SECOND ASSIGNMENT FOR 18.157, SPRING 2022

In this second problem set I would like you to go through some 'symbolic arguments', giving L^2 boundedness of pseudodifferential operators.

0.1. **Schur's criterion.** This is the same Schur as the lemma about irreducibility, hence I just say 'criterion'. This is quite a handy sufficient condition for L^2 boundedness in terms of the Schwartz kernel. It can be generalized to measure spaces (and so manifolds), but for the moment let's think about $\mathbb{R}^n$. Then

Proposition 1. *If $A : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a Lebesgue measurable function which satisfies*

$$\sup_x \int |A(x,y)|dy,\ \sup_y \int |A(x,y)|dx < \infty \tag{1}$$

then the integral operator (say defined initially on $\mathcal{C}_c(\mathbb{R}^n)$)

$$Au(x) = \int_{\mathbb{R}^n} A(x,y)u(y)dy \text{ is a bounded operator on } L^2(\mathbb{R}^n). \tag{2}$$

Proof. You might like to look it up, it is basically just a clever use of Schwarz inequality. □

Problem 2.1

Show that if $A \in \Psi^m(\mathbb{R}^n)$ with $m < -n$ then the Schwartz kernel is continuous and satisfies

$$\sup_{x,y}(1+|x-y|)^N |A(x,y)| < \infty\ \forall\ N. \tag{3}$$

Deduce that Schur's criterion applies and hence conclude L^2 boundedness.

In fact you can push this argument so that it applies for $m < 0$ but not up to $m = 0$ (think of the identity).

Problem 2.2

For $A \in \Psi^0(\mathbb{R}^n)$ construct $Q \in \Psi^0(\mathbb{R}^n)$ such that

$$Q = Q^*,\ Q^2 = C\,\mathrm{Id} - A^*A + E,\ C > 0 \text{ constant},\ E \in \Psi^{-1}(\mathbb{R}^n) \tag{4}$$

'Hint': It is enough to choose $C > \sup |a|^2$ where $A = Q_L(a)$. Then show that $q = (C - |a|^2)^{\frac{1}{2}} \in \mathcal{C}^\infty_\infty(\mathbb{R}^n; S^0(\mathbb{R}^n))$ and the set $Q = \frac{1}{2}(Q_L(a)) + Q_L(a)^*$.

Problem 2.3

Now we want to improve the 'error' in (4). Show that if $E \in \Psi^{-k}(\mathbb{R}^n)$, $k \geq 1$, and $E^* = E$ where $E = Q_L(e)$ then the choice

$$B = Q_L(e/q) + Q_L(e/q)^* \text{ satisfies } (Q-B)^2 = C\,\mathrm{Id} - A^*A - E',\ E' \in \Psi^{-k-1}(\mathbb{R}^n),\ (E')^* = E'. \tag{5}$$

Problem 2.4

Using this show that we may 'correct' Q (by adding a lower order term) so that (4) holds with $E \in \Psi^{-N}(\mathbb{R}^n)$ for any preassigned N. (Using asymptotic summation this works for $N = -\infty$.

Problem 2.5

Finally deduce L^2 boundedness in the sense that $A \in \Psi^0(\mathbb{R}^n)$ extends by continuity from $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ to a bounded operator on $L^2(\mathbb{R}^n$.

'Hint'. This whole argument is due to Hörmander. It follows from (4) that, for $\phi \in \mathcal{S}(\mathbb{R}^n)$, in terms of the L^2 inner product

$$0 \le \langle Q\phi, Q\phi\rangle = \langle Q^2\phi, \phi\rangle = C\|\phi\|^2_{L^2} - \|A\phi\|^2_{L^2} + \langle E\phi, \phi\rangle. \tag{6}$$

So, if we know that boundedness of E (which we do) then

$$\|A\phi\|_{L^2} \le (C + C')^{\frac{1}{2}}\|u\|_{L^2}. \tag{7}$$

where C' comes from E.

Problem 2.6: Sobolev boundedness

The Sobolev space $H^s(\mathbb{R}^n)$ is defined as consisting of those elements of $\mathcal{S}'(\mathbb{R}^n)$ (becauese we are allowing $s \le 0$ such that

$$(1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n). \tag{8}$$

Deduce that the operator $(1 + |D|^2)^t = Q_L((1 + |\xi|^2)^{t/2}) = Q_R((1 + |\xi|^2)^{t/2}) \in \Psi^t(\mathbb{R}^n)$, for any $t \in \mathbb{R}$, is an isomorphism

$$(1 + |D|^2)^{t/2} : H^s(\mathbb{R}^n) \longrightarrow H^{s-t}(\mathbb{R}^n). \tag{9}$$

From this, L^2 boundedness and the properties of the calculus deduce that

$$A \in \Psi^m(\mathbb{R}^n) \Longrightarrow A : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n). \tag{10}$$

'Hint': Consider for instance $(1 + |D|^2)^{-m+s/2}A(1 + |D|^2)^{-s/2}$.

Problem 2.7

For anyone who has read the section on the scattering (Shubin) calculus define the weighted Sobolev spaces

$$H^{s,t}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1 + |x|^2)^{t/2}u \in H^s(\mathbb{R}^n). \tag{11}$$

(1) Show that for any real orders

$$A \in \Psi^{m,k}_{\text{sc}}(\mathbb{R}^n) \Longrightarrow A : H^{s,t}(\mathbb{R}^n) \longrightarrow H^{s-m,t-l}(\mathbb{R}^n). \tag{12}$$

(2) Show that

$$\mathcal{F} : H^{s,t}(\mathbb{R}^n) \longrightarrow H^{t,s}(\mathbb{R}^n),\ \forall\ s, t. \tag{13}$$

(3) Show that, in contrast to the usual Sobolev spaces, the inclusion $H^{s',t'}(\mathbb{R}^n) \hookrightarrow H^{s,t}(\mathbb{R}^n)$ for $s' > s$, $t' > t$ is *compact*.