GEOMETRIC OPTICS AND THE BOTTOM OF THE SPECTRUM

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ABSTRACT. The continuous spectrum, near its infimum, of the Laplacian for certain classes of 'doubly-warped' complete Riemannian metrics on the interiors of compact manifolds with boundary is examined using geometric optics. General conditions on the geometry of an algebra of vector fields leading to a calculus of pseudodifferential operators are described and, once verified in this particular case, the associated notion of wavefront set can be used to show the existence of an absolute scattering matrix. This operator is conjectured to be a Fourier integral operator; the confirmation of this conjecture is discussed in the special case of a 'double-zero' metric where the fibrations are trivial.

INTRODUCTION

In this note the structure of the bottom part, which is to say near 0, of the spectrum of the Laplacian (on functions) is discussed, for a complete metric on the interior of a compact manifold with boundary exhibiting a 'doubly-warped' structure near the boundary. Namely if X is the compact manifold with boundary and ∂X is its boundary, then it is assumed that there is a two-level fibration of the boundary

$$\partial X \xrightarrow{\phi_1} Y_1 \xrightarrow{\phi_2} Y_2,\ \psi = \phi_2 \circ \phi_1.$$

Here both $\phi_1 : \partial X \longrightarrow Y_1$ and $\phi_2 : Y_1 \longrightarrow Y_2$ are fibrations of compact manifolds. By taking a product decomposition of X near its boundary these fibrations can be extended to a neighbourhood of the boundary; denote such a product extension by

$$[0,\delta) \times \partial X \xrightarrow{\tilde\phi_1} [0,\delta) \times Y_1 \xrightarrow{\tilde\phi_2} [0,\delta) \times Y_2,\ \tilde\psi = \tilde\phi_2 \circ \tilde\phi_1.$$

Consider a metric on the interior of X which, for some such choice of extensions takes the form

$$g = \frac{\tilde\psi^* \tilde h_2}{x^4} + \frac{\tilde\phi_1^* \tilde h_1}{x^2} + \tilde h_0, \text{ near } \partial X. \tag{1}$$

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Here, $\tilde{h}_2$, $\tilde{h}_1$ and $\tilde{h}_0$ are, respectively, metrics on $[0,\delta)\times Y_2$, $[0,\delta)\times Y_1$ and X and it is also assumed that, near ∂X, $x = \tilde{\psi}^*\tilde{x}$ where $\tilde{x}$ is the distance from the boundary in the $\tilde{h}_2$ metric. In fact it is enough that $\tilde{h}_1$ and $\tilde{h}_0$ be smooth symmetric 2-cotensors which restrict to metrics on the fibres of ϕ_2 and on the fibres ψ respectively. Under these assumptions near the boundary g, further assumed to be a Riemannian metric on the interior of X, is necessarily complete.

The Laplacian associated to g, Δ, is an essentially self adjoint operator on the domain $\dot{\mathcal{C}}^\infty(X) \subset \mathcal{C}^\infty(X)$ consisting of the smooth functions vanishing to all orders at the boundary. The general structure of the spectrum of such an operator can be described as follows.

Theorem 1. *For any metric of the type* (1)*, with* $\dim Y_1 + \dim Y_2 > 0$*, the essential spectrum of the Laplacian on functions is absolutely continuous of infinite multiplicity on* $[0,\infty)$*.*

In this theorem I do not exclude the possibility of embedded discrete spectrum. In all likelihood it does not occur at all if the fibres of ϕ_1 have dimension zero and in general there should be no embedded spectrum near 0 assuming $\dim Y_1 + \dim Y_2 > 0$. If it occurs, the eigenfunctions corresponding to such discrete spectrum are in the space $\dot{\mathcal{C}}^\infty(X)$.

Although this gives a reasonably full description of the spectrum of Δ, a more detailed description of the generalized eigenfunctions is possible through the approach of 'geometric' scattering theory. This can be described in a variety of ways. One rather transparent approach is through a generalized (or degenerate) boundary problem.

Theorem 2. *For a metric of type* (1)*, if* $\lambda \in \mathbb{R}$, $0 < \lambda^2 < A^2$*, where* A^2 *is the infimum of the positive eigenvalues of the Laplacian on the fibres of* ϕ_1*, then for each* $f \in \mathcal{C}^\infty(Y_1)$ *there is a unique generalized eigenfunction with eigenvalue* λ^2 *of the form*

(2)
$$u = P_\lambda(f) = e^{i\lambda/x}x^q u_+ + e^{-i\lambda/x}x^q u_-,\ u_\pm \in \mathcal{C}^\infty(X),\ u_+ \restriction \partial X = \phi_1^* f,$$
$$q = \frac{1}{2}\dim Y_1 + \dim Y_2,$$

orthogonal in L^2 *to any eigenfunctions in* $\dot{\mathcal{C}}^\infty(X)$ *with eigenvalue* λ^2*.*

The proof of this result is outlined below, with particular emphasis on the newer aspects.

The map whose existence is asserted by this Proposition is the *Poisson operator* for the generalized boundary value problem at 'wave number' λ. Since the Laplacian is real, the uniqueness part of this result shows that

$$P_{-\lambda} f = \overline{P_\lambda \overline{f}}.$$

For this reason it is enough to suppose throughout that $\lambda > 0$. Furthermore the *absolute scattering operator* can be defined by

$$A_\lambda : \mathcal{C}^\infty(Y_1) \longrightarrow \mathcal{C}^\infty(Y_1),\ A_\lambda f = f_-,\ u_- \restriction \partial X = \phi_1^* f_-,$$

where $u_\pm$ are as in (2); this operator is necessarily invertible. Its microlocal properties are described here in some cases and the structure in the general case is conjectured.

Let me give (rather *ad hoc*) nomenclature for some of the special cases of the metrics in (1) and hence discuss some results previously established.

First, in case ϕ_1 is the identity and Y_2 is a point, so $\tilde{h}_2 = d\tilde{x}^2$ and $\tilde{h}_0$ is absent, we arrive at the notion of a scattering metric considered in [8] and [10].

Theorem 3. [10] *In the case of a scattering metric (ϕ_1 has trivial fibres, ϕ_2 has one fibre) A_λ is a Fourier integral operator associated with geodesic flow at time π for the induced metric on the boundary.*

The case in which $\dim Y_2 = 0$ but ϕ_1 can have non-trivial fibres is described in [8] and in more detail in [3] where such metrics are called 'fibred cusp' metrics. The same result is shown there in this case:

Theorem 4. [3] *For a fibred cusp metric the scattering operator, for $0 < \lambda^2 < A^2$, is a Fourier integral operator on $\mathcal{C}^\infty(Y_1)$ associated with geodesic flow at time π, for the metric induced by $\tilde{h}_1$ on Y_1.*

In the special case where ϕ_1 and ϕ_2 are isomorphisms, so it can be assumed that $Y_2 = \partial X$, the metric can be described as a 'double-zero' metric (because the associated Lie algebra of vector fields, discussed below, consists of the vector fields which vanish to second order at the boundary). This is the main new case discussed here. The integration of this case with the 'fibred cusp' case can be expected to allow the full analysis of the general doubly-warped metric in (1).

In the case of a double-zero metric the construction of a parametrix for the Poisson operator is outlined here. Carried to its proper conclusion this yields

Theorem 5. *If h is a non-singular Riemann metric on a compact manifold with boundary X and $x \in \mathcal{C}^\infty(X)$ is equal to the distance from the boundary up to a quadratic term (i.e. modulo $x^2\mathcal{C}^\infty(X)$ near the boundary) then the absolute scattering operator defined in (2) is a pseudodifferential operator.*

Formally combining this result and that from [10] one can give a conjecture in the general case. The identity wavefront relation for the scattering operator implied by Theorem 5 can be understood in terms of the limiting behaviour of complete geodesics on X which tend uniformly to the boundary. This limiting geodesic relation can be analyzed in terms of the rescaled symplectic geometry which arises from the (formal part of) the algebra of pseudodifferential operators discussed below. In this way the various boundary variables in (1) can be seen to move at 'different speeds' according to the associated powers of x. The variables in the fibres of ϕ_1 are 'fast' and the problem is global in these variables. The variables in the fibres of ϕ_2 are 'finite speed' whilst those in Y_2 are 'slow'. The limiting geodesic relation for a doubly-warped metric descends to a relation on T^*Y_1 in which motion is along the fibres, with the distance scaled from π to 0 according to the length of the projection of the initial covector into the fibre normal. Using this formal reasoning (described also in [7]) one can expect the following.

Conjecture. *If g is a doubly-warped Riemann metric on a compact manifold with boundary X and $0 < \lambda^2 < A^2$ the absolute scattering operator A_λ, defined in (2), is a Fourier integral operator associated to the limiting geodesic relation at infinity.*

The initial step in the proof of the existence of generalized eigenfunctions as in (2) is the construction of a suitably tailored algebra of pseudodifferential operators, with its associated wavefront set. I shall use this opportunity to outline what I believe is the 'appropriate' structure to give such an algebra and also a related 'large' calculus, this is done in §1. The doubly-warped metric is then shown, in §2, to correspond to a Lie algebra of vector fields satisfying these general conditions. In the particular case of double-zero metrics the calculus so defined, and its associated wavefront set, are used in §3 to show the existence of generalized eigenfunctions; the general case should follow by combining the methods here with those of [3]. In §4 a more constructive approach is taken, as in [10], which leads to a a parametrix for the Poisson operator and hence a proof of Theorem 5. The same approach in the general setting should yield the Conjecture.

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§1: Boundary structures

To construct an algebra of pseudodifferential operators appropriate to the analytic questions of interest here a general procedure of 'microlocalization' will be invoked. This is based on the differential analysis of manifolds with corners and I shall therefore describe some general definitions and results. These should eventually appear in [9]; in the meantime they can be extracted from various places [1], [4], [5], [6], [7].

Manifolds with corners. An intrinsic definition of a manifold with corners is as a (connected) Hausdorff, paracompact topological manifold with boundary X with a ring of functions $\mathcal{C}^\infty(X)$ which can be embedded as a submanifold with corners of a smooth manifold of the same dimension. That is, there is a continuous map $i : X \longrightarrow \widetilde{X}$ into a $\mathcal{C}^\infty$ manifold $\widetilde{X}$, such that i is a homeomorphism onto its image, $\mathcal{C}^\infty(X) = i^*\mathcal{C}^\infty(\widetilde{X})$ and there are k non-negative functions $\rho_i \in \mathcal{C}^\infty(\widetilde{X})$, $i = 1, \dots, k$ such that

$$i(X) = \{p \in \widetilde{X}; \rho_i(p) \ge 0,\ i = 1, \dots, k\} \text{ and}$$
$$\forall\ p \in \widetilde{X},\ \{d\rho_i(p); \rho_i(p) = 0\} \text{ is an independent set.}$$

This is the same as a coordinate-covering definition, with $\mathcal{C}^\infty$ transitions, based on the local model space $[0,\infty)^r \times \mathbb{R}^{n-r}$ (for fixed n but variable r) except that it ensures that the boundary hypersurfaces, which are the components of the sets $i^{-1}(\{\rho_i = 0\})$, are embedded. These components are each manifolds with corners with the $\mathcal{C}^\infty$ structure given by restriction of $\mathcal{C}^\infty(X)$. The set of these boundary hypersurfaces is denoted $M_1(X)$; each of these necessarily has a defining function $\rho_H \in \mathcal{C}^\infty(X)$, $H \in M_1(X)$, such that

$$\rho_H \ge 0,\ H = \{\rho_H = 0\} \text{ and } d\rho_H(p) \ne 0\ \forall\ p \in H.$$

Here $df(p) \in T^*_pX$ is an element of the intrinsic cotangent bundle which is identified with $T^*_{i(X)}\widetilde{X}$ by i, i.e. it is the pull-back under i of $T^*\widetilde{X}$. Similar definitions apply to any of the usual natural bundles over a manifold without boundary.

p-submanifolds. There are various notions of submanifold of a manifold with corners, the most restrictive of which is a *p-submanifold* where the 'p' stands for product. Notice that a manifold with corners always has a product decomposition near each point p, in the sense that p has a neighbourhood O_p which is diffeomorphic to a product $[0,1)^k \times (-1,1)^{n-k}$. A closed subset $Y \subset X$ is a p-submanifold if each point $p \in Y$ has such a neighbourhood O_p with a diffeomorphism which also (i.e. simultaneously) reduces $Y \cap O_p$ to a product $\{0\}^r \times [0,1)^{k-r} \times \{0\}^s \times (-1,1)^{n-k-s}$. It follows that Y itself is a manifold with corners.

b-maps. A $\mathcal{C}^\infty$ map $f : X \longrightarrow Y$, between manifolds with corners is defined by the usual condition that $f^*\mathcal{C}^\infty(Y) \subset \mathcal{C}^\infty(X)$. It is useful to deal with a restricted class of smooth maps which 'respect the boundaries'. Namely consider the notion of an *interior b-map* defined by the requirement in terms of boundary defining functions that

$$\begin{gathered} f^*\rho'_G = a_G \prod_{H \in M_1(X)} \rho_H^{e(H,G)} = a_G \rho^{e(G)}, \\ 0 < a_G \in \mathcal{C}^\infty(X),\ \forall\ G \in M_1(Y). \end{gathered}$$

Since f is assumed smooth, the *boundary exponents* $e(H,G)$ are non-negative integers for each $H \in M_1(X)$ and $G \in M_1(X)$. Note that $e(H,G)$ is positive if and only if $f(H) \subset G$. The obvious multiindex notation is used here to write the powers of boundary defining functions with $e(G) : M_1(X) \longrightarrow \mathbb{N}_0$ the 'multiindex'.

A general b-map is an interior b-map into one of the boundary faces of the range space.

Tangent vector fields. The smooth vector fields on X, meaning the smooth sections of TX, form a Lie algebra. A more significant Lie subalgebra is the space, $\mathcal{V}_{\mathrm{b}}(X)$, of smooth vector fields tangent to each boundary hypersurface. This has many properties similar to the whole Lie algebra. In particular it is the space of sections of a smooth vector bundle over X,

$$\mathcal{V}_{\mathrm{b}}(X) = \mathcal{C}^\infty(X; {}^bTX).$$

This new bundle is bundle-isomorphic to TX but not in any natural way. There is a natural $\mathcal{C}^\infty$ vector bundle map

$$\iota_{\mathrm{b}} : {}^bTX \longrightarrow TX$$

which is a diffeomorphism over the interior and has fibre rank exactly k at any boundary point of codimension k, i.e. point which lies in precisely k boundary hypersurfaces. The null space of ι_p at such a point is a vector space, ${}^bN_pX$, which is naturally the product of k oriented lines, each corresponding to one of the hypersurfaces passing through p.

b-differential. By continuity from the usual differential in the interior, an interior b-map, as defined above, has both a push-forward and a pull-back b-differential (dual to each other)

$${}^bf_* : {}^bT_pX \longrightarrow {}^bT_{f(p)}Y \text{ and } {}^bf^* : {}^bT^*_{f(p)}Y \longrightarrow {}^bT^*_pX.$$

It follows automatically that at any boundary point

$${}^bf_* : {}^bN_pX \longrightarrow {}^bN_{f(p)}Y.$$

Using these maps, two surjectivity properties for an interior b-map can be identified. Namely f is said to be a *b-submersion* if it is surjective and has surjective b-differential at each point. On the other hand it is said to be *b-normal* if at each boundary point the restricted b-differential on the b-normal spaces is surjective. If it is both b-normal and is a b-submersion it is said to be a *b-fibration.*

Each of the four classes just defined, interior b-maps, b-submersions, b-normal maps and b-fibrations is closed under composition. Some of their analytic properties are described below.

Generalized blow-down maps. Another class of b-maps that typically arise from blow-up constructions have a weaker surjectivity property that b-submersions. Namely a b-map between manifolds with corners of the same dimension is said to be a *generalized blow-down map* if it restricts to a diffeomorphism of the interiors and its pull-back differential has the property

$$\mathcal{C}^\infty(X)\cdot {}^bf^*\mathcal{C}^\infty(Y;{}^bT^*Y)\supset \rho^a\mathcal{C}^\infty(X;{}^bT^*X) \text{ for some multiindex } a.$$

This implies in particular that if $\nu_b(Y)$ is a positive b-density on Y then pull-back of densities induces a map of the form

$$f^*\nu_b(Y)=a\rho^w\nu_b(X),\ w:M_1(X)\longrightarrow\mathbb{N}_0,\ 0<a\in\mathcal{C}^\infty(X).$$

It further follows that a generalized blow-down map always induces an isomorphism on functions and densities

$$f^*:\dot{\mathcal{C}}^\infty(Y)\longleftrightarrow\dot{\mathcal{C}}^\infty(X),\ f_*:\dot{\mathcal{C}}^\infty(X;\Omega)\longleftrightarrow\dot{\mathcal{C}}^\infty(Y,\Omega).$$

This in turn implies that it induces an isomorphism from the space of extendible distributional sections of any bundle

$$f^*:\mathcal{C}^{-\infty}(Y;G)\longleftrightarrow\mathcal{C}^{-\infty}(X;f^*G),\ \mathcal{C}^{-\infty}(Y;G)=\Big(\dot{\mathcal{C}}^\infty(Y;G'\otimes\Omega)\Big)'.$$

Thus a generalized blow-down map can be thought of as a 'distributional isomorphism' in the sense that it changes the structure near the boundary but only to finite order. Notice that the composite of two generalized blow-down maps is again a generalized blow-down map.

The generalized blow-down maps considered here are constructed from radial blow-down maps. Part of the construction shows that the blow down map for blow up of a boundary p-submanifold, i.e. a p-submanifold of a proper boundary face, is always a generalized blow-down map in this sense.

Lifting. If $f : X \longrightarrow Y$ is a generalized blow-down map let $K \subset Y$ be the 'singular set' which is the image of the set near which it is not locally a diffeomorphism. Then the lift of a subset $C \subset Y$ is defined provided *either* C is the closure of its intersection with the complement of K i.e. $C = \mathrm{Clos}(C \cap (Y \setminus K))$, when by definition it is the set

$$f^*(C) = \mathrm{Clos}(f^{-1}(C \cap (Y \setminus K))$$

or in case $C \subset K$ when it is taken to be

$$f^*(C) = f^{-1}(C).$$

This definition reflects the commutation properties of blow ups.

Boundary structure. By a boundary structure (this is a special case of the more general notion of a boundary-fibration structure) on a compact manifold with corners X, I shall mean a Lie subalgebra $\mathcal{W} \subset \mathcal{V}_{\mathrm{b}}(X)$ which can be 'resolved' in a sense made precise in terms of the notions introduced above. These conditions are just formal properties of the products X^2 and X^3 in their relation to X in the boundaryless case.

(1) There must be two manifolds with corners $X^2_{\mathcal{W}}$ and $X^3_{\mathcal{W}}$ and generalized blow-down maps

$$\beta^2 : X^2_{\mathcal{W}} \longrightarrow X^2, \ \beta^3 : X^3_{\mathcal{W}} \longrightarrow X^3$$

such that the symmetry groups $X^2 \ni (x, y) \longrightarrow (y, x) \in X^2$ and the similar exchange maps on X^3 lift to diffeomorphisms of $X^2_{\mathcal{W}}$ and $X^3_{\mathcal{W}}$ respectively.

(2) Under β^2 the diagonal $\Delta = \{(x, x) \in X^2; x \in X\}$, lifts to a p-submanifold $\Delta_{\mathcal{W}} \subset X^2_{\mathcal{W}}$ such that β^2 restricts to a diffeomorphism of $\Delta_{\mathcal{W}}$ to Δ.

(3) The two composite maps

$$\pi_{O,\mathcal{W}} = \pi_O \circ \beta^2_{\mathcal{W}}, \ O = R, L$$

are b-fibrations, where π_R and π_L are the right and left projections from X^2 to X

(4) The elements of $\mathcal{W}$, lifted to X^2 as vector fields on either the left or right factor, lift under β^2 into $\mathcal{V}_{\mathrm{b}}(X^2_{\mathcal{W}})$ to give a space of vector fields transversal to $\Delta_{\mathcal{W}}$.

(5) Let π_O for $O = S, C, F$ be the three projections $\pi_O : X^3 \longrightarrow X^2$ corresponding, respectively, to dropping the left, the central

and the right factor of X. Then there are required to be three b-fibrations each giving a commutative diagramme

$$\begin{tikzcd} X^3_{\mathcal{W}} \arrow[r, "\pi_{O,\mathcal{W}}"] \arrow[d, "\beta^3"'] & X^2_{\mathcal{W}} \arrow[d, "\beta^2"] \\ X^3 \arrow[r, "\pi_O"'] & X^2, \end{tikzcd} \qquad O = F, C, L,$$

such that under $\pi_{O,\mathcal{W}}$ the submanifold $\Delta_{\mathcal{W}} \subset X^2_{\mathcal{W}}$ lifts to a p-submanifold of $\Delta_{O,\mathcal{W}} \subset X^3_{\mathcal{W}}$ to which the other two $\pi_{O',\mathcal{W}}$, $\{S, C, F\} \ni O' \neq O$ restrict to diffeomorphisms.

Given the symmetry condition it suffices to check the last two conditions for one case. Over the interiors of X, X^2 and X^3 all these conditions are trivial except that they ensure that $\mathcal{W}$ must be unconstrained there. At the boundary the existence of such a resolution is distinctly non-trivial. Even in the simplest of cases, such as for $\mathcal{V}_{\mathrm{b}}(X)$ described next, such a resolution need not be unique.

One property which follows from these conditions is that $\mathcal{W} = \mathcal{C}^\infty(X; {}^{\mathcal{W}}TX)$ for a vector bundle ${}^{\mathcal{W}}TX$ over X.

Notation. Essentially by definition a boundary structure arises from some sort of 'uniformity' structure near the boundary. The Lie algebra of vector fields is generally denoted $\mathcal{V}_{\mathrm{n}}(X)$ for some abbreviated name n. Then the spaces and maps are denoted X^2_{n}, β^2_{n} etc.; note that the 'name' refers to the *resolution* rather than just the space of vector fields.

Basic example, $\mathcal{V}_{\mathrm{b}}(X)$. For any compact manifold with corners the space of all tangent vector fields $\mathcal{V}_{\mathrm{b}}(X)$ is a boundary structure on X. This is not however completely trivial! In particular it is certainly not the case that the double space X^2_{b} is the product X^2, rather it is obtained from this product by blowing up all the codimension two boundary faces $H \times H$ where $H \in M_1(X)$. The triple product X^3_{b} is similarly defined from X^3 by first blowing up all the triple products, $H^3 \subset X^3$ and then all the double products of the form $X \times H^2$, $H \times X \times H$ or $H^2 \times X$.

Even in the case of a manifold with boundary this resolution is not always unique. If the boundary has two components, H_1, and $H_2 \in M_1(X)$ then one can also define an 'overblown' resolution with X^2_{ob} defined by blowing up $H_1 \times H_2$ and $H_2 \times H_1$ in addition to H_1^2 and H_2^2.

Conormal distributions at a p-submanifold. Conormal distributions with respect to an interior p-submanifold (one that meets the interior) can be defined just as in the original case of an embedded submanifold of a manifold without boundary as considered by Hörmander in [2]. Indeed it is always possible to 'double' a manifold with corners successively across each boundary hypersurface in such a way that a given p-submanifold extends to a closed embedded submanifold of an extension $\widetilde{X}$. Then conormal distributions with respect to an interior p-submanifold $Y \subset X$ are just the restrictions of conormal distributions for the extension $\widetilde{Y} \subset \widetilde{X}$. The resulting spaces are independent of the choice of extension and their properties can be deduced from the boundaryless case. The space of conormal distributions with respect to a p-submanifold can then be denoted $I^m(X;Y)$. The 1-step polyhomogeneous subspace $I^m_{\mathrm{phg}}(X;Y)$ is similarly well defined, as are the corresponding spaces of sections of any vector bundle. These spaces are always $\mathcal{C}^\infty(X)$ modules with principal symbol maps as in the boundaryless case.

Conormal coefficients. As well as the natural space, $\mathcal{C}^\infty(X)$, of smooth functions on a manifold with corners it is frequently useful to consider larger spaces of distributions conormal with respect to the boundary. Since the boundary may consist of a number of intersecting hypersurfaces these distributions are somewhat more complicated than in the case of a p-submanifold. However, the 'spaces with bounds' are easily defined. Let ν_{b} be a smooth positive b-density on X, assumed compact for convenience. Thus if

$$\rho_{\mathrm{tot}} = \prod_{H \in M_1(X)} \rho_H$$

is a 'total boundary defining function' then $0 < \rho_{\mathrm{tot}} \nu_{\mathrm{b}} \in \mathcal{C}^\infty(X;\Omega)$ is a smooth positive measure. Now, define

$$L^2_{\mathrm{b}}(X) = \left\{ f \in L^2_{\mathrm{loc}}(X^\circ); \int_X |f|^2 \nu_{\mathrm{b}} < \infty \right\}$$

and more generally if $\alpha : M_1(X) \longrightarrow \mathbb{R}$ is a 'multiindex' set

$$\rho^\alpha L^2_{\mathrm{b}}(X) = \left\{ f \in L^2_{\mathrm{loc}}(X^\circ); \int_X |\rho^{-\alpha} f|^2 \nu_{\mathrm{b}} < \infty \right\}$$
$$\text{where } \rho^\alpha = \prod_{H \in M_1(X)} \rho_H^{\alpha(H)}.$$

Since $L^2_{\mathrm{loc}}(X^\circ)$ is a well-defined subspace of the space of distributions on the interior, derivatives can be interpreted distributionally and then

$$\rho^\alpha H^\infty_{\mathrm{b}}(X) = \left\{ f \in \rho^\alpha L^2_{\mathrm{b}}(X); V_1 \dots V_l f \in \rho^\alpha L^2_{\mathrm{b}}(X),\ \forall\ V_i \in \mathcal{V}_{\mathrm{b}}(X),\ \forall\ l \right\}.$$

These are the L^2-based conormal distributions (or functions since they are necessarily $\mathcal{C}^\infty$ in X°.) There are similar L^∞-based conormal functions, obtained by replacing $L^2_{\mathrm{b}}(X)$ by $L^\infty(X)$ throughout. These are more naturally thought of as symbols so we set

$$\begin{aligned}&S^\alpha(X)\\ &= \left\{f \in \rho^{-\alpha} L^\infty(X); V_1 \dots V_l f \in \rho^{-\alpha} L^\infty(X),\ \forall\ V_i \in \mathcal{V}_{\mathrm{b}}(X),\ \forall\ l\right\}.\end{aligned}$$

A form of the Sobolev embedding theorem then gives the 'Sandwich Theorem'

$$S^{-\alpha-\epsilon}(X) \subset \rho^\alpha H^\infty_{\mathrm{b}}(X) \subset S^{-\alpha}(X)\ \forall\ \alpha$$

provided ϵ is a positive multiindex, i.e. all entries are strictly positive.

Polyhomogeneity. As well as these conormal distributions defined by bounds there are polyhomogeneous space. By definition an *index set* is a subset

$$E \subset \mathbb{C} \times \mathbb{N}_0,\ \mathbb{N}_0 = \{0, 1, 2, \dots\}$$

satisfying the following four conditions

$$\begin{gathered}E \text{ is discrete}\\ (z_l, k_l) \in E,\ l \in \mathbb{N},\ |z_l| + k_l \to \infty \Longrightarrow \operatorname{Re}(z_l) \to \infty\\ (z, k) \in E,\ l \in \mathbb{N}_0,\ 0 \le l \le k \Longrightarrow (z, l) \in E\\ (z, k) \in E,\ r \in \mathbb{N} \Longrightarrow (z + r, k) \in E.\end{gathered}$$

An *index family* $\mathfrak{E}$ for X is an assignment of an index set $\mathfrak{E}(H)$ to each $H \in M_1(X)$.

If $\mathfrak{E}$ is an index family for X then the space of polyhomogeneous conormal distributions, $\mathcal{A}^{\mathfrak{E}}(X)$, at the boundary with powers determined by $\mathfrak{E}$ can be defined so that it contains the functions

$$\prod_{H \in M_1(X)} \rho_H^{z(H)} \log^{k(H)} \rho_H,$$

provided $(z(H), k(H)) \in \mathfrak{E}(H)$ and in addition has appropriate completeness properties. These spaces are modules over $\mathcal{C}^\infty(X)$ so one can also define

$$\mathcal{A}^{\mathfrak{E}} I^m(X; Y) = \mathcal{A}^{\mathfrak{E}}(X) \hat{\otimes}_{\mathcal{C}^\infty(X)} I^m(X; Y)$$

as appropriate completions of the tensor products. A similar argument allows the corresponding spaces of sections of any vector bundle over X to be defined.

Kernels and operators. The space of pseudodifferential operators on a compact manifold without boundary, acting on half-densities, is identified with the corresponding space of kernels

$$\Psi^m(X;\Omega^{\frac12}) = I^m(X^2,\varDelta;\Omega^{\frac12}).$$

In the case of a compact manifold with corners an appropriate form of the Schwartz kernel theorem identifies all continuous linear operators

$$A : \dot{\mathcal{C}}^\infty(X;\Omega^{\frac12}) \longrightarrow \mathcal{C}^{-\infty}(X;\Omega^{\frac12})$$

with their kernels which are arbitrary elements of $\mathcal{C}^{-\infty}(X^2;\Omega^{\frac12})$. The assumption that $\beta^2 : X^2_{\mathcal{W}} \longrightarrow X^2$ is a generalized blow-down map implies that the identification over the interior extends by continuity to give an isomorphism

$$(\beta^2)^* : \mathcal{C}^{-\infty}(X^2;\Omega^{\frac12}) \longleftrightarrow \mathcal{C}^{-\infty}(X^2_{\mathcal{W}};\Omega^{\frac12}).$$

Thus a space of operators on X can be defined by specifying the kernels as half-densities on $X^2_{\mathcal{W}}$.

The identity. The relation between $X^2_{\mathcal{W}}$ and X demanded of a resolution implies that β^2 induces a diffeomorphism from the lifted diagonal to the diagonal of X^2. From this, and the assumption that it is a generalized blow-down map, it follows that the kernel of the identity operator, which is a non-vanishing Dirac section of $\Omega^{\frac12}$ for the diagonal, lifts to a distribution

$$\mathrm{Id} \in \mathcal{C}^{-\infty}(X^2_{\mathcal{W}};\Omega^{\frac12}),\ \mathrm{Id} = \rho^\kappa A,\ A \in I^0(X^2_{\mathcal{W}},\varDelta_{\mathcal{W}};\Omega^{\frac12}), A > 0.$$

That is, apart from a boundary weight ρ^κ it too is a positive Dirac section of the half-density bundle for the lifted diagonal. The index κ is only determined for the boundary hypersurfaces of $X^2_{\mathcal{W}}$ which meet the lifted diagonal (they are in one to one correspondence with the boundary hypersurfaces of X).

$\mathcal{W}$-pseudodifferential operators. Using the assumption that the Lie algebra $\mathcal{W}$ is transversal to the lifted diagonal it follows that the algebra of differential operators on half-densities generated by $\mathcal{W}$ and $\mathcal{C}^\infty(X)$ is identified with the space

$$\left\{A \in I^{\mathbb{N}_0}(X^2_{\mathcal{W}},\varDelta_{\mathcal{W}};\rho^\kappa\Omega^{\frac12}); \mathrm{supp}(A) \subset \varDelta_{\mathcal{W}}\right\}.$$

That is, the space of $\mathcal{W}$-differential operators is identified with the space of all smooth Dirac sections over the lifted diagonal of the blown-up space $X^2_{\mathcal{W}}$.

This identification of the $\mathcal{W}$-differential operators makes it natural to define the 'small' calculus of $\mathcal{W}$-pseudodifferential operators in terms of kernels by

$$\Psi^m_{\mathcal{W}}(X;\Omega^{\frac12}) = \\ \left\{A \in I^m(X^2_{\mathcal{W}},\Delta_{\mathcal{W}};\rho^\kappa\Omega^{\frac12}); A \equiv 0 \text{ at } H \in M_1(X^2_{\mathcal{W}}) \text{ if } H\cap\Delta_{\mathcal{W}}=\emptyset\right\}.$$

Larger spaces of $\mathcal{W}$-pseudodifferential operators can be defined for any choice of index set $\mathfrak{E}$ for $X^2_{\mathcal{W}}$ by

$$\Psi^{m,\mathfrak{E}}_{\mathcal{W}}(X;\Omega^{\frac12}) = \mathcal{A}^{\mathfrak{E}}I^m(X^2_{\mathcal{W}},\Delta_{\mathcal{W}};\rho^\kappa\Omega^{\frac12}).$$

If $\mathfrak{E}$ satisfies the compatibility condition that $\mathfrak{E}(H)\subset\mathbb{N}_0\times\{0\}$ if $H\cap\Delta_{\mathcal{W}}\neq\emptyset$, then $\mathcal{C}^\infty(X^2_{\mathcal{W}})\subset\mathcal{A}^{\mathfrak{E}}(X^2_{\mathcal{W}})$ and then

$$\mathrm{Diff}^m_{\mathcal{W}}(X;\Omega^{\frac12})\subset\Psi^m_{\mathcal{W}}(X;\Omega^{\frac12}).$$

These definitions can be extended to operators acting on sections of any vector bundle(s).

Pull-back. Conormal functions can be pulled back under general interior b-maps provided they have a transversality property with respect to the p-submanifold involved (if there is one). Since it suffices for present purposes, attention can be restricted here to b-fibrations. Then each boundary hypersurface $H\in M_1(X)$ is mapped onto either the whole of Y or onto a boundary hypersurface $G=F(H)\in M_1(Y)$. If $\mathfrak{E}$ is an index family for Y define $\mathfrak{F}=f^\#\mathfrak{E}$ by setting $\mathfrak{F}(H)=\mathbb{N}\times\{0\}$ (the index set of $\mathcal{C}^\infty$) in the first case and

$$\mathfrak{F}(H)=\{(z',k); z'=e(H,G)z,\ (z,k)\in\mathfrak{E}(G)\}$$

in the second case.

Proposition. *Suppose $f: X\longrightarrow Y$ is a b-fibration of compact manifolds with corners and that $B\subset Y$ is an interior p-submanifold then $f^*(B)\subset X$ is a p-submanifold and pull back defines linear maps*

$$f^*:\mathcal{A}^{\mathfrak{E}}I^m_{\mathrm{phg}}(Y,B;G)\longrightarrow\mathcal{A}^{\mathfrak{F}}I^{m'}_{\mathrm{phg}}(X,f^*B;f^*G),\ \mathfrak{F}=f^\#\mathfrak{E}$$

here f^*G is the pull-back of the bundle G and $m'-m$ depends on the relative dimensions.

This result follows from a direct characterization of the polyhomogeneous conormal functions in terms of expansions.

Push-forward. There is a similar result for push-forward. For an index set E write $E > 0$ to mean that $(z,k) \in E \Longrightarrow \operatorname{Re} z > 0$. For a b-fibration $f : X \longrightarrow Y$ and an index set $\mathfrak{E}$ on X define an index set $\mathfrak{F} = f_{\#}\mathfrak{E}$ as follows. For each $G \in M_1(Y)$ set

$$\mathfrak{F}(G) = \{(z,k); \exists\ (z_i,k_i) \in \mathfrak{E}(H_i),\ F(H_i) \subset G,\\ z = \sum_i \frac{z_i}{e(H_i,G)},\ k+1 \le \sum_i (k_i+1)\}.$$

Proposition. *Suppose $f : X \longrightarrow Y$ is a b-fibration of compact manifolds with corners and $B \subset X$ is an interior p-submanifold to which f is (b-)normal, then $f(B) \subset Y$ is an interior p-submanifold and provided $\mathfrak{E}(H) > 0$ for all $H \in M_1(X)$ mapped onto Y push-forward defines linear maps*

$$f_* : \mathcal{A}^{\mathfrak{E}} I^m_{\mathrm{phg}}(X, B; f^*G \otimes \Omega_b) \longrightarrow \mathcal{A}^{\mathfrak{F}} I^{m'}_{\mathrm{phg}}(X, f(B); G \otimes \Omega_b),\ \mathfrak{F} = f_{\#}\mathfrak{E};$$

here $m' - m$ depends on the relative dimensions.

This has further extensions to the case, which occurs in the composition of pseudodifferential operators, where B is replaced by two transversally intersecting submanifold, each of which is mapped onto Y but the intersection of the two is mapped onto a proper p-submanifold. A similar result then holds for spaces of conormal distributions defined with respect to the intersecting pair. The important point for the present discussion is that these contain the products of conormal distributions associated to the two submanifolds; again this is just as in the boundaryless case.

Operators. As discussed above the kernels of these $\mathcal{W}$-pseudodifferential operators are defined on $X^2_{\mathcal{W}}$ and hence equivalently as extendible distributions on X^2. They therefore define operators from $\dot{\mathcal{C}}^\infty(X)$ to $\mathcal{C}^{-\infty}(X)$. For the small calculus

$$A \in \Psi^m_{\mathcal{W}}(X) \Longrightarrow A : \rho^a \mathcal{C}^\infty(X) \longrightarrow \rho^a \mathcal{C}^\infty(X)$$

for any multiindex a. Since the calculus is closed under passage to adjoints these operators act on extendible distributions. Elements of the large calculus determined by a choice of index family for $X^2_{\mathcal{W}}$ may not act on any fixed space, depending on the index families, but do map appropriately 'integrable' polyhomogeneous spaces to polyhomogeneous spaces with transformed index families.

Symbol and normal morphisms. The symbol of an element of the small calculus is the symbol of its kernel, as a conormal distribution. This is an equivalence class of functions (or in general sections of a bundle) on the conormal bundle to the lifted diagonal. The hypotheses above identify this bundle with the structure bundle ${}^{\mathcal{W}}T^*X$.

Elements of the small calculus act on $\rho^a\mathcal{C}^\infty(X)$ and hence by restriction define operators on $\mathcal{C}^\infty(H)$ for each $H \in M_1(X)$. These boundary actions are determined by the restriction of the kernels to the corresponding boundary hypersurface of $X^2_{\mathcal{W}}$. There restricted kernels in fact contain more information and can be viewed as the kernels of $\mathcal{W}_H$-pseudodifferential operators (with additional invariance properties) for a boundary(-fibration) structure on ${}^bNH$. Thus even to handle operators associated to boundary structures one needs the more general theory including the fibration part of boundary-fibration structures. For the cases discussed below these normal operators can be handled directly.

Composition. Using the push-forward and pull-back theorems just described, and extensions as in the boundaryless case it follows that the small calculus is always an algebra containing the $\mathcal{W}$-differential operators and the large calculus is a two-sided module over the small calculus. Elements of the larger calculus can only be composed under appropriate conditions on the index families but once these are satisfied the composite is in the calculus.

Sobolev spaces. There is a natural L^2 space, $L^2_{\mathcal{W}}(X)$ defined by a density given locally by the wedge product of a basis of ${}^{\mathcal{W}}T^*X$. Using this basic space and the action of the small calculus on distributions, Sobolev spaces can be defined in terms of weighting by powers of boundary defining functions

$$H^{a,m}_{\mathcal{W}}(X) = \{u \in \mathcal{C}^{-\infty}(X); Au \in \rho^a L2_{\mathcal{W}}(X) \ \forall\ A \in \Psi^{-m}_{\mathcal{W}}(X)\}.$$

Blow up. Essentially by definition the spaces $X^2_{\mathcal{W}}$ and $X^3_{\mathcal{W}}$ are defined by 'blow-up'. The radial blow up of a p-submanifold Y of a manifold with corners X is a well-defined procedure giving a new manifold with corners $[X;Y]$ with natural blow-down map $\beta : [X;Y] \longrightarrow X$. This new manifold has an additional boundary hypersurface compared to X which replaces Y. Polar coordinates in the normal variables to Y in X lift to local coordinates near this 'front face'. If Y is a boundary p-submanifold then β is a generalized blow-down map in the sense described above.

Amongst the important properties of blow up are two commutation properties. If $Y_1 \subset Y_2$ are two p-submanifolds then under blow-up of either of them the other lifts to be a p-submanifold and denoting the lifts by the same letter the two blow ups are naturally isomorphic

$$[X;Y_1;Y_2] \equiv [X;Y_1;Y_2],\ Y_1 \subset Y_2 \text{ p-submanifolds.}$$

Here the naturality is that the isomorphism gives a commutative diagram with the blow-down maps. The same result is true if Y_1 and Y_2 meet transversally, but not in general.

§2: The doubly-warped calculus

The 'elliptic' part of the analysis of the Laplacian associated to a metric as in (1) consists in the construction of an algebra of pseudodifferential operators on X which contains the resolvent away from the spectrum. In this case it is the 'small calculus' discussed above for the appropriate Lie algebra of smooth vector fields which describes the degeneration of the metric in the precise sense that the Laplacian is a (symbolically) elliptic operator in the corresponding enveloping algebra.

Theorem 6. *The space $\mathcal{V}_{\mathrm{dw}}(X)$ consisting of those smooth vector fields on X which have bounded length with respect to the metric g in* (1)*, is a Lie algebra which is a boundary structure in the sense of the previous section.*

Proof. The space $\mathcal{V}_{\mathrm{dw}}(X)$ is easily described. Local coordinates near any boundary point of the form x, z, y, t can be introduced with x being the distance function $\tilde{x}$ lifted from $[0, \delta) \times Y_2$, the t's are coordinates lifted from Y_2, the y's coordinates lifted from the fibre of ϕ_2 in Y_1 through p and the z's are coordinates on the fibre of ϕ_1 through p. A smooth vector field on X is an element of $\mathcal{V}_{\mathrm{dw}}(X)$ if and only if expressed in terms of these coordinates it is a smooth linear combination of the basis vector fields

$$x^2\frac{\partial}{\partial x},\ x^2\frac{\partial}{\partial t},\ x\frac{\partial}{\partial y} \text{ and } \frac{\partial}{\partial z}.$$

Thus the theorem just states that this Lie algebra of vector fields has a resolution in the sense described above. Thus the spaces X^2_{dw} and X^3_{dw} and appropriate maps must be constructed. These spaces will be constructed from X^2 and X^3 respectively by iterated radial blow up.

Let us consider first X^2_{dw}; for simplicity assume that ∂X is connected and that the fibrations have connected fibres. The first stage in its construction is the blow up of the fibre diagonal at the corner of X^2

corresponding to the total fibration ψ. In the corner $(\partial X)^2 \subset X^2$ of X^2 consider the submanifold which is the ψ-fibre diagonal, namely

$$B_2 = \{(p,q) \in (\partial X)^2; \psi(p) = \psi(q)\}.$$

This is a closed embedded submanifold of the corner (which is a manifold without boundary); in terms of local coordinates as discussed above, x, y, t, z and x', y', t', z' in the two factors of X it is $y = y'$, $x = x' = 0$. As such it is a p-submanifold of X^2 so it can be blown up radially to give the new manifold with corners $[X^2; B_2]$ and associated blow-down map $\beta : [X^2; B_2] \longrightarrow X^2$.

The second stage of the blow up involves the 'lift' of the fibre diagonal associated to $\tilde{\phi}_1$. The fibre diagonal, defined near the boundary,

$$B_1' = \{(p,q); \tilde{\phi}_1(p) = \tilde{\phi}_1(q)\}$$

lifts under the first blow up to a p-submanifold near the boundary and B_1 is by definition the intersection of this lift with the boundary. It is again a boundary p-submanifold. Then the stretched double space is given by iterated blow up

$$X^2_{\mathrm{dw}} = [X^2; B_2; B_1].$$

The total blow down map is therefore a generalized blow down map. The conditions in the first four conditions can also be checked.

The construction of X^3_{dw} is similar but more involved since it has four stages. In X^3 there are three analogues of B_2, namely the corresponding fibre diagonals in the three pairs of factors; these can be denoted $B_{2,O}$, $O = S, C, F$. These intersect in the triple fibre diagonal $B_{2,T}$. All are boundary p-submanifolds. The first step is to blow up $B_{2,T}$. Under this blow up the three fibre diagonals lift to disjoint p-submanifolds which can therefore be blown up independently.

The submanifold B_1 similarly has three analogues denote, $B_{1,O}$, for $O = S, C, F$, in the blown up version of X^3 produced so far. They are all p-submanifolds intersecting in pairs in the same p-submanifold $B_{1,T}$. The remaining two stages in the construction of X^3_{dw} are the blow up of $B_{1,T}$ after which the remaining $B_{1,O}$ lift to be disjoint and are then blown up. This defines X^3_{dw}.

The main step in checking the properties required above of a resolution of the Lie algebra $\mathcal{V}_{\mathrm{dw}}(X)$ is to show the existence of the stretched projections $\pi_{O,\mathrm{dw}}$, for $O = S, C, F$. This is done using the commutation properties of blow up outlined above as follows for $O = F$ with

$\tilde{B}_1 = B_{1,C} \cup B_{1,S}$ and $\tilde{B}_2 = B_{2,C} \cup B_{2,S}$:

$$
\begin{aligned}
X^3_{\mathrm{dw}} &= [X^3; B_{2,T}; B_{2,F}; \tilde{B}_2; B_{1,T}; B_{1,F}; \tilde{B}_1] \\
&\xrightarrow{\beta(\tilde{B}_1)} [X^3; B_{2,T}; B_{2,F}; \tilde{B}_2; B_{1,T}; B_{1,F}], \\
&\equiv [X^3; B_{2,F}; B_{2,T}; \tilde{B}_2; B_{1,F}; B_{1,T}],\ B_{2,T} \subset B_{2,F},\ B_{1,T} \subset B_{1,F}, \\
&\xrightarrow{\beta(B_{1,T})} [X^3; B_{2,F}; B_{2,T}; \tilde{B}_2; B_{1,F}] \\
&\equiv [X^3; B_{2,F}; B_{2,T}; B_{1,F}; \tilde{B}_2],\ B_{1,F} \subset \tilde{B}_2 \\
&\xrightarrow{\beta(\tilde{B}_2)} [X^3; B_{2,F}; B_{2,T}; B_{1,F}] \equiv [X^3; B_{2,F}; B_{1,F}; B_{2,T}]],\ B_{2,T} \subset B_{1,F} \\
&\xrightarrow{\beta(B_{2,T})} [X^3; B_{2,F}; B_{1,F}] \equiv [X \times X^2; X \times B_2; X \times B_1] \equiv X \times X^2_{\mathrm{dw}} \\
&\xrightarrow{\pi} X^2_{\mathrm{dw}}.
\end{aligned}
$$

It is then relatively straightforward to check the desired properties of these maps.

The calculus of operators given by the results outlined in the previous section then gives a description of the resolvent.

Proposition. *The Laplacian of the metric g in* (1) *is an element of* $\mathrm{Diff}^2_{\mathrm{dw}}(X)$ *for the corresponding doubly-warped structure and the resolvent* $(\Delta - \sigma)^{-1} \in \Psi^{-2}_{\mathrm{dw}}(X)$ *is a holomorphic function of* $\sigma \in \mathbb{C} \setminus [0, \infty)$.

Proof. This result can be shown constructively, essentially as the same result is obtained in the boundaryless case using the normal operator and the usual symbol map.

§3: Double zero metrics

Next I shall briefly describe the proof of the existence of generalized eigenfunctions as in (2), restricting the discussion to the special case of a double-zero metric, where the two fibrations maps ϕ_i are isomorphisms and can be forgotten. This represents well the changes needed to the proof of the corresponding fact in the scattering calculus ([6]) and the fibred-cusp calculus ([8], [3]).

In terms of the coordinates introduced above there are no z or y variables; the metric becomes

$$g = \frac{h}{x^4}$$

where h is a non-degenerate metric on X and x is a boundary defining function equal to the h-distance to second order.

In the special case that x is the distance for h :

$$(3)\quad \Delta = -(x^2\partial_x)^2 + (2n-4)x^3\partial_x + x^4\Delta_{h(x)} + x^2W,\ W \in \mathcal{V}_{\mathrm{dz}}(X)$$

where $\Delta_{h(x)}$ is the tangential Laplacian of the x-dependent metric. In the general case the same decomposition is valid up to a term in $x^2\operatorname{Diff}^2_{\mathrm{dz}}(X)$ (which annihilates 1). An easy inductive argument then shows that there are formal solutions as in (2). More precisely for any $f \in \mathcal{C}^\infty(\partial X)$ there is a function $\tilde f \in \mathcal{C}^\infty(X)$, unique up to $\dot{\mathcal{C}}^\infty(X)$, i.e. in Taylor series at the boundary, such that

$$(\Delta - \lambda^2)\tilde u_f = e_f \in \dot{\mathcal{C}}^\infty(X),\ \tilde u_f = e^{i\lambda/x}x^{n-1}\tilde f,\ 0 \neq \lambda \in \mathbb{R},\ n = \dim X.$$

The actual eigenfunction in (2) is then constructed as

$$(4)\qquad u_f = \tilde u_f - \lim_{\epsilon\downarrow 0}(\Delta - (\lambda - i\epsilon)^2)^{-1}e_f.$$

Even if there is discrete spectrum this limit exists.

Thus the proof of the existence of u_f reduces to an appropriate form of the limiting absorption principle, the examination of the limit of the resolvent on the real axis. This is not really an 'elliptic' problem but rather has more similarity to hyperbolic equations. To explain this I shall first discuss some properties of the calculus $\Psi^*_{\mathrm{dz}}(X)$ which is defined as a special case of the discussion above.

The double-zero calculus has a global quantization map. Consider the vector bundle ${}^{\mathrm{dz}}T^*X$ which is the dual of the 'structure bundle' ${}^{\mathrm{dz}}TX$ of which $\mathcal{V}_{\mathrm{dz}}(X) = \mathcal{C}^\infty(X; {}^{\mathrm{dz}}TX)$ is the space of all sections. Radial compactification of the fibres of this bundle defines a compact manifold with corners which I shall denote ${}^{\mathrm{dz}}\overline{T}^*X$. Since X is a manifold with boundary (assumed connected for notational simplicity) ${}^{\mathrm{dz}}\overline{T}^*X$ has two boundary hypersurfaces (assuming that $\dim X > 1$ as we shall). Let ρ_∂ and ρ_σ be defining functions for these two boundary hypersurfaces. Then for any integers k, l the spaces $\rho^k_\partial\rho^l_\sigma\mathcal{C}^\infty({}^{\mathrm{dz}}\overline{T}^*X)$ are special case of the polyhomogeneous conormal distributions discussed above. There is then a global quantization map for all k, l which is itself independent of k, l

$$q : \rho^k_\partial\rho^{-l}_\sigma\mathcal{C}^\infty({}^{\mathrm{dz}}\overline{T}^*X) \longrightarrow \rho^k\Psi^l_{\mathrm{dz}}(X).$$

The map q has null space contained in $\dot{\mathcal{C}}^\infty({}^{\mathrm{dz}}\overline{T}^*X)$ and range a complement to $\rho^\infty\Psi^{-\infty}_{\mathrm{dz}}(X)$. In consequence it induces a 'full symbol isomorphism'

$$\tilde q : \rho^k_\partial\rho^{-l}_\sigma\mathcal{C}^\infty({}^{\mathrm{dz}}\overline{T}^*X)/\dot{\mathcal{C}}^\infty({}^{\mathrm{dz}}\overline{T}^*X) \longrightarrow \rho^k\Psi^l_{\mathrm{dz}}(X)/\rho^\infty\Psi^{-\infty}_{\mathrm{dz}}(X)$$

and thus induces a 'star product' on the domain space, the space of Laurent series at the boundary, i.e. an associative product given to any order by bilinear differential operators

$$a \star b \equiv \sum_{j=0}^{\infty} P_j(a,b),\ \ P_0(a,b) = ab.$$

The microlocality of this product arises here from the limiting commutativity of the Lie algebra $\mathcal{W}$. The principal symbol map, defined as the leading part of the inverse of q, is independent of all choices and gives a short exact sequence for each k,l

$$0 \longrightarrow \rho_\partial^{k+1}\Psi_{\mathrm{dz}}^{l-1}(X) \hookrightarrow \rho_\partial^{k}\Psi_{\mathrm{dz}}^{l}(X) \xrightarrow{\sigma_{l,k}} \\ \rho_\partial^k \rho^{-l}\mathcal{C}^\infty({}^{\mathrm{dz}}\overline{T}^* X)/\rho_\partial^{k+1}\rho_\sigma^{-l+1}\mathcal{C}^\infty({}^{\mathrm{dz}}\overline{T}^* X) \longrightarrow 0.$$

This identification (even though it is by no means unique) shows shows that there is an associated notion of wavefront set. Namely for an element of the algebra the wavefront set is the support (in the boundary) of the associated Laurent series:

$$\mathrm{WF}'_{\mathrm{dz}}(A) = \mathrm{supp}([a]) \subset C_{\mathrm{dz}}(X) = \partial^{\mathrm{dz}}\overline{T}^* X, \\ A \in \rho^k \Psi^l_{\mathrm{dz}}(X),\ \tilde{q}([a]) = [A].$$

This wavefront set makes the calculus microlocal in the sense that, $\mathrm{WF}'_{\mathrm{dz}}(A) \subset \mathrm{WF}'_{\mathrm{dz}}(A) \cap \mathrm{WF}'_{\mathrm{dz}}(B)$. Then for extendible distributions on X one can set, as in the boundaryless case,

$$\mathrm{WF}_{\mathrm{dz}}(u) = \bigcap \left\{ \mathrm{WF}'_{\mathrm{dz}}(A); A \in \rho^k \Psi^l_{\mathrm{dz}}(X),\ Au \in \dot{\mathcal{C}}^\infty(X) \right\}$$

with the wavefront set independent of k,l.

This wavefront set has the usual properties, for example

$$u \in \mathcal{C}^{-\infty}(X),\ \mathrm{WF}_{\mathrm{dz}}(u) = \emptyset \Longrightarrow u \in \dot{\mathcal{C}}^\infty(X)$$

and if the elliptic set $\mathrm{Ell}_{k,l}(A)$ of $A \in \rho^k\Psi^l_{\mathrm{dz}}(X)$ is defined to be the set where $\sigma_{k,l}(A)$ is invertible then

$$u \in \mathcal{C}^{-\infty}(X),\ Au \in \dot{\mathcal{C}}^\infty(X) \longrightarrow \mathrm{WF}_{\mathrm{dz}}(u) \cap \mathrm{Ell}_{k,l}(A) = \emptyset.$$

Returning to the question of eigenfunctions, suppose $u \in \mathcal{C}^{-\infty}(X)$ is a tempered eigenfunction of the Laplacian, so

$$(\varDelta - \lambda^2)u = 0.$$

Then it follows immediately that

$$\mathrm{WF}_{\mathrm{dz}}(u) \subset \{\sigma_{0,2}(\Delta) = \lambda^2\} \subset C_{\mathrm{dz}}X.$$

The symbol of the Laplacian is represented in $\rho_\sigma^{-2}\mathcal{C}^\infty({}^{\mathrm{dz}}\overline{T}^*X)$ by the square of the metric length function on ${}^{\mathrm{dz}}T^*X$. Thus the wavefront set of an eigenfunction u must be contained in the characteristic set of $\Delta - \lambda^2$, which is the appropriate sphere bundle over the boundary

$$\mathrm{WF}_{\mathrm{dz}}(u) \subset \{|\Xi|^2 = \lambda^2\} \subset {}^{\mathrm{dz}}T^*_{\partial X}X \longrightarrow C_{\mathrm{dz}}X.$$

To analyze the limit in (4) a uniform (in ϵ) version of this wavefront set, of finite order relative to weighted Sobolev spaces, can be used. Thus, if $u_\epsilon \in \mathcal{C}^{-\infty}(X)$ is a family of distributions depending on $\epsilon \in P$, a parameter space, then a point $Z \in C_{\mathrm{dz}}X$ is not in the uniform wavefront set $\mathrm{WF}^{k,l}_{\mathrm{dz},P}$ relative to the finite order Sobolev space $H^{k,l}_{\mathrm{dz}}(X)$ if it is bounded in some such Sobolev space and there is an operator $A \in \rho^{-l}\Psi^k_{\mathrm{dz}}(X)$ which is elliptic at Z and such that Au_ϵ is bounded in $L^2_{\mathrm{dz}}(X)$.

The starting point is the self-adjointness of Δ from which it follows that u_ϵ, the solution of (4), i.e.

$$(\Delta - (\lambda - i\epsilon)^2)u_\epsilon = g \in \dot{\mathcal{C}}^\infty(X),\ \epsilon > 0,\ 0 \neq \lambda \in \mathbb{R}, \tag{5}$$

has ϵu_ϵ bounded in $L^2_{\mathrm{dz}}(X)$ for $\epsilon \in (0,1]$.

Consider a family v_ϵ which is bounded in $H^{k',l'}_{\mathrm{dz}}(X)$ for $\epsilon \in (0,1] = P$ and some k', l' and suppose that

$$(\Delta - (\lambda - i\epsilon)^2)v_\epsilon = f_\epsilon \text{ is bounded in } \dot{\mathcal{C}}^\infty(X). \tag{6}$$

Directly from the definition of the uniform wavefront set it follows that

$$\mathrm{WF}^{k,l}_{\mathrm{dz},P}(v_\epsilon) \subset \{|\Xi| = \lambda^2\}\ \forall\, k, l,\ P = (0,1].$$

Using commutation arguments as in [6] improved microlocal bounds can also be deduced from (6). First there are no singularities in the 'incoming' subset.

Proposition. *For a family v_ϵ bounded in $H^{k',l'}_{\mathrm{dz}}(X)$ for some k', l' satisfying* (6) *and any $k, l \in \mathbb{R}$*

$$\mathrm{WF}^{k,l}_{\mathrm{dz},P}(v_\epsilon) \cap \{-\lambda\frac{dx}{x^2}\} = \emptyset.$$

Secondly Hörmander's theorem on the propagation of singularities is valid in this context for the 'Hamilton vector field' of the Laplacian

Proposition. *For a family v_ϵ bounded in $H^{k',l'}_{\mathrm{dz}}(X)$ for some k', l' satisfying* (6) *and any $k, l \in \mathbb{R}$, $\mathrm{WF}^{k,l}_{\mathrm{dz},P}(v_\epsilon)$ is a union of maximally extended curves in $\{\tau^2 + |\zeta|^2 = \lambda^2$ of the vector field $\tau\zeta \cdot \partial_\zeta - |\zeta|^2\partial_\tau$.*

Combining these two proposition it follows that $\mathrm{WF}^{k,l}_{\mathrm{dz},P}(v_\epsilon)$ is confined to the 'outgoing' surface $\lambda\frac{dx}{x^2}$. There its order can be bounded,

Proposition. *For a family v_ϵ bounded in $H^{k',l'}_{\mathrm{dz}}(X)$ for some k', l' satisfying* (6) *and any $k, l \in \mathbb{R}$ $\mathrm{WF}^{k,l}_{\mathrm{dz},P}(v_\epsilon)$*

$$\mathrm{WF}^{k,l}_{\mathrm{dz},P}(v_\epsilon) = \emptyset \ \forall \ k \ \textit{and} \ l < -\frac{1}{2}.$$

Now applying these results to $v_\epsilon = \epsilon u_\epsilon$ it follows that ϵu_ϵ is bounded in $H^{k,l}_{\mathrm{dz}}(X)$ for $\epsilon \in P$, any k and any $l < -\frac{1}{2}$. It is therefore a precompact family in the same spaces. In fact this argument applies uniformly in λ near any given non-zero $\lambda' \in \mathbb{R}$. Thus, if $\|\epsilon u_\epsilon\|_{k',l'}$, $l' < -\frac{1}{2}$, does not tend to zero as $\epsilon \downarrow 0$ then along some subsequence it converges to a non-zero limit which is an eigenfunction and has $\mathrm{WF}_{\mathrm{dz}}$ contained in the outgoing set.

Proposition. *If $0 \neq \lambda \in \mathbb{R}$ and $u \in \mathcal{C}^{-\infty}(X)$ satisfies*

$$(\Delta - \lambda^2)u = f \in \dot{\mathcal{C}}^\infty(X), \ \ \mathrm{WF}_{\mathrm{dz}}(u) \in \{\lambda\frac{dx}{x^2}\}$$

then

$$u = e^{i\lambda/x}x^{n-1}u', \ \ u' \in \mathcal{C}^\infty(X). \tag{7}$$

This explicit form of the function allows a version of Green's formula to be proved.

Lemma. *If u is as in the proposition above then*

$$\lim_{\delta\downarrow 0}\int_{x<\delta} \overline{u}(\Delta - \lambda^2)udg = \int_{\partial X} |u'|^2 dh.$$

As a corollary, for an actual eigenfunctions u which is outgoing it follows that $u' = 0$ on ∂X and then by a modified form of the propositions above that $u \in \dot{\mathcal{C}}^\infty(X)$. In fact the proof shows that for $\lambda \in [\delta, 1/\delta]$, $\delta > 0$, the span of such eigenfunctions forms a subspace of $\dot{\mathcal{C}}^\infty(X)$ with bounded sphere, it is therefore finite dimensional.

Proposition. *For the Laplacian of a double-zero metric the embedded L^2 spectrum is discrete and of finite multiplicity.*

Returning now to the family u_ϵ satisfying (5) we can simply project it onto the L^2 eigenspace with eigenvalue λ^2 if this is zero

$$u_\epsilon = u'_\epsilon + u''_\epsilon,$$

where u''_ϵ is the part in the eigenspace. Necessarily, $\epsilon u''_\epsilon$ is bounded in $\dot{\mathcal{C}}^\infty(X)$ for $\epsilon \in (0,1]$. Applying the operator it follows that u'_ϵ satisfies essentially the same condition in that

$$(\Delta - (\lambda - i\epsilon)^2)u'_\epsilon = g_\epsilon \text{ is bounded in } \dot{\mathcal{C}}^\infty(X).$$

Repeating the same arguments as before but using the orthogonality to the eigenfunctions (if there are any) it follows that $u'_\epsilon \longrightarrow u$ is of the form (7). This finally proves the existence of the generalized eigenfunction as in (2) in this case.

§4: Poisson operator and scattering matrix

Construction of the Poisson operator P_λ amounts to the solution of (2) for f a delta function at a general boundary point, $\bar{y}$. This can be done for a double-zero metric much in terms of Legendre distributions, much as in [10] for scattering metrics. In the present case the construction of a parametrix for P_λ is local, it takes place in a small neighbourhood of the point $\bar{y}$.

Take local coordinates near $\bar{y}$ in which $\bar{y} = 0$ and begin with the 'ansatz'

$$v = (2\pi)^{-n+1} \int \exp(i\frac{y\cdot u}{x^2} - i\frac{\alpha(u)}{x})x^l a(x,y,u)du.$$

Here $\alpha(u)$ is a smooth function of $u \in \mathbb{R}^{n-1}$ to be chosen and similarly $a(x,y,u)$ is a smooth function in $x \geq 0$ with support in $|u| < \frac{1}{2}$, $|y-\bar{y}| < \delta$. Thus the integral certainly converges for $x > 0$. Integration against a 'test function' $\phi \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$ and use of stationary phase shows that

$$\begin{aligned}\int v(x,y)\phi(y)dy &= x^l(2\pi)^{-n+1}\int \exp(i\frac{y\cdot u}{x^2} - i\frac{\alpha(u)}{x})a(x,y,u)dudy \\ &= x^{l+2(n-1)}e^{-i\alpha(0)/x}c(x),\ c \in \mathcal{C}^\infty([0,1)),\ c(0) = a(0,0,0)\phi(0).\end{aligned}$$

Thus the 'initial conditions' to give the correct incoming coefficient are

$$l = -(n-1),\ \alpha(0) = \lambda,\ a(0,0,0) = 1.$$

Moreover, using (3) in applying the Laplacian to v and using integration by parts shows that $(\Delta - \lambda^2)v = g$ is of the same form. If the coordinates are chosen to be Riemmanian normal coordinates in the boundary and to first order off the boundary then

$$
\begin{aligned}
g &= (2\pi)^{-n+1} \int \exp(i\frac{y \cdot u}{x^2} - i\frac{\alpha(u)}{x}) x^l b(x, y, u) du, \\
b(x, y, u) &= ((\alpha + 2u \cdot \partial_u \alpha)^2 + |u|^2 - \lambda^2)a + x^2 b', \ b' \ \mathcal{C}^\infty.
\end{aligned} \tag{8}
$$

Thus the 'eikonal equation' for α is radial in the u variables and can be written in polar coordinates in the form

$$(r^{-\frac{1}{2}}\alpha(r))' = \frac{(\lambda^2 - r^2)^{\frac{1}{2}}}{2r^{\frac{3}{2}}}.$$

This has a unique smooth solution in $|u| < \lambda$ with $\alpha(0) = \lambda$.

Choosing α to be this function removes the leading term in (8). The next term, vanishing quadratically with respect to the first, is given by the transport equation and can be made to vanish by demanding that

$$u \cdot \partial_u a(0, 0, u) = 0 \text{ near } u = 0.$$

Here the constant term has been removed by the choice of $l = -(n-1)$. This also has a unique smooth solution with $a(0, 0, 0) = 1$. The presence of constant terms means that lower order transport equations have unique solutions which are smooth at $u = 0$. Thus, at an essentially formal level it can be arranged that b in (8) vanishes to all orders at $x = 0$.

There is an obstruction to continuing the solution in this form, in the singularity in the phase α which occurs at $|u| = \lambda$. Thus in practice it is necessary to insert a cut off function in a to restrict the support to $|u| \le R < \lambda$. This leads to an error term $(\Delta - \lambda^2)v = g$ with g of the form (8) for $b = x^2 b''$ and b'' supported in an annular region $R - \delta < |u| < R + \delta$ where δ can be chosen appropriately small.

This failure amounts to the appearance of a 'caustic' for the Legendre manifold involved. Rather than deal directly with the invariance properties of oscillatory integrals of the form in the ansatz it is convenient to use the locality of the construction (so far) and to blow up the base point. Thus, instead of considering the solution on X it will be lifted to

$$[X, \{\bar{y}\}]$$

which is just X with the initial point blown up. In the local coordinates introduced above, in which $\bar{y} = 0$, this is simply the introduction of polar coordinates in the variables (x, y),

$$(x, y) = \rho(\omega_0, \omega'), \ \omega \in \mathbb{S}^{n-1}, \ \omega_0 \geq 0.$$

Thus the blown up space is a compact manifold with corner of codimension two. However, the initial parametrix is rapidly decreasing near the 'old' boundary (defined by $\omega_0 = 0$) and so attention can be limited to a neighbourhood of the interior of the new face $\{\rho = 0\}$ introduced by the blow up. In this region the simpler projective coordinates

$$x, Y = \frac{y}{x}$$

are valid.

A straightforward computation shows that in these (singular) coordinates the ansatz above gives a Legendre distribution in the sense of [10] and the Laplacian becomes an elliptic scattering differential operator in the sense of [6]. The metric of which it is the Laplacian is not quite a 'scattering metric' in the sense of [6] however, since it does not have the splitting property demanded there. It is still a smooth non-degenerate metric on the scattering tangent bundle. As a result the construction of the Poisson operator given in [10] can be mimicked to give the Poisson operator here, as a Legendre distribution. The structural difference in the metric leads to a difference in the behaviour of the geodesic flow and hence the Legendre manifold. Namely all the rescaled bicharacteristics starting from the point $Y = 0$ on the incoming radial set have end point at $Y = 0$ on the outgoing radial set. It is not the case that the flow remains in the fibre above $Y = 0$, as it does in the original space, rather there is a natural focussing effect.

The construction of the Poisson operator as a Legendre distribution and the geometry of this Legendre manifold leads directly to the conclusion that the scattering matrix is a pseudodifferential operator, just as in [10] it is seen to be a Fourier integral operator.

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