THIRD ASSIGNMENT FOR 18.157, SPRING 2022

Each of these three (so far) little projects is pretty demanding. I am not really expecting anyone to do more than one of them and indeed am happy to give pretty much full credit for serious but partial efforts. I am happy to talk about these, and fix up errors which are surely present. I would actually value TEXfiles so I can maybe incorporate answers into the notes.

1. Problems 3a: Chern character-building

I have laid rather heavy emphasis on the group(s) $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ in the definition of (complex, topological) K-theory. Indeed by fiat I have declared this to be a classifying group for odd K-theory. Here I want you to sort out the map to deRham cohomology leading to the Atiyah-Hirzebruch isomorphism.

In the first part I want you to explain, step by step, the meaning of the 'odd Chern character'

$$\operatorname{Ch}(g) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2\pi i)^k (2k+1)!} \operatorname{Tr}\left((g^{-1}dg)^{2k+1}\right)$$

This is a formal sum of forms in odd degree.

(1) Recall that $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ is an open subset of $\mathcal{S}(\mathbb{R}^{2n})$ and it is 'an infinite matrix group'. Use this to give a clear meaning to the Maurier-Cartan form $g^{-1}dg$ here.

Hint: Perhaps avoid going into a full discussion of Fréchet manifolds! As an open subset of a Fréchet space the tangent space at each point g is $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$. Any element $a \in \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ here defines a curve in the group in the obvious way as $g + ta$ and the identification with the tangent space (with elements equivalence classes of curves) is written 'dg' – meaning I think that $d(g + ta)/dt$ is the tangent vector at g. Then g^{-1} acting on the left on the group maps g to the origin. This is a concrete group in the sense that this gives a linear map

$$g^{-1} : \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n).$$

The push-forward on the tangent spaces would be denoted g^{-1}_* but since the map is linear we can drop the $*$ and then

$$g^{-1}dg : \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$$

is the natural map from the tangent space at g to the tangent space at Id (which is the Lie algebra).

(2) Then the formal product $(g^{-1}dg)^j$ is supposed to be the j-fold exterior product, with composiition thrown in, so it is a j-multilinear, totally anti-symmetric map from j copies of the tangent space at g to the tangent space at Id.

(3) This explains (1) in the sense that the trace functional results in each term being a $(2k + 1)$-multilinear function on the tangent space at each point, which is what a form should be.

(4) The trace property $\operatorname{Tr}([a,b]) = 0$ shows that
$$\operatorname{Tr}((g^{-1}dg)^{2k}) = 0$$
which explains why there are no even terms.

(5) The deRhan differential is easy to define on 1-forms, even with values in an infinite dimensional space, show that
$$d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg$$
in an appropriate sense.

(6) Conclude that the terms in $\operatorname{Ch}(g)$ are all closed.

(7) Suppose $[0,1] \ni t \longmapsto G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ is a smooth curve, show that
$$\frac{d}{dt}\operatorname{Ch}(g_t) = d\operatorname{Et}(g)$$
where Et is the 'Eta' or Chern-Simons form
$$\operatorname{Et}(g) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}k!}{(2\pi i)^k(2k)!}\operatorname{Tr}\left(g^{-1}\frac{dg}{dt}(g^{-1}dg)^{2k}\right)$$
Hint: Make sense of the formula
$$\frac{d}{dt}g^{-1}dg = -g^{-1}\frac{dg}{dt}g^{-1}dg + g^{-1}d\frac{dg}{dt}.$$

(8) Now, suppose M is a compact manifold and $\kappa : M \longrightarrow G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ is a smooth map. Show that the pull-backed form is closed and so defines a map
$$\mathcal{C}^\infty(M; G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)) \ni \kappa \longmapsto \kappa^*\operatorname{Ch} \in H^{\mathrm{odd}}_{\mathrm{dR}}(M).$$

(9) Conclude that this map descends to a map
$$\operatorname{Ch} : K^1(M) \longrightarrow H^{\mathrm{odd}}_{\mathrm{dR}}(M) \text{ the odd Chern character.}$$

(10) Contemplate why this might induce the Atiyah-Hirzebruch isomorphism
$$K^1(M) \otimes \mathbb{C} \longrightarrow H^{\mathrm{odd}}_{\mathrm{dR}}(M).$$
(So this means torsion is killed in the K-group. This does not quite work for general non-compact manifolds).

(11) Now, if you have the energy, do the even version! Start from (7) and show that on the group $\mathcal{C}^\infty(\mathbb{R}; G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n))$, which is used above to define even K-theory,
$$\operatorname{Ch}_{\mathrm{ev}} = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}k!}{(2\pi i)^k(2k)!}\operatorname{Tr}\left(g^{-1}\frac{dg}{dt}(g^{-1}dg)^{2k}\right)dt$$
defines a closed form in each even degree. [I am not sure I have the normalizing constants correct, but here it does not matter.]

(12) Conclude, proceeding as above, that this defines a map
$$K^0(M) \otimes \mathbb{C} \longrightarrow H^{\mathrm{ev}}_{\mathrm{dR}}(M).$$

2. Problems 3b: Isotropic K-pop

P3b.0 Recall to isotropic pseudodifferential algebra, for simplicity of order 0, on $\mathbb{R}^{2n}$ is defined by quantization of the classical symbols joointly in the variables (x,ξ) :

$$\Psi^0_{\mathrm{iso}}(\mathbb{R}^n) = Q_L(\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}).$$

The product defines a smooth bilinear map

$$\Psi^0_{\mathrm{iso}}(\mathbb{R}^n)\times\Psi^0_{\mathrm{iso}}(\mathbb{R}^n)\longrightarrow\Psi^0_{\mathrm{iso}}(\mathbb{R}^n).$$

We can then define $\mathcal{C}^\infty(M;\Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ for any manifold M and in particular

$$\mathcal{S}(\mathbb{R}^{2n'};\Psi^0_{\mathrm{iso}}(\mathbb{R}^n)).$$

Now, show that this allows us to define the 'stabilized algebra'

$$\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$$

by using the product of Schwartz smoothing operators and then the product (2) to define

$$\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\times\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longleftrightarrow\mathcal{S}(\mathbb{R}^{2n'};\Psi^0_{\mathrm{iso}}(\mathbb{R}^n)\times\Psi^0_{\mathrm{iso}}(\mathbb{R}^n))\longrightarrow\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

This is an associative algebra of bounded operators on $L^2(\mathbb{R}^{n+n'})$ which maps $\mathcal{S}(\mathbb{R}^{n+n'})$ to itself.

(1) One reason that this algebra is interestimg is that it has a principal symbol map

$$\sigma_0:\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\mathcal{C}^\infty(\mathbb{S}^{2n-1};\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$$

which is multiplicative and gives a short exact sequence

$$\Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\mathcal{C}^\infty(\mathbb{S}^{2n-1};\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

(2) We need to massage this a little more by identifying

$$\mathbb{R}^{2n-1}\longrightarrow\mathbb{S}^{2n-1}$$

as the 1-point compactification – choose your favourite point (mine is the South Pole). Anyway, this allows us to map

$$\mathcal{S}(\mathbb{R}^{2n-1};\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{n'})\hookrightarrow\mathcal{C}^\infty(\mathbb{S}^{2n-1};\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$$

with image being the subspace of functions vanishing to infinite order at the point at infinity.

(3) Denote by $\dot\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$ which is the inverse image under σ_{iso} of $\mathcal{S}(\mathbb{R}^{2n-1};\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{n'})$

(4) This is an algebra without identity, so add Id to get a ring which we can denote

$$\mathrm{Id}+\dot\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

Check that we now a multiplicative exact sequence

$$\mathrm{Id}+\Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\mathrm{Id}+\dot\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\mathrm{Id}+\mathcal{S}(\mathbb{R}^{2n-1};\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

(5) Finally we can finish the setup by defining

$$\dot{E}(\mathbb{R}^n;\Psi^{-\infty}(\mathbb{R}^{2n'}))=\left\{A\in\operatorname{Id}+\dot{\Psi}^0_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'};\sigma_{\text{iso}}(A)\in\mathcal{C}^\infty(\mathbb{R}^{2n-1};G^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\right\}'$$

and now there is a multliplicative exact sequence

$$\operatorname{Id}+\Psi^{-1}_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\dot{E}_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\mathcal{S}(\mathbb{R}^{2n-1};G^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

3. Problems3c: Bott and Clifford

In lectures I wrote down an explicit family of unipotent matrices on $\mathbb{R}^2$ representing the 'Bott' element (in even K-theory) and showed that the semiclassical quantization of this family is a 1-dimensional shift from the constant family. Here I ask you to do this for $\mathbb{R}^{2n}$. Initially I assumed you knew about the structure of the complex Clifford algebra, I have now added a brief derivation in case you do not.

(1) Deconstruct the matrix I simply wrote down in lectures

$$\mu=\begin{pmatrix}\cos(\chi(r)) & e^{i\theta}\sin(\chi(r))\\ e^{-i\theta}\sin(\chi(r)) & -\cos(\chi(r))\end{pmatrix}.$$

First write it out as

$$\mu=\cos(\chi(r))Z+\sin(\chi(r))\left(\cos\theta E_1+\sin\theta E_2\right).$$

Show that these 2×2 matrices satisfy

$$E_i^2=Z^2=\operatorname{Id},\ E_1E_2+E_2E_1=ZE_1+E_1Z=ZE_2+E_2Z=0.$$

(2) Recall the complexified Clifford algebra on a real Euclidean vector space, V, of even dimension, $2n$, defined as the quotient of the infinte tensor algebra

$$\mathcal{T}(V)=\mathbb{C}\oplus V_{\mathbb{C}}\oplus(V_{\mathbb{C}}\otimes V_{\mathbb{C}})\oplus\cdots=\sum_k V_{\mathbb{C}}^{\otimes k},\ V_{\mathbb{C}}=V\otimes\mathbb{C}$$

by the two-sided ideal (under tensor product) generated by the elements

$$\mathcal{I}(V)\ni\xi\otimes\eta+\eta\otimes\xi-2\langle\xi,\eta\rangle,\ \xi,\ \eta\in V.$$

Here $\langle,\rangle$ is the Euclidean inner product. Thus,

$$\operatorname{Cl}(V)=\mathcal{T}(V)/\mathcal{I}(V).$$

Remind yourself that this has dimension 2^{2n}, the same as the exterior algebra. You should also note that it is isomorphic to the $2^n\times 2^n$ matrix algebra.

The Clifford algebra is closely related to the exterior algebra, but on $\mathbb{C}^n$ rather than $\mathbb{R}^{2n}$. So we are passing to the standard complex structure on $\mathbb{R}^{2n}$ in which a basis, over the complex numbers, is

$$f_i=e_i+ie_{i+n},\ i=1,\dots,n.$$

Show that $\operatorname{Cl}(\mathbb{R}^{2n})$, the complexified Clifford algebra (not the Clifford algebra on the complexification!) acts on $\lambda^*\mathbb{C}^n$ through the formula we saw for the Hodge-Dirac operator

$$\operatorname{cl}(e_k+ie_{k+n})f^\alpha=i\sqrt{2}f_k\wedge f^\alpha,\ \operatorname{cl}(e_k-ie_{k+n})f^\alpha=-i\sqrt{2}\iota(f_k)f^\alpha,\ k\le n$$

where the constants are for length normalization an here α is a strictly increasing sequence in $\{1,\dots,n\}$. This specifies the action of all basis elements and we see that

$$\begin{gathered}\mathrm{cl}(e_k+ie_{k+n})\,\mathrm{cl}(e_k-ie_{k+n})+\mathrm{cl}(e_k-ie_{k+n})\,\mathrm{cl}(e_k+ie_{k+n})=2\,\mathrm{Id},\\ \mathrm{cl}(e_k+ie_{k+n})^2=0=\mathrm{cl}(e_k-ie_{k+n})^2\\ \Longrightarrow \mathrm{cl}(e_k)\,\mathrm{cl}(e_{k'})+\mathrm{cl}(e_{k'})\,\mathrm{cl}(e_k)=2\delta_{kk'}\,\mathrm{Id},\ k,k'=1,\dots,2n.\end{gathered}$$

(3) If e_i is an (oriented) orthonormal basis of V, let E_i be the corresponding elements in the Clifford algebra so

$$E_iE_j+E_jE_i=\delta_{ij}\,\mathrm{Id}\,.$$

Then show that the Clifford algebra has a 'maximal element'

$$Z=i^{n(2n-1)}E_1E_2\dots E_{2n}\Longrightarrow Z^2=\mathrm{Id},\ ZE_i+E_iZ=0.$$

(4) Okay, now observe that (1) is now

(1) $$\mu(\zeta)=\cos(\chi(|\zeta|)Z+\sin(\chi(r))(\hat\zeta\cdot E_*)$$

where $E_*=(E_1,\dots,E_{2n})$ is thought of as a vector with values in matrices and $\cdot$ is the inner product.

(5) Now observe that the definition (1) extends to $V=\mathbb{R}^{2n}$ to give a smooth family of unipotent matrices. Here, as before $\chi(r)$ is decreasing from π near 0 to 0 near ∞.

(6) Review the proof in the notes to check that there is a semiclassical family of idempotents qunatizing μ (so now to smoothing operators on $\mathbb{R}^n$).

(7) For a bonus, show that the quantization is again has relative index 1 (assuming I got the signs right which would be a pleasant accident). You might like to do this by quantizing in two variables repeatedly, so working by induction over n.