Eta invariant on articulated manifolds

Spectral Invariants on Non-compact and Singular Spaces
CRM Montreal

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Outline

- I want to talk today about manifolds with corners. This may come as no great surprise to many of you, but I suspect that I have not talked enough about their basic geometry and analysis.
- In this talk I will concentrate on incomplete metrics and the corresponding Dirac operators.
- In fact I will start by (roughly) stating two related conjectures.
- That there should be such conjectures is well-known but perhaps they have not often been stated precisely (and maybe for good reason . . .).
- I want to at least show you that the tools now exist to check whether these are true or not.
- Maybe someone here would like to take up the challenge.

Conjecture (Eta invariant)

Let Y be an odd-dimensional articulated manifold without boundary and suppose $\eth_0$ is an articulated Dirac operator on a unitary Clifford module, V_0, with respect to a smooth incomplete metric then $\eth_0 : H^1(Y; V_0) \longrightarrow L^2(Y; V_0)$ is self-adjoint with discrete spectrum and the associated eta function and eta invariant are well-defined.

For this to make any sense I need to describe what

- An articulated manifold Y is
- An articulated Dirac operator on it is
- Why it might be true.

The case that I do assert that this is true is when Y has articulation of codimension one.

APS package

Conjecture (APS boundary condition)

Let X be an even-dimensional manifold (with corners) and suppose $\eth$ is a Dirac operator on a unitary ($\mathbb{Z}_2$-graded) Clifford module, V, with respect to a smooth incomplete metric then $\eth_+$ induces an articulated Dirac operator $\eth_0$ on $V_0 = V\big|_{\partial X}$ and

$$\eth_+ : \left\{u \in H^{\frac{3}{2}}(X; V_+); \Pi_+(\eth_0)(u\big|_{\partial X}) = 0\right\} \longrightarrow H^{\frac{1}{2}}(X; V_-)$$

is Fredholm with index given by

$$\operatorname{ind}(\eth_+) = \int_X \widehat{A}\,\mathrm{Ch}'(V) + R - \eta(\eth_0).$$

Here R is supposed to be the sum of integrals of a local differential expressions on the boundary faces. I believe this to be true in codimension two as I will explain below.

Manifolds (with corners)

Here is an extrinsic definition, correct but bad. Of course this is really a theorem, a properly defined manifold (with corners) can always be embedded in this sense.

Definition

An embedded compact manifold (with corners) X is a closed subset of a compact manifold without boundary M of the form

$$X = \{p \in M; \rho_i(p) \geq 0 \;\forall\; i \in \{1, \ldots, N\}\}$$

where $\rho_i \in \mathcal{C}^\infty(M)$ are real-valued functions such that for any $I \subset \{1, \ldots, N\}$ and any $p \in M$

$$\rho_i(p) = 0 \;\forall\; i \in I \Longrightarrow \rho_i(p) \text{ are independent in } T^*_p M, \; i \in I.$$

An (incomplete) metric on X is then by definition the restriction to X of a metric on M. The same is true for bundles, differential operators etc.

Articulated manifolds

Here is a similar, perhaps even worse definition.

Definition

A compact articulated manifold without boundary is a (finite union of) component(s) of the boundary of a compact manifold.

- Again this is really a theorem, that an intrinically defined articulated manifold can be embedded in this way.
- So an articulated manifold is really a finite collection of compact manifolds (with corners of course) with their boundary hypersurfaces identified and consistently in higher codimension.
- The absence of boundary is a completeness condition – there are no unmatched hypersurfaces.
- The important point is that an articulated manifold is a wobbly thing – there are no angles between boundary hypersurfaces or anything like that.

50 years ago – Atiyah and Singer

- For an even-dimensional compact manifold without boundary, a Dirac operator $\eth_+ : \mathcal{C}^\infty(X; V_+) \longrightarrow \mathcal{C}^\infty(X; V_-)$ is an elliptic differential operator of first order, so Fredholm:

$$\mathrm{Nul}(\eth_+) \subset \mathcal{C}^\infty(X; V_+),\ \mathrm{Nul}(\eth_-) = (\mathrm{Ran}(\eth_+))^\perp$$

  are finite-dimensional.
- The index is computable:-

$$\mathrm{ind}(\eth_+) = \dim \mathrm{Nul}(\eth_+) - \dim \mathrm{Nul}(\eth_-) = \int_X \widehat{A}\,\mathrm{Ch}'\,.$$

- In fact in this form, with the twisting Chern character of the Clifford module, the index theorem is due to Berline, Getzler and Vergne[4].

35 years ago – Atiyah, Patodi and Singer

- For a Dirac operator on an odd-dimensional compact manifold, the eta invariant, is well-defined in terms of the heat kernel by

$$\eta(\eth_0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{Tr}\left(t^{-\frac{1}{2}} \eth_0 \operatorname{ext}(-it\eth_0^2)\right) dt.$$

- A Dirac operator on a compact even-dimensional manifold with boundary induces a self-adjoint Dirac operator on the boundary; let $\Pi_+(\eth_0)$ be the projection onto its positive part.
- The operator with APS boundary condition

$$\eth_+ : \left\{u \in \mathcal{C}^\infty(X; V_+); \Pi_+(\eth_+)(u\big|_{\partial X}) = 0\right\} \longrightarrow \mathcal{C}^\infty(X; V_-)$$

is Fredholm with index

$$\operatorname{ind}_{\mathrm{APS}}(\eth_+) = \int \widehat{A}\, \mathrm{Ch}' + R - \eta(\eth_0).$$

- If the operator is a product to first order at the boundary, $R = 0$.

Calderón's sequence

- The work of Calderón on boundary problems gives a very clean approach to understanding the APS theorem.
- Suppose given a linear, elliptic differential operator with smooth coefficients on a compact manifold with boundary $D : \mathcal{C}^\infty(X; V_+) \longrightarrow \mathcal{C}^\infty(X; V_-)$.
- I will assume that all bundles carry inner products and that a metric has been chosen
- In particular D has a formal adjoint $D^* : \mathcal{C}^\infty(X; V_-) \longrightarrow \mathcal{C}^\infty(X : V_+)$.
- Let $\dot{\mathcal{C}}^\infty(X; V) \subset \mathcal{C}^\infty(X; V)$ be the closed subspace of elements which vanish in Taylor series at the boundary then

$$\mathrm{Nul}(D; \mathcal{C}^\infty) \longrightarrow \mathcal{C}^\infty(X; V_+) \xrightarrow{D} \mathcal{C}^\infty(X; V_-) \longrightarrow \mathrm{Nul}(D^*; \dot{\mathcal{C}}^\infty)$$

is exact with $\mathrm{Nul}(D^*; \dot{\mathcal{C}}^\infty) = \mathrm{Nul}\left(D^* : \dot{\mathcal{C}}^\infty(X; V_-) \longrightarrow \dot{\mathcal{C}}^\infty(X; V_+)\right)$

.

Calderón projector

- The null space of the restriction to the boundary of smooth solutions in the interior is finite dimensional

$$\mathrm{Nul}(D;\dot{\mathcal{C}}^\infty) \longrightarrow \mathrm{Nul}(D;\mathcal{C}^\infty) \xrightarrow{\big|_{\partial X}} \mathcal{C}^\infty(\partial X;V_+).$$

- Calderón showed that there is a projection precisely onto the range of this restriction which is a pseudodifferential operator

$$\Pi_C \in \Psi^0(\partial X;V_+),\ \Pi_C : \mathcal{C}^\infty(\partial X;V_+) \longrightarrow \mathrm{Nul}(D;\mathcal{C}^\infty)\big|_{\partial X}.$$

- For instance this is the case for the self-adjoint projection with respect to a choice of metrics and inner products.
- For any choice,

$$\mathrm{Ran}(\sigma_0(\Pi_C)) = \mathrm{Ran}_+(\sigma_1(D_0)),$$

the range of the symbol is always the span of the generalized eigenvectors of the symbol of D_0 in the right half plane where

$$D = N(\partial_x - iD_0) \text{ at } \partial X;\ \ D_0 \in \mathrm{Diff}^1(\partial X;V),\ x=0 \text{ at } \partial X.$$

Jumps formula – boundary case

- Consider the null space on extendible distributions on M

$$\mathrm{Nul}(D;\mathcal{C}^{-\infty}) = \{u \in \mathcal{C}^{-\infty}(X;V_+); Du = 0\},$$
$$\mathcal{C}^{-\infty}(X;V_+) = \dot{\mathcal{C}}^{\infty}(X;V_+)'.$$

- Partial hypoellipticity up to the boundary implies that the restriction to the boundary is well-defined (as are higher normal derivatives),

$$\mathrm{Nul}(D;\mathcal{C}^{-\infty}) \ni u \longmapsto Bu = u\big|_{\partial X} \in \mathcal{C}^{-\infty}(\partial X;V_+).$$

- The 'jumps formula' is also a consequence of this:- There is a unique $v \in \mathcal{C}^{-\infty}(X;V_+)$ such that

$$v = 0 \text{ in } M \setminus x,\ \ v = u \text{ on } X \setminus \partial X$$
$$Pv = w\delta(\rho) \text{ and } w = -i\sigma(D)(d\rho)(Bu).$$

Jumps and projector – boundary case

- Now assume (for simplicity) that $D = \eth_+$, is the restriction of a Dirac operator on the whole of $M \supset X$ and that $\eth : \mathcal{C}^\infty(M; V_+) \longrightarrow \mathcal{C}^\infty(M; V_-)$ is an isomorphism.
- Then we get an explicit Calderón projector as

$$
\begin{array}{ccccc}
\mathcal{C}^\infty(\partial X; V_+) \ni v & & \xrightarrow{\quad \Pi_C \quad} & & \Pi_C v \in \mathcal{C}^{-\infty}(\partial X; V_+) \\
\downarrow & & & & \uparrow B \\
-i\sigma(D)(d\rho)v \otimes \delta(\rho) & \xrightarrow{\eth^{-1}} & \mathcal{C}^{-\infty}(M; V_+) & \xrightarrow{|_{X \setminus \partial X}} & \mathrm{Nul}_X(\eth_+; \mathcal{C}^{-\infty})
\end{array}
$$

- In the general case one needs only do a little more work.

- I want to try to convince you of the existence of such a picture in the general case of a compact manifold with (non-trivial) corners.
- The spaces $\dot{\mathcal{C}}^\infty(X; V)$ with dual $\mathcal{C}^{-\infty}(X; V)$ and $\mathcal{C}^\infty(X; V)$ with dual $\dot{\mathcal{C}}^{-\infty}(X; V)$ are well-defined (metrics everywhere) and in terms of an extension $X \subset M$

$$\dot{\mathcal{C}}^\infty[\text{resp } \dot{\mathcal{C}}^{-\infty}](X; V) = \{u \in \mathcal{C}^\infty[\text{resp } \mathcal{C}^{-\infty}](X; V); \operatorname{supp}(u) \subset X)$$
$$\mathcal{C}^\infty[\text{resp } \mathcal{C}^{-\infty}](X; V) = \mathcal{C}^\infty[\text{resp } \mathcal{C}^{-\infty}](M; V)\big|_{X \setminus \partial X}.$$

- So let $\eth_+ : \mathcal{C}^\infty(X; V_+) \longrightarrow \mathcal{C}^\infty(X; V_-)$ be a Dirac operator, this makes all the pesky finite-dimensional $\operatorname{Nul}(\eth_\pm; \dot{\mathcal{C}}^\infty)$ trivial.
- In particular surjectivity holds

$$\operatorname{Nul}(\eth+; \mathcal{C}^{-\infty}) \longrightarrow \mathcal{C}^{-\infty}(X; V_+) \xrightarrow{D} \mathcal{C}^{-\infty}(X; V_-)$$

- So the whole issue is to define B and Π_C.

- Although partial hypoellipticity fails we can still use a variant of the jumps formula to define B.
- There is a surjective restriction map

$$\dot{\mathcal{C}}^{-\infty}(M; V) \longrightarrow \mathcal{C}^{-\infty}(M; V)$$

  with null space the distributions supported by the boundary; $u \in \operatorname{Nul}(\eth_+; \mathcal{C}^{-\infty})$ can be extended to M to vanish outside X.
- In fact there is always such a 'zero extension' $v \in \dot{\mathcal{C}}^{-\infty}(X; V_+)$ with

$$\eth_+(v) = \sum_H v_H \otimes \delta(\rho_H), \; v_H \in \dot{\mathcal{C}}^{-\infty}(H; V_-) \qquad (1)$$

- Here, each boundary hypersurface H has a defining function ρ_H and the space on the right is a well-defined in $\dot{\mathcal{C}}^{-\infty}(X; V_-)$.
- However, there are two problems, the zero extension – even with this property – is not unique and nor are the 'boundary values' v_H (even fixing the ρ_H which we can. So the *presentation* (1) is also not unique; the crucial question is just how non-unique.

Formal boundary data

- To answer this we now switch to the 'formal smooth theory'.
- Think of ∂X as an articulated manifold – the union of the boundary hypersurfaces with only their boundaries identified in the obvious way. Then the 'smooth' sections of a bundle over ∂X are

$$\mathcal{C}^\infty(\partial X; V) = \left\{ u_i \in \mathcal{C}^\infty(H_i; V); u_i\big|_{H_i \cap H_j} = u_j\big|_{H_i \cap H_j} \right\} = \mathcal{C}^\infty(M; V)\big|_{\partial X}.$$

- As remarked above, this space is 'too big' in the sense that there are no compatibility conditions for the *normal derivatives* at intersections of boundary faces.
- However, a first order elliptic differential operator, gives rise to much smaller subspace of 'compatible' sections

$$\mathcal{C}^\infty_D(\partial X; V_+) = \{u \in \mathcal{C}^\infty(X; V_+); Du \in \dot{\mathcal{C}}^\infty(X; V_+)\}\big|_{\partial X} \subset \mathcal{C}^\infty(Y; V_+).$$

Properties of $\mathcal{C}^\infty_D$.

Lemma

For an elliptic differential operator on a compact manifold (with corners) $D \in \mathrm{Diff}^1(X; V_+, V_-)$ restriction to any one of the of the boundary hypersurfaces defines a surjective map

$$\mathcal{C}^\infty_D(\partial X; V_+) \xrightarrow{\;\big|_H\;} \mathcal{C}^\infty(H; V_+),\ H \in \mathcal{M}_1(M),$$

and there is a natural extension giving an injective map

$$\bigoplus_{H\in\mathcal{M}_1(M)} \dot{\mathcal{C}}^\infty(H; V_+) \hookrightarrow \mathcal{C}^\infty_D(\partial X; V_+). \tag{2}$$

- Note that ∂X can be 'smoothed' (more like annealed!) to a compact manifold without boundary

$$\tilde{H} = \{p \in X; \prod_H \rho_H = \epsilon\}, \ \epsilon > 0 \text{ small.}$$

- Then $\mathcal{C}^\infty_D(\partial X; V_+)$ 'looks' like $\mathcal{C}^\infty(\tilde{H}; V)$ in the sense that the Taylor series at any boundary point coming from one boundary hypersurface determines the Taylor series at any others.
- This new space is not a module of $\mathcal{C}^\infty(\partial X)$.
- On the other hand, it does have a topology very similar to that of $\mathcal{C}^\infty(\tilde{H}; V)$ such that the maps in (2) are continuous.
- The dual space $\mathcal{C}^{-\infty}(\partial X; V_+)$ is similar to $\mathcal{C}^{-\infty}(\tilde{H}; V_+)$.

Properties of $\mathcal{C}_D^{-\infty}$.

Lemma

The topological dual $\mathcal{C}_D^{-\infty}(\partial X; V_+)$ comes equipped with a natural surjection to extendible distributions on the boundary hypersurfaces

$$\mathcal{C}_D^{-\infty}(\partial X; V_+) \longrightarrow \bigoplus_{H\in\mathcal{M}_1(X)} \mathcal{C}^{-\infty}(H; V_+)$$

and injections on supported distributions for each $H \in \mathcal{M}_1(M)$

$$\dot{\mathcal{C}}^{-\infty}(H; V_+) \hookrightarrow \mathcal{C}_D^{-\infty}(\partial X; V_+)$$

such that the collective map is surjective

$$[\cdot] : \bigoplus_{H\in\mathcal{M}_1(X)} \dot{\mathcal{C}}^{-\infty}(H; V_+) \twoheadrightarrow \mathcal{C}_D^{-\infty}(\partial X; V_+).$$

This space answers the question of just how well-defined the boundary data for the null space of an elliptic operator on a compact manifold with corners is where now we have a boundary pairing which gives

$$\mathcal{C}_D^{-\infty}(\partial X; V_+) = (\mathcal{C}_{D^*}^{\infty}(\partial X; V_-))'.$$

Theorem

With the global hypotheses above on the first order elliptic differential operator D, there is a well-defined injective boundary map B giving a commutative diagram

$$\mathrm{Nul}(D; \mathcal{C}^{-\infty}) \xrightarrow{B} \mathcal{C}_D^{-\infty}(\partial X; V_+)$$

$$\left\{ v \in \dot{\mathcal{C}}^{-\infty}(X; V_+),\ \eth_+ v = \sum_H -i\sigma(D)(d\rho_J) w_H \otimes \delta(\rho_H) \right\} \longmapsto [w_H]$$

Calderón projector, corners case

This in turn allows us to define the Calderón projector as in the case of a manifold with boundary except for the extra algebraic overhead

$$\Pi_C : \mathcal{C}_D^{-\infty}(\partial X; V_+) \longrightarrow \mathcal{C}_D^{-\infty}(\partial X; V_+) \text{ by}$$

$$\Pi_C([w_H]) = B\left(D^{-1}\big(\sum_H -i\sigma(d\rho_H)w_H\delta(\rho_H)\big)\Big|_X\right).$$

Theorem

The Calderón projector is a continuous projection on $\mathcal{C}_D^{-\infty}(\partial X; V_+)$ *and has range precisely equal to the range of B which maps* $\mathrm{Nul}(D; \mathcal{C}^{-\infty})$ *injectively into* $\mathcal{C}_D^{-\infty}(\partial X; V_+)$.

- This Calderón projector is as close to being a pseudodifferential operator as one could expect on an articulated manifold. Namely, it consists of pseudodifferential operators on each of the hypersurfaces plus 'Poisson' type operators between them.
- In particular, it preserves $\mathcal{C}^{\infty}_{D}(\partial X; V_{+})$, even though the pseudodifferential pieces do not satisfy the transmission condition. The singularities are cancelled by the Poisson pieces.
- These results should extend to the general case where D is not assumed to either have the extension property or the unique continuation property.
- The extension to higher order systems would be a more serious pain!

- Continuing under the global assumptions, observe that for $t \in \mathbb{R}$, $|t| < \frac{1}{2}$, and on any compact manifold with corners, the extendible and supported Sobolev spaces are identified

$$\dot{H}^t(H; V) = (H^{-t}(H; V))' \equiv H^t(H; V),\ -\frac{1}{2} < t < \frac{1}{2}.$$

- That is, each element of these Sobolev spaces has a unique zero extension with the same regularity (with which it can therefore be identified).
- In view of the properties of the spaces discussed above it follows that

$$\bigoplus_{H \in \mathcal{M}_1(X)} H^t(H; V_-) \subset \mathcal{C}_D^{-\infty}(\partial X; V_+),\ -\frac{1}{2} < t < \frac{1}{2}$$

are well-defined subspaces for any elliptic first-order D.

- The regularity properties of D^{-1} show that that

$$\Pi_C \text{ acts on } \bigoplus_{H \in \mathcal{M}_1(M)} H^t(H; V_+), \ -\frac{1}{2} < t < \frac{1}{2},$$

with range precisely the boundary restrictions of

$$\mathrm{Nul}_s(D) = \{u \in H^s(X; V_+); Du = 0\}, \ s = t + \frac{1}{2}.$$

- Thus, for instance, for $\frac{1}{2} < s < 1$ there is a short exact sequence

$$\{U \in H^{s-\frac{1}{2}}(\partial X; V_+); \Pi_C U = U\} \longrightarrow H^s(X; V_+) \xrightarrow{D} H^{s-1}(H; V_-).$$

where the first map is a Poisson operator.

- For Dirac operators 'restriction' to a boundary hypersurface is functorial - giving a Dirac operator $\eth_H$ on each $H \in \mathcal{M}_1(X)$.
- This involves the product decomposition near a hypersurface in terms of the distance, in which the metric decomposes as

$$g = dx^2 + x^2 h(x),\ h(x) \text{ a family of metric on } H.$$

- There is no (simple) analogue of this in codimension two.
- Nevertheless the double restriction, from $\eth_+$ on X to a boundary face of codimension two is consistent (with change of orientation)

$$(\eth_H)_{H\cap G} + (\eth_G)_{H\cap G} = 0. \tag{1}$$

- This is what is meant above by a Dirac operator on an articulated manifold – on each boundary hypersurface there is a Dirac operator $\eth_H$ associated to a metric and a Clifford module (and unitary Clifford connection). The bundles and metrics must be consistent on the intersection faces of codimension two – from either side one gets the same restriction – and the Clifford modules are consistent in the sense of (1).

This is enough to give sense to the 'Eta invariant' conjecture.

Conjecture (Eta invariant)

Let Y be an odd-dimensional articulated manifold without boundary and suppose $\eth_0$ is an articulated Dirac operator on a unitary Clifford module, V_0, with respect to a smooth incomplete metric then $\eth_0 : H^1(Y; V_0) \longrightarrow L^2(Y; V_0)$ is self-adjoint with discrete spectrum and the associated eta function and eta invariant are well-defined.

- I claim this is true for an articulated manifold with intersection faces only of codimension one – this is close to the boundary case.
- One can get a parametrix, in the sense of an inverse modulo compact errors by summing the generalized inverse of the APS problem on each boundary hypersurface (there is an odd/even switch here).
- In particular the projection onto the positive part makes sense.

APS package

Conjecture (APS boundary condition)

Let X be an even-dimensional manifold (with corners) and suppose $\eth$ is a Dirac operator on a unitary ($\mathbb{Z}_2$-graded) Clifford module, V, with respect to a smooth incomplete metric then $\eth_+$ induces an articulated Dirac operator $\eth_0$ on $V_0 = V\big|_{\partial X}$ and

$$\eth_+ : \left\{u \in H^{\frac{3}{2}}(X; V_+); \Pi_+(\eth_0)(u\big|_{\partial X}) = 0\right\} \longrightarrow H^{\frac{1}{2}}(X; V_-)$$

is Fredholm with index given by

$$\operatorname{ind}(\eth_+) = \int_X \widehat{A}\,\mathrm{Ch}'(V) + R - \eta(\eth_0).$$

The existence of Π_+ follows from the discussion above in case X has boundary of codimension two. The Fredholm property should follow from a symbolic analysis of the two projections, Calderón and APS.

Final remarks

- A lot of this is conjectural, but the case of X of codimension two is surely within reach.
- There is the possibility of induction over boundary codimension.
- If this is all too easy for you, try the 'annealing limit' as $\epsilon \downarrow 0$, passing from a manifold with boundary to the general case.
- I have not given references but there is a large literature related to this subject – but not the Calderón projector as far as I know.

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