18.158, Spring 2013: Analysis on loop spaces

Richard Melrose

Department of Mathematics, Massachusetts Institute of Technology

E-mail address: rbm@math.mit.edu

Abstract. Lecture notes written progressively during Spring 2013.

Contents

Introduction

In this course I will discuss analysis and geometry of loop spaces. These are particular, but as it turns out very special, examples of Fréchet manifolds. One thing I will *not* try to do is discuss analysis in general Fréchet manifolds – for one thing there is very little to discuss! Nevertheless, let me first describe these and try to show what the problem is. Then I will go on to give an overview of loop spaces, why the are of current interest and where I hope to get to by the end of the semester.

Prerequisites:- I will assume familiarity with standard theory of smooth manifolds, but will try to discuss anything substantial that is used. The same applies to Fréchet spaces, groups, operator algebras and the like.

CHAPTER 1

Lectures 1 and 2: An overview, February 5 and 7, 2013

1. Manifolds, maybe big

There is general agreement about what a finite dimensional manifold is, less about the infinite dimensional case. Let me try to indicate what the issues are.

Let's agree from the beginning that a manifold X is a topological space, that it is Hausdorff and that it is paracompact. You might wonder about this last condition but although the infinite-dimensional spaces we consider are big, they are not so big. One significant thing that is lost in passing to the infinite dimensional case is *local compactness*. This makes it very difficult to integrate functions.

In addition to these basic properties, a manifold is supposed to have a 'local regularity structure' given by local coordinate systems.

The local coordinates correspond to the choice of a model space V and X is supposed to have a covering by open sets U_a with homeomorphisms $F_a : U_a \longrightarrow V_a$ where $V_a \subset V$ is open and

$$F_{ab} : V_{ab} = F_a(U_a \cap U_b) \longrightarrow F_b(U_a \cap U_b) = V_{ba}$$

is required to lie in a specified groupoid $\underline{\mathrm{Dff}}(V)$ consisting of homeomorphisms between open subsets of V.

2. Transformation groupoid

This is where the real issues start,

What is $\underline{\mathrm{Dff}}(V)$?

We will want $\underline{\mathrm{Dff}}(V)$ to preserve some space of 'smooth functions' $\underline{\mathcal{C}^\infty}(V)$ on open subsets of V – certainly they are supposed to be continuous. Usually V will be some linear space; in the cases of interest here a Fréchet space – a complete countably normed space. This includes Banach and so Hilbert spaces.

I will usually denote a finite dimensional manifold by M. In this familiar case $V = \mathbb{R}^n$ and the transition groupoid $\underline{\mathrm{Dff}}(\mathbb{R}^n)$ consists of the smooth maps with smooth inverses between open subsets, $F_{ab} : V_{ab} \longrightarrow V_{ba}$ such that $F^*_{ab}(\mathcal{C}^\infty(V_{ba})) = \mathcal{C}^\infty(V_{ab})$. There are restricted choices (for instance symplectic diffeomorphisms) but pretty much everyone agrees that this is the definition of a finite-dimensional smooth manifold. Of course one can vary the local structure by considering $\mathcal{C}^k$ or real-analytic manifolds.

Note however that the transition groupoid, or even the global group of diffeomorphisms $\mathrm{Dff}(\mathbb{R}^n)$ is not so easy to deal with. For any (say compact) finite-dimensional manifold, $\mathrm{Dff}(M)$ is a Fréchet Lie group, with Lie algebra $\mathcal{V}(M)$, the space of real smooth vector fields on M. So $\mathrm{Dff}(M)$ *is* in some sense a manifold modelled on $\mathcal{V}(M)$ as model space. This is the sort of infinite dimensional manifold we will consider, one modelled on a space like the smooth functions on a finite dimensional manifold. I will review the properties of $\mathrm{Dff}(M)$ later, but let me warn you that it is not quite like a finite-dimensional Lie group. For instance, the exponential map (which does exist in the compact case) is not surjective from a neighbourhood of 0. Fortunately the structure of $\underline{\mathrm{Dff}}(\mathbb{R}^n)$ does not arise directly in dealing with finite-dimensional manifolds since we really only consider a finite number of elements at any one time.

3. Riemann manifolds

While I am at it, let me remind you about Riemann manifolds.

With any finite-dimensional smooth manifold we can associate may others, one particularly important one being the tangent bundle which comes with a smooth map $TM \longrightarrow M$. Because it is constructed from M, the groupoid for TM is also $\underline{\mathrm{Dff}}(\mathbb{R}^n)$, but we can expand it to $\underline{\mathcal{C}^\infty}(\mathbb{R}^n; \mathrm{GL}(n))$ (sending a diffeomorphism to its differentials). On the tangent bundle we can choose a Riemann metric – a smooth family of fibre metrics and so shrink the groupoid again to $\underline{\mathcal{C}^\infty}(\mathbb{R}^n; \mathrm{O}(n))$ which we can think of in terms of the principal $\mathrm{O}(n)$-bundle of orthonormal frames in TM. The choice of a Riemann metric also gives a 'nice' covering of M by coordinate systems corresponding to small balls and the exponential map with all (non-trivial) intersections contractible. I will come back to all of this later.

4. Examples

Now, what about really infinite-dimensional manifolds? When the model V is infinite-dimensional, there are many possible choices for $\underline{\mathrm{Dff}}(V)$, many of them quite ugly! I will not try to discuss the zoo of choices, but just mention some important examples. One important Banach manifold is $\mathrm{GL}(H)$ (or $\mathrm{U}(H)$) the space of invertible (or unitary) operators on a separable Hilbert space. The model here is $B(H)$ (or $A(H)$) the bounded (and self-adjoint) operators. An important (and reasonably simple) result is Kuiper's theorem, that $\mathrm{GL}(H)$ is contractible (in the uniform topology). A similar, but 'smoother' manifold is $G^{-\infty}(M)$, the group of invertible smoothing perturbations of the identity acting on functions on a compact finite-dimensional manifold M. The model here is $\mathcal{C}^\infty(M^2)$. This is a Fréchet Lie group, but one where the exponential map is a local diffeomorphism – it is also a classifying group for (odd) K-theory.

The fundamental problem with defining an infinite-dimensional manifold is: What is the regularity we require of the functions on open subsets of the model, V, and what do we require of the transition groupoid which is supposed to preserve this regularity between open sets – and so transfer it to our manifold. The 'obvious' definitions or infinite differentiability of functions and taking the 'maximal groupoid' – which is the direct extension of the finite-dimensional case – do not work at all well. This is the problem. For instance we know almost nothing about the space of homeomorphisms between open subsets of $\mathcal{C}^\infty(\mathbb{R})$ which preserve regularity of function on these sets. And what we do know is not encouraging!

5. Loop spaces

So, we need to be guided by some reasonably sensible examples. The simplest of these really are the loop manifolds. These are the manifolds I want to concentrate on in this course. So, just consider

$$\mathcal{L}M = \mathcal{C}^\infty(\mathrm{U}(1); M)$$

the space of all smooth maps from the circle into M, say a compact finite manifold, of finite dimension. I treat the circle here as the 1-dimensional Lie group for reasons that will become clear. Note that in the very imporant case, that appears often below, that $M = G$ is a Lie group, $\mathcal{L}G$ is also a group under 'pointwise product'.

The model is indeed essentially the smooth functions on a finite-dimensional manifold – in this case it is $\mathcal{C}^\infty(\mathrm{U}(1); \mathbb{R}^n)$ which is just the product of n copies of real-valued functions on U(1). So, what is the structure groupoid? We should perhaps ask what is the precise definition of smooth functions on open subsets of $\mathcal{C}^\infty(\mathrm{U}(1))$, but it is better to postpone that.

One of the first things I will go through carefully is the fact that the structure groupoid of the loop space 'is' meaning it is natural to take it to be, a groupoid which is very like the finite dimensional case. Not some huge thing at all but simply

$$\mathrm{Dff}(\mathcal{L}M) = \mathcal{C}^\infty(\mathrm{U}(1); \underline{\mathrm{Dff}}(\mathbb{R}^n))$$

just the loops in the groupoid for M. Basically this is becase $\mathcal{L}M$ is a space 'associated to M'.

6. Local coordinates

This needs to be properly justified, but let me give an outline of the 'proof' (I have not said exactly what the theorem is of course)! Take a Riemann metric on M and choose $\epsilon > 0$ smaller than the injectivity radius, so all the geodesic balls of this size are really balls. Now, for a loop $l : \mathrm{U}(1) \longrightarrow M$ look at the set of loops

$$N(l;\epsilon) = \{l' : \mathrm{U}(1) \longrightarrow M; d(l'(s), l(s)) < \epsilon \ \forall \ s \in \mathrm{U}(1)\}. \tag{1.6.1}$$

We can use the 'exponential' map of the metric on M to identify $B(l(s), \epsilon)$ with a ball around 0 in the tangent space $T_{l(s)}M$. This gives an identification of $N(l, \epsilon)$ with 'loops in the tangent bundle', meaning sections – with length less than ϵ everywhere – of l^*TM the pull-back to U(1) of the tangent bundle. Well, let's assume that M is orientable. Then this is actually a trivial bundle over the circle and we are really looking at $\mathcal{C}^\infty(\mathrm{U}(1); \mathbb{R}^n)$ as indicated above. What happens when we change base loop and look at the neighbourhood of another? Two $N(l_i, \epsilon)$ $i = 1, 2$ intersect only if there is a loop which is everywhere ϵ close to each – meaning that $d(l_1(s) l_2(s)) < 2\epsilon$ for all $s \in \mathrm{U}(1)$. When this is true the 'coordinate change' is a fibre-preserving map from TM pulled back to one to the pull-back to the other. So, you see it is a loop into the local diffeomorphisms of $\mathbb{R}^n$.

7. Whitehead tower

Now, for the rest of today and probably the next lecture, I want to discuss the question: Why? What makes loop manifolds interesting/important? For us the first reason – and something that I need to discuss in some detail, is because of the difficulties presented by the analytic properties (or lack thereof) of the Whitehead tower. So what is the Whitehead tower? For us it is about successive special properties of manifolds.

Let's go back and talk about what is really interesting, namely finite dimensional manifolds. Let me take the dimension n to be at least 5 so I don't have to keep making qualifying statements.

The tangent bundle of M can be given a Riemann metric, which reduces its 'structure group' to $\mathrm{O}(n)$, the space of orthogonal transformations. This just means that the bundle over M of orthogonal frames in the fibres of TM forms a principal $\mathrm{O}(n)$ bundle over M – the fibres are diffeomorphic to $\mathrm{O}(n)$ and $\mathrm{O}(n)$ acts freely and locally trivially.

The orthogonal group $\mathrm{O}(n)$ has two components, the identity component being $\mathrm{SO}(n)$ for which the first few homotopy groups are

$$\pi_0(\mathrm{O}(n)) = \mathbb{Z}_2,\ \pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2,$$
$$\pi_2(\mathrm{SO}(n) = \{\mathrm{Id}\},\ \pi_3(\mathrm{SO}(n)) = \mathbb{Z},$$
$$\pi_i(\mathrm{SO}(n)) = \{\mathrm{Id}\},\ i = 4, 5,\ pi_6(\mathrm{SO}(n)) = \mathbb{Z}.$$

The non-triviality of $\pi_1(\mathrm{SO}(n))$ corresponds to the existence of a non-trivial double cover – the spin group. In fact Whitehead's theorem says that there is a 'tower' of group homomorphisms – in general only topological groups

$$\mathrm{O}(n) \longleftarrow \mathrm{SO}(n) \longleftarrow \mathrm{Spin}(n) \longleftarrow \mathrm{String}(n) \dots \tag{1.7.1}$$

where for each successive group the bottom homotopy group is killed but the higher ones survive unscathed.

The first three groups here are (meaning have realizations as) Lie groups, the last cannot since $\pi_3(G) = \mathbb{Z}$ for any finite-dimensional connected Lie group.

8. Orientation

As I am confident is well-known to you, the first inclusion corresponds to the orienatility of M. Said formally:

When can the structure group of TM be reduced from $\mathrm{O}(n)$ to $\mathrm{SO}(n)$?

In this case we can associate with M another manifold $\mathrm{O}\,M$ which is just M with one of the two possible choices of orientation of T_pM at each point – and hence nearby. This double cover is trivial, has two components, if and only if M is orientable. As is very well known, this can be expressed by saying that M is orientable if and only if $w_1 \in H^1(M; \mathbb{Z}_2)$ the first Stiefel-Whitney class, vanishes.

9. Spin structure

As is also well-known, the second homomorphism in (1.7.1) corresponds to the question of the existence of a *spin structure* on M. Now we assume that M is oriented – connected, orientable and with an orientation chosen and now ask

Can the structure group $\mathrm{SO}(n)$ of TM be lifted to $\mathrm{Spin}(n)$?

This can be made precise by looking at $F_{\mathrm{SO}(n)}M$, the bundle of oriented orthonormal frames of M – by the choice of an orientation the structure group has been reduced from $\mathrm{O}(n)$ to $\mathrm{SO}(n)$ so this is a principal $\mathrm{SO}(n)$-bundle. The precise question then is – does there exists a principal $\mathrm{Spin}(n)$ bundle $F = F_{\mathrm{Spin}}$ (I denote the spin frame bundle just as F since it will appear often in the sequel) with fibre covering map

$F \longrightarrow F_{\mathrm{SO}(n)}$ giving a commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}(n) & \text{———} & F \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) & \text{———} & F_{\mathrm{SO}(n)} \\ & & \downarrow \\ & & M. \end{array}$$

Again the answer is well-known, that this is possible – there exists a spin structure – if and only if $w_2 \in H^2(M; \mathbb{Z}_2)$, the second Stieffel-Whitney class, vanishes. There seems to be a pattern! There is, but it is not as simple as that.

10. Spin Dirac operator

Why should you be interested in spin structures? From the point of view of differential equations (remember this is, at least in principle, a course on differential equations) one important consequence of the existence of a spin structure is the existence of the spin Dirac operator. I will talk about this and the fact that it is elliptic and that its index is the $\hat{A}$-genus of the manifold. This is a number associated to an oriented compact manifold which is generally rational but in the case of a spin manifold is an integer – this was actually known before the index of the Dirac operator was understood by Atiyah and Singer and the existence of the Dirac operator serves as an explanation of the integrality of the genus

$$\hat{A} = \mathrm{ind}(\eth_{\mathrm{Spin}}). \tag{1.10.1}$$

Here the spin Dirac operator is defined by associating a vector bundle to F – using the spin representation of Spin – observing that it inherits the Levi-Civita connection and combining that with Clifford multiplication to get

$$\eth_{\mathrm{Spin}} = \mathrm{cl} \circ \nabla. \tag{1.10.2}$$

11. Spin and loops

What is the relationship between spin structures and $\mathcal{L}M$? The most obvious relationship with loops is that $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$ and this can be constructed as usual for the universal cover using loops. We do something similar for the spin structure.

Now, we are assuming that M is oriented, but *not* that it has a spin structure. The orientation implies that the orthonormal frame bundle is trivial over each loop in M, so the loops in $F_{\mathrm{SO}(n)}$ form a bundle, $\mathcal{L}F_{\mathrm{SO}(n)}$ over $\mathcal{L}M$ which is actually a principal bundle with structure group $\mathcal{L}\,\mathrm{SO}(n)$. This loop group has two components, so there is an orientation question – can the structure group be reduced to the connected component of $\mathcal{L}\,\mathrm{SO}(n)$? The universal cover construction shows that the connected component of $\mathcal{L}\,\mathrm{SO}(n)$ is canonically $\mathcal{L}\,\mathrm{Spin}(n)$ and so there is a natural choice of an 'orientation map' giving a short exact sequence

$$\mathcal{L}\,\mathrm{Spin} \longrightarrow \mathcal{L}\,\mathrm{SO} \longrightarrow \mathbb{Z}_2. \tag{1.11.1}$$

It was observed by Atiyah that if M is spin then $\mathcal{L}M$ is orientable in this sense of having a reduction of the structure group to $\mathcal{L}\,\mathrm{Spin}$. Conversely, it was proved by McLaughlin [**2**] that the orientability of $\mathcal{L}M$ implies that M is spin, provided M

is simply connected. However, in general it is not the case that orientability in the sense of the reduction of the structure group of $\mathcal{L}F_{\mathrm{SO}(n)}$ to the component of the identity is equivalent to the existence of a spin structure on M. One needs another condition on the orientation.

What one needs to add is the idea of *fusion*, introduced at least in this context by Stolz and Teichner; I will talk about this quite a bit. The combination, 'a fusion orientation' of $\mathcal{L}M$ ensures that it does correspond precisely to a spin structure on M. To see where the fusion condition comes from, suppose that M does have a spin structure; we proceed to construct an 'orientation' on $\mathcal{L}M$. Namely, a loop in $\mathcal{L}F_{\mathrm{SO}(n)}$ is 'positively oriented' if it can be covered by a section of $\mathcal{L}F$. This constructs a map [1]

$$o_F : \mathcal{L}F_{\mathrm{SO}(n)} \longrightarrow \mathbb{Z}_2 = \{1, -1\}. \tag{1.11.2}$$

It should be continuous, and that is what an orientation of $\mathcal{L}M$ is – a continuous map (1.11.2) which restricts to a fibre [2] of $\mathcal{L}F_{\mathrm{SO}(n)} \longrightarrow \mathcal{L}M$ to give an orientation of $\mathcal{L}\,\mathrm{SO}$ –meaning (1.11.1) or its opposite.

Instead of thinking about loops, consider paths in M; these are just smooth maps $[0, \pi] \longrightarrow M$. I take this interval because it is 'half a circle' but it really does not matter. In fact we will consider only flat-ended segments, meaning that all derivatives of the path vanish at the end-points, denote the collection of these as $\mathcal{I}M$. Clearly we have a map

$$\mathcal{I}M \longrightarrow M^2 \tag{1.11.3}$$

mapping to the two end-points of a segment. In fact this is a fibre bundle in an appropriate sense and we can consider the fibre-product (pairs of loops with the same ends). From this there is a 'join' map

$$J : \mathcal{I}^{[2]}M \longrightarrow \mathcal{L}M$$

obtained by following the first segment by the reverse of the second which maps into loops. Note that the flatness of the paths at the end-points means that they join up smoothly to a loop – not an arbitrary loop of course because it is 'flat' at the points 1 and -1 on the circle; other than that it is arbitrary.

From $\mathcal{I}M^{[3]}$ – triples of segments all three with the same ends – there are three maps into $\mathcal{L}M$, taking (f_1, f_2, f_3) to $J(f_1, f_2)$, $J(f_2, f_3)$ and $J(f_1, f_3)$ respectively.

This discussion applies to any manifold, so we can consider $\mathcal{I}F_{\mathrm{SO}(n)}$, the flat-ended loops into $F_{\mathrm{SO}(n)}$ with its map to $F^2_{\mathrm{SO}(n)}$. So 'join' becomes a map

$$J : \mathcal{I}^{[2]}F_{\mathrm{SO}(n)} \longrightarrow \mathcal{L}F_{\mathrm{SO}(n)}. \tag{1.11.4}$$

As noted above, a spin structure on M allows us to assign an orientation (1.11.2) to each element of $\mathcal{L}F_{\mathrm{SO}(n)}$. This assignment is by 'holonomy' – lift the intial point into the spin frame bundle and then travel around the curve (there is a unique local lift to F because it is just a double cover) and ask whether you have come back to the same point or to the other lift. The three paths determined by a triple, an element of $\mathcal{I}^{[3]}F_{\mathrm{SO}(n)}$, each have an orientation and the construction by holonomy means that the product of the orientation of two of them is the orientation of the

---

[1]This is where $F_{\mathrm{SO}(n)}$ became F in v3

[2]If this happens on one fibre it happens on all

third. We can think of this in a fancier way, useful for later generalization, that there are simplicial projection maps

$$\pi_{12},\ \pi_{23},\ \pi_{13} : \mathcal{I}^{[3]}F_{\mathrm{SO}(n)} \longrightarrow \mathcal{I}^{[2]}F_{\mathrm{SO}(n)}, \tag{1.11.5}$$

where π_{ij} drops the missing index. Now the compatibility condition on the orientation is that the product of the three maps

$$(\pi_{12}\circ J)^* o_F \cdot (\pi_{23}\circ J)^* o_F \cdot (\pi_{13}\circ J)^* o_F \equiv 1. \tag{1.11.6}$$

[There should really be an inverse on the third factor but of course here it makes no difference.]

Now, we can say a loop-orientation of $\mathcal{L}M$ – meaning a continuous map (1.11.2) with the right behaviour on each fibre, is 'fusion' if it satisfies (1.11.6).

THEOREM 1 (Stolz and Teichner [**8**], 2005). *There is a 1-1 correspondence between fusion orientations and equivalence classes (up to smooth principal-bundle isomorphism) of spin structures on M and each is classified by $H^1(M;\mathbb{Z})$.*[3]

PROOF. [4] You can of course consult [**8**]. In fact this proof is quite illustrative.

The passage from spin structures to fusion orientations is discussed above, but let me repeat it briefly. The principal $\mathrm{Spin}(n)$ bundle, F, associated to a spin structure is a double cover of $F_{\mathrm{SO}(n)}$, the oriented orthonormal frame bundle. So, given a loop $l \in \mathcal{L}F_{\mathrm{SO}(n)}$ we may lift the initial point $l(1)$ to a point $l'(1) \in F$ above it; there are two choices. Once the initial point is chosen there is a unique path in F covering the loop in $F_{\mathrm{SO}(n)}$ and we assign $o(l) = \pm 1$ corresponding to whether the lifted path does, or does not, return to its starting point. It follows that this assignment is independent of the choice of initial point, it is the *holonomy* of the curve corresponding to the $\mathbb{Z}_2$ bundle which is $F \longrightarrow F_{\mathrm{SO}(n)}$. The fact that the fibre F_m above $m \in M$ has a spin action covering the $\mathrm{SO}(n)$ action on $(F_{\mathrm{SO}(n)})_m$ means that the map o on loops in a given fibre takes both signs, corresponding to the fact that $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ but that $\mathrm{Spin}(n)$ is simply connected. Thus o is an orientation of $\mathcal{L}M$ in the sense of (1.11.2). That the spin structure leads to a *fusion* loop-orientation follows from the definition of fusion. Namely at any point in $\mathcal{I}^{[3]}F_{\mathrm{SO}(n)}$ the three loops in (1.11.6) come from three paths i_1 with the same endpoints. The orientation of the first loop is the holonomy along i_1 followed by the reverse of i_2, the orientation of the second is obtained by going along i_2 and then i_3 reversed. We can certainly use as initial point in the second case the end-point of the lift used in the first case. Then the holonomy along the 'fused' loop is obtained by lifting above i_1 and then i_3 in reverse. However, adding the detour along i_2 and then reversed along i_2 does not do anything so the holonomy along the fusion is indeed the product which gives (1.11.6).

Now, to go in the opposite direction, i.e. to construct a spin structure from a fusion loop-orientation, choose a base point in $F_{\mathrm{SO}(n)}$ and consider the 'pointed paths' $\dot{\mathcal{I}}F_{\mathrm{SO}(n)} \subset \mathcal{I}F_{\mathrm{SO}(n)}$, just those flat-ended paths which start at the base point. Evaluating at π gives a map $\dot{\mathcal{I}}F_{\mathrm{SO}(n)} \longrightarrow F_{\mathrm{SO}(n)}$ which is surjective by the connectedness of M and $\mathrm{SO}(n)$.

Now, look at the product $\dot{\mathcal{I}}F_{\mathrm{SO}(n)} \times \mathbb{Z}_2$ and define a relation on it:

$$(i,s) \sim (i',s') \Longleftrightarrow i(\pi) = i'(\pi),\ o(J(i,i'))s' = s. \tag{1.11.7}$$

[3] Also a typo here in v3

[4] There were missing $\mathrm{SO}(n)$ subscripts in v3

The fusion condition is exactly what is needed to see that this is an equivalence relation. First, we need to know that $(i,s) \sim (i,s)$ which reduces to $o(J(i,i)) = 1$. That is, the orientation of a 'there-and-back' loop is $+1$. This follows from (1.11.6) since $(o(J(i,i)))^3 = 1$. Similarly symmetry reduces to $o(J(i,i')) = o(J(i',i))$. Again this follows from (1.11.6) applied to the three paths i, i, i' since

$$o(J(i,i))o(J(i,i')) = o(J(i',i)). \tag{1.11.8}$$

The transitivity of $\sim$ is the full fusion condition.

Now, the quotient $F = \dot{\mathcal{I}}F_{\mathrm{SO}(n)} \times \mathbb{Z}_2/\sim$ is a double cover $F \longrightarrow F_{\mathrm{SO}(n)}$ since locally near a point of $F_{\mathrm{SO}(n)}$ it is just the product of a neighbourhood with $\mathbb{Z}_2$. In fact this construction applied to $\mathrm{SO}(n)$ is one of the standard constructions of $\mathrm{Spin}(n)$.

Note that there is a slightly sticky issue here, that the loop-orientation construction applied to the spin structure just associated to a fusion loop-orientation should reconstruct the original loop-orientation. This is certainly true on loops obtained by joining two flat-ended paths since a path in the new F is really just a path in $F_{\mathrm{SO}(n)}$ with some choice of $\pm \in \mathbb{Z}_2$. However, an orientation is determined by its restriction to the image of the joint map. This is a strengthened form of the independence of the parameterization and we need to deal with it seriously as 'flattening' below.

Note that we are also using the fact that the holonomy is unchanged under a principal bundle isomorphism but this is clear from the definition. The last statement in the theorem, follows from the fact that spin structures are so classified. Namely a spin structure is a $\mathbb{Z}_2$ bundle over $F_{\mathrm{SO}(n)}$ which encodes the $\mathrm{Spin}(n)$ action. It follows that the tensor product of these two $\mathbb{Z}_2$ bundles is the pull-back of a $\mathbb{Z}_2$ bundle from M under the projection and conversely. That $H^1(M;\mathbb{Z}_2)$ may be identified with the equivalence classes of $\mathbb{Z}_2$ bundles over M is standard. $\square$

It is worth thinking a little about the last part of this proof to see how the classification of loop-orientation structures by $H^1(M;\mathbb{Z})$ arises directly. Namely for two loop-orientations the product is the pull-back of a continuous map

$$\mathcal{L}M \longrightarrow \mathbb{Z}_2 \tag{1.11.9}$$

under the projection map $\mathcal{L}F_{\mathrm{SO}(n)} \longmapsto \mathcal{L}M$ since it is constant on the fibres. This map also has the fusion property for loops on M since this follows by looking at lifts to paths and loops in $F_{\mathrm{SO}(n)}$ for loop-orientations – i.e. follows by taking the product of the identity (1.11.6) for the two loop-orientations (so you need to check that each element of $\mathcal{L}^{[3]}M$ does have a lift to $\mathcal{L}^{[3]}F_{\mathrm{SO}(n)}$. So what this comes down to is the identification

$$H^1(M;\mathbb{Z}_2) = \{\text{Fusion maps (1.11.9)}\}. \tag{1.11.10}$$

Exercise 1. Check (1.11.10) – it is a simpler version of the proof above.

This is all part of the general principal that 'transgression' and 'regression' which are maps from objects on M to objects on $\mathcal{L}M$ and conversely become isomorphisms (or functors) provided the correct 'fusion' condition is added on the loop side. You might, by the way, complain that the notation is messed up and that (1.11.10) should really be the space of 'loop-orientations' on $\mathcal{L}M$ (or M) but it is too late to try to reverse history.

12. Regularity

Let me just note some things about Theorem 1. The fact that we can away without having to worry about regularity is due to the discreteness of $\mathbb{Z}_2$. A map into $\mathbb{Z}_2$ is as smooth as it can be as soon as it is continuous – which is to say it is locally constant. The same applies to the spin structure, the bundle F when it exists has the same regularity as $F_{\mathrm{SO}(n)}$ since it is locally the same. When we go to more serious questions we will have to tackle regularity head on.

Just think for a moment what a continuously differentiable function on some Fréchet manifold modelled on say $\mathcal{C}^\infty(M)$ for a smooth manifold M should be. There are different notions of derivative but in any case such a function should have a derivative at each point. What should that be? The minimal condition would that it be a linear function on the tangent space at each point. However, the tangent space – given the patching definition outlined briefly above – would usually be interpreted as essentially the model space, $\mathcal{C}^\infty(M)$, although not canonically. Still, this means the derivative should be a functional on $\mathcal{C}^\infty(M)$ which is to say a distribution – given that it is continous which we would surely want. So, hidden in a continously differentiable function is a distribution at each point. This is a bit of a problem as soon as we try to do something as is implicit in (1.10.2) which requires some sort of multiplication operation.

Of course there is an implicit bias at work here, 'preferring' the tangent to the cotangent space. It would be reasonable to demand (and we shall) much more regularity than this and claim that the derivative should itself be (or if you want to think in terms of distributions, be given by) an element of $\mathcal{C}^\infty(M)$. Such a function would be 'very smooth'. Still, in order for this to make sense we have to make sure that such notions, and corresponding issues for higher derivatives, transform correctly under the transformation groupoid.

13. Reparameterization

Another property of the holonomy definition of the orientation map on $\mathcal{L}F_{\mathrm{SO}(n)}$ defined from a spin structure is that it is independent of the parameterization of a loop. This is easy to see from the definition and suggests what is natural anyway, that the 'best' objects on a loop space will be independent (in some sense) of the parameterization of the curves. In fact the group $\mathrm{Dff}^+(\mathrm{U}(1))$ of oriented diffeomorphisms of the circle (or the unoriented ones for that matter) act on say $\mathcal{L}M$ by reparameterizing loops and we can therefore think about the quotient

$$\mathcal{L}M/\,\mathrm{Dff}^+(\mathrm{U}(1)). \tag{1.13.1}$$

The problem is that this is quite singular, since for instance the constant loops are fixed points for the action which is very much non-free. Still, it is natural to look for invariance, or equivariance, under this action as we certainly have in the case of the orientation.

There is however a tension between reparameterization and fusion, both of which are clearly important. Namely, the fusion operation from paths to loops only really makes sense if we have flat-ended paths so that the resulting loops are smooth. One can go along way with piecewise-smooth loops, which is what you get by joining smooth but not flat-ended paths with the same ends, but then the topology is going to get out of hand since one needs to allow the breaks to occur anywhere.

These are the sort of issues that I hope to nail down properly. They do not arise above, but they immediately come up below and that is one reason these problems have remained open for quite a long time.

14. String structures

Now, on to the main topic of the first part of these lectures. Namely the next structure in the Whitehead tower, the notion of a *string structure* on M. That is, when does a manifold with a spin structure have a $\mathrm{String}(n)$ principal bundle covering the $\mathrm{Spin}(n)$ bundle:-

(1.14.1)
$$\begin{array}{ccc} \mathrm{String}(n) & \text{———} & F_{\mathrm{String}} \\ \downarrow & & \downarrow \\ \mathrm{Spin}(n) & \text{———} & F \\ & & \downarrow \\ & & M. \end{array}$$

The existence of a decent model for $\mathrm{String}(n)$ is not so trivial.

In fact the answer is also well-established. The spin structure defines a characteristic class in $H^4(M;\mathbb{Z})$ twice which is the usual p_1, the first Pontryagin class. So it is generally denoted $\frac{1}{2}p_1$ but it is integral. The statement here depends a bit on finding a decent model for String so I will not go into it at precisely at this stage but a lift (1.14.1) exists if and only if $\frac{1}{2}p_1 = 0$. Note that the pattern

Orientation — Spin — String — (Fivebrane)

does continue in the sense that the extension to successive groups in the Whitehead tower is obstructed by a cohomology class, not the class involved in the third step (and in the next too) is an integral cohomology class, not a $\mathbb{Z}_2$ class.

At this point you might well have a couple of big questions. Why would we care about string structures? And in any case, what have they to do with loops? What is this thing that comes next? I will not get into the next step!

From looping behaviour,

$$\pi_2(\mathcal{L}\,\mathrm{Spin}(n)) = \pi_3(\mathrm{Spin}(n)) = \mathbb{Z}.$$

In the absence of other groups nearby this implies that $H^2(\mathcal{L}\,\mathrm{Spin}(n)) = \mathbb{Z}$, so there is a line bundle over $\mathcal{L}\,\mathrm{Spin}(n)$. This line bundle is in fact 'primitive', i.e. corresponds to a central extension by a circle

$$\mathrm{U}(1) \longrightarrow E\mathcal{L}\,\mathrm{Spin}(n) \longrightarrow \mathcal{L}\,\mathrm{Spin}(n).$$

At the level of Lie algebras this is the Kac-Moody extension.

15. Loop-spin structures

The existence of a string structure on M is then related to the existence of a 'loop-spin' structure on $\mathcal{L}M$ in the sense of a covering of the $\mathcal{L}\,\mathrm{Spin}(n)$ principal bundle by an $E\mathcal{L}\,\mathrm{Spin}(n)$-principal bundle

$$
\begin{array}{ccc}
E\mathcal{L}\,\mathrm{Spin}(n) & \text{———} & D\mathcal{L}F \\
\downarrow & & \downarrow \\
\mathcal{L}\,\mathrm{Spin}(n) & \text{———} & \mathcal{L}F \\
 & & \downarrow \\
 & & M,
\end{array}
\tag{1.15.1}
$$

where D is an 'appropriate' circle bundle over $\mathcal{L}F$.

The situation is similar to spin and loop-orientation above. There is an obstruction (Dixmier-Douady) class in $H^3(\mathcal{L}M;\mathbb{Z})$ to the existence of an extension principal bundle. Recently Waldorf has shown that there is a notion of 'fusion' structure for a loop-spin structure and the existence of such a structure is equivalent to the existence of a string structure. With any luck (i.e. it does not blow up in the writing) Chris Kottke and I have shown that there is a 1-1 correspondence between string and fusive (this is a strengthening of the fusion condition that we will get to) loop-spin structures up to natural equivalence; these are both classified by $H^3(M;\mathbb{Z})$. Moreover there are analogues in this case of the direct constructions outlined above.

16. Bundle gerbes

What more do we need to carry through this construction? One thing we will use is a geometci realization of 3-dimensional integral cohomology, in the form of bundle gerbes [**3**]. I will develop the theory of these as needed. In particular they apply directly to analyse extensions of principal bundles corresponding to central extensions of their structure groups as in (1.15.1). In fact we need some sort of geometric realization of 4-dimensional integral cohomology, to capture the obstructure class $\frac{1}{2}p_1$. These are '2-gerbes' in this case the Brylinski-McLaughlin bundle 2-gerbe.

17. Witten genus

So, why is this interesting? The basic 'claim' is that analysis is much easier on the loop-spin side of this correspondence than on the string side. In particular, Witten has given a Physical discussion of the index of a differential operator, the Dirac-Raymond operator, on $\mathcal{L}M$, which is associated to the loop-spin bundle. This index is not a number, but is rather a formal power series with integer coefficients – the Witten genus. Again this is analogous to the $\hat{A}$ genus, with integrality of the coefficients a consequence of the existence of a spin structure. To discuss all this properly we would need an analytic-geometric theory of elliptic cohomology, in which the Witten genus resides. I can hope to do this during the semester, but I do not know how to do it now!

18. In the sky

What else would I *like* to (be able to) do? Full analysis of the Dirac-Ramond operator to derive the Witten genus as the equivariant index. Discuss the relationship to quantum field theory and topological quantum field theory. Give a geometric realization of elliptic cohomology and topological modular forms. Don't hold your breath on this.

CHAPTER 2

Lectures 3 and 4, 12th and 14th February: The circle

After the overview last week I will start at the beginning, indeed this week I will talk about analysis (and even a little geometry!) of the circe, U(1). I have moved the general discussion of compact manifolds to an appendix.

1. Functions and Fourier series

Now, we specialize to the circle, U(1) which naturally enough plays a fundamental role in the discussion of loop spaces. The most obvious model for the circle for the discussion of basic regularity is as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ so that $\mathcal{C}^\infty(\mathrm{U}(1))$ is identified with the subspace of $\mathcal{C}^\infty(\mathbb{R})$ consisting of the 2π-periodic functions – and similarly for pretty much every other function space between $\mathcal{C}^{-\infty}(\mathrm{U}(1))$ and $\mathcal{C}^\infty(\mathrm{U}(1))$.

In particular we have a very natural differential operator on functions, $\frac{1}{i}\frac{d}{dt}$ and the spectral theory of this operator corresponds to Fourier series. Thus there are identifications

$$\begin{gathered}\mathcal{C}^\infty(\mathrm{U}(1)) \longrightarrow \mathcal{S}(\mathbb{Z})\\ L^2(\mathrm{U}(1)) \longrightarrow l^2(\mathbb{Z})\\ \mathcal{C}^{-\infty}(\mathrm{U}(1)) \longrightarrow \mathcal{S}'(\mathbb{Z})\\ u(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikt},\ c_k = \frac{1}{2\pi}\int_{\mathrm{U}(1)} u(t)e^{-ikt}|d\theta|\end{gathered} \tag{2.1.1}$$

where the spaces on the right consists of the rapidly decreasing, square summable, and polynomially increasing sequences:-

$$\begin{gathered}\mathcal{S}(\mathbb{Z}) = \{c.:\mathbb{Z}\longrightarrow\mathbb{C}; \sum_k (1+|k|)^l|a_k| < \infty\ \forall\ l\in\mathbb{N}\},\\ l^2(\mathbb{Z}) = \{c.:\mathbb{Z}\longrightarrow\mathbb{C}; \sum_k |a_k|^2 < \infty\},\\ \mathcal{S}'(\mathbb{Z}) = \{c.:\mathbb{Z}\longrightarrow\mathbb{C}; \sum_k (1+|k|)^{-N}|a_k| < \infty \text{ for some } N\in\mathbb{N}\}\end{gathered} \tag{2.1.2}$$

where in the last case N depends on the sequence.

The use of Fourier series allows much of the general discussion above to be made explicit on the circle, but does lead to difficulties when it comes to coordinate-invariance. For example Schwartz kernel theorem in this case becomes the (relatively simply checked) statement that any continuous linear map $\mathcal{C}^\infty(\mathrm{U}(1)) \longrightarrow \mathcal{C}^{-\infty}(\mathrm{U}(1))$ is given by an infinite matrix of polynomial growth on the Fourier transform side

$$(Ac)_k = \sum_j A_{kj} c_j,\ |A_{jk}| \le C(1+|j|+|k|)^N \text{ for some } N. \tag{2.1.3}$$

Smoothing operators correspond to matrices in $\mathcal{S}(\mathbb{Z}^2)$ in the obvious sense. Thus, is $A \in \Psi^{-\infty}(\mathrm{U}(1))$ is a smoothing operator

$$\begin{gathered} A(\sum_k c_k e^{ikt}) = \sum_j d_j e^{ijt},\ d_j = \sum_k A_{jk} c_k, \\ A_{jk} \in \mathcal{S}(\mathbb{Z}^2),\ \text{i.e. } \forall\ N,\ |A_{jk}| \le C_N(1+|j|+|k|)^{-N}. \end{gathered} \tag{2.1.4}$$

Conversely any such rapidly decreasing matrix defines a smoothing operator.

Exercise 2. Write out the relationship of Fourier series on the 2-torus between the matrix A_{jk} and the kernel $A \in \mathcal{C}^\infty(\mathrm{U}(1)^2)$ of a smoothing operator.

2. Hardy space

Of course the circle is extremely special among manifolds! In particular its identification with the unit circle in $\mathbb{C}$ with respect to the Euclidean metric, via the exponential function, means that there is a special (really a whole lot of special) subspaces of $\mathcal{C}^\infty(\mathrm{U}(1))$ of 'half dimension'. The usual choice is the (smooth) Hardy space

$$\mathcal{C}^\infty_H(\mathrm{U}(1)) = \{u \in \mathcal{C}^\infty(\mathrm{U}(1)); c_k = 0 \text{ for } k < 0 \text{ in } (2.1.1)\}. \tag{2.2.1}$$

The elements of $\mathcal{C}^\infty_H(\mathrm{U}(1))$ are precisely those smooth functions which are the restriction to the unit circle of a smooth function on the closed ball, $\{|z| \le 1\}$ in $\mathbb{C}$ which is holomorphic in the interior.

Consider the projection onto this subspace, which can be written explicitly in terms of the expansion in Fourier series

$$P_H u = \sum_{k \ge 0} c_k e^{ikt},\ u = \sum_k c_k e^{ikt},\ P_H^2 = P_H. \tag{2.2.2}$$

As such it is clear self-adjoint with respect to the L^2 inner product

$$\langle u, v\rangle = \int_{\mathrm{U}(1)} u(t)\overline{v(t)} dt. \tag{2.2.3}$$

In fact P_H is probably the simplest example of non-differential pseudodifferential operator. To see this of course, you need to know what a pseudodifferential operator is, but for the moment this does not matter.

The most crucial, perhaps non-obvious, property of P_H is that it 'almost commutes' with multiplication. Thus suppose $a \in \mathcal{C}^\infty(\mathrm{U}(1))$ (maybe complex-valued), this defines a multiplication operator which I will also denote a :

$$a : \mathcal{C}^\infty(\mathrm{U}(1)) \ni u \longmapsto au \in \mathcal{C}^\infty(\mathrm{U}(1)). \tag{2.2.4}$$

Lemma 1. *For any $a \in \mathcal{C}^\infty(\mathrm{U}(1))$, the commutator with P_H is a smoothing operator*

$$[a, P_H] \in \Psi^{-\infty}(\mathrm{U}(1)). \tag{2.2.5}$$

PROOF. We only need find expressions for aP_H and $P_H a$ in terms of Fourier series, which amounts to finding its action on e^{ikt}. First observe that multiplication becomes convolution in the sense that

$$\begin{gathered} a = \frac{1}{2\pi}\sum_l a_l e^{ilt} \in \mathcal{C}^\infty(\mathrm{U}(1)) \Longrightarrow \\ au = \frac{1}{2\pi}\sum_j b_j e^{ijt} \text{ where } b_j = \sum_l a_{j-l}c_l \text{ if } u = \frac{1}{2\pi}\sum_k c_k e^{ikt}. \end{gathered} \tag{2.2.6}$$

It follows that

$$\begin{aligned} (aP_H)u &= \frac{1}{2\pi}\sum_j \left(\sum_{l\geq 0} a_{j-l}c_l)\right) e^{ijt} \\ (P_H a)u &= \frac{1}{2\pi}\sum_{j\geq 0}\left(\sum_l a_{j-l}c_l)\right) e^{ijt}. \end{aligned} \tag{2.2.7}$$

Thus the commutator is given by the difference, which means that

$$\begin{gathered} u = \sum_j \sum_l B_{jl}c_l e^{ijt}, \text{ where} \\ B_{j,l} = \begin{cases} a_{j-l} & \text{if } j < 0, l \geq 0 \\ -a_{j-l} & \text{if } j \geq 0, l < 0 \\ 0 & \text{otherwise.} \end{cases} \end{gathered} \tag{2.2.8}$$

By assumption, a is smooth, so $|a_{j-l}| \leq C_N(1 + |j-l|)^{-N}$ for each N. For the non-zero terms in (2.2.8), $|j-l| = |j| + |l|$ since the signs are always opposite. Thus

$$|B_{j,l}| \leq C_N(1 + |j| + |l|)^{-N}$$

is rapidly decreasing in all directions on $\mathbb{Z}^2$. □

Note that this behaviour can be attributed to one oddity of U(1), the circle, that distinguishes it from other connected compact manifolds. Namely its cosphere bundle has two components.

3. Toeplitz operators

This behaviour of commutators allows us to define the Toeplitz algebra. Clearly $\mathcal{C}^\infty(\mathrm{U}(1))$ forms an algebra of multiplication operators. However, if we project this onto $\mathcal{C}^\infty_H(\mathrm{U}(1))$ by defining

$$a_H = P_H a P_H : \mathcal{C}^\infty_H(\mathrm{U}(1)) \longrightarrow \mathcal{C}^\infty_H(\mathrm{U}(1)). \tag{2.3.1}$$

This is a Toeplitz operator, the 'projection' of a multiplication operator onto the Hardy space. However, these operators do not form an algebra since the composite is instead

$$P_H a P_H b P_H = P_H ab P_H + P_H a[P_H, b]P_H. \tag{2.3.2}$$

The second term here is a smoothing operator – this is one of the basic properties of pseudodifferential operators, that the composite of a pseudodifferential operator and a smoothing operator is smoothing. In particular this means that the composites with the Hardy projection or multiplication by a smooth function are again smoothing

$$P_H A,\ AP_H,\ aA,\ Aa \in \Psi^{-\infty}(\mathrm{U}(1)) \text{ if } A \in \Psi^{-\infty}(\mathrm{U}(1)),\ a \in \mathcal{C}^\infty(\mathrm{U}(1)). \tag{2.3.3}$$

Exercise 3. Prove (2.3.3).

Definition 1. The Toeplitz operators (with smooth coefficients) consist of the sum

$$\Psi_{\mathrm{To}}(\mathrm{U}(1)) = P_H \mathcal{C}^\infty(\mathrm{U}(1)) P_H + P_H \Psi^{-\infty}(\mathrm{U}(1)) P_H \tag{2.3.4}$$

as an algebra of operators on $\mathcal{C}^\infty_H(\mathrm{U}(1))$.

Now, the decomposition (2.3.4) of a Toeplitz operator is unique and moreover the multiplier can be recovered from

$$\begin{gathered} P_H a P_H + \Psi^{-\infty}_{\mathrm{To}}(\mathrm{U}(1)),\ a \in \mathcal{C}^\infty(\mathrm{U}(1)). \\ \Psi^{-\infty}_{\mathrm{To}}(\mathrm{U}(1)) = \{P_H A P_H; A \in \Psi^{-\infty}(\mathrm{U}(1))\}. \end{gathered} \tag{2.3.5}$$

To recover a from $a_H = P_H a P_H$ it suffices to look at $a_H e^{ikt}$ for k large. Indeed from (2.2.7)

$$\begin{gathered} a_H e^{ikt} = \sum_{j \ge 0} a_{j-k} e^{ijt},\ k \ge 0 \Longrightarrow \\ \int_{\mathrm{U}(1)} e^{-i(p+k)t} \left(a_H e^{ikt}\right) dt = a_p,\ k+p \ge 0. \end{gathered} \tag{2.3.6}$$

So, by taking k large enough we can recover a_p from $a_H e^{ikt}$ and hence we can recover $a \in \mathcal{C}^\infty(\mathrm{U}(1))$. More generally, if $A \in \Psi^{-\infty}_{\mathrm{To}}(\mathrm{U}(1))$ then

$$\lim_{k \to \infty} \int_{\mathrm{U}(1)} e^{-i(p+k)t} \left(A e^{ikt}\right) dt = 0 \tag{2.3.7}$$

Proposition 1. *The Toeplitz-smoothing operators form an ideal in the Toeplitz operators (with smooth coefficients) and gives a short exact sequence of algebras*

$$\Psi^{-\infty}_{\mathrm{To}}(\mathrm{U}(1)) \longrightarrow \Psi_{\mathrm{To}}(\mathrm{U}(1)) \xrightarrow{\ \sigma\ } \mathcal{C}^\infty(\mathrm{U}(1)). \tag{2.3.8}$$

The second homomorphism here is a special case of the 'symbol map' for pseudodifferential operators.

Proof. Basically this is proved above. □

Although this is what I will call the Toeplitz algebra, the name is applied to several closely related algebras (particularly the norm closure of this algebra as bounded operators on the L^2 version of the Hardy space). Even keeping things 'smooth' there is another algebra which is important here, at least as an aid to understanding. Namely, we can simply 'compress' the pseudodifferential operators on the Hardy space and define

$$\Psi^{\mathbb{Z}}_H(\mathrm{U}(1)) = P_H \Psi^{\mathbb{Z}}(\mathrm{U}(1)) P_H. \tag{2.3.9}$$

The usual definition of a Toeplitz algebra in higher dimensions is derived from this. Since multiplication operators and smoothing operators are pseudodifferential operators,

$$\Psi_{\mathrm{To}}(\mathrm{U}(1)) \subset \Psi^0_H(\mathrm{U}(1)). \tag{2.3.10}$$

The question then is:- What else is in the space on the right – we could call it the 'extended Toeplitz algebra'.

The main extra terms are the compressions of pseudodifferential operators of negative integral order. As matix operator on the Fourier series side these are of the form, for $m \in \mathbb{N}$

(2.3.11)
$$Ae^{ikt} = \sum_{j=0}^{\infty} A_{jk}e^{ijt}, A_{jk} = a_{j-k}(k+1)^{-m},\ j,k \geq 0,\ \sup_j |a_j|(1+|p|)^N < \infty\ \forall\ N.$$

This is in fact just the composite of the compression of a multiplication operator, by $a \in \mathcal{C}^\infty(\mathrm{U}(1))$ with Fourier coefficients a_j and the convolution operator given by multiplication by $(1+k)^{-m}$ on the Fourier series side. Note that the $k+1$ is just to avoid problems at $k=0$. Beyond this the extended Toeplitz algebra has a completeness property.

4. Toeplitz index

How is this related to loops? One way to see a relationship is to observe that

$$\mathcal{L}\,\mathrm{U}(1) \subset \mathcal{C}^\infty(\mathrm{U}(1)) \tag{2.4.1}$$

consisting of the maps with values in $\mathrm{U}(1) \subset \mathbb{C}$. These form a group and so we can look for the invertible elements in $\Psi_{\mathrm{To}}(\mathrm{U}(1))$ which map into this group. The answer is at first a bit disappointing!

Lemma 2. *The elements of* $\mathcal{L}\,\mathrm{U}(1) \subset \mathcal{C}^\infty(\mathrm{U}(1))$ *which lift to invertible elements of* $\Psi_{\mathrm{To}}(\mathrm{U}(1))$ *are precisely those with winding number* 0, *i.e. are contractible to constant in* $\mathcal{L}\,\mathrm{U}(1)$.

This you might say is the simplest form of the Atiyah-Singer index theorem, long predating it of course since it was known to Toeplitz (in the 1920s I think). If we do a little more we get a true index theorem:-

Theorem 2. *If* $a \in \mathcal{C}^\infty(\mathrm{U}(1);\mathbb{C}^*)$ *(i.e. is non-zero) then, as an operator on* $\mathcal{C}^\infty_H(\mathrm{U}(1))$, *any element* $B \in \Psi_{\mathrm{To}}(\mathrm{U}(1))$ *with* $\sigma(B) = a$ *is Fredholm – has finite dimensional null space and closed range of finite codimension – and*

$$\operatorname{ind}(B) = \dim\operatorname{null}(B) - \dim\left(\mathcal{C}^\infty_H(\mathrm{U}(1))/B\mathcal{C}^\infty_H(\mathrm{U}(1))\right) = -\operatorname{wn}(a) \tag{2.4.2}$$

is determined by the winding number of a.

5. Diffeomorphisms and increasing surjections

6. Toeplitz central extension

Now let me get a little closer to the core topic and show one construction of the central extension of the loop group on any connected and simply connected Lie group. There are quite a few other constructions (see for instance that of Mickelsson).

We can embed $\mathrm{Spin}(n)$ in the (real) Clifford algebra $\mathrm{Cl}(n)$ and so think of it concretely as a group in an algebra. This means that $\mathcal{L}\,\mathrm{Spin}(n)$ sits inside the algebra $\mathcal{C}^\infty(\mathrm{U}(1);\mathrm{Cl}(n))$. There is a 'Hardy' subalgebra of $H \subset \mathcal{C}^\infty(\mathrm{U}(1);\mathrm{Cl}(n)$ consisting of the functions with vanishing negative Fourier coefficients and a natural projection Π_H onto it. The Toeplitz operators $\Pi_H\mathcal{C}^\infty(\mathrm{U}(1);\mathrm{Cl}(n))\Pi_H$, do not form an algebra, but do so when extended by the Toeplitz smoothing operators with values in $\mathrm{Cl}(n)$ – because $[H, \mathcal{C}^\infty(\mathrm{U}(1); \mathrm{Cl}(n)]$ consists of such smoothing operators. We can call this the Clifford-Toeplitz algebra.

Invertible elements in $\mathcal{C}^\infty(\mathrm{U}(1); \mathrm{Cl}(n))$ give Fredholm operators and the fact that $\mathrm{Spin}(n)$ is simply connected means that the index vanishes and so there is a group of invertible operators in the Clifford-Toeplitz algebra and a subgroup of unitary extension. This gives a short exact sequence of groups

$$G^{-\infty}_H(\mathrm{U}(1); \mathrm{Cl}(n)) \longrightarrow \mathcal{G}_H \longrightarrow \mathcal{L}\,\mathrm{Spin}(n)$$

with kernel the unitary smoothing perturbations of Π_H. The Fredholm determinant is defined on $G^{-\infty}_H(\mathrm{U}(1); \mathrm{Cl}(n))$ so we may take subgroup of determinannt one; it is also a normal subgroup of $\mathcal{G}_H$. Finally then the quotient group

$$E\mathcal{L}\,\mathrm{Spin}(n) = \mathcal{G}/\{A \in G^{-\infty}_H(\mathrm{U}(1); \mathrm{Cl}(n)); \det(A) = 1\}$$

is the desired (basic) central extension of $\mathcal{L}\,\mathrm{Spin}(n)$. This construction can be modified to work even for non-simply connected groups such as $\mathrm{U}(n)$.

APPENDIX A

Finite-dimensional manifolds

First let me recall basic facts about compact manifolds, mainly to set up notation before specializing to the circle. Some of this I do not really need, but it seems a good idea to put things in context. In fact as you can see I got rather carried away and wrote down almost everything I could think of which might be relevant. Don't worry if you do not get ALL of this, since I generally do not need this much but if you want me to explain any of it a little more I am happy to do so. I do not plan to lecture on most of what is in this first section but am open to counter-proposals. In Lecture 3 I plan to start directly with §1.

Perhaps the most basic object associated with a compact $\mathcal{C}^\infty$ manifold, M, is the space $\mathcal{C}^\infty(M)$ of (real- or complex-valued, if necessary I can use the notation $\mathcal{C}^\infty(M;\mathbb{R})$, $\mathcal{C}^\infty(M;\mathbb{C})$) functions on M. From this one can construct the tangent and cotangent bundles, TM, T^*M – for instance the fibre of T^*M at $m \in M$ can be identified with $\mathcal{I}_m/\mathcal{I}_m^2$, $\mathcal{I}_m \subset \mathcal{C}^\infty(M)$ being the ideal of functions vanishing at m and $\mathcal{I}_M^2$ being the finite span of at two factors from $\mathcal{I}_m$.

I assume we are familiar with the notion of a vector bundle, V, real or complex, and of the operations on them giving the dual, V^*, the tensor product of two $V \otimes W$, the bundle of fibre homomorphisms from V to W, $\hom(V,W)$, that this is canonically $W \otimes V^*$, exterior products as antisymmetric parts of tensor products etc.

There is one construction which is maybe slightly less familiar than these standard ones, and which only works for rank one real bundles. Namely, if L is such a real line bundle then it can be identified (as in general) with $(L^*)^*$, so elements of L_m are linear maps

$$l : (L^*)_m \longrightarrow \mathbb{R}. \tag{A.0.1}$$

Instead one can consider maps which are absolutely homogeneous of any given degree $a \in \mathbb{R}$:

$$w : (L^*)_m \setminus \{0\} \longrightarrow \mathbb{R},\ w(s\mu) = |s|^a w(\mu),\ \forall\ s \in \mathbb{R}\setminus\{0\},\ \mu \in (L^*)_m \setminus \{0\}. \tag{A.0.2}$$

Clearly w is determined by its value at any one point and moreover the space of such w is actually linear and extends to give a smooth bundle. One could denote this bundle, somewhat confusingly, as $|L|^a$. It is *always trivial* so nothing much is going on with this construction, but note that $|L|^a \otimes |L|^b = |L|^{a+b}$ canonically and $|L|^0$ is canonically the trivial bundle. The notation comes from the fact that if $l \in \mathcal{C}^\infty(M;L)$ is a smooth section then $|l|^a \in \mathcal{C}^0(M;|L|^a)$ is a well-defined constinuous section. However the bundle $|L|^a$ always has a global smooth positive section.

The reason for interest is that we wish to set $\Omega^a(M) = |\Lambda^{\dim M} M|^a$ and call this the bundle of a-densities on M. Then the Riemann integral is well defined

$$\int : \mathcal{C}^0(M;\Omega) \longrightarrow \mathbb{R},\ \Omega M = \Omega^1 M. \tag{A.0.3}$$

These are the objects that can be integrated. One version of an orientation of M is that it is an isomorphisms $o : \Lambda^{\dim M} M \longrightarrow \Omega M$ such that any positive global section of the density bundle is the absolute value of its pull-back.

To be definite I will take $TM = (T^*M)^*$ to be the dual bundle of T^*M. The fibre T_mM can also be identified with the space of derivations on $\mathcal{C}^\infty(M)$ at m. A Riemann metric on M is a smooth positive-definite quadratic form on the fibres of TM, so in particular a section of the symmetric part of $T^*M \otimes T^*M$. Any vector bundle (over a compact manifold) can be embedded in a trivial bundle.

The topology on $\mathcal{C}^\infty(M)$ is the Fréchet topology given by all the $\mathcal{C}^k$ semi-norms on compact subsets of coordinate patches, or equivalently the Sobolev norms on balls. It is countably normed and a Montel space – there is a sequence of norms $\|\cdot\|_k$, $k \in \mathbb{N}$, giving the topology such that a bounded set with respect to $\|\cdot\|_{k+1}$ is precompact with respect to $\|\cdot\|_k$. The space $\mathcal{C}^\infty(M;V)$ of sections of a vector bundle has a similar topology which is the same as the product topology in the trivial case and consistent with embedding.

Now, once we have introduced the density bundle we can recall the standard notation for distributions. Namely 'distributional functions' form the dual space

$$\begin{gathered}\mathcal{C}^{-\infty}(M) = (\mathcal{C}^\infty(M;\Omega))' \text{ or more generally}\\ \mathcal{C}^{-\infty}(M;V) = (\mathcal{C}^\infty(M;\Omega\otimes V^*))'.\end{gathered} \tag{A.0.4}$$

This is done so that the smooth sections map naturally into the 'distributional sections' (not quite sections of course)
(A.0.5)

$$\mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^{-\infty}(M;V),\ \phi \longmapsto i(\phi) \in (\mathcal{C}^\infty(M;\Omega\otimes V^*))',\ i(\phi)(\psi) = \int \langle \phi, \psi \langle$$

since the pointwise pairing of $\phi \in \mathcal{C}^\infty(M;V)$ and $\psi \in \mathcal{C}^\infty(M;\Omega\otimes V^*)$ gives a section of the density bundle.

The space of smooth vector fields

$$\mathcal{V}(M) = \{V : M \longrightarrow TM; V(m) \in T_mM\} \tag{A.0.6}$$

is a Lie algebra and its universal enveloping algebra consists of the (smooth, linear) differential operators on $\mathcal{C}^\infty(M)$. There is a corresponding space of differential operators, $\mathrm{Diff}^k(M;V,W)$, between the sections of any two smooth (finite-dimensional) vector bundles V and W over M with $\mathrm{Diff}^0(M;V,W) = \hom(V,W)$. Note that this is not a vector bundle over M (if $k > 0$) because the transition maps are not bundle maps.

Although seldom really used it is good to know the basic theorems about differential operators and distributions. Namely the embedding (A.0.5) of smooth functions into distributins extends to map $L^1(M;V)$ injectively into $\mathcal{C}^{-\infty}(M;V)$ (and we regard this injection as an identification). In particular $\mathcal{C}^0(M;V) \longrightarrow L^2(M;V) \longrightarrow L^1(M;V)$ are all identified with subspaces of distributions. Schwartz' representation theorem gives a partial inverse of this. Namely the action of smooth differential operators extends uniquely

$$\begin{array}{ccc} \mathcal{C}^\infty(M;V) & \xrightarrow{P} & \mathcal{C}^\infty(M;W) \\ \downarrow & & \downarrow \\ \mathcal{C}^{-\infty}(M;V) & \xrightarrow[P]{} & \mathcal{C}^{-\infty}(M;W) \end{array} \quad \forall\ P \in \operatorname{Diff}^k(M;V,W). \tag{A.0.7}$$

Then for any $u \in \mathcal{C}^{-\infty}(M;V)$ there exists $P \in \operatorname{Diff}^k(M;V) = \operatorname{Diff}^k(M;V,V)$ (where k depends on u) such that $u = Pv$, $v \in L^2(M;V)$. Pushing this a bit further one can define global Sobolev spaces so that

$$\begin{gathered} \mathcal{C}^{-\infty}(M;V) = \bigcup_{m\in\mathbb{R}} H^m(M;V),\ \bigcap_{m\in\mathbb{R}} H^m(M;V) = \mathcal{C}^\infty(M;V), \\ P \in \operatorname{Diff}^k(M;V,W) \Longrightarrow P : H^m(M;V) \longrightarrow H^{m-k}(M;W). \end{gathered} \tag{A.0.8}$$

The other big, related, theorem is the Schwartz kernel theorem. It can be interpreted in terms of completion of tensor products (and spawned a big industry in the 1960s along these lines). To state it we need to give $\mathcal{C}^{-\infty}(M;V)$ a topology – the weak (or is it weak*) topology is given by the seminorms $|u(\cdot)|$ for $u \in \mathcal{C}^\infty(Ml\Omega \otimes V^*)$ acting through the duality pairing. There are other topologies but this is okay for this purpose. Then we know what a continuous linear map

$$Q : \mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^{-\infty}(M;W) \tag{A.0.9}$$

is. Schwartz' kernel theorem says there is a bijection (tological too) between such continuous linear 'operators' and $\mathcal{C}^{-\infty}(M^2;\operatorname{Hom}(W,V) \otimes \pi_R^*\Omega)$. Here, π_R is the projection onto the right factor of M and $\operatorname{Hom}(V,W)$ is the two-point homorphism bundle, with fibre over $(m,m') \in M^2$ the space $\hom(V_{m'},W_m)$. The map Q assocatiated to $\tilde{Q} \in \mathcal{C}^{-\infty}(M^2;\operatorname{Hom}(W,V) \otimes \pi_R^*\Omega)$ is determined by the condition

$$Q(v)(w) = \tilde{Q}(w \boxtimes v) \tag{A.0.10}$$

where you need to sort out the pairing over M^2 on the right to get the pairing over M on the left.

One class of operators we will consider (mostly over the circle as with everything else) are the smoothing operators. These correspond to the subspace

$$\mathcal{C}^\infty(M^2;\operatorname{Hom}(W,V) \otimes \pi_R^*\Omega) \subset \mathcal{C}^{-\infty}(M^2;\operatorname{Hom}(W,V) \otimes \pi_R^*\Omega) \tag{A.0.11}$$

in Schwartz' theorem and form an algebra. The action then is really give by an integral so that $\tilde{Q} \in \mathcal{C}^\infty(M^2;\operatorname{Hom}(W,V) \otimes \pi_R^*\Omega)$ defines

$$Q : \mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^\infty(M;W),\ (Qv)(m) = \int_M \langle \tilde{Q}(m,\cdot), v(\cdot)\rangle \tag{A.0.12}$$

where again the pairing leading to a density needs to be sorted out. I will denote the algebra of smoothing operators on V by $\Psi^{-\infty}(M;V)$ for reasons that will become clear below – the module of smoothing operators between sections of different bundles will be denoted $\Psi^{-\infty}(M;V,W)$.

One important (fairly elementary) result is that the group of operators

$$\begin{aligned} G^{-\infty}(M;V) = \big\{ \operatorname{Id} + A,\ A \in \Psi^{-\infty}(M;V); \\ \exists\ B \in \Psi^{-\infty}(M;V),\ (\operatorname{Id} + A)(\operatorname{Id} + B) = \operatorname{Id} \big\} \end{aligned} \tag{A.0.13}$$

is a classifying space for odd K-theory. For our purposes it is more important for the moment that it carries the Fredholm determinant. In fact this is a well-defined (entire analytic) function on $\Psi^{-\infty}(M;V)$, which we write as $\det_{\mathrm{Fr}}(\mathrm{Id}+A)$, $A \in \Psi^{-\infty}(M;V)$, and which has many of the properties of the finite-dimensional determinant so

$$\begin{gathered}\mathrm{Id}+A \in G^{-\infty}(M;V) \Longleftrightarrow \det_{\mathrm{Fr}}(\mathrm{Id}+A) \neq 0,\\ \det_{\mathrm{Fr}}\left((\mathrm{Id}+A)(\mathrm{Id}+B)\right) = \det_{\mathrm{Fr}}(\mathrm{Id}+A)\det_{\mathrm{Fr}}(\mathrm{Id}+B),\\ \frac{d}{ds}\det_{\mathrm{Fr}}(\mathrm{Id}+sA) = \det_{\mathrm{Fr}}(\mathrm{Id}+sA)\,\mathrm{Tr}(A)\end{gathered} \tag{A.0.14}$$

where the trace functional $\mathrm{Tr} : \Psi^{-\infty}(M;V) \longrightarrow \mathbb{C}$ is the unique continuous linear map vanishing on commutators and such that on finite rank projections (or idempotents) in $\Psi^{-\infty}(M;V)$ it reduces to the rank.

The relationship between $\det_{\mathrm{Fr}}$ and the K-theoretic statement above is that $\frac{1}{2\pi i}\det_{\mathrm{Fr}} : G^{-\infty}(M;V) \longrightarrow \mathbb{C}^*$ generates the 1-dimensional integral cohomology of $G^{-\infty}(M;V)$ and this is the bottom part of the odd Chern character – the rest can be written down similarly. Remember that one of the things that this course is at least related to is 'smooth cohomology' and this is somewhat epitomized by $G^{-\infty}(M;V)$ which carries smooth universal odd Chern classes. The even version is not much more complicated.

Although we will not need (at least I don't think it will come up) the general case, it seems appropriate to understand a little about the space of pseudodifferential operators over M. These are operators like the smoothing operators in that they map $\mathcal{C}^\infty(M;V)$ to $\mathcal{C}^\infty(M;W)$ for any two vector bundles. In fact it is useful to consider (classical) pseudodifferential operators of complex order $\Psi^z(M;V,W)$. These compose sensibly

$$\Psi^z(M;V_2,V_3) \circ \Psi^{z'}(M;V_1,V_2) = \Psi^{z+z'}(M;V_1,V_3) \tag{A.0.15}$$

and have lots of other properties too. The main point though is that they, as a space of operators, are much more like $\mathcal{C}^\infty$ rather than $\mathcal{C}^{-\infty}$.

More precisely, there is a symbol map giving a short exact sequence for any z :

$$\Psi^{z-1}(M;V,W) \longrightarrow \Psi^z(M;V,W) \longrightarrow \mathcal{C}^\infty(S^*M;\pi^*\hom(V,W)\otimes N^z). \tag{A.0.16}$$

Here $S^*M = (T^*M \setminus O_M)/\mathbb{R}^+$ is the cosphere bundle of M and N^z is the bundle with sections over S^*M which are functions on $T^*M \setminus O_M$ which are positively (not absolutely) homogeneous of degree z. So this is a trivial bundle with section given by a metric for instance, $|\xi|^z$.

The main point about this sequence, which will show up mostly when $z \in \mathbb{Z}$, is that we can 'iterate' it and the notation is consistent

$$\Psi^{k-1}(M;V,W) \subset \Psi^k(M;V,W) \tag{A.0.17}$$

is the subspace of operators with vanishing symbol of order k and

$$\Psi^{-\infty}(M;V,W) = \bigcap_{k\in\mathbb{Z}} \Psi^k(M;V,W), \tag{A.0.18}$$

the 'residual' space is indeed the space of smoothing operators.

The pseudodifferential operators can be characterized quite explicitly in terms of their Schwartz kernels, and I will talk more about this in the case of the circle. However, in brief, the Schwartz kernels of the elements of say $\Psi^k(M;V,W)$ are sections over M^2 of the appropriate bundle $\mathrm{Hom}(V,W)\otimes\pi_R^*\Omega$ with the following special properties

(1) The kernels are smooth away from the diagonal
(2) A neighbourhood of the diagonal is diffeomorphic to a neighbourhood of the zero section of TM and the bundles to the pull-backs of their restrictions to the diagonal, which is identified with the zero section so trivial on the fibres. Cutting off the kernels by a smooth function of compact support in the neighourhood the Fourier t.ransform in the fibre directions reduces the kernels to 'Laurent' sections of the bundles over $\overline{T^*M}$ – the radial compactification of the cotangent bundle to a (closed) ball bundle. This means that these Fourier transforms are precisely of the form $|\xi|^{-z}a$ where a is a smooth section of the bundle including up to the boundary of the ball. The symbol is (taking care of densities correctly) the restriction of a to 'infinity'.

The differential operators are the subspace of the pseudodifferential operators with Schwartz kernels supported in the diagonal (this is 'locality' of differential operators). Their symbols are homogeneous polynomial sections of the appropriate bundle on T^*M.

Now, I went as far as including classical pseudodifferential operators of complex order so as to be able to describe the residue trace introduced by Wodzicki and later by Guillemin. One can actually find an entire family

$$R(z)\in\Psi^z(M;V) \tag{A.0.19}$$

which are invertible, with $R(-z)$ the inverse of $R(z)$. In fact Seeley did this, just find a positive-definite self-adjoint differential operator, P, of order 2 on sections of V with respect to some Hermitian inner product – such a thing is like a quantize metric and always exists (nothing very natural about the choice). Then the complex powers exists and $R(z)=P^{z/2}$ is of the form (A.0.19).

Now, if we take an element $A\in\Psi^k(M;V)$ of *integral* order we can form

$$R(z)A\in\Psi^{z+k}(M;V). \tag{A.0.20}$$

The trace functional $\mathrm{Tr}(A)$ discussed above on smoothing operators actually extends by continuity to $\Psi^z(M;V)$ provided $z<-\dim M$ – the elements of this space are trace class operators on $L^2(M;V)$. Seeley already observed that the trace has an analytic extension so one can say

(A.0.21)
$$\mathrm{Tr}(R(z)A)\text{ is meromorphic with poles only at } z=-\dim M-k-j,\ j\in\mathbb{Z},\ j\geq 0.$$

The first poles just corresponds to the point where the operator stops being trace class (assuming its symbol doesn't vanish). What we are particularly interested in is the pole at $z=0$ which might occur by (A.0.21) if $k\geq-\dim M$.

The pole at $z=0$ is always simple and the *residue*

$$\mathrm{Tr}_{\mathrm{R}}(A)=\lim_{z\to 0}z\,\mathrm{Tr}(R(z)A) \tag{A.0.22}$$

is called the residue trace. It does not depend on the choice of R with the properties above – not so surprising since $R(0)=\mathrm{Id}$. By construction it vanishes on $\Psi^{-\dim M-1}(M;V)$. It can be explicitly computed as

$$\mathrm{Tr}_{\mathrm{R}}(A)=c\int_{S^*M}\sigma_{-\dim M}(A),\ A\in\Psi^{-\dim M}(M;V) \tag{A.0.23}$$

and really for all the integral-order pseudodifferential operators.

The residue trace *is* a trace. We will also be interested in the functional, the 'regularized trace'

$$\mathrm{Tr}_R(A) = \lim_{z \to 0} (\mathrm{Tr}(R(z)A) - \mathrm{Tr}_{\mathrm{R}}(A)/z) \tag{A.0.24}$$

which does depend on R and which is not a trace, but does restrict to Tr on operators of order $< -\dim M$.

We will use the regularized trace later (for the circle) to define connections on bundles.

Bibliography

[1] Jean-Luc Brylinski, *Representations of loop groups, dirac operators on loop space and modular forms*, Topology **29** (1990), no. 4, 461–480.

[2] Dennis A. McLaughlin, *Orientation and string structures on loop space*, Pacific J. Math. **155** (1992), no. 1, 143–156.

[3] M.K. Murray, *Bundle gerbes*, J. London Math. Soc. **54** (1996), 403–416.

[4] A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1986.

[5] Corbett Redden, *String structures and canonical 3-forms*, arXiv:0912.2086.

[6] Graeme Segal, *Elliptic cohomology*, Sminaire N. Bourbaki, no. exp. no 695, Nundam, 1987-1988, http://www.numdam.org, pp. 187–201.

[7] D. Stevenson, *Bundle 2-gerbes*, ArXiv Mathematics e-prints (2001).

[8] Stephan Stolz and Peter Teichner, *The spinor bundle on loop spaces*, Preprint.

[9] Konrad Waldorf, *Spin structures on loop spaces that characterize string manifolds*, arXiv:1209.1731.

[10] E. Witten, *The index of the dirac operator on loop space*, Elliptic curves and modular forms in algebraic topology, Lecture Notes in Math, vol. 1326, Springer Verlag, 1988.