Due date: The sooner you get in solutions the sooner you will get them returned. I am hoping that you will do them by Feb 26 but am open to discussion – on the problems too of course!

I detected some resistance to the idea of radial compactification of $\mathbb{R}^n$ in class so the main part of the first problem set is to work out some of the details. Quite a bit of this is already in the notes.

0 First recall, for background if nothing else, the basis of projective geometry (which seems to have disappeared as a subject taught to undergraduates not long before I started studying Mathematics). Define complex projective space as a quotient

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{T}, \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \mathbb{S}^{2n+1} = (\mathbb{R}^{2n+2} \setminus \{0\}) / \mathbb{R}^+ = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^+.$$ 

Check that this is a complex manifold and that $\mathbb{C}^n$ is identified with an open dense subset by the inclusion

$$\mathbb{C}^n \ni z \mapsto (z, 1) \in \mathbb{C}^{n+1} \setminus \{0\} \hookrightarrow \mathbb{P}^n$$

and that the complement of the image may be identified with $\mathbb{P}^n - 1$.

1 Now sort out the real (or more correctly a) real analogue of this. Take the embedding

$$\mathbb{R}^n \ni x \mapsto (x, 1) \in \mathbb{R}^n \times [0, \infty) \subset \mathbb{R}^{n+1}$$

and consider the quotient map

$$\iota : \mathbb{R}^n \longrightarrow (\mathbb{R}^n \times [0, \infty) \setminus \{0\}) / \mathbb{R}^+ = \mathbb{S}^n = \mathbb{R}^{n+1}$$

mapping into the upper half-sphere (see picture below); the last equation defines $\mathbb{R}^n$. I take this as the definition of the radial compactification; show that the embedding is given explicitly by

$$\iota(x) = \left( \frac{x}{\sqrt{1 + |x|^2}}, \frac{1}{\sqrt{1 + |x|^2}} \right)$$

and deduce that $t = (1 + |x|^2)^{-\frac{1}{2}} \in C^\infty(\mathbb{S}^n)$ is a boundary defining function (vanishes only on the boundary and has differential non-zero there).

2 Derive the Taylor series of $a \in S^0_0(\mathbb{R}^n) = C^\infty(\mathbb{S}^n)$ (this is the definition of the space of classical symbols of order 0 from lectures) in the form

$$\sum_{k=0}^\infty |x|^{-k} a_k \left( \frac{x}{|x|} \right), \quad a_k \in C^\infty(\mathbb{S}^{n-1}), \quad |x| > 2 (\iff t < \sqrt{2}).$$

Deduce that Taylor series with remainder gives

$$|a - \sum_{k=0}^N |x|^{-k} a_k \left( \frac{x}{|x|} \right)| \leq C_N |x|^{-N-1}.$$
We want similar estimates for derivatives too.

3. Introduce projective coordinates on $\mathbb{R}^n$ given by $2n + 1$ coordinate patches on $\mathbb{S}^n$. The first one is $x \in \mathbb{R}^n$ defining the compactification. Then for each $k = 1, \ldots, n$ set

\[ D^+_k = \{ x \in \mathbb{R}^n; \pm x_k > 0 \}, \quad C^+_k = \{ p = (p_1, \ldots, p_{n+1}) \in \mathbb{S}^n; \pm p_k > 0 \} \]

(note that this includes part of the boundary of $\mathbb{S}^n$) and show that the diffeomorphisms

\[ C^+_k \ni x \mapsto \left( \frac{1}{\pm x_k}, \frac{x}{x_k} \right) \in (0, \infty) \times \mathbb{R}^{n-1} \]

extend to diffeomorphisms $D^+_k \to [0, \infty) \times \mathbb{R}^{n-1}$.

4. Show that these projective coordinate systems give a coordinate cover of $\mathbb{R}^n$.

5. Write out formulæ for the images of the vector fields

\[ \partial_{x_j}, \quad x_l \partial_{x_j} \]

in these projective coordinate systems (note these span the Lie algebras of the translation group and $\text{GL}(n, \mathbb{R})$ respectively).

6. Show that the $x_l \partial_{x_j}$ extend to be smooth on $\mathbb{R}^n$ (meaning smooth up to the boundary) and that they are elements of the Lie algebra

\[ \mathcal{V}_b(\mathbb{R}^n) = \{ V \text{ a } C^\infty \text{ vector field tangent to the boundary}\}. \]

Note that tangency to the boundary means $V t = 0$ at $t = 0$.

7. Show that the images of the $x_l \partial_{x_j}$ span $\mathcal{V}_b(\mathbb{R}^n)$ over $C^\infty(\mathbb{R}^n)$.

8. Show that the $\partial_{x_j}$ are also smooth up to the boundary of $\mathbb{R}^n$ and span, over $C^\infty(\mathbb{R}^n)$ the space

\[ \mathcal{V}_b(\mathbb{R}^n) = \{ W = tV, \text{ } V \in \mathcal{V}_b(\mathbb{R}^n)\}. \]

9. Show that the space $\mathcal{S}^0(\mathbb{R}^n)$ of ‘symbols with bounds’ is identified with the space

\[ \{ u \in L^\infty(\mathbb{R}^n); V_1 \ldots V_N u \in L^\infty(\mathbb{R}^n) \forall V_i \in \mathcal{V}_b(\mathbb{R}^n) \forall N \}. \]

[Don’t get hung up on worrying about distributions on a manifold with boundary, we will come back to this later.]

10. Putting some of these things together show that

\[ \mathcal{S}_{cl}^0(\mathbb{R}^n) \subset \mathcal{S}^0(\mathbb{R}^n). \]

11. Deduce that an element $a \in \mathcal{S}^0(\mathbb{R}^n)$ is in $\mathcal{S}_{cl}^0(\mathbb{R}^n)$ if and only if

\[ a \sim \sum_k (1 - \phi)(\xi)|\xi|^{-k}a_k \left( \frac{\xi}{|\xi|} \right) \]

where $\phi \in C^\infty_c(\mathbb{R}^n)$ is equal to one near 0 (to make everything smooth) and the $a_k \in C^\infty(\mathbb{S}^{n-1})$. 