THIRD ASSIGNMENT FOR 18.157, SPRING 2022

Each of these three (so far) little projects is pretty demanding. I am not really expecting anyone to do more than one of them and indeed am happy to give pretty much full credit for serious but partial efforts. I am happy to talk about these, and fix up errors which are surely present. I would actually value TeXfiles so I can maybe incorporate answers into the notes.

1. PROBLEMS 3A: CHERN CHARACTER-BUILDING

I have laid rather heavy emphasis on the group(s) $G^{-\infty}_S(\mathbb{R}^n)$ in the definition of (complex, topological) K-theory. Indeed by fiat I have declared this to be a classifying group for odd K-theory. Here I want you to sort out the map to deRham cohomology leading to the Atiyah-Hirzebruch isomorphism.

(1) First I want you to explain the meaning of the ‘odd Chern character’

$$\text{Ch}(g) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2\pi i)^k (2k + 1)!} \text{Tr} \left((g^{-1}dg)^{2k+1}\right)$$

This is a formal sum of forms in odd degree.

(2) Recall that $G^{-\infty}_S(\mathbb{R}^n)$ is an open subset of $S(\mathbb{R}^{2n})$ and it is ‘an infinite matrix group’. Use this to give a clear meaning to the Maurier-Cartan form $g^{-1}dg$ here.

Hint: Perhaps avoid going into a full discussion of Fréchet manifolds! As an open subset of a Fréchet space the tangent space at each point $g$ is $\Psi^{-\infty}_S(\mathbb{R}^n)$. Any element $a \in \Psi^{-\infty}_S(\mathbb{R}^n)$ here defines a curve in the group in the obvious way as $g + ta$ and the identification with the tangent space (with elements equivalence classes of curves) is written ‘$dg$’ – meaning $\text{d}(g + ta)/\text{d}t$ is the tangent vector at $g$. Then $g^{-1}$ acting on the left on the group maps $g$ to the origin. This is a concrete group in the sense that this gives a linear map

$$g^{-1} : \Psi^{-\infty}_S(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_S(\mathbb{R}^n).$$

The push-forward on the tangent spaces would be denoted $g^{-1} \ast$ but since the map is linear we can drop the $\ast$ and then

$$g^{-1} dg : \Psi^{-\infty}_S(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_S(\mathbb{R}^n)$$

is the natural map from the tangent space at $g$ to the tangent space at $\text{Id}$ (which is the Lie algebra).

(3) Then the formal product $(g^{-1}dg)^j$ is supposed to be the $j$-fold exterior product, with composition thrown in, so it is a $j$-multilinear, totally antisymmetric map from $j$ copies of the tangent space at $g$ to the tangent space at $\text{Id}$.

(4) This explains (1) in the sense that the trace functional results in each term being a $(2k + 1)$-multilinear function on the tangent space at each point, which is what a form should be.
(5) The trace property \( \text{Tr}([a, b]) = 0 \) shows that
\[
\text{Tr}((g^{-1}dg)^{2k}) = 0
\]
which explains why there are no even terms.

(6) The deRham differential is easy to define on 1-forms, even with values in an infinite dimensional space, show that
\[
d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg
\]
in an appropriate sense.

(7) Conclude that the terms in \( \text{Ch}(g) \) are all closed.

(8) Suppose \([0, 1] \ni t \mapsto g^{-1}dg\) is a smooth curve, show that
\[
\frac{d}{dt}\text{Ch}(g_t) = \frac{d}{dt}\text{Et}(g)
\]
where \( \text{Et} \) is the ‘Eta’ or Chern-Simons form
\[
\text{Et}(g) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}k!}{(2\pi i)^k(2k)!} \text{Tr} \left( \frac{d}{dt}(g^{-1}dg)^{2k} \right)
\]
Hint: Make sense of the formula
\[
\frac{d}{dt}g^{-1}dg = -g^{-1}d\frac{dg}{dt}g^{-1}dg + g^{-1}\frac{d}{dt}g^{-1}dg.
\]

(9) Now, suppose \( M \) is a compact manifold and \( \kappa : M \to G^{-\infty}_S(\mathbb{R}^n) \) is a smooth map. Show that the pull-backed form is closed and so defines a map
\[
C^\infty(M; G^{-\infty}_S(\mathbb{R}^n)) \ni \kappa \mapsto \kappa^* \text{Ch} \in H^{\text{odd}}_{dR}(M).
\]

(10) Conclude that this map descends to a map
\[
\text{Ch} : K^1(M) \to H^{\text{odd}}_{dR}(M) \text{ the odd Chern character}.
\]

(11) Contemplate why this might induce the Atiyah-Hirzebruch isomorphism
\[
K^1(M) \otimes \mathbb{C} \to H^{\text{odd}}_{dR}(M).
\]
(So this means torsion is killed in the K-group. This does not quite work for general non-compact manifolds).

(12) Now, if you have the energy, do the even version! Start from (8) and show that on the group \( C^\infty(\mathbb{R}; G^{-\infty}_S(\mathbb{R}^n)) \), which is used above to define even K-theory,
\[
\text{Ch}_{\text{ev}} = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}k!}{(2\pi i)^k(2k)!} \text{Tr} \left( \frac{d}{dt}(g^{-1}dg)^{2k} \right) dt
\]
defines a closed form in each even degree. [I am not sure I have the normalizing constants correct, but here it does not matter.]

(13) Conclude, proceeding as above, that this defines a map
\[
K^0(M) \otimes \mathbb{C} \to H^{\text{ev}}_{dR}(M).
\]
2. Problems 3b: Isotropic K-pop

P3b.0 Recall to isotropic pseudodifferential algebra, for simplicity of order 0, on $\mathbb{R}^{2n}$ is defined by quantization of the classical symbols jointly in the variables $(x, \xi)$:

$$\Psi^0_{\text{iso}}(\mathbb{R}^n) = Q_L(C^\infty(\mathbb{R}^{2n})).$$

The product defines a smooth bilinear map

$$\Psi^0_{\text{iso}}(\mathbb{R}^n) \times \Psi^0_{\text{iso}}(\mathbb{R}^n) \rightarrow \Psi^0_{\text{iso}}(\mathbb{R}^n).$$

We can then define $C^\infty(M; \Psi^0_{\text{iso}}(\mathbb{R}^n))$ for any manifold $M$ and in particular $\mathcal{S}(\mathbb{R}^{2n}; \Psi^0_{\text{iso}}(\mathbb{R}^n))$.

Now, show that this allows us to define the 'stabilized algebra' $\Psi^0_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n))$ by using the product of Schwartz smoothing operators and then the product (2) to define

$$\Psi^0_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n)) \rightarrow \mathcal{S}(\mathbb{R}^{2n}; \Psi^0_{\text{iso}}(\mathbb{R}^n) \times \Psi^0_{\text{iso}}(\mathbb{R}^n)) \rightarrow \Psi^0_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n)).$$

This is an associative algebra of bounded operators on $L^2(\mathbb{R}^{n+n'})$ which maps $\mathcal{S}(\mathbb{R}^{n+n'})$ to itself.

(1) One reason that this algebra is interesting is that it has a principal symbol map

$$\sigma_0 : \Psi^0_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n)) \rightarrow C^\infty(S^{2n-1}; \Psi^{-\infty}_S(\mathbb{R}^n))$$

which is multiplicative and gives a short exact sequence

$$\Psi^{-1}_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n)) \rightarrow \Psi^0_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n)) \rightarrow C^\infty(S^{2n-1}; \Psi^{-\infty}_S(\mathbb{R}^n)).$$

(2) We need to massage this a little more by identifying

$$\mathbb{R}^{2n-1} \rightarrow S^{2n-1}$$

as the 1-point compactification – choose your favourite point (mine is the South Pole). Anyway, this allows us to map

$$\mathcal{S}(\mathbb{R}^{2n-1}; \Psi^{-\infty}_S(\mathbb{R}^n)) \rightarrow C^\infty(S^{2n-1}; \Psi^{-\infty}_S(\mathbb{R}^n))$$

with image being the subspace of functions vanishing to infinite order at the point at infinity.

(3) Denote by $\hat{\Psi}^0_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n))$ which is the inverse image under $\sigma_{\text{iso}}$ of $\mathcal{S}(\mathbb{R}^{2n-1}; \Psi^{-\infty}_S(\mathbb{R}^n))$.

(4) This is an algebra without identity, so add $\text{Id}$ to get a ring which we can denote

$$\text{Id} + \hat{\Psi}^0_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n)).$$

Check that we now a multiplicative exact sequence

$$\text{Id} + \Psi^{-1}_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n)) \rightarrow \text{Id} + \hat{\Psi}^0_{\text{iso}}(\mathbb{R}^n; \Psi^{-\infty}_S(\mathbb{R}^n)) \rightarrow \text{Id} + \mathcal{S}(\mathbb{R}^{2n-1}; \Psi^{-\infty}_S(\mathbb{R}^n)).$$
(5) Finally we can finish the setup by defining
\[
\dot{E}(\mathbb{R}^n; \Psi^{-\infty}(\mathbb{R}^{2n})) = \left\{ A \in \text{Id} + \dot{\Psi}_0(\mathbb{R}; \Psi^{-\infty}(\mathbb{R}^{2n}) ; \sigma_{\text{iso}}(A) \in C^\infty(\mathbb{R}^{2n-1} ; G_S^{-\infty}(\mathbb{R}^{2n})) \right\}
\]
and now there is a multiplicative exact sequence
\[
\text{Id} + \dot{\Psi}_0(\mathbb{R}^n; \Psi^{-\infty}(\mathbb{R}^{2n})) \longrightarrow \dot{E}_0(\mathbb{R}^n; \Psi^{-\infty}(\mathbb{R}^{2n})) \longrightarrow \dot{S}(\mathbb{R}^{2n-1} ; G_S^{-\infty}(\mathbb{R}^{2n})).
\]

3. Problems 3c: Bott and Clifford

In lectures I wrote down an explicit family of unipotent matrices on \(\mathbb{R}^2\) representing the ‘Bott’ element (in even K-theory) and showed that the semiclassical quantization of this family is a 1-dimensional shift from the constant family. Here I ask you to do this for \(\mathbb{R}^{2n}\).

(1) Deconstruct the matrix I simply wrote down in lectures
\[
\mu = \begin{pmatrix}
\cos(\chi(r)) & e^{i\theta} \sin(\chi(r)) \\
e^{-i\theta} \sin(\chi(r)) & -\cos(\chi(r))
\end{pmatrix}.
\]
First write it out as
\[
\mu = \cos(\chi(r)) Z + \sin(\chi(r)) (\hat{\zeta} \cdot E^*)
\]
where \(E^* = (E_1, \ldots, E_{2n})\) is thought of as a vector with values in matrices and \(\cdot\) is the inner product.

(2) Recall the complexified Clifford algebra on a real Euclidean vector space, \(V\), of even dimension defined as the quotient of the infinite tensor algebra
\[
T(V) = \mathbb{C} \oplus V_C \oplus V_C \otimes V_C \oplus \cdots, V_C = V \otimes \mathbb{C}
\]
by the two-sided ideal (under tensor product) generated by the elements
\[
\mathcal{I}(V) = \xi \otimes \eta + \eta \otimes \xi - (\xi, \eta), \xi, \eta \in V.
\]
Here \((,\)\) is the Euclidean inner product. Thus,
\[
\text{Cl}(V) = T(V)/\mathcal{I}(V).
\]
Remind yourself that this has dimension \(2^{2n}\), the same as the exterior algebra. You should also note that it is isomorphic to the \(2^n \times 2^n\) matrix algebra (no proof needed).

(3) If \(e_i\) is an (oriented) orthonormal basis of \(V\), let \(E_i\) be the corresponding elements in the Clifford algebra so
\[
E_i E_j + E_j E_i = \delta_{ij} \text{Id}.
\]
(4) The Clifford algebra has a ‘maximal element’
\[
Z = \tau^{(2n-1)} E_1 E_2 \cdots E_{2n} \Rightarrow Z^2 = \text{Id}, \ Z E_i + E_i Z = 0.
\]
(5) Okay, now observe that (1) is now
\[
\mu(\zeta) = \cos(\chi(|\zeta|)) Z + \sin(\chi(r)) (\zeta \cdot E_*)
\]
where \(E_* = (E_1, \ldots, E_{2n})\) is thought of as a vector with values in matrices and \(\cdot\) is the inner product.

(6) Now observe that the definition (1) extends to \(V = \mathbb{R}^{2n}\) to give a smooth family of unipotent matrices. Here, as before \(\chi(r)\) is decreasing from \(\pi\) near 0 to 0 near \(\infty\).
(7) Review the proof in the notes to check that there is a semiclassical family of idempotents quantizing $\mu$ (so now to smoothing operators on $\mathbb{R}^n$).

(8) For a bonus, show that the quantization is again has relative index 1 (assuming I got the signs right which would be a pleasant accident). You might like to do this by quantizing in two variables repeatedly, so working by induction over $n$. 