gathered} \tag{11}$$

As an algebra this is isomorphic to the composition product on $\mathcal{S}(\mathbb{R}^{2n})$ considered as the kernels of operators on $\mathcal{S}(\mathbb{R}^n)$, in such a way that the continuous extension gives the algebra of isotropic pseudodifferential operators on $\mathbb{R}^n$:

$$\rho^{-k}\mathcal{C}^\infty(\bar{W}) \equiv \Psi^k_{\mathrm{iso}}(\mathbb{R}^n).$$

Except that we have the parabolic compactification of U and we need to take into account the effect of order, this gives the product on the Heisenberg principal symbol; some further comments on the necessary normalizations are made below.

Proposition 3. *Let $\Psi^{\mathbb{Z}}_{\frac12 \mathrm{iso}}(U)$ be the bundle of istropic algebras, with symbols having half-integral step expansions, on the fibres of the bundle U over X_L then*

$$_{\mathrm{H}}\sigma_k : \Psi^k_{\mathrm{H}}(X)/\Psi^{k-1}_{\mathrm{H}}(X) \longrightarrow \Psi^{-2k}_{\frac12 \mathrm{iso}}(U)/\sim_\infty$$

is an isomorphism of algebras, where the equivalence relation $\sim_\infty$ requires that under the orientation isomorphism of U_+ and U_- the operator is invariant modulo $\Psi^{-\infty}_{\mathrm{iso}}$.

That is, the leading symbol product in the Heisenberg calculus is ${}_{\mathrm{H}}\sigma_{k+k'}(AA') = {}_{\mathrm{H}}\sigma_k(A)\#{}_{\mathrm{H}}\sigma_{k'}(A')$.

Thus the principal symbol composition for the Heisenberg algebra can be reduced to the global composition of isotropic pseudodifferential operators. The product for the extended Heisenberg algebra is essentially the same, except that the 'link' between the two parts of ${}^{\mathrm{L}}\mathbb{S}T^*X \setminus Z$ given by $\sim_\infty$ is broken. Namely the induced product

$$\mathcal{C}^\infty({}^{\mathrm{L}}\mathbb{S}^*X; R_{\mathrm{eH}}^{-k,-l}) \times \mathcal{C}^\infty({}^{\mathrm{L}}\mathbb{S}^*X; R_{\mathrm{eH}}^{-k',-l'}) \longrightarrow \mathcal{C}^\infty({}^{\mathrm{L}}\mathbb{S}^*X; R_{\mathrm{eH}}^{-k-k',-l-l'})$$

is given by $\#$, the isotropic product on the Heisenberg boundary hypersurface, and by pointwise product on the radial hypersurface S. The compatibility condition between these products is just the multiplicativity of the principal symbol in the isotropic calculus.

4. Kernels

The notion of parabolic blow up (or iterated radial blow up) already alluded to allows the kernels of Heisenberg, and extended Heisenberg, pseudodifferential operators to be reduced to polyhomogeneous conormal distributions.

For a contact manifold X the difference of the lift of the contact line bundle from the two factors defines a subbundle $\tilde{L} \subset N^* \operatorname{Diag}$ of the conormal bundle to the diagonal $\operatorname{Diag} \subset X^2$. The parabolic blow up of the diagonal with respect to this subbundle, as discussed in [4], gives a manifold with boundary

$$X^2_{\mathrm{H}} = [X^2; \operatorname{Diag}, \tilde{L}] \longrightarrow X^2.$$

The kernels of Heisenberg pseudodifferential operators of negative order are locally integrable functions (really right densities) on X^2, smooth away from the diagonal and lifting to polyhomogeneous conormal distributions, with respect to the boundary, on X^2_H. The kernels can be precisely characterized in terms of their expansions (with logarithmic terms occuring only for operators of integral order).

This description of the kernels is completely analogous to the description of pseudodifferential operators as singular integral operators in terms of their kernels. The two cases can be combined to give a similar description of the kernels of extended Heisenberg operators. The appropriate space can be obtained from X^2_H by the blow up of a submanifold, Z', of the boundary

$$X^2_{\mathrm{eH}} = [X^2_{\mathrm{H}}; Z'] \longrightarrow X^2_{\mathrm{H}} \longrightarrow X^2. \tag{12}$$

Here Z' is defined as the boundary of the lift of a submanifold through the diagonal with normal $\tilde{L}$.

Theorem 1. *The kernels of extended Heisenberg pseudodifferential operators with both orders negative lift to be polyhomogeneous conormal on* H^2_{eH}.

Once again the distributions which occur can be precisely characterized in terms of their orders and the coefficients of logarithmic terms.

5. The $\overline{\partial}$-Neumann problem

In joint work with Rafe Mazzeo this explicit description of the kernels of extended Heisenberg pseudodifferential operators is exploited to give a similarly explicit description of the kernel of the solution operator to the $\overline{\partial}$-Neumann problem of J.J. Kohn, on a strictly pseudoconvex domain. This is done by construction a class of operators which combines the features of the Poisson operator for an elliptic boundary problem with the parabolic operators discussed in [4]. This construction is quite analogous to the combination above of standard and Heisenberg pseudodifferential operators to give the extended Heisenberg operators. In particular appropriate composition theorems show the Poisson operator for the $\overline{\partial}$-Neumann operator to be in this class.

6. Boundedness, compactness and closure

The symbolic properties of the Heisenberg algebras allow Hörmander's elegant proof of L^2 boundedness to be invoked in this setting.

Proposition 4. *The elements of* $\Psi^{\mathbb{Z}}_{\mathrm{H}}(X)$ *and* $\Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{eH}}(X)$ *which extend to bounded operators on* $L^2(X)$ *are precisely those in* $\Psi^0_{\mathrm{H}}(X)$ *and* $\Psi^{0,0}_{\mathrm{eH}}(X)$ *respectively. The elements extending to be compact operators on* $L^2(X)$ *are those in* $\Psi^{-1}_{\mathrm{H}}(X)$ *and* $\Psi^{-1,-1}_{\mathrm{eH}}(X)$*; those giving Fredholm operators ar distinguished by the invertibility of their symbols.*

As usual these results extend directly to operators acting on sections of vector bundles.

The Heisenberg algebra has a natural ideal. In terms of the definition through Riemann-Weyl quantization it consists of the sums of elements of the smoothing ideal together with operators given by (4) where

$$a \in \rho^{-k}\mathcal{C}^\infty({}^L\bar{T}^*X),\ \rho^k a \equiv 0 \text{ at } Z,$$

i.e. the symbol vanishes to infinite order at Z. In fact this is easily understood in terms of the extended Heisenberg algebra since it is simply

$$\Psi_{\mathrm{eH}}^{\mathbb{Z},-\infty}(X) \subset \Psi_{\mathrm{eH}}^{\mathbb{Z},\mathbb{Z}}(X)$$

in which it is also an ideal.

Let $\mathcal{A}_{\mathrm{H}}(X)$, $\mathcal{J}_{\mathrm{H}}(X)$ and $\mathcal{K}(X)$ be the C^* algebras on $L^2(X)$ obtained by closure of $\Psi^0_{\mathrm{H}}(X)$, $\Psi_{\mathrm{eH}}^{0,-\infty}(X)$ and $\Psi^{-\infty}(X)$; similarly let $\mathcal{A}_{\mathrm{eH}}(X)$ be the closure of $\Psi^{0,0}_{\mathrm{eH}}(X)$. Thus $\mathcal{K}(X)$ is just the Kalkin algebra.

Proposition 5. *There are short exact sequences of algebras*

$$\mathcal{K}(X) \longrightarrow \mathcal{J}_{\mathrm{H}}(X) \longrightarrow \mathcal{C}^0(X;\mathcal{K}(\mathbb{R}^n)) \text{ and } \mathcal{J}_{\mathrm{H}}(X) \longrightarrow \mathcal{A}_{\mathrm{H}}(X) \longrightarrow \mathcal{C}^0(\mathbb{S}^*H)$$

where $H = T^*X/L$ is the contact hyperplane bundle.

These ideal decompositions can be used to compute the K-theory of the completed algebras.

7. Hochschild homology

The (continuous) Hochschild homology of an algebra such as $\mathcal{A} = \Psi^{\mathbb{Z}}_{\mathrm{H}}(X)$ is determined by the Hochschild differential acting on the completed tensor products

$$b_k : \mathcal{A}^{\hat{\otimes}(k+1)} \longrightarrow \mathcal{A}^{\hat{\otimes}k},\ \mathrm{HH}_k(\mathcal{A}) = \frac{\operatorname{null}(b_k)}{b_{k+1}\mathcal{A}^{\hat{\otimes}(k+2)}}.$$

On $\mathcal{A}\hat{\otimes}\mathcal{A}$, b_1 gives the commutator and on higher chain spaces gives 'higher commutators'. The dual Hochschild cohomology therefore represents higher traces. In particular $\mathrm{HH}^0(\mathcal{A})$ is the space of trace functionals on $\mathcal{A}$.

For the algebra of smoothing operators $\mathrm{HH}^0(\Psi^{-\infty}(X)) = \mathbb{C}$ with all other cohomology vanishing (it is homologous to a matrix algebra). The same is true for the residual calculus of the isotropic algebra on any vector space. The full symbol algebra of the isotropic algebra has Hochschild cohomology

$$\mathrm{HH}^k(\Psi^{\mathbb{Z}}_{\mathrm{iso}}(V)/\Psi^{-\infty}_{\mathrm{iso}}(V)) \simeq H^{2n-k}(\mathbb{S}(V\times V')\times\mathbb{S}^1). \tag{13}$$

Here $\mathbb{S}(V\times V')$ is the sphere of the cotangent space $V\times V'$, $\mathbb{S}^1$ is the circle and $n=\dim V$. This identification can be seen from the work of Brylinski and Getzler [2].

Using these ideas the homology of subquotients can be identified

$$\mathrm{HH}^k(\Psi^{-\infty,\mathbb{Z}}_{\mathrm{eH}}(X)/\Psi^{-\infty}(X)) \simeq H^{2n-k}(\mathbb{S}L\times\mathbb{S}^1) \tag{14}$$

where now $\dim X = 2n-1$ is a contact manifold with contact bundle L of which $\mathbb{S}L$ is the associated 'circle' (i.e. orientation) bundle. Similarly

$$\mathrm{HH}^k(\Psi^{-\mathbb{Z},\infty}_{\mathrm{eH}}(X)/\Psi^{-\infty}(X)) \simeq H^{2n-k}(\mathbb{S}T^*X\times\mathbb{S}^1;\partial L\times\mathbb{S}^1) \tag{15}$$

is the cohomology relative to the boundary of $\bar{L}\subset\mathbb{S}T^*X$.

8. Index formulæ

Using the properties of the calculus and these homological computations the index of extended Heisenberg operators, acting on bundles, can be computed. There are two constituent index theorems.

First, the Atiyah-Singer index theorem gives the index for standard pseudodifferential operators. Secondly, for elements of the Heisenberg calculus of the type $\mathrm{Id} + A$ where $A \in \Psi_{\mathrm{H}}^{-\infty}(X); E)$ and the non-commutative symbol of $\mathrm{Id} + A$ is invertible at each point, the index can be deduced from the index formula of Boutet de Monvel [3]. The combination of these to formulæ to give a general index formula uses a deformation argument which allows the operator to be retracted into the standard calculus, modulo a term for which the index can be computed using the Toeplitz index theorem of Boutet de Monvel. This deformation is possible because the ellipticity of the operator, in the Heisenberg sense, implies the vanishing of its index as a family of isotropic operators – see also the family index theorem of Fedosov; see [5].

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Department of Mathematics, MIT
E-mail address: `rbm@math.mit.edu`