# FIRST ASSIGNMENT FOR 18.157, SPRING 2022

V2: Two corrections from Benjy.

Due date:- The sooner you get in solutions the sooner you will get them returned. I am hoping that you will do them by Feb 26 but am open to discussion – on the problems too of course!

I detected some resistance to the idea of radial compactification of $\mathbb{R}^n$ in class so the main part of the first problem set is to work out some of the details. Quite a bit of this is already in the notes.

0 First recall, for background if nothing else, the basis of projective geometry (which seems to have disappeared as a subject taught to undergraduates not long before I started studying Mathematics). Define complex projective space as a quotient

(1)
$$\mathbb{P}^n = (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^* = \mathbb{S}^{2n+1}/\mathbb{T},\ \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ \mathbb{S}^{2n+1} = (\mathbb{R}^{2n+2}\backslash\{0\})/\mathbb{R}^+ = (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^+.$$

Check that this is a complex manifold and that $\mathbb{C}^n$ is identified with an open dense subset by the inclusion

$$\mathbb{C}^n \ni z \longmapsto (z,1) \in \mathbb{C}^{n+1} \setminus \{0\} \hookrightarrow \mathbb{P}^n \tag{2}$$

and that the complement of the image may be identified with $\mathbb{P}^{n-1}$.

(1) Now sort out the real (or more correctly a) real analogue of this. Take the embedding

$$\mathbb{R}^n \ni x \longmapsto (x,1) \in \mathbb{R}^n \times [0,\infty) \subset \mathbb{R}^{n+1} \tag{3}$$

and consider the quotient map

$$\iota : \mathbb{R}^n \longrightarrow (\mathbb{R}^n \times [0,\infty) \setminus \{0\})/\mathbb{R}^+ = \mathbb{S}^n_+ = \overline{\mathbb{R}^n} \tag{4}$$

mapping into the upper half-sphere (see picture below); the last equation defines $\overline{\mathbb{R}^n}$. I take this as the definition of the radial compactification; show that the embedding is given explicitly by

$$\iota(x) = \left(\frac{x}{\sqrt{1+|x|^2}}, \frac{1}{\sqrt{1+|x|^2}}\right) \tag{5}$$

and deduce that $t = (1+|x|^2)^{-\frac{1}{2}} \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ is a boundary defining function (vanishes only on the boundary and has differential non-zero there).

(2) Derive the Taylor series of $a \in S^0_{\mathrm{cl}}(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{S}^n_+)$ (this is the definition of the space of classical symbols of order 0 from lectures) in the form

$$\sum_{k=0}^{\infty} |x|^{-k} a_k(\frac{x}{|x|}),\ a_k \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ |x| > 2(\Longleftrightarrow t < \sqrt{2}). \tag{6}$$

Deduce that Taylor series with remainder gives

$$|a - \sum_{k=0}^{N} |x|^{-k} a(\frac{x}{|x|})| \le C_N |x|^{-N-1}. \tag{7}$$

[We want similar estimates for derivatives too].

(3) Introduce projective coordinates on $\overline{\mathbb{R}^n}$ given by $2n+1$ coordinate patches on $\mathbb{S}^n_+$. The first one is $x \in \mathbb{R}^n$ defining the compactification. Then for each $k = 1, \dots, n$ set

$$D_k^{\pm} = \{x \in \mathbb{R}^n; \pm x_k > 0\},\ C_k^{\pm} = \{p = (p_1, \dots, p_{n+1}) \in \mathbb{S}^n_+; \pm p_k > 0\} \tag{8}$$

(note that this includes part of the boundary of $\mathbb{S}^n_+$) and show that the diffeomorphisms

$$C_k^{\pm} \ni x \longmapsto (\frac{1}{\pm x_k}, \pm\frac{x_j}{x_k}) \in (0, \infty) \times \mathbb{R}^{n-1} \tag{9}$$

extend to diffeomorphism $D_k^{\pm} \longrightarrow [0, \infty) \times \mathbb{R}^{n-1}$.

(4) Show that these projective coordinate systems give a coordinate cover of $\overline{\mathbb{R}^n}$.

(5) Write out formulæ for the images of the vector fields

$$\partial_{x_j},\ x_l\partial_{x_j} \tag{10}$$

in these projective coordinate systems (note these span the Lie algebras of the translation group and $\mathrm{GL}(n, \mathbb{R})$ respectively).

(6) Show that the $x_l\partial_{x_j}$ extend to be smooth on $\overline{\mathbb{R}^n}$ (meaning smooth up to the boundary) and that they are elements of the Lie algebra

$$\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n}) = \{V \text{ a } \mathcal{C}^\infty \text{ vector field tangent to the boundary}\}. \tag{11}$$

Note that tangency to the boundary means $Vt = 0$ at $t = 0$.

(7) Show that the images of the $x_l\partial_{x_j}$ span $\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n})$ over $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$.

(8) Show that the $\partial_{x_j}$ are also smooth up to the boundary of $\overline{\mathbb{R}^n}$ and span, over $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ the space

$$t\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n}) = \{W = tV,\ V \in \mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n})\}. \tag{12}$$

(9) Show that the space $S^0(\mathbb{R}^n)$ of 'symbols with bounds' is identified with the space

$$\{u \in L^\infty(\overline{\mathbb{R}^n}); V_1 \dots V_N u \in L^\infty(\overline{\mathbb{R}^n})\ \forall\ V_i \in \mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n})\ \forall\ N\}. \tag{13}$$

[Don't get hung up on worrying about distributions on a manifold with boundary, we will come back to this later.]

(10) Putting some of these things together show that

$$S^0_{\mathrm{cl}}(\mathbb{R}^n) \subset S^0(\mathbb{R}^n). \tag{14}$$

(11) Deduce that an element $a \in S^0(\mathbb{R}^n)$ is in $S^0_{\mathrm{cl}}(\mathbb{R}^n)$ if and only if

$$a \sim \sum_k (1-\phi)(\xi)|\xi|^{-k} a_k(\frac{\xi}{|\xi|}) \tag{15}$$

where $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to one near 0 (to make everything smooth) and the $a_k \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$.