Analysis on Loop Spaces

Richard Melrose

Department of Mathematics, Massachusetts Institute of Technology

Email address: rbm@math.mit.edu

February 8, 2021

Contents

Preface

A main objective of these lectures, originally planned for Fall 2020, but 'delivered' in the difficult conditions of the Covid19 pandemic of Winter/Spring 2021, is to answer the question:- Does the Dirac-Ramond operator exist and can we work with it? I have thought about this question over a period of some years and I hope to relate here what I know. My usual joke is that the Dirac-Ramond operator is like the 'Tasmanian Tiger' () – there have been many claimed sightings but most if these turn out to be dogs. I leave it to you, gentle reader, to come to your own conclusion.

The answer to the first question is yes, although the definition is still 'weak' (in a technical sense). The answer to the second is still a little unclear but my hope is that these notes will shed a little light on that too. To define *the* Dirac-Ramond operator – which is on the loop space of a string manifold – I need to go through a rather substantial preparation which includes differential analysis (function spaces, operators), differential topology (string structures, gerbes, transgression), representation theory (loop and diffeomorphism groups), index theory (including the Witten genus) and more.

Introduction

Initially I will give a 'colloquim style' discussion of the background leading to the definition of the (or really a) Dirac-Ramond operator. If you understand everything here you should probably be giving the lectures. By the end of the semester I hope everything will be adequately described.

The idea of a differential operator on an infinite-dimensional space arose quite early in the development of string theory as an analogue of the spin Dirac operator; in this case it is intended to describe 'spinning strings.' However there are substantial mathematical difficulties which have obstructed the precise definition of this as an operator, and some of these issues remain. I hope to convince you that some progress has been made and that there is interesting Mathematics in what might otherwise be thought of as a quixotic enterprise.

Let me start with a 'topological description of geometric structures', in particular spin structures. Consider the Whitehead (my erstwhile colleague George) tower for the group $\mathrm{O}(n)$. Here $n > 2$ and it creeps up a bit below, take $n \geq 5$ throughout if you want to be safe from low dimensional annoyances. At some point n might be even as well. So the tower in question is

27.2.2020.1 (1)
$$\begin{tikzcd}
\mathbb{Z}_2 & & \mathbb{Z}_2 \arrow[d] & K(\mathbb{Z},2) \arrow[d] & \\
\mathrm{O}(n) \arrow[u, "\det"] & \mathrm{SO}(n) \arrow[l] & \mathrm{Spin}(n) \arrow[l] & \mathrm{String}(n) \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

The successive maps here 'remove' the lowest homotopy group while keeping the higher ones unchanged. In the first step the map is injective but in higher steps it is surjective. Thus $\mathrm{O}(n)$ has two components, '$\pi_0 = \mathbb{Z}_2$,' then $\pi_1 = \mathbb{Z}_2$ as well then $\pi_2 = \{\mathrm{Id}\}$, $\pi_3 = \mathbb{Z}$ and I'm not going to talk about the higher groups (look up 'fivebrane' if you want to know). All the spaces here are, or really can be taken to be, topological groups but they are actually only determined up to homotopy equivalence.

What is the relation of this to geometry? A smooth (finite-dimensional) manifold, which is really what we am interested in throughout, has a tangent bundle which, being a vector bundle, has a frame bundle – the elements F_p at each point

Version of February 8, 2021

$p \in M$, are just the bases of the tangent space T_pM. Here F is a principal $\mathrm{GL}(n,\mathbb{R})$-bundle with the action being change of basis

27.2.2020.2 (2)
$$\begin{tikzcd} \mathrm{GL}(n,\mathbb{R}) \arrow[r, dash] & F \arrow[d] \\ & M. \end{tikzcd}$$

We can recover TM as the bundle associated to the standard representation of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$

27.2.2020.3 (3)
$$TM = FM \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n.$$

Equipping M with a Riemann metric, as we always can, reduces the structure group from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$ by taking the orthonomal frames

27.2.2020.4 (4)
$$\begin{tikzcd} \mathrm{O}(n) \arrow[r, dash] & F_{\mathrm{O}} \arrow[r, hook] \arrow[d] & F \arrow[dl] \\ & M. & \end{tikzcd}$$

Thus we are at the bottom of the tower.

Now the first horizontal arrow in (27.2.2020.1) corresponds to the existence, and choice, of an orientation on the manifold. Let me spell this out explicitly in one way for later reference. An orientation is a reduction of the structure group from $\mathrm{O}(n)$ to $\mathrm{SO}(n)$, a subbundle of consistently oriented orthonormal frames

27.2.2020.5 (5)
$$\begin{tikzcd} \mathrm{SO}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{SO}} = o^{-1}(1) \arrow[d] & \\ \mathrm{O}(n) \arrow[r, dash] & F_{\mathrm{O}} \arrow[r, "o"] \arrow[d] & \mathbb{Z}_2 \\ & M. & \end{tikzcd}$$

One way to specify an orientation is to give a continuous map o to $\mathbb{Z}_2$, as indicated, with the property that it takes both values on each fibre. An orientation exists if and only if the first Stieffel-Whitney class vanishes and then, assuming as I will, that M is connected, there are two choices.

The next step in the Whitehead tower corresponds to a spin structure on the manifold. Here $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$ and is simply connected; it is a compact Lie group. Let me pause to indicate one relevant construction of it. Namely consider the path and loop groups of $\mathrm{SO}(n)$, these are major characters in this story. For the moment we can take continuous 'pointed' paths and loops – and we might as well consider a general (connected) Lie group

27.2.2020.6 (6)
$$\begin{gathered} \dot{\mathcal{P}}(G)) = \{\chi : [0,\pi] \longrightarrow G, \text{ continuous and with } \chi(0) = \mathrm{Id}\} \\ \dot{\mathcal{L}}(G) = \{\lambda : \mathbb{S} \longrightarrow G, \text{ continuous and with } \lambda(0) = \mathrm{Id}\}. \end{gathered}$$

Here I am thinking of the circle as $\mathbb{R}/2\pi\mathbb{Z}$ so the loops λ are 2π-periodic maps from the line. The reason I take π to be the parameter length for paths will show up below, it is simply a normalization.

Now $\dot{\mathcal{P}}(G)$ and $\dot{\mathcal{L}}(G)$ are groups under pointwise composition and there is a short exact sequence of groups

27.2.2020.7 (7)
$$\begin{tikzcd} \dot{\mathcal{L}}(G) \arrow[r] & \dot{\mathcal{P}}(G) \arrow[d] \\ & G \end{tikzcd}$$

where the last map is evaluation at the endpoint π. The kernel of this homorphism is the subgroup of pointed paths with endpoint at Id; halving the parameter allows this to be identified with $\dot{\mathcal{L}}(G)$; so my choice of normalization is not for this reason!

The path space is contractible, through path shortening, and as a result (27.2.2020.7 7) is a classifiying sequence so

27.2.2020.24 (8)
$$G = B\dot{\mathcal{L}}(G) \text{ and } \pi_j(\dot{\mathcal{L}}(G)) \simeq \pi_{j+1}(G).$$

Returing to the orthogonal group, the statement that $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ therefore means that $\dot{\mathcal{L}}(\mathrm{SO}(n)$ has two components and then

27.2.2020.8 (9)
$$\mathrm{Spin}(n) = \dot{\mathcal{P}}\,\mathrm{SO}(n))/\dot{\mathcal{L}}_{\mathrm{Id}}(\mathrm{SO}(n))$$

identifies paths if they are homotopic through paths with the same endpoint.

The question of the existence of a spin structure on M is the search for an extension of the oriented frame bundle

27.2.2020.9 (10)
$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{Spin}} \arrow[d] \\ \mathrm{SO}(n) \arrow[r, dash] & F_{\mathrm{SO}} \arrow[d] \\ & M. \end{tikzcd}$$

Such a principal bundle exists if and only if the second Stieffel-Whitney class vanishes.

Since the objective is generalize it, let me now remind you of the Spin Dirac operator – let's take the dimension to be even, $n = 2m$. The spin group has a fundamental representation of dimension 2^m coming from the identification $\mathrm{Spin}(2m) \subset \mathrm{Cl}_{\mathbb{C}}(2m) \simeq M(2^m)$ of the complexified Clifford algebra with the corresponding matrix algebra. This induces a bundle, the spinor bundle, $S = S^+ \oplus S^-$ over M with grading coming from the two irreducible parts of the spin representation.

Now the bundle F_{spin} is a double cover, so the Levi-Civita connection lifts from F_{SO} to a connection and induces a connection ∇ on S. The spin action corresponds to an action of the bundle of Clifford algebras $\mathrm{Cl}_{\mathbb{C}}(T^*M)$ on S,

27.2.2020.10 (11)
$$\mathrm{cl} : T^*M \longrightarrow \mathrm{GL}(S)$$

and combining these leads to the definition of the spin Dirac operator

27.2.2020.11 (12)
$$\eth_{\mathrm{Spin}} = \begin{pmatrix} 0 & \eth^- \\ \eth^+ & 0 \end{pmatrix} : \mathcal{C}^\infty(M;S) \longrightarrow \mathcal{C}^\infty(M;S),\ \eth_{\mathrm{Spin}} = \mathrm{cl} \circ \nabla.$$

The spin Dirac operator is elliptic, hence Fredholm and its graded index, computed as part of the Atiyah-Singer index theorem,

27.2.2020.12
$$\operatorname{ind}(\eth_{\mathrm{Spin}}) = \dim \operatorname{Nul}(\eth^+) - \dim \operatorname{Nul}(\eth^-) = \int_M \widehat{A} \tag{13}$$

is the $\widehat{A}$ genus of M. This was one of the early achievements of Atiyah and Singer, explaining the integrality of the $\widehat{A}$ genus for spin manifolds (which was known previously).

So, it is this we are trying to 'emulate' at the next step up, for string structures. Before proceeding in this way, let me describe the 'trangression' of spin structures.

Consider now the free (rather than the pointed) loop and path spaces spaces now for a general manifold M :

27.2.2020.13
$$\begin{gathered} \mathcal{P}(M) = \{\chi : [0, \pi] \longrightarrow M,\ \text{continuous}\} \\ \mathcal{L}(M) = \{\lambda : \mathbb{S} \longrightarrow M,\ \text{continuous}\}. \end{gathered} \tag{14}$$

Each path has two endpoints and (M being assumed connected) and the loop space has a similar map by evaluation at 1 and $-1 \in \mathbb{S}$

AnLoSp.1
$$\mathcal{P}(M) \longrightarrow M^2,\ \mathcal{L}(M) \longrightarrow M^2. \tag{15}$$

Both are fibre bundles; these are important later.

Loop and path spaces have functorial properties arising by pull-back along the defining maps. For instance, if F is a principal G-bundle over M then there are corresponding principal bundles arising from pull-back

27.2.2020.14
$$\begin{tikzcd} \mathcal{P}(G) \arrow[r] & \mathcal{P}(F) \arrow[d] & \mathcal{L}(G) \arrow[r] & \mathcal{L}(F) \arrow[d] \\ & \mathcal{P}(M) & & \mathcal{L}(M). \end{tikzcd} \tag{16}$$

Here the fibre at a path or loop consists of all the paths/loops into F which 'cover' the given map into M.

In particular for an oriented manifold, as was observed by Atiyah in the 1980s,

A spin structure on M induces an 'orientation' on $\mathcal{L}(M)$.

Note that the latter notion is not clearly defined (because whatever the tangent space to the loop space is, it is infinite-dimensional) but we find a picture very reminiscent of the finite dimensional case

27.2.2020.15
$$\begin{tikzcd} \mathcal{L}(\mathrm{Spin}(n)) \arrow[r, dash] \arrow[d] & \mathcal{L}(F_{\mathrm{Spin}}) \arrow[d] & \\ \mathcal{L}(\mathrm{SO}(n)) \arrow[r, dash] & \mathcal{L}(F_{\mathrm{SO}}) \arrow[r, "o_{\mathrm{Spin}}"] \arrow[d] & \mathbb{Z}_2 \\ & \mathcal{L}(M). & \end{tikzcd} \tag{17}$$

Here o_{Spin} is ± 1 on a given loop in F_{SO} as it is, or is not, the image of a loop in $\mathcal{L}(F_{\mathrm{Spin}})$ – it is a continuous map which takes both values on each fibre.

Conversely, it was show by McLaughlin that if M is 2-connected, connected with $\pi_1(M) = \{0\}$, then the converse is true, but not without some such restriction. The relationship was finally clarified by Stolz and Teichner around 2005. They observed

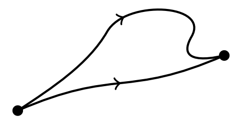

Figure 1. Joining paths to a loop F:Join

that the orientation in (17) induced by a spin structure has an additional property corresponding to the endpoint maps (15). To see this, form the fibre products with respect to this map

27.2.2020.17 (18)
$$\mathcal{P}^{[k]}(M)=\{(\chi_1,\dots,\chi_k)\in(\mathcal{P}(M))^k;\chi_1(0)=\cdots=\chi_k(0),\ \chi_1(\pi)=\cdots=\chi_k(\pi)\}.$$

These form a simplicial space where the 'face' maps are just the maps forgetting one of the paths

27.2.2020.18 (19)
$$\begin{tikzcd}\mathcal{P}(M) & \mathcal{P}^{[2]}(M) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, leftrightarrow] & \mathcal{P}^{[3]}(M) \arrow[l, shift left=2, "f_*"'] \arrow[l] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=3] \arrow[l, shift left] \arrow[l, shift right] \arrow[l, shift right=3] \\ & \mathcal{L}(M). & & \end{tikzcd}$$

So this consists of all the k-tuples of paths with the same initial and terminal endpoints.

The vertical bijection here is by 'joining' paths. If two paths χ_1,χ_2 have the same endpoints then traversing the first and then, in reverse and with parameter renormalized, the second gives a loop

LC.4 (20)
$$\mathcal{P}^{[2]}_{\mathcal{C}}M\ni(\chi_1,\chi_2)\longmapsto\lambda_{12}\in\mathcal{L}_{\mathcal{C}}(M)$$

and conversely a loop can be divided into 2 paths with the same endpoints by shifting and reversing the second half of a loop..

Applying this construction to $\mathcal{L}F_{\mathrm{SO}}$ gives three pull-back maps and hence the simplicial differential

LC.5 (21)
$$\begin{gathered}\mathcal{C}(\mathcal{L}F_{\mathrm{SO}};\mathbb{Z}_2)\ni o_{\mathrm{Spin}}\longmapsto f_i^*o_{\mathrm{Spin}}\in\mathcal{C}(\mathcal{L}F_{\mathrm{Spin}};\mathbb{Z}_2)\\ \delta o_{\mathrm{Spin}}=f_3^*o_{\mathrm{Spin}}(f_2^*o_{\mathrm{Spin}})^{-1}f_1^*o_{\mathrm{Spin}}=o_{\mathrm{Spin}}(\lambda_{12})(o_{\mathrm{Spin}}(\lambda_{13}))^{-1}o_{\mathrm{Spin}}(\lambda_{23})=1\end{gathered}$$

(where in the case of a $\mathbb{Z}_2$-valued function inversion does nothing). Stolz and Teichner showed that

> Spin structures on M are in 1-1 correspondence with fusion orientation structures on $\mathcal{L}(M)$, in the sense that $\delta o_{\mathrm{Spin}}=1$ on $\mathcal{L}(M)$.

QSpin

Even though this is correct, it is a little misleading as regards subsequent developments because of the discreteness of $\mathbb{Z}_2$. The important point is that 'objects' on M are often in 1-1 correspondence with 'fusive' objects on $\mathcal{L}(M)$.

So far I have considered continuous paths and loops but it is important to consider other regularity, with

1.3.2020.1 (22)
$$\begin{gathered}\mathcal{P}_\infty(M)=\{\lambda:[0,\pi]\longrightarrow M;\ \text{infinitely differentiable}\}\\ \mathcal{L}_\infty(M)=\{\lambda:\mathbb{S}\longrightarrow M;\ \text{infinitely differentiable}\}\end{gathered}$$

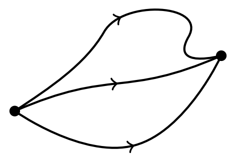

FIGURE 2. Fusion of three paths to three loops F:figure

perhaps the most natural. In fact the Sobolev-regular, for $s > \frac{1}{2}$, and Lipschitz-regular paths and loops are also significant. These are well-defined as spaces of maps into a finite-dimesional manifold because the corresponding spaces of functions on $[0, \pi]$ or $\mathbb{S}$ are '$\mathcal{C}^\infty$ algebras'. Not only are they closed under the product of functions but if f_i are n of these maps into $\mathbb{R}$ collectively taking values in an open set $\Omega \subset \mathbb{R}^n$ and $F : \Omega \longrightarrow \mathbb{R}$ is a smooth function on Ω then the composite $F \circ f_*$ is in the space. This allows one to define the varous path and loop spaces with dense inclusions

1.3.2020.2 (23)
$$\begin{tikzcd} & \mathcal{P}_s(M) \arrow[dr, hook] & \\ \mathcal{P}_\infty(M) \arrow[ur, hook] \arrow[dr, hook] & & \mathcal{P}_{\mathcal{C}}(M) \\ & \mathcal{P}_\Lambda(M) \arrow[ur, hook] & \end{tikzcd}$$

Then $\mathcal{P}_{\mathcal{C}}(M)$ and $\mathcal{P}_\Lambda(M)$ are Banach manifolds, the $\mathcal{P}_s(M)$ are Hilbert manifolds and $\mathcal{P}_\infty(M)$ is a Fréchet manifold. Assuming M to be smooth they are in fact all $\mathcal{C}^\infty$ manifolds in the appropriate sense. To see why this is so, take a metric on M and, for any $\epsilon > 0$ smaller than the injectivity radius, consider the covering by these open metric balls. The exponential map at each $p \in M$ identifies the ball with the ball of the same radius around the origin in T_pM and the composites $F_{pq} = \exp_p^{-1} \exp_q$ give smooth transition maps.

Now for a given element $\chi \in \mathcal{P}_{\mathcal{C}}(M)$ consider the paths

1.3.2020.3 (24)
$$\mathcal{N}(\chi) = \{\chi' \in \mathcal{P}_{\mathcal{C}}(M); d_M(\chi'(t), \chi(t)) < \epsilon \ \forall \ t \in [0, \pi]\}.$$

These are open subsets and pull-back along the base curve gives a bijection

1.3.2020.4 (25)
$$\sigma' = \exp^*_{\chi(t)} \chi', \ \operatorname{Exp}_\chi : \mathcal{N}(\chi) \longleftrightarrow \{\sigma' \in \mathcal{C}([0, \pi]; \chi^* TM); |\sigma'(t)|_g < \epsilon\}$$

with the continuous sections of the pull-back of the tangent bundle under the curve and similarly for the other regularities. Two of the $\mathcal{N}(\chi)$ intersect if and only if for the corresponding base curves $d(\chi_1(t), \chi_2(t)) < \epsilon$ for all $t \in [0, \pi]$. Then the induced transition map

1.3.2020.5 (26)
$$\mathcal{F}_{\chi_2,\chi_1} : \operatorname{Exp}_{\chi_2} \circ \operatorname{Exp}_{\chi_1}^{-1} : \operatorname{Exp}_{\chi_1}(\mathcal{N}(\chi_1) \cap \mathcal{N}(\chi_2)) \longrightarrow \operatorname{Exp}_{\chi_2}(\mathcal{N}(\chi_1) \cap \mathcal{N}(\chi_2))$$

is a diffeomorphism, i.e. it is infinitely differentiable on these open subsets of Banach spaces.

Indeed, the transition maps are non-linear bundle maps. The derivatives of the transition maps on M are symmetric $|\alpha|$-multilinear maps

LC.6 (27)
$$F^\alpha_{pq}(m) : T_qM \times T_qM \times \cdots \times T_qM \longrightarrow T_pM$$

1.3.2020.5

depending smoothly on m in the intersection of the coordinate balls. From (26) the corredponding (weak) derivatives exist for the transition maps on the path or loop spaces given by pointwise action:

LC.7
$$\begin{aligned}(28)\quad \mathcal{F}^{\alpha}_{\lambda_2,\lambda_1} : \mathcal{C}^{\infty}(\mathbb{S}; \lambda_1^* TM) \times \cdots \times \mathcal{C}^{\infty}(\mathbb{S}; \lambda_1^* TM) \ni (\sigma_2', \dots, \sigma'_{|\alpha|}) \\ \longmapsto F_{\lambda_1(*),\lambda_2(*)}(\sigma_1'(*), \dots, \sigma'_{|\alpha|})(*)) \in \mathcal{C}^{\infty}(\mathbb{S}; \lambda_2^* TM)\end{aligned}$$

This corresponds to the fact that the only differentiation which arises is of the transition maps for exp on M. Note that $\chi^* TM$ is trivial as a bundle over the interval. This carries over to the other regularities and to the loop spaces, with the triviality statement for the pull-back under loops being orientability.

It is rather natural to think of these spaces being 'thickenings' of $\mathcal{P}_\infty(M)$ or $\mathcal{L}_\infty(M)$ in which it is dense. In all cases the tangent space at a path or loop is naturally the space of sections, of the corresponding regularity, of $\chi^* TM$. In the standard approach to manifolds the cotangent space would be defined as the dual of the tangent space. Since it is rather natural to think of this as a space of sections of the cotangent space on the manifold and the most natural pairing between sections of TM and sections of T^*M pulled back is

AnLoSp.1a
$$(29)\qquad \int_{\mathbb{S}} \lambda^*(v(s) \cdot w(s)) ds$$

this would realize the cotangent space as the dual space of sections of $\lambda^* T^* M$, i.e. measures for $\mathcal{L}_{\mathcal{C}}(M)$ and distributions for $\mathcal{L}_\infty(M)$. Not only is this 'handist' but it is unwisely prescriptive since in fact on $\mathcal{L}_s(M)$ all the spaces

AnLoSp.2
$$(30)\qquad H^t(\mathbb{S}; \lambda^* TM),\ H^t(\mathbb{S}; \lambda^* T^* M) \text{ for } -s < t < s$$

make invariant sense. Thus the pointwise value of a 'vector field' of a '1-form' on $\mathcal{L}_s(M)$ can be reasonably taken to lie in any one of these spaces. From this it is already clear that there are many notions of regularity of objects over the loop spaces. Note in particular that continuity of a function on $\mathcal{L}_s(M)$ is a stronger statement than continuity on the dense subspace $\mathcal{L}_\infty(M)$.

One reason that the Lipshitz paths and loops are relevant is that (affine) arclength (re-)parameterization of a curve (so the parameter length is still 2π) is well-defined as a map

AnLoSp.3
$$(31)\qquad \mathcal{L}_1(M) \longrightarrow \mathcal{L}_\Lambda(M).$$

Let me make an apparent digression. As Mathematicians we can ask a question that perhaps the Physists do not feel bound to ask themselve. Namely, what precisely is a String? Clearly it is related to a loop, an element of the free loop space, say smooth

27.2.2020.19
$$(32)\qquad \mathcal{L}_\infty(M) = \{\lambda : \mathbb{S} \longrightarrow M,\ \mathcal{C}^\infty\}.$$

The diffeomorphism group of the circle acts on this by reparameterization and one (not quite ideal) definition of a String is that it is an element of the quotient space

27.2.2020.20
$$(33)\qquad \mathcal{L}_\infty(M)/\operatorname{Dff}^+(\mathbb{S})$$

given as the orbits under the action of the orientation-preserving diffeomorphisms of the circle. Thus a String is (perhaps) an 'unparameterized' loop. Of course taking a quotient like this is dangerous since the diffeomorphism group does not act freely, so the quotient is bound to be rather singular. Instead, as is done in many contexts, we look for $\operatorname{Dff}^+(\mathbb{S})$-invariant or -equivariant objects on $\mathcal{L}(M)$ and think of them as objects on the quotient space. So, it is important for us to understand the action of the reparameterization group.

All this suggests that we can 'transgress' things from M to $\mathcal{L}(M)$ without loss of information if we are careful. One version of this is coholomogy. There is an evaluation map

27.2.2020.21 (34)
$$\mathrm{ev} : \mathbb{S} \times \mathcal{L}(M) \ni (\theta, \lambda) \longmapsto \lambda(\theta) \in M$$

which allows cohomology to be pulled back and then integrated over the circle to define transgression from the upper left part of the diagram for each $k \geq 1$

27.2.2020.22 (35)
$$\begin{tikzcd} H^k(\mathbb{S} \times \dot{\mathcal{L}}(M); \mathbb{Z}) \arrow[d, "\pi_*"'] & H^k(M; \mathbb{Z}) \arrow[l, "\mathrm{ev}^*"'] \arrow[dl, "\tau"] \arrow[d, leftrightarrow, "\tau_{\mathrm{fus}}"] \\ H^{k-1}(\dot{\mathcal{L}}(M); \mathbb{Z}) & H^{k-1}_{\mathrm{fu}}(\dot{\mathcal{L}}(M), \mathbb{Z}). \arrow[l, "\mathrm{fg}"'] \end{tikzcd}$$

However the diagonal trangression map loses information, in general it is neither injective nor surjective. For this reason Chris Kottke and I introduced 'fusive' cohomology (in Čech cohomology) by imposing fusion and a second (figure-of-eight) reuirements at the chain level. This makes the corresponding cohomology spaces invariant under reparameterization and gives an isomorphism as indicated on the right, with a forgetful map to ordinary cohomology.

So a general 'principle' here is that

> Objects can be trangressed, without loss, to fusive objects on
> 27.2.2020.23 the loop space.

One particular, and fundamental, case of this is the notion of a *string structure*. This corresponds to the third step in the Whitehead tower (27.2.2020.1 1); now we are getting to the heart of the matter.

The question then is analogous to (27.2.2020.9 10). Now we ask about the existence of a lift of the spin frame bundle

28.2.2020.1 (36)
$$\begin{tikzcd} \mathrm{String}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{String}} \arrow[d] \\ \mathrm{Spin}(n) \arrow[r, dash] & F_{\mathrm{Spin}} \arrow[d] \\ & M \end{tikzcd}$$

to a principal bundle with structure group $\mathrm{String}(n)$.

This string group is not well-defined as a group, only up to homotopy equivalence, but the existence of a string structure in the sense of (28.2.2020.1 36) is independent of any choice and the final word here (as always there is a lot of history I am suppressing) is due to Redden:

> A string structure exists if and only $\frac{1}{2}p_1 = 0$ and then the equivalence classes are parameterized by $H^3(M; \mathbb{Z})$.

The somewhat confusingly denoted obstruction, $\frac{1}{2}p_1 \in H^4(M; \mathbb{Z})$, is the Pontryagin class of the *Spin* principal bundle F_{Spin}, it is an integral class and $2 \times \frac{1}{2}p_1$ is the 'usual' Pontryagin class of F_{SO}.

Whatever realization of String(n) one takes, it cannot be a finite-dimensional Lie group, since it must have trivial π_3. It is quite difficult to contemplate doing analysis direcctly on F_{String} or related spaces. However the principle in (27.2.2020.23) holds, as was understood, with some caveats, in the 1980s and we should 'trangress' to the loop space.

Since $\pi_3(\text{Spin}(n)) = \mathbb{Z}$, we know from (27.2.2020.24. 8) that $\pi_2(\dot{\mathcal{L}}(\text{Spin})) = \mathbb{Z}$ and that this loop group is simply-connected. It follows that there is a circle bundle with Chern class a generator of $H^2(\dot{\mathcal{L}}(\text{Spin}))$. In fact this class is equivariant and the circle bundle corresponds to a central extension

28.2.2020.2
$$\mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}}(\text{Spin}) \longrightarrow \mathcal{L}(\text{Spin}). \tag{37}$$

There is a $\mathbb{Z}$ of such extensions, but all may be obtained from the 'basic' one (28.2.2020.2 37) by covering. The corresponding element of $H^2(\dot{\mathcal{L}}(\text{Spin}))$ is called the *level* of the central extension.

So now we can see a corresponding lifting question over the loop space. The principal spin bundle, the top part of (27.2.2020.9 10), pulls back to a principal $\mathcal{L}(\text{Spin})$ bundle over the loop space, as in (27.2.2020.14 16), and we can ask whether this has a 'lift' to a principal bundle for the basic central extension

28.2.2020.3
$$\begin{tikzcd}
\widehat{\mathcal{L}}(\text{Spin}(n)) \arrow[r, dash] \arrow[d] & \mathcal{F} \arrow[d] \\
\mathcal{L}(\text{Spin}(n)) \arrow[r, dash] & \mathcal{L}(F_{\text{Spin}}) \arrow[d] \\
 & \dot{\mathcal{L}}(M);
\end{tikzcd} \tag{38}$$

The projections from the top line here correspond to circle bundles.

The situation is very similar to the spin to orientation transgression already discussed. Here the existence of a string structure implies the existence of a extension. One way of seeing this is to note that such a U(1) lifting problem corresponds to the triviality of a 'lifting bundle gerbe' in the sense of Murray. Thinking more abstractly of a principal bundle with a (possibly large but topological) group with a central extension $\mathrm{U}(1) \longrightarrow \widehat{\mathcal{G}} \longrightarrow \mathcal{G}$ we can ask the same existence question – we look for a diagram

28.2.2020.4
$$\begin{tikzcd}
\widehat{\mathcal{G}} \arrow[r, dash] \arrow[d] & \widehat{\mathcal{P}} \arrow[d] & L \arrow[d] & s : \delta L \simeq \mathrm{U}(1) \arrow[d] & \delta s = 1 \arrow[d] & \\
\mathcal{G} \arrow[r, dash] & \mathcal{P} \arrow[d] & \mathcal{P}^{[2]} \arrow[l] & \mathcal{P}^{[3]} \arrow[l] & \mathcal{P}^{[4]} \arrow[l] & \cdots \arrow[l] \\
 & M & & & &
\end{tikzcd} \tag{39}$$

Here $\mathcal{P}$ is the total space of the principal bundle and to the right are the various fibre products over M forming a simplicial space (so there are k maps from $\mathcal{P}^{[k]}$ to $\mathcal{P}^{[k-1]}$). Then L is a circle bundle defined from the central extension of $\mathcal{G}$. Namely the fibre of $\mathcal{P}^{[2]}$ over m is the set of pairs $(f_1, f_2) \in \mathcal{P}_m \times \mathcal{P}_m$ so the $\mathcal{G}$ action induces a shift map $\mathcal{P}^{[2]} \longrightarrow \mathcal{G}$ mapping $(f_1, f_2 = gf_1)$ to g. Then L is the pull-back of the circle bundle given by the central extension. This circle bundle is simplicial (in the sense of Brylinski), as indicated – the simplicial differential gives a line bundle over $\mathcal{P}^{[3]}$ as the tensor product of the three pull-backs of L. From its definition this has a section, s trivializing it. Again the simplicial differential of a section is a section of the simplicial differential of the new circle bundle, where now we are over $\mathcal{P}^{4]}$. Here the line bundle is canonically trivial and the pulled back section is given by this canonical trivialization. This is Murray's notion of a bundle gerbe.

> A bundle gerbe induces a Dixmier-Douady class $\mathbb{D}(L) \in H^3(M;\mathbb{Z})$ and for a lifting bundle gerbe the vanishing of this class is equivalent to the existence of a lifted principal bundle $\widehat{\mathcal{P}}$ such that as a circle bundle over $\mathcal{P}$ its image under δ is L.

28.2.2020.5

Now, the Dixier-Douady class of this bundle gerbe is the trangression of the obstruction to the existence of a string structure.

28.2.2020.6

$$\mathbb{D}(\dot{\mathcal{L}}F_{\mathrm{Spin}} = \tau(\frac{1}{2}p_1) \tag{40}$$

Thus again, the existence of a string structure implies the existence of a 'spin structure' in the sense of a lift of the principal bundle as in (28.2.2020.4 39). However, just as before this transgressed spin structure has additional properties, fusion, figure-of-eight and equivariance. We can these more refined structures 'loop spin structures'. To make real progress also need to consider regularity, continuous objects do not suffice.

Ignoring these niceties, it is shown by Waldorf and in [K-M ?] that

> String structures on M are in 1-1 correspondence with loop spin structures.

Of course the automorphisms of these objects need to be taken into account too.

Bibliography