Lectures on Microlocal Analysis
18.157, Spring 2022

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Outline and Practicalities

[Revised: 24 January, 2022.]

In this course I hope to cover four (types of) theorems which involve microlocal analysis and in particular the theory of pseudodifferential operators. Namely

(1) Hörmander's theorem on the propagation of singularities
(2) Weyl's law for the distribution of eigenvalues
(3) The Atiyah-Singer index theorem and K-theory
(4) Hodge theory and boundaries

As a first step I will proceed to discuss the algebras of pseudodifferential operators on Euclidean space and on a compact manifold and then similar algebras (and related modules) on manifolds with boundary and for fibrations and more

157.74 (0.1) $$\Psi^*(\mathbb{R}^n),\ \Psi^*(M),\ \Psi^*_*(M)$$

where the upper star is an order and the lower star is some sort of structural information.

To me the four results listed above are fundamental, and I like them! The first two are relatively closely related and both give realization of the 'semiclassical limit', the interplay between the non-commutative theory of (pseudo-)differential operators and the more familiar behaviour of analysis of functions. The latter two are more global but both involve the essential invertibility of (pseudo-)differential operators.

Let me briefly indicate what these theorems are about.

Hörmander's theorem on the propagation of singularities is a precise version, and massive generalization, of 'Huyghen's Principle'. The latter describes the spreading of the singular edge of solutions of the wave equation. The precise version is one of the consequences of 'microlocalization', transferring analysis from 'space' to 'phase space' interpreted concretely as a manifold and its cotangent bundle respectively.

Weyl's asymptotic formula describes, at 'high energy', the number of eigenvalues of a self-adjoint elliptic operator, on a compact manifold, in terms of the volume inside the energy surface in the cotangent bundle. The original theorem was actually about the eigenvalues of the Dirichlet problem on a domain in $\mathbb{R}^2$.

Elliptic (pseudo-)differential operators on a compact manifold are Fredholm – they are invertible modulo finite dimensional null space and complement of the range. The index, the difference of these two dimensions, is a very stable number in the sense that it only depends on the 'topology' defined by the leading part of the operator and the theorem gives a formula for it. One classical version of this is the Riemann-Roch theorem for the $\overline{\partial}$ operator on (line bundles over) a compact Riemann surface. This already requires some effort to understand! There is a one-dimensional real version of the theorem, due to Toeplitz, which states that the index of an elliptic Toeplitz operator on the circle (the projection onto the Hardy space, consisting of the functions smooth on and holomorphic on the interior of the disk, of multiplication by a non-vanishing smooth function) is equal to (minus) the winding number of the function.

You probably do know the Hodge theorem for a compact manifold without boundary as the identification of the deRham cohomology with the space of harmonic forms. For non-compact manifolds there is no simple generalization, rather there are many corresponding to structures 'at infinity' (meaning near the boundary).

Clearly, each of these theorems could easily expand to take the whole semester. Still I hope to show how they can be approached using pseudodifferential operators and 'quantization'. In fact an alternative title for this course might be 'Smooth quantization'. So most of the time will be devoted to preparing the background material, specifically pseudodifferential operators on $\mathbb{R}^n$, pseudodifferential operators on a manifold, families of pseudodifferential operators and then rings of pseudodifferential operators quantizing a Lie algebroid.

I plan to give 26 one-hour lectures in the 9:30-10:30 slot on Tuesdays and Thursdays and leave 20 minutes for questions and discussions (even short presentations by students); if there is sufficient interest I will organize another 'discussion' time, perhaps on Wednesdays in the afternoon. There will be notes for each topic (the precise correspondence to the individual lectures will depend on various things), which will include topics I will not have time to cover and will certainly include further references – to books, lecture notes and papers. With any luck at least some of the lectures should appear on my webpage before the beginning of the semester.

Problem sets: There will be approximately 5, every two weeks. Grading may be by discussion with me.

Grades: Graduate students are expected to participate actively. That is what 'A' means to me. By this I mean that I expect people to attend lectures and to ask questions. For undergraduates this course might be heavy lifting, it is for me, so please talk to me by early in the semester at the latest. We can discuss what you should expect. There are no exams.

Prerequisites: I will assume familiarity with manifolds and distributions, essentially as in 18.155 but plan to review pretty much everything.

Why don't I just follow a book or my earlier lecture notes? This probably reflects some personal failing and general dissatisfaction with how things are done! I find it difficult to think through things without seeing some other way of approaching them. If it is not to your taste, I am sorry but that is the way it is. I may not get to all the results listed above, but I expect to at least get to the point where they are all within reach and that is really what I want to do – try to put these results in a general context that maybe encourages them to be exploited (i.e. applied) and extended.

In the interim, feel free to contact me with questions or comments.

Richard Melrose, 17 November, 2021.

CHAPTER 1

Pseudodifferential operators, manifolds and compactification

The main aim of this course is to describe various algebras of pseudodifferential operators. Let me start with a traditional 'crypto-historical' description of the 'standard' algebra of pseudodifferential operators on $\mathbb{R}^n$. I recall notation for functions below. Let's assume you know about the spaces of smooth functions on Euclidean space and the successively larger subspaces of compactly supported functions, Schwartz functions and of functions with all derivatives bounded L1

157.1 (1.1)
$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$$

maybe including their topologies and duals.

For any multiindex $\alpha \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ being the non-negative integers, the corresponding iterated partial derivative acts on each of these spaces

157.2 (1.2)
$$u \longmapsto D^\alpha u,\ D^\alpha u(x) = i^{-|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1} \dots \frac{\partial^{\alpha_n}}{\partial x_n} \dots u(x),\ |\alpha| = \alpha_1 + \dots \alpha_n$$

where the normalizing power of i is inserted to help with notation for the Fourier transform.

These generate the commutative ring of *differential operators with constant coefficients* with general element

157.42 (1.3)
$$p(D) = \sum_{|\alpha| \le m} c_\alpha D^\alpha,\ c_\alpha \in \mathbb{C}.$$

This is a filtered ring which is isomorphic to the ring of polynomials in n variables.

Similarly, each of the spaces in (1.1) is a ring, so multiplication of functions is defined. Combining these we consider *linear partial differential operators* which are given by sums

157.3 (1.4)
$$P(x, D)u = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha u.$$

In each case, when the coefficients are in one of the spaces (1.1), we get an operator – a continuous linear map – on the corresponding space.

Whilst this is probably very familiar, and the *operator product* is given explicitly by Leinbiz' formula, it is very significant that these form a ring (and algebra) with product

157.43 (1.5)
$$P(x, D)Q(x, D) = \sum_{\gamma \le \alpha, \beta} p_\alpha(x)(D_x^\gamma q_\beta(x)) D^{\alpha+\beta-\gamma},\ Q(x, D) = \sum_{|\beta| \le m'} q_\beta(x) D^\beta.$$

It is worth thinking a little more about what is going on here. First note that (1.4) is not as 'natural' as (1.3) in so far as we have chosen to write the 'coefficients', the function $p_\alpha(x)$, on the left. This is true in (1.3) as well but there the constants commute with the differentiation operators. Of course this is reflected in the fact that the product (1.5) is not commutative.

Now, let's concentrate Schwartz space. For this we have the Fourier transform

157.4 (1.6) $$\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n),\ \mathcal{F}u(\xi)=\hat{u}(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u(x)dx.$$

It is a linear isomorphism. We know that

157.5 (1.7) $$u\in\mathcal{S}(\mathbb{R}^n)\Longrightarrow\mathcal{F}(D^\alpha u)(\xi)=\xi^\alpha\hat{u}(\xi).$$

The Fourier transform conjugates differentiation to multiplication. Whilst a monomial such as ξ^α is not in the Schartz space, but it does define an operator on it by multiplication.

So the inverse Fourier transform, $u(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}\hat{u}(\xi)$, allows us to write

157.6 (1.8) $$D^\alpha u(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\xi^\alpha\hat{u}(\xi)d\xi.$$

A linear partial differential operator, (1.4), is given by a finite sum so we can combine (1.8) with (1.4) and write

157.7 (1.9) $$Pu(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}p(x,\xi)\hat{u}(\xi)d\xi,\ p(x,\xi)=\sum_{|\alpha|\le m}p_\alpha(x)\xi^\alpha d\xi.$$

Since $\hat{u}\in\mathcal{S}(\mathbb{R}^n)$, the integral converges absolutely. If we just assume that the coefficients are in $\mathcal{C}^\infty(\mathbb{R}^n)$ then the integral converges uniformly on compact subsets in $x\in\mathbb{R}^n$, with all its formal derivatives in x because of the obvious estimates

157.8 (1.10) $$|D_x^\gamma p(x,\xi)|\le C_{K,\gamma}(1+|\xi|)^m,\ x\in K\Subset\mathbb{R}^n,\ \xi\in\mathbb{R}^n.$$

We can actually define the 'standard' space of pseudodifferential operators of order $m\in\mathbb{R}$ by considering those functions $a\in\mathcal{C}^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$ which satisfy the *symbol estimates*

157.9 (1.11) $$|D_x^\gamma D_\xi^\beta a(x,\xi)|\le C_{\beta,\gamma}(1+|\xi|)^{m-|\beta|},\ \forall\ \gamma,\beta\in\mathbb{N}_0^n.$$

Notice that p in (1.9) satisfies these estimates for an *integer* m if the coefficients are in the space

157.10 (1.12) $$\mathcal{C}^\infty_\infty(\mathbb{R}^n)=\{f\in\mathcal{C}^\infty(\mathbb{R}^n);\sup|D_x^\gamma f(x)|<\infty\ \forall\ \gamma\},$$

consisting of the smooth functions with all derivatives bounded.

The space of functions satifying estimates (1.11) is often written $S^m_{1,0}$ as part of a more general class of spaces $S^m_{\rho,\delta}$ where the exponent $m-|\beta|$ is replaced by $m-\rho|\beta|+\delta|\alpha|$. Since we will not need the general ρ, δ spaces – in fact there are many variants of such estimates (see for instance [**5**]) – and we will already have enough things to think about I will use the shorter notation, for the moment just $S^m=S^l_{1,0}$ for the functions satisfying (1.11).

It follows directly from (1.11) that if $a\in S^m$ then the direct generalization of (1.9),

157.11 (1.13)
$$A(x,D_x)u=Au(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}a(x,\xi)\hat{u}(\xi)d\xi\Longrightarrow a:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{C}^\infty_\infty(\mathbb{R}^n).$$

In fact much more is true:

157.12 THEOREM* 1.1.[1] *The space of operators, $\Psi^m(\mathbb{R}^n) = \Psi^m(\mathbb{R}^n)$ defined by symbols, a, satisfying (1.11) act on $\mathcal{S}(\mathbb{R}^n)$ and form a filtered $*$-closed ($*$ for adjoint here) ring*

157.13
$$\Psi^m(\mathbb{R}^n) \circ \Psi^{m'}(\mathbb{R}^n) \subset \Psi^{m+m'}(\mathbb{R}^n),\ \forall\ m, m' \in \mathbb{R}. \tag{1.14}$$

This is the main content of the first chapter of [**6**], see also Grigis-Sjöstrand [**3**]. Probably the first place this result appeared in this form is [**4**].

It is not that it is so hard to prove such a result, it is rather that I prefer to approach it from a position of strength, so somewhat indirectly, in the sense that I want to give a good deal of background before proving it.

Still[2] it is important to see what is straightforward to prove and what may require some more thought. First let's make sure we do have (1.13).

PROOF OF (1.13). If $u \in \mathcal{S}(\mathbb{R}^n)$ then the product

$$a(x,\xi)\hat{u}(\xi) \in S^{-\infty} = \bigcap_{M\in\mathbb{R}} S^M \qquad \text{157.44} \tag{1.15}$$

meaning that the estimates in (1.11) hold for all m. Indeed this is just the product rule for differentiation. Written out fully in terms of Leibniz' formula

$$D_x^\gamma D_\xi^\beta \left(a(x,\xi)\hat{u}(\xi)\right) = \sum_{\gamma\le\beta} \binom{\beta}{\gamma} D_x^\alpha D_\xi^\gamma a(x,\xi) \cdot D_\xi^{\beta-\gamma}\hat{u}(\xi). \qquad \text{157.45} \tag{1.16}$$

Then one can apply the more obvious fact the product is rapidly decaying in ξ :

$$S^m \cdot \mathcal{S}(\mathbb{R}^n_\xi) \subset S^{m-k}\ \forall\ k \in \mathbb{R}. \qquad \text{157.46} \tag{1.17}$$

The integral (1.13) is therefore convergent. Again, if you like to be precise, you can see that

$$S^m \subset \mathcal{C}^0_\infty(\mathbb{R}^n; L^1(\mathbb{R}^n_\xi)),\ m < -n \qquad \text{157.47} \tag{1.18}$$

since $(1+|\xi|)^{-n-\epsilon} \in L^1(\mathbb{R}^n)$ if $\epsilon > 0$. Now we can use standard properties of Lebesgue (or improper Riemann) integrals to see that $Au \in \mathcal{C}^0_\infty(\mathbb{R}^n)$ is a bounded continuous function and the same holds for all derivatives giving (1.13). □

We should check a couple of other statements, towards Theorem 1.1. First the stronger mapping property that

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \qquad \text{157.48} \tag{1.19}$$

This is a matter of getting 'decay'. Namely we need to show that for any monomial and any derivative

$$x^\gamma D_x^\alpha Au \in \mathcal{C}^0_\infty(\mathbb{R}^n). \qquad \text{157.49} \tag{1.20}$$

We can approach this one step at a time, asking just about $x_j Au$. Note that we can certainly multiply by x_j but the operator $x_j A$ is *not* in general in $\Psi^m(\mathbb{R}^n)$ (for any m) since $x_j a(x,\xi)$ is not

[1]The * in the header of the theorem is to indicate that I will not prove it immediately but a full proof, and more, will follow later.

[2]Narrowed parts of a lecture are things I don't expect to have time to cover.

bounded as $|x_j| \to \infty$ even for fixed ξ. However the integral in (1.13) still converges rapidly in ξ for x in a compact set if we replace a by $x_j a$ so

$$x_j A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \qquad \boxed{157.50} \qquad (1.21)$$

for instance.

157.51 Lemma 1.1. *In the sense of operators* (1.21)

$$[x_j, A] = x_j A - A x_j \in \Psi^{m-1}(\mathbb{R}^n). \qquad \boxed{157.52} \qquad (1.22)$$

Proof. We use 'integration by parts'. Consider the operator Ax_j. The Fourier transform of $x_j u$, $u \in \mathcal{S}(\mathbb{R}^n)$ is $i\partial_{\xi_j}\hat{u}$ so

$$Ax_j u = (2\pi)^{-n} \int a(x,\xi) e^{ix\cdot\xi} i\partial_{\xi_j}\hat{u}(\xi) d\xi = x_j A(x,D)u + b_j(x,D)u,$$

$$b_j(x,\xi) = -i\partial_{\xi_j} a(x,\xi) \in S^{m-1}.$$

The rapid decay of $a(x,\xi)\hat{u}(\xi)$ in ξ means that

$$\int \partial_{\xi_j}\left(a(x,\xi)e^{ix\cdot\xi}\hat{u}(\xi)\right) d\xi = 0. \qquad \boxed{157.55} \qquad (1.23)$$

Proceeding by induction we conclude that

$$x^\gamma A(x,D) = \sum_{\delta \le \gamma} B_\delta(x,D) x^\delta, \ B_\delta(x,D) \in \Psi^{m-|\gamma|+|\phi|}(\mathbb{R}^n). \qquad \boxed{157.56} \qquad (1.24)$$

$\square$

We will be most interested in a smaller space which I will denote $\Psi^m_{\text{cl}}(\mathbb{R}^n)$ often called the ring (with the composition property (1.14)) of 'classical' pseudodifferential operators where the symbols a have the additional property:

157.453 Definition 1.1. A symbol in the sense of (1.11) is *classical* (also 'polyhomogeneous') if there exists a sequence $a_i \in \mathcal{C}^\infty(\mathbb{R}^n_x \times (\mathbb{R}^n_\xi \setminus \{0\}))$ of homogeneous functions of degree $m - i$ (in the ξ variables)

157.14 (1.25) $$a_i(x, t\xi) = t^{m-i} a(x,\xi),\ t > 0,\ (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$$

such that for (any) cutoff $\chi \in \mathcal{C}^\infty_{\text{c}}(\mathbb{R}^n_\xi)$ with $\chi = 1$ near 0

157.15 (1.26) $$a(x,\xi) - \sum_{i=0}^{N} (1-\chi(\xi)) a_i(x,\xi) \in S^{m-N-1}.$$

These 'classical' symbols form a filtered subring $S^m_{\text{cl}} \subset S^m = S^m$. The relationship (1.26) (see Problem set 1) is often written

157.33 (1.27) $$a \simeq \sum_i a_i$$

and a is then said to have a complete asymptotic expansion. Such 'asymptotic summation' (the existence of a given the a_i) is discussed below, it is closely related to E. Borel's 'Lemma' on Taylor series. There is no statement of *convergence* of the series in (1.26) (although there is one lurking in the background) but you should be able to see that the a_i, assuming they *exist* are determined by the relations (1.26).

When we insert such classical symbols in (1.13) (or if you prefer, restrict to classical symbols) the resulting space of constitutes a filtered subring $\Psi^m_{\text{cl}}(\mathbb{R}^n) \subset$

$\Psi^m(\mathbb{R}^n)$ which for positive integral m includes the differential operators of order m discussed above.

These two rings have many important properties but one of the most important is that one can recover the terms a_i in (1.26) from the operator A and the leading term defines *the principal symbol,* σ_m, as a map

157.16 (1.28)

$$\Psi^m_{\mathrm{cl}}(\mathbb{R}^n) \xrightarrow{\sigma_m} \{a_0 \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \text{ homogeneous of degree } m \text{ in } \xi\}.$$

and this map is surjective, multiplicative and defines a short exact sequence

$$\sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B),\ A \in \Psi^m_{\mathrm{cl}}(\mathbb{R}^n),\ B \in \Psi^{m'}_{\mathrm{cl}}(\mathbb{R}^n)$$

157.17 (1.29)
$$\Psi^{m-1}_{\mathrm{cl}}(\mathbb{R}^n) \hookrightarrow \Psi^m_{\mathrm{cl}}(\mathbb{R}^n) \longrightarrow \{a_0 \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \text{ homogeneous of degree } m \text{ in } \xi\}$$

Here I have stuck with a cumbersome notation for the homogeneous space which will be refined below. There are similar exact sequences for the larger algebra $\Psi^*(\mathbb{R}^n)$ but the principal symbol lies in a quotient space.

So, we want to prove all these things and a lot more! However, I do not want to go there directly but rather map out the territory a bit first, in particular discussing the 'symbol spaces' concretely. L1-end

1. Manifolds with corners

This might appear to be a serious non-sequitor but I hope you will get used to the idea of these sections on background material and see a bit later why I am proceeding this way.

Both for 'local' analysis and the formulation of global results it is very convenient to focus on *manifolds with corners* as our basic 'category of spaces' (which it is as will be made precise later). There are several reasons to introduce these. An immediate one is to understand the symbol spaces and their generalizations. This I will get to next time. This allows me to introduce the spaces of conormal distributions which arise as the Schwartz kernels of the pseudodifferential operators we are interested in. Thinking about the kernels abstractly will allow us to generalize readily later. This involves manifolds with boundary, but then products will get you to manifolds with corners.

So, this is one of the basic settings for the course – analysis on manifolds with corners – but only taken as far as we need for the moment. Let me start with an explicit definition and then explain all the terms used in it. I'm assuming familiarity with the standard definition of a manifold *without boundary.*

157.34 Definition 1.2. A manifold, M, is a metrizable, separable (so second countable) topological space with an open covering giving a (maximal) atlas of $\mathcal{C}^\infty$-related coordinate patches modeled on $[0,\infty)^n$ and with embedded boundary hypersurfaces.

I will not assume connectedness without explicitly saying so, but the definition then requires all the components to have the same dimension.

So we are given a separable metric space, M, but the 'metrizable' means we do not take the actual metric seriously, just the open sets it defines as the unions of open balls. A coordinate patch in such a topological space is a triple (F, U, V) consisting of a homeomorphism $F : U \longrightarrow V$ of an open subset $U \subset M$ onto a (relatively) open subset $V \subset [0,\infty)^n$. So this means there exists an open subset

$V' \subset \mathbb{R}^n$ such that the range $V = V' \cap [0,\infty)^n$. The coordinates on the coordinate patch are the pull-backs of the coordinate functions x_i on $\mathbb{R}^n$.

To make clear what '$\mathcal{C}^\infty$-related' for two such coordinate patches means, we need to define $\mathcal{C}^\infty(V)$ (I will not bother with lower regularity than $\mathcal{C}^\infty$):

157.35
$$\begin{gathered}\mathcal{C}^\infty(V) = \{u : V \longrightarrow \mathbb{R}(\text{ or } \mathbb{C});\\ \exists\, V' \subset \mathbb{R}^n \text{ open } V = V' \cap [0,\infty)^n,\ u' \in \mathcal{C}^\infty(V') \text{ and } u = u'\big|_V\}.\end{gathered} \tag{1.30}$$

So I am assuming you know about $\mathcal{C}^\infty(V')$ for open subsets of $\mathbb{R}^n$.

Now the $\mathcal{C}^\infty$-compatibility of two coordinate patches (F_i, U_i, V_i), $i = 1,2$, as introduced above, means that either $U_1 \cap U_2 = \emptyset$ or else the transition maps

157.36
$$\begin{gathered}F_{12} = F_1 \circ F_2^{-1} : F_2(U_1 \cap U_2) \longrightarrow F_1(U_1 \cap U_2) \text{ and}\\ F_{21} = F_2 \circ F_1^{-1} : F_1(U_1 \cap U_2) \longrightarrow F_2(U_1 \cap U_2)\end{gathered} \tag{1.31}$$

are $\mathcal{C}^\infty$ in the sense that $F_{12}^* : \mathcal{C}^\infty(F_1(U_1 \cap U_2)) \longrightarrow \mathcal{C}^\infty(F_2(U_1 \cap U_2))$ and $F_{21}^* : \mathcal{C}^\infty(F_2(U_1 \cap U_2)) \longrightarrow \mathcal{C}^\infty(F_1(U_1 \cap U_2))$; this is equivalent to saying either pull-back map is an isomorphism. This is also equivalent to saying that the pull-backs of the coordinate functions, under either of the maps F_i, restrict to $U_1 \cap U_2$ to be $\mathcal{C}^\infty$ functions of the other coordinates.

So now an atlas is a covering by such (pairwise) $\mathcal{C}^\infty$-compatible coordinate patches. If some coordinate patches are compatible with all the elements of an atlas then the combined collection is still an atlas – they are necessarily compatible amongst themselves as well. Hence any atlas is contained in a unique maximal atlas – all this is as in the boundaryless case.

If we just stop at this point then M is what I call a *tied manifold* although there is no general agreement on this. The missing point is the additional condition that 'boundary hypersurfaces are embedded'. A point in a coordinate patch is a boundary point of codimension k if exactly k of the coordinate functions vanish on it (note that coordinate patches map into $[0,\infty)^n$ so by fiat all coordinates are non-negative – I will actually drop this requirement later but it makes things easier to state initially). By considering the differential of the transition map it follows that the codimension is well-defined at each point, it is independent of the coordinate patch used. This means that M has a stratification, a decomposition into disjoint pieces, based on the codimension

157.37
$$M = M_0 \cup M_1 \cup \cdots \cup M_n \tag{1.32}$$

where the M_j can be empty (from some $k > 0$ onward). The points of boundary codimension zero are the interior points of the manifold (there is a slight inconsistency between openness of subsets of $[0,\infty)^n$ and this, so the interior there is $(0,\infty)^n$, of course otherwise there would be no point in talking about the interior of a relatively open subset).

Each M_j itself is a manifold without boundary and the *closures* of the components of the M_j are called the boundary faces of codimension j; the set of these boundary faces I will write as $\mathcal{M}_j(M)$. In particular the boundary faces of codimension one, the $H_i \in \mathcal{M}_1(M)$ are called the boundary *hypersurfaces*. The 'boundary hypersurfaces are embedded' part of the definition is just the statement that the restrictions of the coordinate patches to each H_i given them $\mathcal{C}^\infty$-compatible atlases. One consequence of this is functorial, that the boundary hypersurfaces (and in consequence all boundary faces) are themselves manifolds with corners. There are several useful ways to restate this condition but note how it fails for a 'tear-shaped region' in the plane.

The $\mathcal{C}^\infty$ functions on M are those that are $\mathcal{C}^\infty$ in each coordinate patch, meaning

157.38 (1.33) $$f \in \mathcal{C}^\infty(M) \Longleftrightarrow (F^{-1})^*(f\big|_U) \in \mathcal{C}^\infty(V) \text{ for each coordinate patch.}$$

This is equivalent to the same condition for any one compatible atlas.

The direct consequence of the 'embedded' requirement is that the boundary hypersurfaces have defining functions:

157.39 (1.34) $$\begin{gathered} H_i \in \mathcal{M}_1(M) \Longrightarrow \ \exists\ \rho_i \in \mathcal{C}^\infty(M),\ \rho_i \ge 0,\ H_i = \{\rho_i = 0\}, \\ d((F^{-1})^*\rho_i)(F(p)) \neq 0\ \forall\ p \in H_i \text{ for all coordinate patches containing } p. \end{gathered}$$

This last condition means that for each $p \in H_i$ there is a coordinate patch containing p in which ρ_i is a coordinate function.

If $\tilde{M}$ is a manifold without boundary, i.e. $\tilde{M}_1 = \emptyset$, then $M \subset \tilde{M}$ is a(n embedded) submanifold if M has a covering by coordinate patches of $\tilde{M}$ which restrict to give it the structure of a manifold with corners.

157.40 Theorem 1.2. *For any manifold with corners there exists a manifold without boundary $\tilde{M}$ of the same dimension in which M is embedded as a submanifold; if M is compact then $\tilde{M}$ can be taken to be compact.*

Although there is no quite canonical way of constructing such an extension, $\tilde{M}$, all the standard constructions of the tangent, cotangent, form bundles and other bundles associated to the frame bundle, pass over to the case of a manifold with corners in such a way that the restrictions for an extension of this type are canonical

157.41 (1.35) $$TM = T\tilde{M}\big|_M,\ T^*M = T^*\tilde{M}\big|_M \text{ etc.}$$

However, there are important additional structures which arise from the boundary faces as I will discuss later.

Why work in this degree of generality? Manifolds with corners are the smooth (i.e. $\mathcal{C}^\infty$) analogue of smooth algebraic varieties with divisors and they occur for similar reasons. One place manifolds with corners arise is through 'compactification'.

Remark 1. From now on the default meaning for 'manifold' is a manifold with corners.

As you are no doubt aware, it is bad form to specify a class of objects, in this case manifolds without specifying the 'morphisms' between them. For manifolds without boundary the morphisms, forming a category, are just the smooth maps. This also works when there are corners but it is not a good idea. The problem is that a smooth map $F : M \longrightarrow N$ in general does not 'respect' the boundary. For instance a smooth curve, so $M = (0, 1)$ might have a single point of tangency to a boundary hypersurface.

What we want are *b-maps*, where as always the b- just stands boundary. Each boundary hypersurface is specified by the ideal of functions which vanish on it; this is a principal ideal because we are assuming boundary hypersurfaces to be embedded. So if ρ_H is a defining function for $H \in \mathcal{M}_1(M)$ then

157.965 (1.36) $$\mathcal{I}(H) = \rho_H \mathcal{C}^\infty(M).$$

We do want our maps to be smooth of course, and then

157.966 (1.37) $$F^*\mathcal{I}(H) \subset \mathcal{C}^\infty(M).$$

157.967 DEFINITION 1.3. A smooth map $F : M \longrightarrow N$ between manifolds (with corners) is a b-map if for each $H \in \mathcal{M}_1(N)$ one of the following is true

(1)

157.968 (1.38) $$F^*\mathcal{I}(H) = \{0\}$$

(2) There is a map $F_\#(H) : \mathcal{M}_1(N) \longrightarrow \mathbb{N}_0$, not identically zero, such that

157.969 (1.39) $$F^*\mathcal{I}(H) = \prod_{K\in\mathcal{M}_1(M)} \mathcal{I}(K)^{F_\#(H,K)},\ F_\#(H,K) = F_\#(H)(K).$$

(3)

157.970 (1.40) $$F^*\mathcal{I}(H) \text{ contains a non-vanishing element.}$$

The third case corresponds to $F_\#(H)$ as in the second, with all entries 0. This is more evident if one chooses a defining functiona ρ'_H for H and ρ_K for the boundary hypersufaces of M and then the three cases correspond to $F^*\rho'_H = 0$,

157.971 (1.41) $$F^*\rho'_H = a \prod_{K\in\mathcal{M}_1(M)} \rho_K^{F_\#(H,K)},$$

and $F^*\rho'_H > 0$.

157.972 PROPOSITION 1.1. *Manifolds with corners are the objects of a category with b-maps as morphisms.*

Just as for manifolds without boundary, the tangent bundle is a manifold well-defined manifold with a smooth bundle projection

157.973 (1.42) $$\pi : TM \longrightarrow M.$$

157.974 EXERCISE 1. Check that π is a b-map!

Smooth vector fields, sections of TM, are important for several reasons, for one they are natural operators

157.977 (1.43) $$V : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M),\ V(uv) = uVv + vVu.$$

Perhaps the most fundamental is that real vector fields can be integrated. The usual local existence theorem fails on manifolds with corners as soon as the boundary is non-trival – since the integral curves will often tend to 'leave the manifold'. This is one reason to consider the space

157.975 (1.44) $$\mathcal{V}_{\mathrm{b}}(M) = \{V \in \mathcal{C}^\infty(M;TM); V \text{ is tangent to the boundary}\}.$$

Tangency to the boundary can be interpreted in terms of the action (157.977 1.43) and the ideals $\mathcal{I}(H)$ for $H \in \mathcal{M}_1(M)$, as the statement

157.976 (1.45) $$V : \mathcal{I}(H) \longrightarrow \mathcal{I}(H)\ \forall\ H \in \mathcal{M}_1(M).$$

Tangency to the corners is then automatic.

We will consider $\mathcal{V}_{\mathrm{b}}(M)$ more closely below but for $\mathcal{V}_{\mathrm{b}}(M)$ the standard local existence theorem holds.

157.978 Lemma 1.2. *For a real-valued element $V \in \mathcal{V}_b(M)$ and each $p \in M$ there exists a maximal integral curve*

157.979
$$\chi_p : I_p \longrightarrow M,\ 0 \in I_p \subset \mathbb{R} \text{ open and connected, } \chi_p(0) = p,$$
$$^\circ A(\chi_p)_*(\frac{d/dt}{)} = V(\chi(t)) \text{ with } I_p \text{ maximal.} \tag{1.46}$$

As usual, if M is compact then $I = \mathbb{R}$ always. In any case these integral curves always define a smooth map on the open set

157.980
$$F : O = \bigcup_{p \in M} (I_p \times \{p\}) \ni (t, p) \longmapsto \chi_p(t) \in M. \tag{1.47}$$

If M is compact this is a 1-parameter family of diffeomorphisms of M.

With a little work you can see that

157.982 Proposition 1.2. *On a compact manifold the Lie algebra of the groupd of diffeomorphisms is $\mathcal{V}_b(M)$.*

Note that this infinite-dimensional group is missing some properties of a Lie group, for one the diffeomorphism obtained by integrating the Lie algebra do not contain a neighbourhood of the identity. This can be obviated by considering parameter-dependent vector fields.

If W is a t-depedent smooth vector field on M, so a vector field in $\mathcal{V}_b(U)$ for an open subset $U \subset \mathbb{R}_t \times M$ such that $Vt = 0$ then Lemma 1.2 yields a 1-parameter family of diffeomorphisms, G_t defined on an appropriate open subset, and satisfying

157.981
$$\frac{d}{dt}(G_t)^* f = (G_t)^*(Wf + \frac{df}{dt}),\ f \in \mathcal{C}^\infty(\mathbb{R} \times M). \tag{1.48}$$

This invariant formulation of the chain rule, and its generalizations, are used below in various constructions.

2. Compactification

T1.Compactification

Although we will deal with non-compact manifolds, the ones that arise below have some 'structure at infinity'. One way to describe what this means is through the notion of compactification.

157.18 Definition 1.4. A *compactification* of a manifold M is a compact manifold $\overline{M}$ and a smooth injection $\iota : M \longrightarrow \overline{M}$ which is a diffeomorphism to a (relatively of course) open dense submanifold.

Here, both M and $\overline{M}$ may have corners. As always when introducing a new notion, we should specifiy when two compactifications are to be regarded as 'the same'.

157.19 Definition 1.5. Two compactifications $\iota_i : M \longrightarrow \overline{M}_i$ are *equivalent* if there exists a diffeomorphism $e : \overline{M}_1 \longrightarrow \overline{M}_2$ giving a commutative diagramme

157.20 (1.49)
$$
\begin{tikzcd}
 & \overline{M}_1 \arrow[dd, "e"] \\
M \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & \\
 & \overline{M}_1.
\end{tikzcd}
$$

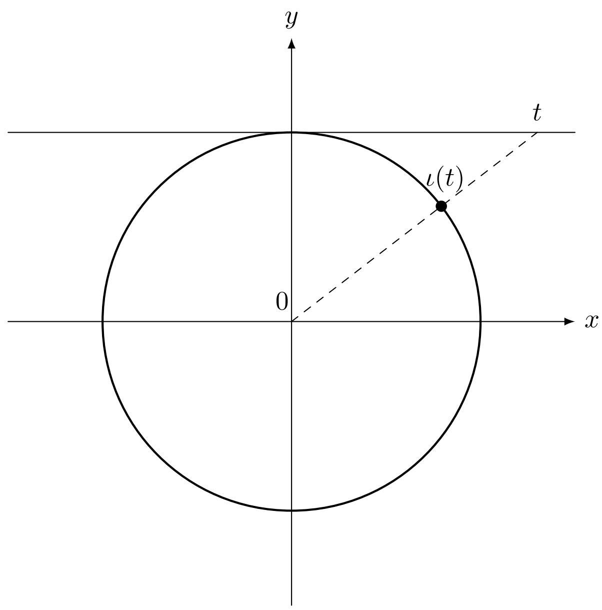
Notice that the equivalence map e is unique if it exists since it is fixed on an open dense subset by (1.49). We also say that one compactification is *finer* than another if there is a smooth map e giving a commutative digaramme; again it if it exists it is determined. This defines a partial order on compactifications – as we shall see below there can be non-comparable compactifications.

If M is compact it is a compactification of itself and it is unique in this sense of equivalence.

We might well want more structure for the compactification – for instance if M is a complex manifold then we might want $\overline{M}$ to be complex and all maps to be holomorphic. There are important examples from algebraic geometry here. Most relevant at the moment is the projective compactification of a complex vector space $W \hookrightarrow \mathbb{P}W$ which I mention below but there are much more sophisticated examples to check out. There is the Deligne-Mumford compactification of the Riemann moduli spaces $\mathcal{M}_{g,n}$ (okay I hear a complaint from someone that the $\mathcal{M}_{g,n}$ are not quite manifolds, they are orbifolds in general, but take the number of punctures n large compared to the genus $g \geq 0$). Also there is the deConcini-Procesi 'wonderful' compactification of complex adjoint Lie groups [**2**] (there is a real version of this compactification in [**1**]). Also, compactificaiton of 'Gravitational Instantons' (aren't the Physicists good at inventing names!)

The examples I will consider immediately are more prosaic, namely of a real finite-dimensional vector space V. This is both to illustrate the notion and for later reference. I will discuss

(1) The one-point compactification(s) given by a sphere $\overline{V}^o$.
(2) The parabolic compactification gives a closed ball $\overline{V}^p$.
(3) The radial compactification also given by a closed ball $\overline{V} = \overline{V}^R$.



From the notation you can see that I have a preference for the radial compactification – I hope the discussion below shows why. Only the radial compactification is really used subsequently.

These can all be constructed using variants of stereographic projection. So, let's start with $V = \mathbb{R}^n$, i.e. choose a basis. We embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as the hyperplane

157.21 (1.50) $$\mathbb{R}^n \ni x \longmapsto (x, 1) \in P \subset \mathbb{R}^{n+1}.$$

In the first, case consider the the sphere $\mathbb{S}_o$ of radius $\frac{1}{2}$ centred at $(0, \frac{1}{2})$ and in the second and third cases take the sphere $\mathbb{S}_R$ of radius 1 centred at the origin. In both these latter cases a point of $\mathbb{R}^n$ determines a unique line $L_o(x)$ or $L_R(x)$ through the image of x in P and the centre of the corresponding sphere then

157.22 (1.51)
$$\begin{gathered} I_o : \mathbb{R}^n \longrightarrow \mathbb{S}_o,\ I_o x \text{ is the other point in } \mathbb{S}_o \cap L_o(x) \\ I_R : \mathbb{R}^n \longrightarrow \mathbb{S}_R^+, \\ I_R x \text{ is the other point in } \mathbb{S}_R \cap L_1(x) \subset \mathbb{S}_R^+ = \mathbb{S}_R \cap \{x_{n+1} \geq 0\} \\ I_p : \mathbb{R}^n \longrightarrow \mathbb{B}_p \subset \mathbb{R}^n, \\ I_p x \text{ is the projection of } L_R x \text{ onto the closed unit ball in } \mathbb{R}^n \times \{0\}. \end{gathered}$$

In all three cases the full orthogonal group $\mathrm{O}(n)$, acting on the first factor of $\mathbb{R}^n \times \mathbb{R}^n$ satisfies $I_\bullet A x = A L_\bullet x$ for all $A \in \mathrm{O}(n)$, effectifely reducing the discussion to the case $n = 1$. Explicit formulæ for the maps are easily derived:

157.23 (1.52)
$$\begin{gathered} I_o x = \left(\frac{x}{1 + |x|^2}, \frac{1}{1 + |x|^2} \right) \in \mathbb{S}_o \subset \mathbb{R}^{n+1}_+ \\ I_R = \left(\frac{x}{(1 + |x|^2)^{\frac{1}{2}}}, \frac{1}{(1 + |x|^2)^{\frac{1}{2}}} \right) \in \mathbb{S}_R^+ \subset \mathbb{R}^{n+1}_+ \\ I_p x = \frac{x}{(1 + |x|^2)^{\frac{1}{2}}} \in \mathbb{R}^n. \end{gathered}$$

Thus, for the radial compactification $(1+|x|^2)^{-\frac{1}{2}}$ is a boundary defining function and hence $|x|^{-1}$, which is a smooth function of it away from $x=0$, is a defining function near the boundary. It follows that

157.27 (1.53)
$$\{|x|>\epsilon>0\}\ni x\longmapsto\left(\frac{1}{|x|},\frac{x}{|x|}\right)\in[0,1)\times\mathbb{S}^{n-1}$$

extends to a smooth product decomposition of $\overline{\mathbb{R}^n}^R$ near the boundary. For the parabolic compactification it follows similarly that

157.28 (1.54)
$$\{|x|>\epsilon>0\}\ni x\longmapsto\left(\frac{1}{|x|^2},\frac{x}{|x|}\right)\in[0,1)\times\mathbb{S}^{n-1}$$

is a product decomposition near the boundary.

It can be seen directly that

157.24 (1.55)
$$I_o\left(\frac{x}{|x|^2}\right)=SI_o \text{ where } S:\mathbb{S}_o\setminus\{(0,1),(0,0)\}\longrightarrow\mathbb{S}_o\setminus\{(0,1),(0,0)\},$$
$$\text{with } S(y,y_n)=(y,-y_n+1)$$

is equatorial reflection on $\mathbb{S}_o$.

In all cases it is clear either geometrically, or from the forumlæ (157.23 1.52), that the action of $\mathrm{O}(n)$ extends smoothly from $\mathbb{R}^n$ to the compactification. Similarly the scaling action by $\mathbb{R}^+$, with generator on $\mathbb{R}^n$

157.25 (1.56)
$$\sum_i x_i\frac{\partial}{\partial x_i}$$

extends smoothly. For the one-point compactification this follows from (157.24 1.55) and in the other two cases

157.26 (1.57)
$$\lim_{|x|\to\infty}\frac{tx}{(1+t|x|^2)^{\frac{1}{2}}}=\frac{x}{(|x|^2)^{\frac{1}{2}}} \text{ and } \lim_{|x|\to\infty}\frac{1}{(1+t|x|^2)^{\frac{1}{2}}}=0.$$

Thus in all cases the action of the conformal group $\mathrm{O}(n)\times\mathbb{R}^+$ extends smoothly to the compactification.

157.29 Proposition 1.3. *The action of the general linear group extends smoothly from $\mathbb{R}^n$ to the radial and parabolic compactifications, but not to the one-point compactification; the translation action of $\mathbb{R}^n$ extends smoothly to the radial and the one-point compactifications, but not to the parabolic compactification and there are smooth surjective maps, which are not diffeomorphisms, giving a commutative diagramme*

157.30 (1.58)
$$\begin{tikzcd}
 & & \mathrm{GL}(n,R)\ltimes\mathbb{R}^n \arrow[d, dash] & & \\
 & & \mathbb{S}^{n+}_R \arrow[dl] \arrow[dr] & & \\
\mathrm{O}(n)\ltimes(\mathbb{R}^+\times\mathbb{R}^n) \arrow[r, dash] & \mathbb{S}^n_0 & \mathbb{R}^n \arrow[l, "I_o"'] \arrow[u, "I_R"] \arrow[r, "I_p"] & \mathbb{B}^n_p \arrow[r, dash] & \mathrm{GL}(n,R).
\end{tikzcd}$$

Outline of proof. That the group actions extend as indicated follows by noting that the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ consists of vector fields homogeneous of degree 0 and similarly the translations are homogeneous of degree -1. Similar arguments show that the groups shown are the maximal subgroups of $\mathrm{GL}(n, R) \ltimes \mathbb{R}^n$ which extend to act smoothly on the one-point and parabolic compactifications. $\square$

157.31 Corollary 1. *The one-point compactification is defined for a vector space with conformal-Euclidean structure, the radial comactification is well-defined for an affine space and the parabolic compactification is well-defined for a vector space.*

Both the radial and the parabolic compactifications have boundaryless variants, in which the bounding sphere is replaced by an embedded projective space $\mathbb{S}^{n-1}/\pm$ by doubling across the boundary. The apparent advantage of this smaller compactification does not seem to be realized in practice.

157.32 Conjecture 1. *The five compactifications are minimal in their respective categories (i.e. as manifolds with/without boundary) among compactifications with the invariance properties in* (157.30 1.58).

Although, as noted above, it is the radial compactification which mostly appears below, other variants are relevant. In particular none of these compactifications are natural for products – the radial compactification of $V_1 \times V_2$ is not 'comparable' to the products of the radial compactifications. Still, the relationship between the radial compactification of the product of vector spaces and the product of the radial compactifications is significant and will be examined later.

3. Collar neighbourhood

Sect.collar

157.447 Remark. Edited by Paige Dote

This theorem provides a rather precise description of a neighbourhood of a closed embedded submanifold, $Y \subset M$ where M is an n-dimensional manifold. The usual proof exploits the geodesic flow for a metric on M, but here we give a related approach using the notation of a radial vector field for Y. This is very closely related to the linearization theorem of Sternberg (Sternberg [?]). We also carry it out in the context of manifolds with corners.

A closed p-submanifold (p- if for 'product' it really is the natural notion of a submanifold in the case of a manifold with corners) is a closed subset $Y \subset M$ such that for each point $\overline{y} \in Y$, there exists adapted coordinates based at $\overline{y}$ in M in terms of which Y is linear. Precisely, in a neighbourhood $U_{\overline{y}} \subset M$ of $\overline{y}$, there exists coordinate functions $x_i \geq 0$, $y_j \in C^\infty(U_{\overline{y}})$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n-k$ with independent differentials and

157.448 (1.59)
$$Y \cap U_{\overline{y}} = \{x_1 = \cdots = x_{d'} = 0,\ y_1 = \cdots = y_{n'-d'} ==\}.$$

It follows that Y is itself a submanifold, with these as *adapted coordinates* for Y.

Note that if $d' > 0$ here then Y is actually contained in a boundary face (assuming it is connected, in any case one component is). Boundary faces themselves are clearly p-submanifolds but quite a few natural submanifolds are not. For instance the diagonal in M^2, where $\partial M \neq \emptyset$, is not a p-submanifold. This turns out to be rather significant.

We note the following

Definition 1.6. A smooth vector field, $V \in \mathcal{V}_{\mathrm{b}}(M)$ is *tangent* to Y if $Vf\big|_Y = 0$ for any $f \in C^\infty(M)$ with $f\big|_Y = 0$. A smooth vector field *vanishes* on Y if $Vf\big|_Y = 0$ for all $f \in C^\infty(M)$.

In adapted coordinates for Y, V is tangent to Y if it takes the form

$$V = \sum_{i,j=1}^{d} x_i a_{i,j} \partial_{x_j} + \sum_{k=1}^{n-d} b_k \partial_{y_k}$$

where $a_{i,j}$, b_k are smooth coefficients.

Similarly, V vanishes on Y if

157.449
$$V = \sum_{i,j=1}^{d} x_i a_{i,j} \partial_{x_j} + \sum_{i=1}^{d} \sum_{k=1}^{n-d} x_i b_{i,k} \partial_{y_j} = \sum_{i=1}^{d} x_i W_i \tag{1.60}$$

Where the W_i are smooth as are the $a_{i,j}$, $b_{i,k}$.

Definition 1.7. The differentials of functions which vanish on Y define the *conormal bundle* $N^*Y \subset T^*_Y M$.

Hence, in adapted coordinates N^*Y has the basis

$$\mathrm{d}x_1, \dots, \mathrm{d}x_d.$$

For each $y \in Y$, $N^*_y Y$ is the annihilator, in $T^*_y M$, of $T_y Y \subset T_y M$. By duality, the *normal bundle* $NY = T_Y M/TY$ is spanned by

$$\partial_{x_1}, \dots, \partial_{x_d}.$$

Theorem 1.3 (Collar, or normal, neighbourhood). *For a closed embedded submanifold $Y \subset M$ there is a neighbourhood, T, of the zero section $O_N \subset NY$, a neighbourhood $\Omega \subset M$ of Y, and a diffeomorphism $F : T \to \Omega$ satisfying the additional conditions*

I1 (1) *$F(0_N) = Y$ is the natural identification and*
I2 (2) *$F_* : N(0_N) \to NY$ is the identity.*

Here, the bundle projection from NY to Y restricts to a diffeomorphism $\pi : O_N \to Y$ giving meaning to (1). Then, the tangent space to 0_N is mapped to TY as the identity. It follows that the normal bundle to 0_N in NY is mapped to the normal bundle NY by F. Additionally, for any vector bundle, the normal bundle to the zero section is naturally identified with the bundle itself, so F_* lifts to a bundle map for NY to NY which is required to be the identity in (2).

Adapted coordinates give such a collar neighbourhood locally, so the challenge is to make this global.

A vector field on M which vanishes on Y induces a linear map at each point of Y:

$$L(V, y) : N^*_y Y \to N^*_y Y$$

through

$$N(V, y)\xi = \mathrm{d}(Vf)(y)$$

where, for $\xi \in NY$, $f \in C^\infty(M)$ has $f\big|_Y = 0$ and $df(y) = \xi$. In terms of (1.60),

$$N(V, y) dx_j = \sum_{i=1}^{d} a_{i,j}(y) dx_i.$$

Definition 1.8. A smooth (real) vector field, R, on M is *radial* on Y if it vanishes at Y and

$$L(V, y) = \mathrm{Id}$$

for all $y \in Y$.

Lemma 1.3. *There is a radial vector field for any closed embedded submanifold, and the difference between two such radial vector fields is locally a sum* $\sum x_i W_i$ *where the* W_i *are tangent to* Y.

The existence of a radial vector field follows from a 'patching' argument. Locally, in adapted coordinates, we have an obvious radial vector field in the Euler field

$$R_0 = \sum_{i=1}^{d} x_i \partial_{x_i}.$$

which is the generator of the scaling action

157.961 (1.61) $$(x, y) \longmapsto (tx, y),\ t > 0.$$

There is no constraint on a radial vector field away from Y at all. So one can take functions $\varphi_a \in \mathcal{C}^\infty_{\mathrm{c}}(M)$ with locally finite supports such that each $\operatorname{supp} \varphi_a$ is contained in an adapted coordinate patch for Y and $\sum \varphi_a = 1$ in a neighbourhood of Y. Then,

$$R = \sum_a \varphi_a R_{0,a}$$

is radial with the $R_{0,a}$ being the coordinate Euler vector fields.

As a smooth real vector field, there is a unique integral curve of R through each point of M. For R_0 these are curves with $x_i = e^t \overline{x_i}$ and $y_i = \overline{y_i}$ so that $x_i \downarrow 0$ as $t \to -\infty$. For a general radial vector field the same is true near Y in the sense that

Lemma 1.4. *If* R *is a radial vector field for* Y *then* Y *has an open neighbourhood such that the integral curves of* R*, as* $t \to -\infty$*, approach* R *smoothly in* e^t *with a non-vanishing limiting tangent vector at* Y*.*

As remarked at the beginning, this is a linearization theorem in the spirit of Sternberg.

Proposition 1.4. *Near any point of* Y *there are adapted coordinates in terms of which in which a given radial vector is the Euler vector field.*

Proof. This can be seen using the homotopy method of Möser. In any adapted coordinates, the radial vector field has the form

$$R = R_0 + \sum_{i,j,p} x_i x_j a_{i,j,p} \partial_{x_p} + \sum_{i,k} x_i b_{i,k} \partial_{y_k}$$

for smooth coefficients $a_{i,j,p}$ and $b_{i,k}$. Pushing R forward under the inverse of the scaling diffeomorphism (1.61), we obtain the one parameter family of vector fields

157.450 (1.62) $$R_t = R_0 + \sum_{i,j,p} t x_i x_j a_{i,j,p}(tx) \partial_{x_p} + \sum_{i,j} t x_i b_{i,k}(tx, y) \partial_{y_k}.$$

This family is smooth down to $t = 0$, and $R_t\big|_{t=0} = R_0$. Since this is 'exponential scaling' in terms of R_0, we see that

157.452 (1.63)
$$t\frac{\mathrm{d}}{\mathrm{d}t}R_t = [R_0, R_t],$$

which can easily be checked directly. The t-dependent vector field

157.451 (1.64)
$$W_t = \frac{1}{t}(R_t - R_0)$$

is smooth and vanishes at Y. It follows that integration of W_t defines a 1-parameter family of local diffeomorphisms, G_t, defined in a neighbourhood U of the point in Y for all $t \in [0, 1]$ fixing each point of Y. Thus $G_0 = \mathrm{Id}$ and

$$\frac{d}{dt}G_t^* f = G_t^*(W_t f),$$

for all $f \in C^\infty(U)$.

The standard variation formula (really the chain rule) shows that

$$\frac{d}{dt}(G_t)_*(R_t) = (G_t)_*\left(\frac{dR_t}{dt} + [W_t, R_t]\right).$$

From (1.64), it follows that $[W_t, R_t] = -\frac{1}{t}[R_0, R_t]$. So by (1.63),

$$\frac{dR_t}{dt} + [W_t, R_t] = 0.$$

Thus, $(G_t)_* R_t$ is constant in t as a vector field near Y, but by assumption $(G_0)_* R_0 = R_0$ so

$$(G_1)_* R = R_0.$$

This gives a diffeomorphism locally fixing Y which reduces R to the Euler vector field. $\square$

The preceding lemma is an immediate consequence since the inegral curves of R_0 are of the form $x \mapsto tx$ and $y = y$.

This shows the existence of a diffeomorphism as required for the Collar Neighbourhood theorem, with the inverse mapping $p \in U$ to the tangent vector at the end point.

CHAPTER 2

Symbols and conormal distributions at a point

1. Schwartz kernels L2

Before tackling the properties of the ring $\Psi^*(\mathbb{R}^n)$ of pseudodifferential operators on $\mathbb{R}^n$, I want to look into the properties of the Schwartz kernels of these operators, so we can get a picture of them. We can 'guess' (it is easy to justify) that the Schwartz kernel of an operator $A \in \Psi^m(\mathbb{R}^n)$, defined by a symbol a satisfying (1.11), is

157.94
$$A(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi) e^{i(x-y)\cdot\xi} d\xi. \tag{2.1}$$

Here I use the same letter for the operator and its Schwartz kernel – since the Schwartz kernel theorem (which I will talk a little about later) shows that they determine each other.

We can think of (2.1) in a couple of different ways – in general it is *not* a convergent integral. We can make a (formal at this stage) linear change of variables on $\mathbb{R}^{2n}$ from (x,y) to (x,z), $z = x - y$ and then

157.95
$$A(x,y) = \alpha(x, x-y) \text{ where } \alpha(x,z) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi) e^{iz\cdot\xi} d\xi. \tag{2.2}$$

Now the integral is a partial inverse Fourier transform. In fact, since a is smooth in x we can interpret the definition of α in (2.2) as the inverse Fourier tranform from ξ to z for each fixed x. This in fact is what I will do today. Alternatively one can just check that the partial Fourier transform with respect to a decomposition of Euclidean space into a product behaves 'correctly'.

So, for the moment, we have dispensed with the 'coefficients' and just look at the (commutative) algebra of constant-coefficient pseudodifferential operators where the composition operation is convolution.

Recall the convolution of distributions on $\mathbb{R}^n$. On cannot define the convolution of arbitrary distributions, even arbitrary tempered distributions – this however is an issue of 'growth' rather than singularities. In particular the convolution

157.57
$$u * v \text{ is defined if either } u \text{ or } v \text{ has compact support} \tag{2.3}$$

but can be defined in other cases too. I will denote the space of distributions of compact support as

157.61
$$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) = (\mathcal{C}^\infty(\mathbb{R}^n))'. \tag{2.4}$$

So the space of distributions of compact support is actually a commutative ring, since $u * v = v * u$ and the support of a convolution as in (2.3) satisfies

157.58
$$\operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v). \tag{2.5}$$

It is also the case that $\mathcal{S}(\mathbb{R}^n)$ is closed under convolution and we know that the Fourier transform satisfies

157.59 (2.6) $$\mathcal{F}(u * v) = \mathcal{F}(u)\mathcal{F}(v),\ u,\ v \in \mathcal{S}(\mathbb{R}^n).$$

The ring we are interested in is contained in

157.60 (2.7) $$\mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n)$$

for which the identity (2.6) still holds. Note that

157.62 (2.8) $$\mathcal{F}(\mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n)) \subset \mathcal{C}^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n).$$

So, we are looking for are some interesting spaces of smooth functions on the dual $\mathbb{R}^n$ which are closed under multiplication. You might ask, in view of the identification of the convolution kernels here with the inverse Fourier transforms of symbols, why is there any problem at all? There isn't a problem for convolution as such because of (2.6) but recall that the Fourier transform does not 'behave well' on say the space $L^\infty(\mathbb{R}^n)$. Of course the Fourier tranform maps this to a well-defined linear subspace of the tempered distributions – which includes for instance the delta functions at any point – but it is quite hard, in a certain sense I think impossible, to give a 'direct' characterization of the Fourier image of L^∞ and the same is true for our symbols which are modeled on L^∞ in the sense that they are defined by bounds. We will in fact 'sandwich' the image between spaces characterized directly (meaning without the Fourier tranform), but this still loses information which is rather vital to us!

In the notes related to the first lecture, I discussed the radial compactification of a real, finite-dimensional, vector space V, to a ball $\overline{V}$. Ignoring all the niceties, for Euclidean space, $\mathbb{R}^n$ with the standard Euclidean norm, we can identify the complement of the origin with the product

157.63 (2.9) $$\mathbb{R}^n \setminus \{0\} \ni x \longmapsto \left(|x|, \frac{x}{|x|}\right) = (r, \omega) \in (0,\infty) \times \mathbb{S}^{n-1}.$$

The inversion map $r \longrightarrow 1/r$ is a diffeomorphism of $(0,\infty)$ to itself 'switching the ends'. This allows us to add the sphere at infinity of $\mathbb{R}^n$ setting

157.64 (2.10) $$\overline{\mathbb{R}^n} = \left(\mathbb{R}^n \sqcup [0,\infty) \times \mathbb{S}^{n-1}\right)/I$$

where

157.65 (2.11) $$I : \mathbb{R}^n \setminus \{0\} \ni x \longmapsto \left(\frac{1}{|x|}, \frac{x}{|x|}\right) \in (0,\infty) \times \mathbb{S}^{n-1}$$

identifies the complement of the origin with the interior of the second part.

Thus $\overline{\mathbb{R}^n}$ is a compact manifold with boundary 'obtained by introducing inverted polar coordinates near infinity'. The interior is $\mathbb{R}^n$ and the boundary is 'the sphere at infinity'.

This immediately gives us a ring of functions on $\mathbb{R}^n$, namely

157.66 (2.12) $$\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^n).$$

I can write inclusion here for what is really the restriction from $\overline{\mathbb{R}^n}$ to its interior since this map is injective.

This is the space of 'classical symbols on $\mathbb{R}^n$ of order zero' which I will write as

157.67 (2.13) $$S^0_{\mathrm{cl}}(\mathbb{R}^n) = \mathcal{C}^\infty(\overline{\mathbb{R}^n}).$$

I will approach the issue of characterizing this space precisely on $\mathbb{R}^n$ below.

As a consequence of the discussion of radial compactification in § T1.Compactification 2, or directly, we can see that the coordinate vector fields on $\mathbb{R}^n$ extend to be smooth on $\overline{\mathbb{R}^n}$. In fact

157.78 Proposition 2.1. *The coordinate vector fields on $\mathbb{R}^n$ extend to smooth vector files on $\overline{\mathbb{R}^n}$ and span, over $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$, all the smooth vector fields which are of the form*

157.79 (2.14) $$\rho W,\ W \text{ smooth and tangent to the boundary of } \overline{\mathbb{R}^n}.$$

Here $\rho \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$ vanishes at the boundary.

See Project P-Comp 1

157.80 Corollary 2. *The space $S^0_{\mathrm{cl}}(\mathbb{R}^n)$ consists of smooth functions which satisfy the estimates*

157.81 (2.15) $$\sup_{\xi\in\mathbb{R}^n} |(1+|\xi|)^{|\alpha|}\partial^\alpha_\xi a(\xi)| < \infty\ \forall\ \alpha.$$

Note that I do not say that this characterizes $S^0_{\mathrm{cl}}(\mathbb{R}^n) = \mathcal{C}^\infty(\overline{\mathbb{R}^n})$, because it does not.

157.83 Definition 2.1. We denote the subspace of $\mathcal{C}^\infty(\mathbb{R}^n)$ of functions satisfying all the estimates (157.81 2.15) by

157.84 (2.16) $$S^0(\mathbb{R}^n) \supset S^0_{\mathrm{cl}}(\mathbb{R}^n).$$

These are the 'symbols with bounds' containing the classical symbols.

More generally, consider the function

157.68 (2.17) $$(1+|x|^2)^{z/2} \text{ on } \mathbb{R}^n,\ z \in \mathbb{C}.$$

This is certainly smooth on $\mathbb{R}^n$. It is rather clear that

157.69 (2.18) $$(1+|x|^2)^{z/2} \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}) \text{ iff } z \in -\mathbb{N}_0.$$

Indeed, in $x \neq 0$ it can be written

157.70 (2.19) $$t^{-z}(1+t^2)^{z/2},\ t = 1/|x|.$$

This is smooth down to $t = 0$, the boundary of $\overline{\mathbb{R}^n}$, if and only if $-z$ is a non-negative integer.

We define the space of classical symbols of (complex) order z to be the products

157.71 (2.20) $$S^z_{\mathrm{cl}}(\mathbb{R}^n) = (1+|x|^2)^{z/2}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) = (1+|x|^2)^{z/2}S^0_{\mathrm{cl}}(\mathbb{R}^n).$$

The space of symbols (with bounds) or real order m is similarly defined to be

157.85 (2.21) $$S^m(\mathbb{R}^n) = (1+|x|^2)^{m/2}S^0(\mathbb{R}^n).$$

Why no complex order in the second case?

157.86 Exercise 2. Show that in terms of Definition 157.83 2.1

157.87 (2.22) $$(1+|x|^2)^{is/2} \in S^0(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}.$$

This in turn implies that

157.88 (2.23) $$S^z_{\mathrm{cl}}(\mathbb{R}^n) \subset S^{\operatorname{Re} z}(\mathbb{R}^n)\ \forall\ z \in \mathbb{C}.$$

157.72 DEFINITION 2.2. The space of (Schwartz-) conormal distributions on $\mathbb{R}^n$, with respect to the origin is

157.73
$$I_{\mathcal{S}}^{m+n/4}(\mathbb{R}^n) = \mathcal{F}^{-1}\left(S^m(\mathbb{R}^n)\right). \tag{2.24}$$

The corresponding spaces of *classical* (Schwartz-) conormal distributions at the origin where now z is allowed to be complex, are

157.89
$$I_{\mathrm{cl},\mathcal{S}}^{z+n/4}(\mathbb{R}^n) = \mathcal{F}^{-1}\left(S_{\mathrm{cl}}^z(\mathbb{R}^n)\right). \tag{2.25}$$

So

157.90
$$I_{\mathrm{cl},\mathcal{S}}^{z}(\mathbb{R}^n) \subset I_{\mathcal{S}}^{\operatorname{Re} z}(\mathbb{R}^n). \tag{2.26}$$

Why the weird normalization of the order with the $n/4$? This is part of a bigger scheme that I hope will be explained later. It is the standard notion with the n interpreted as the codimension of the submanifold, here the origin, with respect to which we are defining conormality.

So, apart from the issue with the order these are just the inverse Fourier tranforms of our 'classical symbols'.

157.75 THEOREM 2.1. *If $u \in I_{\mathcal{S}}^m(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ then*

157.77
$$\begin{gathered} \operatorname{singsupp}(u) \subset \{0\} \\ (1-\phi)u \in \mathcal{S}(\mathbb{R}^n) \text{ if } \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n),\ 0 \notin \operatorname{supp}(1-\phi). \end{gathered} \tag{2.27}$$

The conditions in (2.27) *do not* characterize the conormal distributions.

PROOF. By definition, a smooth function on $\mathbb{R}^n$ is a 'symbol with bounds' of order m if it satisfies all the estimates (2.15). We can reëxpress these in the form

157.96
$$\xi^\beta \partial_\xi^\alpha a = (1+|\xi|)^m b,\ b \in L^\infty(\mathbb{R}^n)\ \forall\ \beta \text{ with } |\beta| \le |\alpha|. \tag{2.28}$$

It follows that if $|\alpha| > m+n+N+1$ then

R.9
$$\partial_\xi^\alpha a = (1+|\xi|)^{-n-N-1} b_N,\ b_N \in L\infty(\mathbb{R}^n). \tag{2.29}$$

Since then $\mathcal{F}^{-1}\left((1+|\xi|)^{-n-N-1} b_N\right)$ is bounded with its first N derivatives it follows that the same is true for

R.10
$$|x|^{2k} u,\ 2k > m+n+N+1. \tag{2.30}$$

From this (2.27) follows. □

I have made a rather mixed definition of classical and non-classical symbols here. The classical ones defined in terms of the radial compactification and the non-classical ones in terms of estimates on $\mathbb{R}^n$ more directly, let me try to unravel this.

157.91 LEMMA 2.1. *The 'residual symbol spaces' are*

157.92
$$S^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) = \dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n}) \subset S_{\mathrm{cl}}^z(\mathbb{R}^n)\ \forall\ z \in \mathbb{C}. \tag{2.31}$$

Here I am using the notation for any manifold with corners

157.93
$$\dot{\mathcal{C}}^\infty(M) = \{u \in \mathcal{C}^\infty(M); u \text{ vanishes to infinite order at } \partial M\}. \tag{2.32}$$

L2-end So these are the 'trivial' symbols in the case of $\overline{\mathbb{R}^n}$.

L3

Last time I talked about the symbol spaces $S^m(\mathbb{R}^n)$ and the space of distributions conormal at 0 defined as

157.97 (2.33)
$$I_{\mathcal{S}}^{m+\frac{n}{4}}(\mathbb{R}^n;\{0\}) = \mathcal{F}^{-1}(S^m(\mathbb{R}^n)).$$

Since we know that the symbol spaces form a filtered ring under multiplication we deduce a corresponding result for convolution of the conormal spaces

157.98 (2.34)
$$I_{\mathcal{S}}^{M}(\mathbb{R}^n;\{0\}) * I_{\mathcal{S}}^{M'}(\mathbb{R}^n;\{0\}) = I_{\mathcal{S}}^{M+M'-\frac{n}{4}}(\mathbb{R}^n;\{0\}).$$

2. Homogeneous distributions

The quesiton naturally arises as to precisely what classical conormal distributions 'look like'. The answer is given, almost, in terms of homogoneous distributions. Then the first question is, What is a homogeneous distribution?

We know what a homogeneous function on $\mathbb{R}^n \setminus \{0\}$ is – it is a function satisfying

$$u(tx) = t^z u(x)\ \forall\ t > 0,\ x \in \mathbb{R}^n \setminus \{0\}. \quad (2.35)$$ 157.983

We do not want to include the origin here since the degree z could be negative, so the function need not be defined at 0. In fact we are interested in smooth functions away from 0 which are homogeneous. So this is straihgtforward

157.984 Lemma 2.2. *The smooth functions on $\mathbb{R}^n \setminus \{0\}$ which are homogeneous of degree $z \in \mathbb{C}$ and smooth outside the origin may be identified with $\mathcal{C}^\infty(\mathbb{S}^{n-1})$ via*

$$u(x) = v(\frac{x}{|x|})|x|^z,\ v \in \mathcal{C}^\infty(\mathbb{S}^{n-1}). \quad (2.36)$$ 157.985

When $\operatorname{Re} z > -n$ it follows that such a function is locally integrable across the origin, and so defines a tempered distribution. In this case we can express the condition 'weakly' by noting

$$\int u(tx)\phi(x)dx = t^{-n}\int u(x)\phi(x/t)dx \Longrightarrow$$
$$\langle u, \phi(x/t)\rangle = t^{n+z}\langle u,\phi\rangle,\ \forall\ t > 0,\ \phi \in \mathcal{S}(\mathbb{R}^n). \quad (2.37)$$ 157.986

157.987 Definition 2.3. A (tempered) distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree z if it satisfies the second part of (2.37).

The argument leading to (2.37) can be reversed, at least in part.

157.988 Lemma 2.3. *A distribution which is homogeneous of degree z with $\operatorname{Re} z > -n$ and which is smooth outside the origin is of the form (2.36).*

Proof. The identity (2.37) can be differentiated with respect to t and at $t = 1$ this shows that Euler's identity holds for homogeneous distributions

$$(x \cdot \partial_x - z)u = 0. \quad (2.38)$$ 157.989

Since it is assumed that u is smooth away from the origin it follows that u must be of the form (2.36) in $\mathbb{R}^n \setminus \{0\}$. The difference of u and the locally integrable function so defined is therefore a distribution supported at the origin, and hence a sum of derivatives of the delta function. These all have homogeneity $-n-k$ for some $k \in \mathbb{N}_0$ so cannot be homogeneous of degree z with $\operatorname{Re} z > -n$. $\square$

To pass to the case of general z we may use the Fourier transform.

157.991 Lemma 2.4. *The Fourier transform of a distribution which is homogeneous of degree $z \in \mathbb{C}$ is homogeneous of degree $-n-z$.*

Proof. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ change of variable shows that

$$\widehat{\phi(\cdot/t)}(\xi) = t^n \hat{\phi}(t\xi) \quad \boxed{157.992} \tag{2.39}$$

so the identity for $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \hat{u}, \psi \rangle = \langle u, \hat{\psi} \rangle \quad \boxed{157.993} \tag{2.40}$$

implies the result. $\square$

It follows that for $\operatorname{Re} z < 0$ a distribution, v, homogeneous of degree z is the inverse Fourier transform of one homogeneous of degree $-z-n$. If v is assumed to be smooth outside the origin then it is the sum of a symbol of degree z and a distibution of compact support. Thus its Fourier transform is singular only at the origin and in this case we conclude that

$$v(x) = \mathcal{F}^{-1}\left(|\xi|^{-n-z} u(\frac{\xi}{|\xi|})\right), \ u \in \mathcal{C}^\infty(\mathbb{S}^{n-1}). \quad \boxed{157.994} \tag{2.41}$$

Conversely the functions in (2.36) are the sums of symbols and compactly supported distribution so v defined by (2.41) is indeed smooth outside the origin.

Thus the space of distribution homogeneous of any given degree $z \in \mathbb{C}$ is always isomorphic to $\mathcal{C}^\infty(\mathbb{S}^{n-1})$. However for $z = -n-k$, $k \in \mathbb{N}_0$ this isomorphism takes a different form.

157.995 Proposition 2.2. *For any degree $z \notin \mathbb{C} \setminus (-n - \mathbb{N}_0)$ the space of distributions homogeneous of degree z and smooth outside the origin is identified with $\mathcal{C}^\infty(\mathbb{S}^{n-1})$ by restriction to $\mathbb{R}^n \setminus \{0\}$ and (2.36). For $z = -n-k$, $k \in \mathbb{N}_0$ the space of distributions homogeneous of degree z and smooth outside the origin maps surjectively, by restriction to $\mathbb{R}^n \setminus \{0\}$ to*

$$\{u \in \mathcal{C}^\infty(\mathbb{S}^{n-1}); \int_{\mathbb{S}^{n-1}} u(\omega)\omega^\alpha d\omega = 0 \ \forall \ |\alpha| = k \quad \boxed{157.996} \tag{2.42}$$

with null space

$$\operatorname{sp}\{D^\alpha_x \delta(x); |\alpha| = k\}. \quad \boxed{157.997} \tag{2.43}$$

In fact the proof above can be modified to identify all homogenous distributions with $\mathcal{C}^{-\infty}(\mathbb{S}^{n-1})$ in essentially the same way.

However, this is not quite what we really want. The aim is to understand the structure of classical conormal distributions at the origin. These are readily described, as we do below, in terms of homogeneous distributions and the inverse Fourier transforms of functions of the form

$$a(\xi) = (-\chi(\xi))|\xi|^z u\left(\frac{\xi}{|\xi|}\right), \quad u \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ \chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ \chi(\xi) = 1 \text{ in } |\xi| < 1. \tag{2.44}$$

Except for the case $z \in -n - \mathbb{N}_0$ such a function is, using Proposition 2.2, the sum of a homogeneous distribution and a distribution of compact support. Conversely this is not the case for $z = -n - k$, $k \in \mathbb{N}_0$, unless u is in the space (2.42). The extra distributions we need are quasi-homogeneous in the sense that they satisfy

$$(x \cdot \partial_x - z)u \in \mathcal{S}(\mathbb{R}^n) \tag{2.45}$$

The new case that arises is for $z \in \mathbb{N}_0$

Lemma 2.5. *The inverse Fourier transform of*

$$b_\alpha(x) = \chi(x))x^\alpha \log|x|,\ \alpha \in \mathbb{N}^n_o,\ \chi \in \mathcal{C}^\infty_c(\mathbb{R}^n),\ \chi(\xi) = 1 \textit{ in } |x| < 1 \tag{2.46}$$

is quasi-homogeneous of degree $-n - |\alpha|$ and of the form

$$c_{n,\alpha}\frac{\xi^\alpha}{|\xi|^{n+2|\alpha|}} + e'_\alpha,\ e'_\alpha \in \mathcal{S}(\mathbb{R}^n) \textit{ in } |\xi| > 1. \tag{2.47}$$

with $c_{n,\alpha} \neq 0$.

Proof. Consider the case $\alpha = 0$. Then b_0 is a function of $|x|$ and by the $\mathrm{O}(n)$-invariance of the Fourier transform the same is true of $\hat{b}_0$. Since

$$x \cdot \partial_x b_0 = \chi(x) + (x \cdot \partial_x \chi)\log|x| \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \tag{2.48}$$

it follows that

$$(\xi \cdot \partial_\xi + n)\hat{b}_0 = e \in \mathcal{S}(\mathbb{R}^n). \tag{2.49}$$

Using polar coordinates we can integrate the Schwartz term 'inwards from infinity' and conclude that

$$\hat{b}_0 = c_n(1 - \chi(\xi))|\xi|^{-n} + e',\ e' \in \mathcal{S}(\mathbb{R}^n) \tag{2.50}$$

where $c_n \neq 0$ involves the measure of the sphere. It cannot vanish, since otherwise b_0 would be smooth.

Since $b_\alpha = x^\alpha b_0$ has compact support its inverse Fourier transform, $\hat{b}_\alpha(-\xi) \in \mathcal{S}'(\mathbb{R}^n)$ is smooth and from (2.50)

$$\hat{b}_\alpha = D^\alpha_\xi b_0 = c_{n,\alpha}\frac{\xi^\alpha}{|\xi|^{n+2|\alpha|}} + e'_\alpha,\ e'_\alpha \in \mathcal{S}(\mathbb{R}^n) \tag{2.51}$$

where again $c_{n,\alpha} \neq 0$ since b_α is not smooth. These constants are readily computed. $\square$

Finally then we see the structure of classical conormal distributions.

157.999 PROPOSITION 2.3. *A classical conormal distribution at the origin in* $\mathbb{R}^n$ *of order* $z+\frac{n}{4}$,

$$u=\mathcal{F}^{-1}(a),\ a\in|\xi|^z\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \quad \boxed{157.1102} \tag{2.52}$$

has an expansion near the origin, modulo smooth terms,

$$u(x)\sim\sum_k u_k(x) \quad \boxed{R.8} \tag{2.53}$$

where all terms are smooth outside the origin and

$$u_k\begin{cases} \text{is homogeneous of degree } -z=-n+k & \text{if } -z-n+k\notin\mathbb{N}_0\\ \text{is quasi-homogeneous of degree } -z=-n+k & \text{if } -z-n+k\in\mathbb{N}_0. \end{cases} \quad \boxed{157.1000} \tag{2.54}$$

From (2.54) the u_k become increasingly smooth as $k\to\infty$ and the meaning of (2.53) is that, for given N the difference of u and the sum for $k<K$ for K sufficiently large is in $\mathcal{C}^N(\mathbb{R}^n)$. The appearance of the quasi-homogoneous terms in (2.54) corresponds to the fact that the polynomials of degree k do not contribute to the singularity at the origin (nor of course to the asymptotic expansion of the symbol).

There are other ways to construct the homogeneous, and quasi-homogeneous distributions, for instance by analytic continuation in z.

3. Topology and asymptotic summation

The topology on the symbols space $S^m(\mathbb{R}^n)$ is the Fréchet topology given by the norms defining the space

157.99 (2.55)
$$\|a\|_{m,N}=\sup_{\mathbb{R}^n,|\beta|<N}(1+|\xi|)^{-m+|\beta|}|\partial_\xi^\beta a(\xi)|,N\in\mathbb{N}_0.$$

Certainly, $a\in S^m(\mathbb{R}^n)$ if and only if $a\in\mathcal{C}^\infty(\mathbb{R}^n)$ and all these norms are finite.

Recall that a metric on a countably normed space, such as this, is defined by

157.100 (2.56)
$$d(u,v)=\sum_N 2^{-N}\frac{\|u-v\|_{m,N}}{1+\|u-v\|_{m,N}}.$$

So the topology is metric, generated by the open balls with respect to (2.56). I say 'a metric' because replacing the sequence 2^{-N} by an positive, summable, sequence gives the same topology.

157.114 PROPOSITION 2.4. *The spaces* $S^m(\mathbb{R}^n)$ *are Fréchet spaces, so complete with respect to the translation-invariant distance* (2.56).

If it matters to you, they are Montel spaces. They are *not* projective limits of Hilbert spaces.

PROOF. Convergence with respect to this distance is the same as convergence with respect to *each* of the norms $\|\cdot\|_{m,N}$ (without any uniformity in N). Thus a Cauchy sequence with respect to the metric (2.56) is Cauchy with respect to each of these norms and conversely. So all the derivatives converge locally uniformly and with respect to the distance with the limit in the space. □

There is a topology on $S^\infty(\mathbb{R}^n) = \bigcup_m S^m(\mathbb{R}^n)$ but I will leave you to figure it out:

157.454 Exercise 3. Try to sort out (or look up) the inductive limit topology on $S^\infty(\mathbb{R}^n)$ defined by taking a set to be open if its intersection with each of the $S^m(\mathbb{R}^n)$ is open and show that the inclusions $S^m(\mathbb{R}^n) \longrightarrow S^\infty(\mathbb{R}^n)$ are then continuous.

So the Fréchet topology on the symbol spaces induces a Fréchet topology on $I^M_{\mathcal{S}}(\mathbb{R}^n, \{0\})$, since the Fourier transform identifies this with $S^{M-\frac{n}{4}}(\mathbb{R}^n)$. This means we know what a continuous map into the conormal space (and also what a smooth map into it) means.

Now to density. The intersection of the symbol spaces is the space of Schwartz functions

157.101 (2.57)
$$\mathcal{S}(\mathbb{R}^n) = S^{-\infty}(\mathbb{R}^n) = \bigcap_m S^m(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n).$$

157.102 Proposition 2.5. *The 'residual space' $\mathcal{S}(\mathbb{R}^n)$ is dense in $S^m(\mathbb{R}^n)$ in the topology of $S^{m+\epsilon}(\mathbb{R}^n)$ for any $\epsilon > 0$. More precisely, there exist a sequence of 'regularizing operators' which are linear maps*

157.103 (2.58)
$$\Phi_k : S^\infty(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

such that

157.104 (2.59)
$$a \in S^m(\mathbb{R}^n) \Longrightarrow \Phi_k a \text{ is bounded in } S^m(\mathbb{R}^n)$$
$$\text{and } \Phi_k a \to a \text{ in the topology of } S^{m+\epsilon}(\mathbb{R}^n)\ \forall\ \epsilon > 0.$$

Proof. The Φ_k can be defined by cut-off. Take $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\phi(\xi) = 1$ in $\{|\xi| < 1\}$ and set

157.105 (2.60)
$$\Phi_k a(\xi) = \phi(\xi/k)a(\xi) \in \mathcal{S}(\mathbb{R}^n).$$

The difference

157.106 (2.61)
$$(\mathrm{Id} - \Phi_k)a = (1 - \phi(\xi/k))a(\xi)7 \in S^m(\mathbb{R}^n)$$

since $1 - \phi(\xi/k) \in S^0(\mathbb{R}^n)$.

Certainly $1 - \phi(\xi/k)$ is uniformly bounded and the derivatives are

157.107 (2.62)
$$\partial^\beta_\xi(1 - \phi(\xi/k)) = -k^{-|\beta|}(\partial^\beta\phi)(\xi/k),\ |\beta| > 0.$$

Since this function is supported in $|\xi| < Ck$, for some constant C, the product satisfies

157.108 (2.63)
$$\sup_\xi |\partial^\beta_\xi(1 - \phi(\xi/k))| \le C_\beta(1 + Ck)^{|\beta|}k^{-|\beta|} < \infty$$

This shows that $1 - \phi(\xi/k)$ is bounded with respect to all the seminorms for $S^0(\mathbb{R}^n)$. It follows that $\Phi_k a$ is bounded in $S^m(\mathbb{R}^n)$.

The seminorms on $S^\epsilon(\mathbb{R}^n)$ on the difference $1 - \Phi_k$, which has support in $|\xi| > k$, have an extra factor of $(1 + |\xi)^{-\epsilon}$

157.136 (2.64)
$$(1 + |\xi|)^{-\epsilon+|\beta|}\partial^\beta_\xi(1 - \phi(\xi/k)) = k^{-|\beta|}(1 + |\xi|)^{-\epsilon+|\beta|}(\partial^\beta(1 - \phi))(\xi/k)$$
$$\Longrightarrow \|1 - \Phi_k\|_{\epsilon,N} \le C_N k^{-\epsilon}$$

where C_N depends on ϕ and N. Thus $1-\phi(\xi/k)\to 0$ with respect to each seminorm on $S^\epsilon(\mathbb{R}^n)$ for $\epsilon>0$. It follows that

157.137
$$\Phi_k a \to a \text{ in } S^\epsilon(\mathbb{R}^n)\ \forall\ \epsilon>0. \tag{2.65}$$

This in turn implies (2.59). ☐

We record the norm estimate which underlies (2.65) for use below

157.138 Lemma 2.6. *For $a\in S^m(\mathbb{R}^n)$ and any $m'>m$*

157.139
$$\|(\mathrm{Id}-\Phi_k)a\|_{m',N}\le C_{N,m,m'}\|a\|_{m,N}k^{m-m'} \tag{2.66}$$

where the constant is independent of a and k.

4. Integration

157.109 Lemma 2.7. *Integration in one of the variables, say the last, gives a continuous linear map*

157.110
$$\int_{\mathbb{R}}dx_n : I^m_{\mathcal{S}}(\mathbb{R}^n)\longrightarrow I^{m-\frac14}_{\mathcal{S}}(\mathbb{R}^{n-1}),\ \mathcal{F}\left(\int_{\mathbb{R}}u(x)dx_n\right)=\mathcal{F}(u)\big|_{\xi_n=0}. \tag{2.67}$$

Of course we can iterate this, integrating over k variables to get a conormal distribution with order decreased by $k/4$.

Proof. The integral is defined since integration of both Schwartz functions and distributions of compact support is well-defined. Using density we can suppose that $a\in\mathcal{S}(\mathbb{R}^n)$ and then

157.111
$$\begin{aligned}\int_{\mathbb{R}}dx_n(\mathcal{F}^{-1}a)(x',x_n)dx_n&=(2\pi)^{-n}\int_{\mathbb{R}}dx_n\int_{\mathbb{R}^n}e^{ix'\cdot\xi}a(\xi)d\xi\\&=(2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}}e^{ix'\cdot\xi'}a(\xi',0)d\xi',\ x=(x',x_n),\ \xi=(\xi',\xi_n).\end{aligned} \tag{2.68}$$

by the Fourier inversion formula in one dimension. Thus

157.112
$$\int_{\mathbb{R}}dx_n\mathcal{F}^{-1}(a)=\mathcal{F}^{-1}(a\big|_{\xi_n=0}),\ a\in\mathcal{S}(\mathbb{R}^n). \tag{2.69}$$

Clearly, with Φ_k as defined above

157.113
$$\Phi_k(a\big|_{\xi_n=0})=(\Phi_k a)\big|_{\xi_n=0} \tag{2.70}$$

so the general case follows. ☐

5. Wavefrontset

The support of a function or distribution on $\mathbb{R}^n$ is defined by

$$\mathrm{supp}(u)=\left(\bigcup\{U\subset\mathbb{R}^n; U\text{ is open and }u=0\text{ on }U\}\right)^{\complement}. \tag{2.71}$$ 157.115

This is really a notion defined for *sheaves* (the theory of which I will outline below in case you have not seen it). We define a related notion for symbols which is to do with the directions of growth at infinity.
If $V\subset\mathbb{S}^{n-1}$ is open then the set

$$\mathbb{R}^+V=\{\xi\in\mathbb{R}^n\setminus\{0\};\frac{\xi}{|\xi|}\in V\} \tag{2.72}$$ 157.116

is what we mean by an open cone – an open subset of $\mathbb{R}^n \setminus \{0\}$ which is invariant under the radial $\mathbb{R}^+$ action. If $\psi \in \mathcal{C}^\infty_c(V)$ and $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ is identically equal to 1 near 0 then

$$\chi(\xi) = (1-\phi)(|\xi|)\psi(\frac{\xi}{|\xi|}) \in S^0(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n) \qquad \boxed{157.117} \qquad (2.73)$$

where we define it to be identically zero in $|\xi| < \epsilon$ where $\phi(|\xi|) = 1$ on $|\xi| < \epsilon$. Thus, ϕ is only there to cut out the singularity the homogeneous function $\psi(\frac{\xi}{|\xi|})$ is almost certain to have at the origin.

157.118 Lemma 2.8. *An element of $\mathcal{C}^\infty(\mathbb{R}^n)$ which is homogeneous of complex degree z vanishes identically unless $z \in \mathbb{N}_0$, in which case it is necessarily a polynomial.*

Proof. The homogeneity statement is that

$$u(t\xi) = t^z u(\xi) \ \forall\ t > 0,\ \xi \in \mathbb{R}^n. \qquad \boxed{157.455} \qquad (2.74)$$

Consider the derivatives of u at the origin. From ([157.455] 2.74) it follows that

$$\partial^\alpha_\xi u(0) = t^{|\alpha|}\partial^\alpha_\xi u(0) = t^z \partial^\alpha_\xi u(0). \qquad \boxed{157.456} \qquad (2.75)$$

Thus either $z = |\alpha|$ or $\partial^\alpha_\xi u(0) = 0$. So, if z is not a non-negative integer then u must vanish to infinite order at 0. But then

$$t^{-z}u(tu) \to 0 \text{ as } t \downarrow 0 \Longrightarrow u \equiv 0. \qquad \boxed{157.457} \qquad (2.76)$$

If $z = k$ then u is the sum of a polynomial and a function which vanishes to infinite order at 0 and the same argument shows that such a homogeneous function vaniahes identically. □

The product of $a \in S^m(\mathbb{R}^n)$ and a function χ as in ([157.117] 2.73) is always in $S^m(\mathbb{R}^n)$. However, it might be much smaller.

157.119 Definition 2.4. The cone-support of a symbol $a \in S^m(\mathbb{R}^n)$ is the (relatively) closed subset of $\mathbb{R}^n$

$$\operatorname{conesupp}(a) = \left(\bigcup\{\mathbb{R}^+V; \chi a \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \psi \in \mathcal{C}^\infty_c(V)\}\right)^{\complement}. \qquad \boxed{157.120} \qquad (2.77)$$

In fact the union of all sets V for which $\psi a \in \mathcal{S}(\mathbb{S}^{n-1})$ when $\psi \in \mathcal{C}^\infty_c(V)$ is still a V for which this holds – i.e. there is a maximal such V. Clearly the cone-support is a cone, so the information it contains is the same as the corresponding closed subset of the sphere. It is traditional to think of it as a cone, partly because of the definition, but also because it has a little content as we will see later.

157.121 Exercise 4. Show that $\operatorname{conesupp}(a) = \emptyset$ iff $a \in \mathcal{S}(\mathbb{R}^n) = S^{-\infty}(\mathbb{R}^n)$.

157.122 Definition 2.5. For $u \in I^M_{\mathcal{S}}(\mathbb{R}^n; \{0\}) = \mathcal{F}^{-1}(S^{M-\frac{n}{4}}(\mathbb{R}^n))$ we set

$$\operatorname{WF}(u) = \operatorname{conesupp}(a),\ u = \mathcal{F}^{-1}(a) \subset T^*_0\mathbb{R}^m \setminus \{0\} = \mathbb{R}^n \setminus \{0\} \qquad \boxed{157.123} \qquad (2.78)$$

It matters here that this is the inverse Fourier transform not the Fourier transform, otherwise there is a reflection. The identification of $\mathbb{R}^n \setminus \{0\}$ as the cotangent fibre at 0 on $\mathbb{R}^n$ might appear somewhat arbitrary but is justified by results below on coordinate-invariance. In any case, by definition the wavefront set (that is what WF stands for) of a conormal distribution at the origin in $\mathbb{R}^n$ is a closed cone in $T^*_0\mathbb{R}^n \setminus \{0\}$.

6. Restriction

What is this notion of wavefrontset good for? Notice in ([157.110] 2.67) that integration and restriction are dual under Fourier transform, at least in this special case. In general we cannot expect to restrict a conormal distribution to $x_n = 0$ – for instance this is not reasonable for the delta function at the origin. Dually, we cannot expect to integrate a symbol, it may just be too large at infinity.

157.124 LEMMA 2.9. *If $a \in S^m(\mathbb{R}^n)$ then*

$$\pm e_n \notin \operatorname{conesupp}(a) \Longrightarrow \int_{\mathbb{R}} d\xi_n a(\xi, \xi_n) \in S^{m+1}(\mathbb{R}^{n-1}). \quad \boxed{157.125} \tag{2.79}$$

Here $e_n = (0, \dots, 0, 1)$ is the unit vector and this is not in $\operatorname{conesupp}(a)$ if and only if the half-line $\mathbb{R}^+ e_n$ does not meet $\operatorname{conesupp}(a)$. So you can interpret the condition in (2.79) as saying

$$\mathbb{R} \cdot e_n \cap \operatorname{conesupp}(a) = \emptyset. \quad \boxed{157.126} \tag{2.80}$$

PROOF. The condition on $\operatorname{conesupp}(a)$ means that we can find a cut-off function $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ on the sphere which is non-vanishing at $\pm e_n$ and such that

$$(1 - \phi|\xi|)\psi\left(\frac{\xi}{|\xi|}\right) a \in \mathcal{S}(\mathbb{R}^n). \quad \boxed{157.129} \tag{2.81}$$

So this means that in a conic region

$$\Gamma_\epsilon = \{\xi \in \mathbb{F}^n; |\xi'| \le \epsilon|\xi_n|\},\ \epsilon > 0 \quad \boxed{157.130} \tag{2.82}$$

the symbol a is rapidly decreasing with all its derivatives. In fact we can can assume that $\psi = 1$ near $\pm e_n$ and then write

$$\begin{gathered} a = a' + \phi(|\xi|) + (1 - \phi(|\xi|))\psi\left(\frac{\xi}{|\xi|}\right) \in S^m(\mathbb{R}^n), \\ a = 0 \text{ in } \Gamma_\epsilon,\ a - a' \in \mathcal{S}(\mathbb{R}^n). \end{gathered} \quad \boxed{157.131} \tag{2.83}$$

Since integration certainly maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^{n-1})$ it suffices to consider a' in place of a and look at

$$b(\xi) = \int_{\mathbb{R}} d\xi_n a(\xi', \xi_n). \quad \boxed{157.132} \tag{2.84}$$

This integral certainly exists since for each ξ' the integrand is supported in $|\xi_n| \le \epsilon^{-1}|\xi'|$. Thus from the leading symbol estimate for a we see that in $|\xi'| > 1$

$$|b(\xi')| \le C \int_{|\xi_n| \le \epsilon^{-1}|\xi'|} (|\xi'| + |\xi_n|)^m d\xi_n. \quad \boxed{157.133} \tag{2.85}$$

Now, changing the variable of integration to $\tau = \xi_n/|\xi'|$ it follows that

$$|b(\xi')| \le C \int_{|\tau| \le \epsilon^{-1}} |\xi'|^{m+1}(1 + |\tau|)^m d\tau \le C'|\xi'|^{m+1}. \quad \boxed{157.134} \tag{2.86}$$

The same argument applies to all the ξ' derivatives, so

$$b \in S^{m+1}(\mathbb{R}^{n-1}). \quad \boxed{157.135} \tag{2.87}$$

□

157.127 COROLLARY 3. *Restriction to the coordinate hyperplane is well-defined as a linear map*

$$\begin{gathered} \big|_{x_n=0} : \{u \in I^M_{\mathcal{S}}(\mathbb{R}^n; \{0\}); \{dx_n, -dx_n\} \cap \mathrm{WF}(u) = \emptyset\} \\ \longrightarrow I^{M+\frac{3}{4}}(\mathbb{R}^{n-1}; \{0\}). \end{gathered} \quad \boxed{157.128} \tag{2.88}$$

7. Multiplicativity

One of the applications of the notion of wavefront – for general distributions not just in the conormal case – is to provide conditions under which operations extend to distributions.

R.11 EXERCISE 5. Show that if $\Gamma_i \subset \mathbb{R}^n \setminus \{0\}$, $i = 1, 2$, are (relatively of course) closed cones with the property

R.12 (2.89)
$$\xi_i \in \Gamma_i,\ 1, 2 \Longrightarrow \xi_1 + \xi_2 \neq 0$$

then the product of functions can be extended to

R.13 (2.90)

$$\{u_1 \in I^{m_1}_{\mathcal{S}}(\mathbb{R}^n;\{0\}); \mathrm{WF}(u_1)\cup\Gamma_1\} \times \{u_2 \in I^{m_2}_{\mathcal{S}}(\mathbb{R}^n;\{0\}); \mathrm{WF}(u_2)\cup\Gamma_2\} \longrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Hint: One way to define the product is to use the fact that these conormal spaces are modules over $\mathcal{S}(\mathbb{R}^n)$. So it is enough to define the pairing of the product with 1 provided there is some continuity. If $u_2 \in \mathcal{S}(\mathbb{R}^n)$ with the integrals interpreted as pairings

R.14
$$\int_{\mathbb{R}^n} u_1(x)u_2(x)dx = \frac{1}{(2\pi)^n}\int \hat{u}_1(\xi)\hat{u}_1(-\xi)d\xi. \tag{2.91}$$

Now you can see where the condition (2.89) comes from. Using a partition of unity to divide $\mathbb{R}^n$ into two conic regions (and a compact part) the 'integral' on the right can be divided into parts in each of which one of the terms is Schwartz. To really justify (2.90) one needs to do something about continuity – find a topology on the spaces in (2.90) with $\mathcal{S}(\mathbb{R}^n)$ dense and such that convergence is consistent with the decomposition.

You might also ask whether the product so defined is conormal. In general it is not.

8. Asymptotic completeness

The main interest in symbols on $\mathbb{R}^n$ is their behaviour 'at infinity' (which is the boundary of the radial compactification). This allows for a notion of 'convergence' which corresponds to the 'asymptotic completeness' in the following sense.

157.140 THEOREM 2.2. *If $a_j \in S^{m_j}(\mathbb{R}^n)$ is a sequence (we think of it as a series) of symbols with $m_j \to -\infty$ as $j \to \infty$ then there exists a symbol $a \in S^M(\mathbb{R}^n)$, $M = \sup m_j$ such that for every k*

157.141
$$a - \sum_{j\le k} a_j \in S^{M(k)},\ M(k) = \sup_{j>k} m_j \tag{2.92}$$

and a is determined up to an error in $S^{-\infty}(\mathbb{R}^n)$ by these conditions.

The relationship (2.92) between a and the a_j is interpreted as 'a complete asymptotic expansion' and written

157.142
$$a \sim \sum_j a_j. \tag{2.93}$$

Note that we are certainly *not* saying that the series on the right converges in any sense (well people say it converges asymptotically, just meaning the order $m_j \to -\infty$).

I have been a little vague here about the range of j, usually one takes $j \in \mathbb{N}_0$, so starting off at 0, but this is just a convention.

PROOF. The 'uniqueness' (modulo $S^{-\infty}(\mathbb{R}^n)$) is immediate from (2.92) – given two such 'asymptotic sums' a and a' the difference satisfies

157.153
$$a' - a = (a' - \sum_{j\le k} a_j) - (a - \sum_{j\le k} a_j) \in S^{M(k)}(\mathbb{R}^n)\ \forall\ k \Longrightarrow \tag{2.94}$$
$$a' - a \in S^{-\infty}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n).$$

For existence, I will assume, as discussed below, without loss of generality that the m_j are strictly decreasing, just to simplify notation.

I will use the 'approximation' operators $(\mathrm{Id} - \Phi_l)$ discussed above, where Φ_l is multiplication by $\phi(\xi/l)$ for $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ equal to 1 near 0. So these are cutoffs near infinity. Here l will vary with j so we are looking for a sequence of integers

157.144 (2.95)
$$l(j) \to \infty \text{ in } \mathbb{N}.$$

What we want these integers to satisfy – they depend of course on the given sequence a_j – is

157.145 (2.96)
$$\sum_{j>k} \|(\mathrm{Id} - \Phi_{l(j)})a_j\|_{m_k,N} < \infty \ \forall\ k, N.$$

So this is a countable set of conditions we need to satisfy.

Let's just examine one of the conditions (2.96). It makes sense, since the terms are in $S^{m_j} \subset S^{m_k}$ and $m_k - m_j > 0$ for $j > k$ by assumption. This is often called 'absolute summability' of the sequence with respect to the norm. It implies that the series is Cauchy with respect to this norm, and that is what we are after. That is we will ensure that the series

157.146 (2.97)
$$\sum_{j>k} (\mathrm{Id} - \Phi_{l(j)})a_j \text{ is Cauchy with respect to } \|\cdot\|_{m_k,N}.$$

For the moment of course just for one N and k.

We have, from Lemma 2.6, an estimate on each of these norms in (2.96)

157.147 (2.98)
$$\|(\mathrm{Id} - \Phi_{l(j)})a_j\|_{m(k),N} \le C_{N,k}(l(j))^{m_k - m_j}\|a_j\|_{m_k,N}$$

(ultimately because the $m_j < m_k$). Here the constant does not depend on $l(j)$ – the dependence is the power. To make the series converge absolutely it suffices to arrange that

157.148 (2.99)
$$\|(\mathrm{Id} - \Phi_{l(j)})a_j\|_{m(k),N} \le j^{-2}$$

for instance. In fact convergence is a property of the 'tail' of the sequence – the behaviour of any finite number of terms is irrelevant – so it is enough to arrange (2.99) from some j onwards. From (2.98) we see that we can ensure this by choosing $l(j)$ so that

157.149 (2.100)
$$l(j) > L(N, k, j)$$

where for this N and k, $L(N, k, j)$ is some explicit sequence which depends on the norms of the a_j.

Now, this shows we can choose the $l(j)$ so that any one of the series (labelled by k) converges absolutely with respect to any one of the norms $\|\cdot\|_{m_k,N}$. In fact by a 'diagonalization' procedure we can ensure that *all* the series are Cauchy with respect to *all* the norms (and hence converge in the corresponding symbol space). To do this, just arrange all the (N, k) as a sequence, parameterized by p, and demand that (2.100) hold for $j > p$.

So we can choose the integers $l(j)$ such that each of the series

157.150 (2.101)
$$\sum_{j>k} (\mathrm{Id} - \Phi_{l(j)})a_j \text{ converges in } S^{m_k}(\mathbb{R}^n)$$

in the strong sense that it converse absolutely with respect to each of the seminorms. Now set

157.151 (2.102)
$$a = a_0 + \sum_{j\ge 1} (\operatorname{Id} - \Phi_{l(j)}) a_j \in S^{m_0}(\mathbb{R}^n).$$

This is our asymptotic sum. To check this observe that the difference with a finite sum can be written

157.152 (2.103)
$$a - \sum_{j\le k+1} a_j = - \sum_{j\le k+1} \Phi_{l(j)} a_j + \sum_{j>k+1} (\operatorname{Id} - \Phi_{l(j)}) a_j.$$

The last sum here is in $S^{m_{k+1}}$ and the finite sum is actually of compact support, so in $S^{-\infty}(\mathbb{R}^n)$. The last term on the left is in the same space, $S^{m_{k+1}}(\mathbb{R}^n)$ so (157.141 2.92) follows.

If we do not have a strictly decreasing sequence of orders, we can rearrange the sequence so that the order is weakly decreasing and then sum up an finite sequences of fixed order. This reduces the problem to the strictly decreasing case and, since we have arranged absolute convergence, we recover (157.141 2.92) in general. □

Except that the topology is a little dubious, we have shown that a series with elements in $S^{m_j}(\mathbb{R}^n)/S^{-\infty}(\mathbb{R}^n)$ 'always converges' if the $m_j \to -\infty$. What this really means is that *there exist* representatives of the elements in $S^{m_j}(\mathbb{R}^n)$ such that the series does converge and gives a well-defined limit in $S^{\sup m_j}(\mathbb{R}^n)/S^{-\infty}(\mathbb{R}^n)$. L3-end

L4

9. $I^m_{\mathcal{S}}(\mathbb{R}^n;\{0\})$ as a module

First let's fix some notation.

R.15 Definition 2.6. The *full symbol map* for conormal distributions at the origin is

157.158 (2.104)
$$\sigma : I^m_{\mathcal{S}}(\mathbb{R}^n;\{0\}) \ni u \longrightarrow [\mathcal{F}(u)] \in S^{m-\frac{n}{4}}(\mathbb{R}^n)/S^{-\infty}(\mathbb{R}^n).$$

So the full symbol in this sense is the Fourier transfrom – which would be called the amplitude – $\mathcal{F}(u) \in S^{m-\frac{n}{4}}(\mathbb{R}^n)$ modulo the Schwartz space.

This means there is a short exact sequence for each m

R.16 (2.105)
$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow I^m_{\mathcal{S}}(\mathbb{R}^n;\{0\}) \xrightarrow{\sigma} S^{m-\frac{n}{4}}(\mathbb{R}^n)/S^{-\infty}(\mathbb{R}^n).$$

The smooth functions of 'slow growth' form a (not very pleasant) linear space which I will denote by $\mathcal{O}(\mathbb{R}^n)$ which is a space of multipliers on $\mathcal{S}(\mathbb{R}^n)$. A smooth function is an element $\psi \in \mathcal{O}(\mathbb{R}^n)$ if for each multiindex $\alpha \in \mathbb{N}^n_0$ there exists m_α such that

157.154 (2.106)
$$\sup_\xi (1+|\xi|)^{-m_\alpha} |D^\alpha_\xi \psi(\xi)| < \infty.$$

Thus multiplication gives a bilinear map

157.155 (2.107) $\mathcal{O}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ which restricts to $\mathcal{O}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$.

157.156 Proposition 2.6. *For any m, multiplication defines a map*

157.157 (2.108)
$$\mathcal{O}(\mathbb{R}^n) \times I^m_{\mathcal{S}}(\mathbb{R}^n;\{0\}) \longrightarrow I^m_{\mathcal{S}}(\mathbb{R}^n;\{0\}) \text{ with } \sigma(\psi u) \sim \sum_{\alpha\in\mathbb{N}^n_0} \frac{1}{\alpha!} \partial^\alpha_x \psi(0) D^\alpha_\xi \sigma(u).$$

The sum of the right does determine a unique element of $S^{m-\frac{n}{4}}/S^{-\infty}(\mathbb{R}^n) = S^{m-\frac{n}{4}}(\mathbb{R}^n)/S^{-\infty}(\mathbb{R}^n)$ as we showed last time using asymptotic summation. If we take a representative $a \in S^{m-\frac{n}{4}}(\mathbb{R}^n)$ – such as the actual Fourier transform of u – then the terms in the infinite sum are of orders $m - \frac{n}{4} - |\alpha|$ so the sum 'converges asymptotically'.

PROOF. Since we know that $u \in I^m_{\mathcal{S}}(\mathbb{R}^n; \{0\})$ can be written as the sum of a compactly supported term and one in $\mathcal{S}(\mathbb{R}^n)$ on which $\psi \in \mathcal{O}(\mathbb{R}^n)$ acts it suffices to suppose that both u and ψ are compactly supported. The Taylor series expansion

157.458 (2.109) $$\psi = \sum_{|\alpha| \le N} \frac{1}{\alpha!} \partial_x^\alpha x^\alpha \psi(0) + \sum_{|\alpha|=N+1} x^\alpha \mu_\alpha(x),\ \mu_\alpha \in \mathcal{C}^\infty(\mathbb{R}^n)$$

can then be multiplied by a cut-off of compact support equal to 1 on $\operatorname{supp}(u) \cup \operatorname{supp}(\psi)$ showing that

157.459 (2.110) $$\psi u = \sum_{|\alpha| \le N} \frac{1}{\alpha!} \partial_x^\alpha \psi(0) x^\alpha u(x) + \sum_{|\alpha|=N+1} \mu'_\alpha(x)(x^\alpha u)\mu'_\alpha(x),\ \mu'_\alpha \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$$

The terms in the first sum are are in $I^{m-|\alpha|}_{\mathcal{S}}(\mathbb{R}^n; \{0\})$ as the inverse Fourier transforms of the

157.460 (2.111) $$\frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha \psi(0) \partial_\xi^{|\alpha|} \hat{u}(\xi).$$

Similarly the terms in the second sum consists of products in

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \cdot I^{m-N-1}_{\mathcal{S}}(\mathbb{R}^n; \{0\}).$$

Let $v \in I^m_{\mathcal{S}}(\mathbb{R}^n; \{0\})$ be an asymptotic sum of this series (2.111) which can be taken to have compact support. Then from (2.110)

157.461 (2.112) $$\psi u - v = v_N + \sum_{|\alpha|=N+1} \mu'_\alpha(x)(x^\alpha u)\mu'_\alpha(x),\ v_N \in I^{m-N-1}_{\mathcal{S}}(\mathbb{R}^n; \{0\}).$$

For for $N > m + k + n$

157.463 (2.113) $$I^{m-N-1}_{\mathcal{S}}(\mathbb{R}^n; \{0\}) \subset \mathcal{C}^k(\mathbb{R}^n) \Longrightarrow \psi u - v \in \bigcap \mathcal{C}^k(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n).$$

So in fact $\psi u - v \in \mathcal{S}(\mathbb{R}^n)$.

The asymptotic formula (2.108) is a restatement of the conclusion that ψu is an asymptotic sum of the terms (2.111). $\square$

10. Action of Ψ^* on I^*

157.464 PROPOSITION 2.7. *Pseudodifferential operators act on conormal distributions giving a bilinear map*

157.465 (2.114) $$\Psi^m(\mathbb{R}^n) \times I^{m'}_{\mathcal{S}}(\mathbb{R}^n; \{0\}) \longrightarrow I^{m+m'}_{\mathcal{S}}(\mathbb{R}^n; \{0\}) \text{ with}$$
$$\sigma_{m+m'}(Au) \sim \sum_\alpha \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha (a\sigma(u)),\ A = Q_L(a)$$

To take care of the 'error terms' below we need first consider a coarser mapping property; here there is no need for precision as regards orders and regularity.

R.17 Lemma 2.10. *If $k \in \mathbb{N}_0$ and*

R.18
$$m + m' < -n - k \tag{2.115}$$

then

R.20
$$P \in \Psi^m(\mathbb{R}^n),\ u \in I_{\mathcal{S}}^{m'}(\mathbb{R}^n;\{0\}) \Longrightarrow Pu \in \mathcal{C}_\infty^k(\mathbb{R}^n), \tag{2.116}$$

meaning the derivatives up to order k are bounded.

Proof. We can use density of $\mathcal{S}(\mathbb{R}^n)$ in symbol spaces, with a little loss of order. The inequality (2.115) implies that the products

R.21
$$|D_x^\beta a(x,\xi)\hat{u}(\xi)| \le C_\beta(1+|\xi|)^{-n-k}. \tag{2.117}$$

So, in particular are actually L^1 functions of ξ and so the integral

R.22
$$Pu(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} a(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi \tag{2.118}$$

converges uniformly in x. Differentiation by x to any order β with $|\beta| \le k$ still leads to absolutely convergent integrals because of (2.117) so (2.116) follows. $\square$

Proof of Proposition (2.7). We already know a special case of (2.114) because we have an inclusion map

R.23
$$I_{\mathcal{S}}^{m+\frac{n}{4}}(\mathbb{R}^n;\{0\}) \longrightarrow \Psi^m(\mathbb{R}^n),\ u(x) \longmapsto A(x,y) = u(x-y) \tag{2.119}$$

in which conormal distributions act as convolution operators – this after all is one reason I are talking about conormal distributions (although not the only one). We will exploit this ruthlessly! $\square$

L4-end

11. Radial compactification and symbols

Project 1 P-Comp

V2: Two corrections from Benjy.

I detected some resistance to the idea of radial compactification of $\mathbb{R}^n$ in class so the main part of the first problem set is to work out some of the details. Quite a bit of this is already in the notes.

0 First recall, for background if nothing else, the basis of projective geometry (which seems to have disappeared as a subject taught to undergraduates not long before I started studying Mathematics). Define complex projective space as a quotient

P1.1 (2.120)
$$\mathbb{P}^n = (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^* = \mathbb{S}^{2n+1}/\mathbb{T},\ \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ \mathbb{S}^{2n+1} = (\mathbb{R}^{2n+2}\backslash\{0\})/\mathbb{R}^+ = (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^+.$$

Check that this is a complex manifold and that $\mathbb{C}^n$ is identified with an open dense subset by the inclusion

P1.2
$$\mathbb{C}^n \ni z \longmapsto (z,1) \in \mathbb{C}^{n+1}\setminus\{0\} \hookrightarrow \mathbb{P}^n \tag{2.121}$$

and that the complement of the image may be identified with $\mathbb{P}^{n-1}$.

(1) Now sort out the real (or more correctly a) real analogue of this. Take the embedding

P1.3
$$\mathbb{R}^n \ni x \longmapsto (x,1) \in \mathbb{R}^n \times [0,\infty) \subset \mathbb{R}^{n+1} \tag{2.122}$$

and consider the quotient map

P1.4
$$\iota : \mathbb{R}^n \longrightarrow (\mathbb{R}^n \times [0,\infty)\setminus\{0\})/\mathbb{R}^+ = \mathbb{S}^n_+ = \overline{\mathbb{R}^n} \tag{2.123}$$

mapping into the upper half-sphere (see picture below); the last equation defines $\overline{\mathbb{R}^n}$. I take this as the definition of the radial compactification; show that the embedding is given explicitly by

P1.5
$$\iota(x) = \left(\frac{x}{\sqrt{1+|x|^2}}, \frac{1}{\sqrt{1+|x|^2}}\right) \tag{2.124}$$

and deduce that $t = (1+|x|^2)^{-\frac{1}{2}} \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ is a boundary defining function (vanishes only on the boundary and has differential non-zero there).

(2) Derive the Taylor series of $a \in S^0_{\mathrm{cl}}(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{S}^n_+)$ (this is the definition of the space of classical symbols of order 0 from lectures) in the form

P1.6
$$\sum_{k=0}^{\infty} |x|^{-k} a_k(\frac{x}{|x|}),\ a_k \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),\ |x| > 2(\Longleftrightarrow t < \sqrt{2}). \tag{2.125}$$

Deduce that Taylor series with remainder gives

P1.7
$$|a - \sum_{k=0}^{N} |x|^{-k} a(\frac{x}{|x|})| \le C_N |x|^{-N-1}. \tag{2.126}$$

[We want similar estimates for derivatives too].

(3) Introduce projective coordinates on $\overline{\mathbb{R}^n}$ given by $2n+1$ coordinate patches on $\mathbb{S}^n_+$. The first one is $x \in \mathbb{R}^n$ defining the compactification. Then for each $k = 1, \dots, n$ set

P1.8
$$D^\pm_k = \{x \in \mathbb{R}^n; \pm x_k > 0\},\ C^\pm_k = \{p = (p_1, \dots, p_{n+1}) \in \mathbb{S}^n_+; \pm p_k > 0\} \tag{2.127}$$

(note that this includes part of the boundary of $\mathbb{S}^n_+$) and show that the diffeomorphisms

P1.9
$$C^\pm_k \ni x \longmapsto (\frac{1}{\pm x_k}, \pm\frac{x_j}{x_k}) \in (0,\infty) \times \mathbb{R}^{n-1} \tag{2.128}$$

extend to diffeomorphism $D^\pm_k \longrightarrow [0,\infty) \times \mathbb{R}^{n-1}$.

(4) Show that these projective coordinate systems give a coordinate cover of $\overline{\mathbb{R}^n}$.

(5) Write out formulæ for the images of the vector fields

P1.10
$$\partial_{x_j},\ x_l \partial_{x_j} \tag{2.129}$$

in these projective coordinate systems (note these span the Lie algebras of the translation group and $\mathrm{GL}(n,\mathbb{R})$ respectively).

(6) Show that the $x_l \partial_{x_j}$ extend to be smooth on $\overline{\mathbb{R}^n}$ (meaning smooth up to the boundary) and that they are elements of the Lie algebra

P1.11
$$\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n}) = \{V \text{ a } \mathcal{C}^\infty \text{ vector field tangent to the boundary}\}. \tag{2.130}$$

Note that tangency to the boundary means $Vt = 0$ at $t = 0$.

(7) Show that the images of the $x_l \partial_{x_j}$ span $\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n})$ over $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$.

(8) Show that the ∂_{x_j} are also smooth up to the boundary of $\overline{\mathbb{R}^n}$ and span, over $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ the space

P1.14
$$t\mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n}) = \{W = tV,\ V \in \mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n})\}. \tag{2.131}$$

(9) Show that the space $S^0(\mathbb{R}^n)$ of 'symbols with bounds' is identified with the space

P1.12 (2.132)
$$\{u \in L^\infty(\overline{\mathbb{R}^n}); V_1 \dots V_N u \in L^\infty(\overline{\mathbb{R}^n}) \ \forall \ V_i \in \mathcal{V}_{\mathrm{b}}(\overline{\mathbb{R}^n}) \ \forall \ N\}.$$

[Don't get hung up on worrying about distributions on a manifold with boundary, we will come back to this later.]

(10) Putting some of these things together show that

P1.13 (2.133)
$$S^0_{\mathrm{cl}}(\mathbb{R}^n) \subset S^0(\mathbb{R}^n).$$

(11) Deduce that an element $a \in S^0(\mathbb{R}^n)$ is in $S^0_{\mathrm{cl}}(\mathbb{R}^n)$ if and only if

P1.15 (2.134)
$$a \sim \sum_k (1-\phi)(\xi)|\xi|^{-k} a_k(\frac{\xi}{|\xi|})$$

where $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to one near 0 (to make everything smooth) and the $a_k \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$.

CHAPTER 3

The ring $\Psi^*(\mathbb{R}^n)$

C.PsiRn L5

I will start today with the coordinate-invariance of conormal distributions at a point and proceed to discuss the fact that the formal adjoint of a pseudodifferential operator is also a pseudodifferential operator. These might seem to be rather unrelated results, but as we shall see the proofs are closely related.

1. Coordinate invariance of $I^m_{\mathrm{c}}(\mathbb{R}^n;\{0\})$

Since we do not want to worry about the global behaviour of diffeomorphisms we will work locally near $0 \in \mathbb{R}^n$. If $\Omega \subset \mathbb{R}^n$ is an open neighbourhood of 0 set

157.159
$$I^m_{\mathrm{c}}(\Omega;\{0\}) = \{u \in I^m_{\mathbb{S}}(\mathbb{R}^n;\{0\}); \operatorname{supp}(u) \Subset \Omega\}. \tag{3.1}$$

Here of course we are thinking of Ω as an open subset of $\mathbb{R}^n$ but we can also think of it as a manifold. For the conormal functions to make sense on a manifold we need:

157.160 Proposition 3.1. *If $F : \Omega \longrightarrow \Omega'$ is a diffeomorphism of open neighbourhoods of $0 \in \mathbb{R}^n$ with $F(0) = 0$ then*

157.161
$$F^* : I^m(\Omega';\{0\}) \longrightarrow I^m(\Omega;\{0\}). \tag{3.2}$$

You should recall that the pull-back of distributions is well-defined under a diffeomorphism (not under a general smooth map). I will remind you of the 'issues' arising in the proof of this by duality – namely the need to think about densities – below. Using the density of $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ in $\mathcal{C}^{-\infty}_{\mathrm{c}}(\Omega)$ for any open set Ω and the fact that F^* extends by continuity I claim that (3.2) already has meaning.

Proof. First recall that the 'full symbol map' is still surjective if we restrict supports as in (3.1) since any $u' \in I^m(\mathbb{R}^n;\{0\})$ differs from an element of $I^m_{\mathrm{c}}(\Omega;\{0\})$ by an element of $\mathcal{S}(\mathbb{R}^n)$. We will use this in the proof.

First we start with a simple case, when $F \in \mathrm{GL}(n,\mathbb{R})$ is actually an invertible linear map. Then there is no problem with supports.

157.162 Lemma 3.1. *Under pull-back by $L \in \mathrm{GL}(n,\mathbb{R})$*

157.163
$$L^* : I^m_{\mathcal{S}}(\mathbb{R}^n;\{0\}) \longrightarrow I^m_{\mathcal{S}}(\mathbb{R}^n;\{0\}). \tag{3.3}$$

Proof. This corresponds to the fact the Fourier transform behaves 'well' under linear change of coordinates. For $u \in \mathcal{S}(\mathbb{R}^n)$ it follows directly that

157.164
$$\begin{aligned}\mathcal{F}(L^*u)(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(Lx)dx = \int_{\mathbb{R}^n} e^{-i(L^{-1}y)\cdot\xi}u(y)|\det L|^{-1}dy \\ &= \int_{\mathbb{R}^n} e^{-iy\cdot(L^{-1})^t\xi}u(y)|\det L|^{-1}dy = |\det L|^{-1}\hat{u}((L^{-1})^t\xi).\end{aligned} \tag{3.4}$$

So the only issue is the constant factor, but this does not affect the fact that $\mathcal{F}(L^*u) \in S^{m-\frac{n}{4}}(\mathbb{R}^n)$ if $\hat{u} \in S^{m-\frac{n}{4}}(\mathbb{R}^n)$. $\square$

Now, this allows us to simplify the general case in Proposition 3.1. Namely we can write $F = LG$ where L is the Jacobian matrix of F at 0 and $G : \Omega \longrightarrow L^{-1}\Omega'$ still has $G(0) = 0$ and now has Jacobian equal to the identity at 0. Replacing G by F again we can therefore assume that

157.165 (3.5) $$F(x) = x + \sum_{ij} x_i x_j G_{ij}(x) \text{ in } |x| < \epsilon,\ G_{ij} \in \mathcal{C}^\infty.$$

There is no problem in shrinking supports to a smaller neighbourhood of 0 since the conormal distributions are all smooth away from 0.

We can exploit the triviality of the Jacobian at 0 by observing that

157.166 (3.6) $$F_t(x) = x + t\sum_{ij} x_i x_j G_{ij}(x),\ t \in [0,1]$$

is a smooth family of diffeomorphisms of a fixed neighbourhood of 0 with image containing some fixed neighbourhood of 0 and with

157.167 (3.7) $$F_0(x) = x,\ F_1(x) = F(x).$$

This allows us to replace the problem by a deformation problem, meaning we can get from beginning to end along a path (you might think this is actually harder). However we can now use the variation formula (really just the chain rule) that

157.168 (3.8) $$\frac{d}{dt}(F_t^* u) = F_t^*(V_t u),\ V_t = \sum_{k,j,i} x_i x_j a_{ij,k}(t,x)\partial_{x_k}.$$

Here V_t is the t-dependent vector field which defines F_t by integration – F_t is the unique 1-parameter family of local diffeomorphisms which satisfies (3.8) (and $F_0 = \mathrm{Id}$). It is important here that V_t vanishes to second order at 0 (meaning its coefficients vanish quadratically at 0 of course).

So how does this help us? We need another idea, which I learnt from Jürgen Moser in a rather different context. Namely we can suppose that $u = u_t$ actually depends smoothly on t as a parameter (with values in the conormal distributions). Then the variation formula (3.8) becomes

157.169 (3.9) $$\frac{d}{dt}(F_t^* u_t) = F_t^*(V_t u_t + \frac{d}{dt}u_t).$$

Now, the idea is that we try to choose u_t with $u_1 = u$ so that $V_t u_t + \frac{d}{dt}u_t = 0$. We cannot manage this directly but what we can do is to choose $u_t \in \mathcal{C}^\infty([0,1]; I^m_{\mathrm{c}}(\Omega''))$ so that

157.170 (3.10) $$V_t u_t + \frac{d}{dt}u_t \in \mathcal{C}^\infty([0,1]; \mathcal{C}^\infty_{\mathrm{c}}(\Omega''));$$

here Ω'' is some suitably small open neighbourhood of 0.

The idea is to solve (3.10) by successive steps and the crucial point here is that

157.171 (3.11) $$V_t : I^m_{\mathrm{c}}(\Omega''; \{0\}) \longrightarrow I^{m-1}_{\mathrm{c}}(\Omega''; \{0\})\ \forall\ m \in \mathbb{R}.$$

This follows from Proposition 2.6 and the fact that

157.172 (3.12) $$\times x_i : I^m_{\mathrm{c}}(\Omega''; \{0\}) \longrightarrow I^{m-1}_{\mathrm{c}}(\Omega''; \{0\}),\ \partial_{x_k} : I^m_{\mathrm{c}}(\Omega''; \{0\}) \longrightarrow I^{m+1}_{\mathrm{c}}(\Omega''; \{0\}).$$

So, we look for u_t as a formal, for the moment, sum

157.173 (3.13)
$$u_t \sim \sum_j v_j,\ v_j \in \mathcal{C}^\infty([0,1]; I^{m-j}(\Omega'', \{0\})).$$

Take RBM:Clarify

157.174 (3.14)
$$v_0 = u,\ v_j = \int_t^1 V_t v_{j-1},\ j \geq 1.$$

Then, slightly generalizing the asymptotic summation result (see Problems 2) to include the 'parameter' $t \in [0,1]$, we can find u_t satisfying (3.13) and hence (3.10). However, we do know that $\mathcal{C}_c^\infty$ is coordinate-invariant so we have proved the Proposition. □

The question arises as to what the full symbol of F^*u might be. The answer is that it is not so simple to write out because of the iteration. We will deduce a few things about this complicated formula below, but for the moment notice that the 'prinicpal symbol' is given by a relatively simple formula in terms of the Jacobian.

157.175 (3.15)
$$\sigma_m(F^*u)(\xi) = |\det L|^{-1}\sigma_m(u)((L^{-1})^t\xi) \in S^{m-\frac{n}{4}}/S^{m-1-\frac{n}{4}}(\mathbb{R}^n),\ L = D_0F.$$

157.176 Lemma 3.2. *The transformation (3.15) is that of a density on $T_0^*\mathbb{R}^n$.*

2. Left/right invariance

For differential operators it is conventional to write the coefficients 'on the left'

157.178 (3.16)
$$P = \sum_{|\alpha|\leq m} p_\alpha(x) D_x^\alpha,\ p_L(x,\xi) = \sum_{|\alpha|\leq m} p_\alpha(x)\xi^\alpha.$$

However one can just as well write them on the right

157.179 (3.17)
$$P = \sum_{|\alpha|\leq m} D_x^\alpha q_\alpha(x),\ p_R(x,\xi) = \sum_{|\alpha|\leq m} q_\alpha(x)\xi^\alpha.$$

157.180 Lemma 3.3.

157.181 (3.18)
$$p_L(x,\xi) = \sum_{\beta\in\mathbb{N}_0^n} \frac{1}{\beta!}\partial_x^\beta D_\xi^\beta p_R(x,\xi),$$
$$p_R(x,\xi) = \sum_{\beta\in\mathbb{N}_0^n} \frac{(-1)^{|\beta|}}{\beta!}\partial_x^\beta D_\xi^\beta p_L(x,\xi).$$

Here of course only a finite number of terms are non-zero. The formal power series here are those of an exponential so we can write

157.182 (3.19)
$$p_L = \exp(D_x\cdot\partial_x)p_R,\ p_R = \exp(-D_x\cdot\partial_x)p_L$$

to see that one is the inverse of the other.

Proof. Leibniz' formula. □

For pseudodifferential operators we can do 'the same thing' but it is then not so clear that we get the same space of operators. For a differential operator with coefficients written on the right we see, again using the Fourier inversion formula on $\mathcal{S}$ that the operator is given by the formula

157.183 (3.20)
$$\mathcal{F}(Pu)(\xi) = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} p_R(y,\xi)u(y)dy,\ u \in \mathcal{S}(\mathbb{R}^n).$$

157.184 Proposition 3.2. *If $p_R \in \mathcal{C}^\infty_\infty(\mathbb{R}^n_y; S^m(\mathbb{R}^n_\xi))$ the operator defined by (3.20) is an element of $\Psi^m(\mathbb{R}^n)$ with left-reduced symbol, p_L, given asymptotically by (3.18).*

Proof. We proceed very much as in the proof above. Namely, the Schwartz kernel of the operator P in (3.20) is

157.185
$$P(x,y) = B(y, x-y),\ B(y,z) = (2\pi)^{-n}\int_{\mathbb{R}^n} p_R(y,\xi)e^{i(x-y)\cdot\xi}d\xi \tag{3.21}$$

where we may assume that p_R is of very low order to ensure absolute convergence of the integral (and sort the general case out by continuity). Thus the kernel is given by introducing the coordinates y and $z = x-y$ in $\mathbb{R}^{2n}$ and taking the partial inverse Fourier transform in z.

So this is very similar to the original 'left-reduced' formula except we have switched x and y as the variable independent of $z = x - y$ on $\mathbb{R}^{2n}$. Using the same idea as above we can consider a 1-parameter family of 'quantization maps' including left and right as extreme cases

RBM:Check Q_L is earlier

157.186
$$Q_t(a_t) = (2\pi)^{-n}\int_{\mathbb{R}^n} a_t(tx + (1-t)y, \xi)e^{i(x-y)\cdot\xi}d\xi,\ t \in [0,1] \tag{3.22}$$

and again allow a to vary smoothly with t. The full estimates we are considering on a are therefore

157.187
$$\sup(1+|\xi|)^{-m+|\beta|}|\partial_t^k\partial_x^\alpha\partial_\xi a(t,x,\xi)| < \infty \text{ on } [0,1]_t \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi. \tag{3.23}$$

So again, the claim is that the space of kernels, distributions that is, on $\mathbb{R}^{2n}$ defined by (3.22) is actually independent of t.

To see this we compute, as before, the derivative in t and note that it can be written

157.188
$$\frac{d}{dt}Q_t(a_t) = Q_t(i\sum_j \partial_{\xi_j}\partial_{x_j}a_t + \frac{d}{dt}a_t) \tag{3.24}$$

where the first term comes from the chain rule and integration by parts since $\frac{d}{dt}(tx + (1-t)y) = x - y$ and $x - y = -i\partial_\xi i(x-y)\cdot\xi$. So, now we want to choose a_t so that

157.189
$$i\sum_j \partial_{\xi_j}\partial_{x_j}a_t + \frac{d}{dt}a_t \text{ is of order } -\infty. \tag{3.25}$$

In this case we can solve (3.25) explicitly by taking

157.192
$$a_t(x,\xi) \sim \sum_j \frac{t^k}{k!}(D_x\cdot\partial_\xi)^k a(x,\xi) = \exp(tD_x\cdot\partial_\xi)a \tag{3.26}$$

in the sense of formal power series at $t = 0$.

If we choose a_t to be an asymptotic sum (uniform in the other variables) as in (3.26) then the 'error term' is

157.193
$$\frac{d}{dt}Q_t(a_t) = Q_t(e_t(x,\xi)),\ \sup(1+|\xi|)^{-N}|\partial_t^k\partial_x^\alpha\partial_\xi^\beta e_t| < \infty\ \forall\ k,\alpha,\beta. \tag{3.27}$$

So we can unload the last step in the proof on the following lemma. □

L5-end

Remark. Undelivered Lecture 6

I did not finish the proof of left/right equivalence last time. Let me not start at precisely the place I left off, but instead consider the 'residual' operators.

157.194 LEMMA 3.4. *For each $t \in [0,1]$ the quantization Q_t, in (3.22) applied to the residual symbols, which satisfy*

157.195 (3.28)
$$\sup(1+|\xi|)^N |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| < \infty \;\forall\; N, \alpha, \beta$$

gives the space of kernel of elements of $\Psi^{-\infty}(\mathbb{R}^n)$ are precisely those smooth functions which satisfy

157.196 (3.29)
$$\sup(1+|x-y|)^N |\partial_x^\alpha \partial_y^\beta A(x,y)| < \infty \;\forall\; N, \alpha, \beta.$$

PROOF. The functions satisfying (3.28) are exactly the elements of the space $\mathcal{C}^\infty_\infty(\mathbb{R}^n_x; \mathbb{S}(\mathbb{R}^n_\xi))$. Consider the case $t=1$, which is the 'left quantization' we started with. Since the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$, the space of kernels of elements of $\Psi^{-\infty}(\mathbb{R}^n)$ consists of the functions

157.198 (3.30)
$$A(x,y) = B(x, x-y),$$

where $B(x,z)$ is the (partial) inverse Fourier transform $\xi \longrightarrow z$

157.199 (3.31)
$$B(x,z) = (2\pi)^{-n} \int e^{iz\cdot\xi} a(x,\xi) d\xi \Longrightarrow B \in \mathcal{C}^\infty_\infty(\mathbb{R}^n_x; \mathcal{S}(\mathbb{R}^n_z)).$$

Thus, after this change of variable, the space of B's satisfy the same estimates as the symbols they are defined by

157.201 (3.32)
$$\sup_{x,z}(1+|z|)^N |\partial_x^\alpha \partial_z^\beta B| < \infty.$$

The general quantization for $t \in [0,1]$ replaces these kernels by the

157.200 (3.33)
$$A_t(x,y) = B(tx + (1-t)y, z)$$

for the same space of B's. In terms of the B's themselves this corresponds to the change of coordinates $(x,z) \longrightarrow (X = x - (1-t)z, z)$. This is invertible and the coordinate vector fields tranform to

157.202 (3.34)
$$\partial_z = \partial_z - (1-t)\partial_x,\; \partial_x = \partial_X$$

from which it is clear that the estimates (3.32) are invariant under such transformations. Thus

157.203 (3.35)
$$Q_t\left(\mathcal{C}^\infty_\infty(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^n))\right) = \Psi^{-\infty}(\mathbb{R}^n) \;\forall\; t \in [0,1].$$

□

This takes care of the residual terms.

So, going back to the proof of Proposition 3.2 we are proceeding to construct a family $a_t \in \mathcal{C}^\infty_\infty([0,1]\times\mathbb{R}^n; S^m(\mathbb{R}^n))$ so that

157.204 (3.36)
$$\frac{da_t}{dt} + i\partial_\xi \cdot \partial_x a_t \in \mathcal{C}^\infty_\infty([0,1]_t \times \mathbb{R}^n; S^{-\infty}(\mathbb{R}^n_\xi)),\; a_1 \in \mathcal{C}^\infty_\infty(\mathbb{R}^n; S^m(\mathbb{R}^n)) \text{ given.}$$

To do this we choose successive families $v_j \in \mathcal{C}^\infty_\infty([0,1]\times\mathbb{R}^n; S^{m-j}(\mathbb{R}^n))$ by

157.205 (3.37)
$$v_0 = a,\; v_j = \frac{(1-t)^j}{j!}(i\partial_\xi \cdot \partial_x)^j a.$$

Here I have done the integrals explicitly, so these satisfy

157.206 (3.38)
$$\frac{dv_0}{dt} = 0,\; \frac{dv_j}{dt} + i\partial_\xi \cdot \partial_x v_{j-1} = 0 \; j \geq 1,\; v_0 = a,\; v_j\big|_{t=1} = 0,\; j \geq 1.$$

Now we choose a_t as an asymptotic sum of the v_j's. This goes beyond the earlier summation because of the presence of the parmeters $t \in [0,1]$ and $x \in \mathbb{R}^n$. What we want to do is to ensure that the cutoff series

157.207
$$\sum_{j>k} \Phi_{n_k} v_j \tag{3.39}$$

should converge absolutely with respect to the seminorms of $\mathcal{C}^\infty_\infty([0,1]\times\mathbb{R}^n; S^{m_j}(\mathbb{R}^n))$. All the terms have lower order than this. The point is that there are still only a countable number of norms, even though they now involve the supremum over $[0,1]\times\mathbb{R}^n$ as well. So absolute convergence can be ensured for each series (3.39) by choosing the n_k large enough. Again this only involves a finite number of conditions on each n_k.

Once we choose a_t to be such an asymptotic sum then we get (3.27) and. following the discussion of the residual terms above, the complete the proof of Proposition 3.2.

2.1. Composition. So, finally we are in a position to prove the multiplicativity of (standard) pseudodifferential operators as in Theorem 1.1:

157.197
$$\begin{gathered} A \in \Psi^m(\mathbb{R}^n),\ B \in \Psi^{m'}(\mathbb{R}^n) \Longrightarrow A\circ B \in \Psi^{m+m'}(\mathbb{R}^n) \text{ and} \\ \sigma(AB) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma(A) D_x^\alpha \sigma(B) \end{gathered} \tag{3.40}$$

where σ is the left-reduced full symbol.

157.614 Remark 2. The asymptotic formula in (3.40) is one version of 'Moyal's formula'.

Proof. First I suggest the standard proof, which I will not quite follow through. The idea is to write A in left-reduced form and B is right-reduced form – now that we know they are equivalent. Thus

157.208
$$\begin{aligned} Au(x) &= (2\pi)^{-n}\int a_L(x,\xi)e^{ix\cdot\xi}\hat{u}(\xi)d\xi, \\ \mathcal{F}(Bv)(\xi) &= \int b_R(y,\xi)e^{-iy\cdot\xi}v(y)dy,\ u,\ v\in\mathcal{S}(\mathbb{R}^n). \end{aligned} \tag{3.41}$$

We can assume that the symbols themselves are of order $-\infty$ and use density. The composite is then

157.209
$$(AB)v(x) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a_L(x,\xi)b_R(y,\xi)v(y)dyd\xi \tag{3.42}$$

This is almost what we want, except the 'amplitude' in the integral depends explicitly on both x and y, as well as ξ. So we need to show that the kernel of the composite

157.210
$$K(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a_L(x,\xi)b_R(y,\xi)d\xi \in \Psi^{m+m}(\mathbb{R}^n). \tag{3.43}$$

This can be proved by an argument very similar to the left/right reduction. I leave the details to you!

Let's return for a moment to the spaces of conormal distributions at the origin, $I_{\mathcal{S}}^m(\mathbb{R}^n; \{0\})$. We can easily define conormal distributions at another point simply by translation. Thus, if $p \in \mathbb{R}^n$,

157.211 (3.44) $$I_{\mathcal{S}}^m(\mathbb{R}^n; \{p\}) = \{u \in \mathcal{S}'(\mathbb{R}^n); u(x+p) = T_{-p}^* u \in I_{\mathcal{S}}^m(\mathbb{R}^n)\}.$$

This is made more convincing by the proof of coordinate invariance. Here T_q is translation by $q \in \mathbb{R}^n$, $T_q x = x + q$.

157.212 Exercise 6. Define, and the formulate (and prove) the coordinate-invariance of, the spaces $I_{\mathrm{c}}^m(\Omega, \{p\})$ for $p \in \Omega \subset \mathbb{R}^n$ open.

157.213 Lemma 3.5. *The Schwartz kernels of elements of $\Psi^m(\mathbb{R}^n)$ may be identified with the space*

157.214 (3.45) $$\mathcal{C}_\infty^\infty(\mathbb{R}_y^n; I_{\mathcal{S}}^{m+\frac{n}{4}}(\mathbb{R}^n; \{0\})) \text{ by } A(x,y) \longmapsto A(x-y, y).$$

Proof. □

Using earlier results we have another method. What we have shown above, in left/right reduction is that the kernel of an element of $\Psi^m(\mathbb{R}^n)$ □

If there is a little time left today I want to introduce another algebra of pseudodifferential operators. This is a sign of things to come. I have been rather hard on the 'coefficient ring' $\mathcal{C}_\infty^\infty(\mathbb{R}^n)$ which is involved in the ring $\Psi^m(\mathbb{R}^n)$. What is a 'nicer' possibility? The one I have in mind is the symbol space itself. We can easily introduce the space of 'symbol-valued symbols' (in either direction)

157.215 (3.46) $$S^{m,k}(\mathbb{R}^n; \mathbb{R}^n) = \{a \in \mathcal{C}^\infty(\mathbb{R}_{x,\xi}^{2n}); \sup(1+|x|)^{-k+|\alpha|}(1+|\xi|)^{-m+|\beta|}|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| < \infty,\ \forall\ \alpha, \beta\}.$$

What I mean by symbol-valued symbols is that as a smooth map from the first set of variables,

$$a : \mathbb{R}_x^n \longrightarrow S^m(\mathbb{R}^n).$$

Moreover, if $\|\cdot\|_{m,N}$ are the seminorms on $S^m(\mathbb{R}_\xi^n)$ then the estimates (157.215 3.46) are equivalent to

157.616 (3.47) $$\sup_x (1+|x|)^{-m+|\alpha|}\|\partial_x^\alpha a(x,\cdot)\|_{k,N} < \infty\ \forall\ \alpha.$$

These defining conditions give seminorms. Here there are two orders and differentiation with respect to x lowers the second (but not the first) and differentiation with respect to ξ lowers the first but not the second.

Directly from (157.215 3.46) we see that

157.216 (3.48) $$k \le 0 \Longrightarrow S^{m,k}(\mathbb{R}^n; \mathbb{R}^n) \subset \mathcal{C}_\infty^\infty(\mathbb{R}^n; S^m(\mathbb{R}^n)).$$

It is also the case that

157.217 (3.49) $$\begin{gathered}(1+|x|^2)^{k/2} \in S^{0,k}(\mathbb{R}^n; \mathbb{R}^n) \text{ and} \\ S^{m,k}(\mathbb{R}^n; \mathbb{R}^n) \cdot S^{m',k'}(\mathbb{R}^n; \mathbb{R}^n) = S^{m+m',k+k'}(\mathbb{R}^n; \mathbb{R}^n).\end{gathered}$$

Combining these two observations we see that

157.218 (3.50) $$(1+|x|^2)^{-k/2} S^{m,k}(\mathbb{R}^n; \mathbb{R}^n) \subset \mathcal{C}_\infty^\infty(\mathbb{R}^n; S^m(\mathbb{R}^n)).$$

So, we can quantize these double symbols using left quantization and define

157.219 (3.51)
$$\Psi^{m,k}_{\mathrm{sc}}(\mathbb{R}^n) = \{A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n);\\ A = (1+|x|^2)^{k/2} Q_L((1+|x|^2)^{-k/2}a),\ a \in S^{m,k}(\mathbb{R}^n;\mathbb{R}^n)\}.$$

This algebra was introduced by Shubin, MR0273463 [**8**], but I call it the scattering algebra, which is the subscript sc, because it has a direct extension to compact manifolds with boundary (as I hope we will see).

So, I am getting ahead of myself here:

157.220 PROPOSITION* 3.3. *The scattering psedudofferential operators from a double-filtered algebra*

157.221 (3.52)
$$\Psi^{m,k}_{\mathrm{sc}}(\mathbb{R}^n) \circ \Psi^{m',k'}_{\mathrm{sc}}(\mathbb{R}^n) \subset \Psi^{m+m',k+k'}_{\mathrm{sc}}(\mathbb{R}^n)$$

with residual space

157.222 (3.53)
$$\Psi^{-\infty,-\infty}_{\mathrm{sc}}(\mathbb{R}^n) = \bigcap_{m,k} \Psi^{m,k}_{\mathrm{sc}}(\mathbb{R}^n)$$

equal to the space of operators with kernels $A \in \mathcal{S}(\mathbb{R}^{2n})$.

Maybe you would like to try your hand at proving this! You can easily see why it *should* be true because Moyal's formula for the composite of two such operators gives

$$\sigma_L(AB) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi \sigma_L(A) \cdot D_x \sigma_L(B) \quad \text{157.223} \quad (3.54)$$

and the individual terms here are in

$$S^{m-|\alpha|,k}(\mathbb{R}^n;\mathbb{R}^n) \cdot S^{m',k'-|\alpha|}(\mathbb{R}^n;\mathbb{R}^n) \subset S^{m+m'-|\alpha|,k+k'-|\alpha|}(\mathbb{R}^n;\mathbb{R}^n) \quad \text{157.224} \quad (3.55)$$

which is decreasing in both orders. It takes a little thought to prove that everything 'works' correctly; here is an outline of one approach – where I will use Kumano-go's double symbols.

First go through the left/right reduction argument in this case. For convenience I take $k \le 0$ because we can always recover the general case by multiplying by $(1+|x|^2)^{k/2}$. So, we want to choose a 1-parameter family of double symbols,

$$a_t \in \mathcal{C}^\infty([0,1]_t; S^{m,k}(\mathbb{R}^n;\mathbb{R}^n)) \quad \text{157.225} \quad (3.56)$$

so that the identity 157.189 (3.25) holds in the new sense. Looking at 157.205 (3.37) we can see that if $v_0 = a$ is chosen in $S^{m,k}(\mathbb{R}^n;\mathbb{R}^n)$ then the $v_j \in \mathcal{C}^\infty([0,1]; S^{m-j,k-j}(\mathbb{R}^n;\mathbb{R}^n)$ have both orders decreasing. Going back to the asymptotic summation lemma we now need to do a little more with our cutoffs. So in this case we would consider all the series

$$\sum_{j>l}(1-\phi(x/n_j))\phi(\xi/n_j)v_j(t,x,\xi) \quad \text{157.226} \quad (3.57)$$

where we cut out the region where both $|x|$ and $|\xi|$ are less than n_j. So on the support of $(1-\phi(x/n_j)\phi(\xi/n_j))$ *either* $|x| > n_j$ or $|\xi| > n_j$ but the symbol lies in the space $S^{m-j,k-j}(\mathbb{R}^n;\mathbb{R}^n)$ with $j > l$. So this is small in the symbol space $S^{m-l,k-l}(\mathbb{R}^n;\mathbb{R}^n)$ if we choose n_j large enough. This means we can make all the series converge by an appropriate choice of the integers n_j and then the error term and then the error term

$$\frac{d}{dt}Q_t(a_t) \in \mathcal{S}(\mathbb{R}^{2n}) \quad \text{157.227} \quad (3.58)$$

is a residual operator in the new sense. So we win and we see that $Q_t(S^{m,k}(\mathbb{R}^n;\mathbb{R}^n) = \Psi^{m,k}_{\mathrm{sc}}(\mathbb{R}^n)$ for all $t \in [0,1]$.

Now to Kumano-go's result. Suppose we 'overspecify' the amplitude of the pseudodifferential operator by taking a 'triple symbol'

$$b(x,y,\xi)\in S^{m,k,k}(\mathbb{R}^n;\mathbb{R}^n;\mathbb{R}^n)=S^k(\mathbb{R}^n_x);S^k(\mathbb{R}^n_y);S^m(\mathbb{R}^n_\xi). \quad \boxed{157.228} \tag{3.59}$$

This means we consider smooth functions on $\mathbb{R}^{3n}$ which satisfy

$$\sup(1+|x|)^{-k+\alpha}(1+|y|)^{-k+|\gamma|}(1+|\xi|)^{-m+|\beta|}|\partial_x^\alpha\partial_y^\gamma\partial_\xi^\beta b(x,y,\xi)|<\infty \quad \boxed{157.229} \tag{3.60}$$

This defines a countably normed space. Notice that these are symbols 'separately' in all the variables, there is no joint decay.

157.230 Proposition 3.4. *[Kumano-go] The 'overspecified' quantization map*

$$Q:b\longrightarrow\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi}b(x,y,\xi)d\xi\in\Psi^{m,k}_{\mathrm{sc}}(\mathbb{R}^n). \quad \boxed{157.231} \tag{3.61}$$

Proof. We can think of the double symbols $a(x,\xi)$ as special cases of the triple symbols which are indpendent of y. Then (3.61) is standard quantization, so the range certainly contains $\Psi^{m,k}_{\mathrm{sc}}(\mathbb{R}^n)$. To see that it contains nothing more, we can use the same deformation argument as above and try to construct a family of triple symbols b_t so that the intermediate quantization maps

$$Q_t:b_t\longrightarrow\int_{\mathbb{R}^n}e^{i(x-y)\cdot\xi}b_t(x,tx+(1-t)y,\xi)d\xi \quad \boxed{157.232} \tag{3.62}$$

have derivative a smoothing, here meaning Schwartz, kernel. □

3. Isotropic algebra

S.Isotropic There is a text on this by Parmeggiani [7].

157.468 Remark. Edited by Paige Dote.

The calculus $\Psi^{*,*}_{\mathrm{sc}}(\mathbb{R}^n)$, due to Shubin, is described above. Let me introduce yet another algebra of pseudodifferential operators on $\mathbb{R}^n$. This one, as we will see later, has direct topological applications, whereas the scattering algebra is more of geometric significance. The symbols considered are the 'pure symbols' on $\mathbb{R}^{2n}=\mathbb{R}^n_x\times\mathbb{R}^n_\xi$,

$$S^m(\mathbb{R}^{2n})=\{a\in C^\infty(\mathbb{R}^{2n});\forall\alpha,\beta\in\mathbb{N}_0^n,\sup(1+|x|+|\xi|)^{-m+|\alpha|+|\beta|}|\partial_x^\alpha\partial_\xi^\beta a|<\infty\}.$$

So there is <u>no</u> difference in behavior between the x and ξ variables. That is what 'isotropic' is supposed to indicate here, meaning 'the same in all directions'.

Rather than repeat the basic constructions again we can use the properties of $\Psi^{*,*}_{\mathrm{sc}}(\mathbb{R}^n)$ in view of the following Lemma:

157.471 Lemma 3.6. *The pure symbols and symbol-valued symbols are related by the following:*

157.470
$$\begin{cases}S^m(\mathbb{R}^{2n})\subset S^m(\mathbb{R}^n_x;S^m(\mathbb{R}^n_\xi)), & m\ge 0\\ S^m(\mathbb{R}^{2n})\subset S^{m/2}(\mathbb{R}^n;S^{m/2}(\mathbb{R}^n)), & m\le 0\ .\\ S^{-\infty}(\mathbb{R}^{2n})=S^{-\infty}(\mathbb{R}^n;S^{-\infty}(\mathbb{R}^n)) & \end{cases} \tag{3.63}$$

Proof. These results follow from

$$\max\{1+|x|,1+|\xi|\}\le(1+|x|+|\xi|)\le(1+|x|)(1+|\xi|).$$

This gives the leading estimates in the first two statements in (3.63) since $a \in S^m(\mathbb{R}^{2n})$ satisfies

$$|a(x,\xi)| \le (1+|x|)^m(1+|\xi|)^m,\ m \ge 0$$
$$|a(x,\xi)| \le (1+|x|)^{m/2}(1+|\xi|)^{m/2},\ m \ge 0.$$

In $S^m(\mathbb{R}^{2n})$ an x-derivative corresponds to an extra decay factor of $(1+|x|+|\xi|)^{-1} \le (1+|x|)^{-1}$ and a ξ-derivative does as well but $(1+|x|+|\xi|)^{-1} \le (1+|\xi|)^{-1}$, giving all of the 'double' symbol estimates. The last statement follows from the second last. $\square$

Thus, using left quantization we can define

$$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) = Q_L(S^m(\mathbb{R}^{2n}_{(x,\xi)})) \tag{3.64}$$

and it follows that

$$\begin{cases} \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^{m,m}_{\mathrm{sc}}(\mathbb{R}^n), & m \ge 0 \\ \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^{m/2,m/2}_{\mathrm{sc}}(\mathbb{R}^n), & m \le 0 \end{cases}. \tag{3.65}$$

157.472 THEOREM 3.1. *The isotropic operators form an $*$-closed filtered algebra of operators with a well-defined principal symbol map giving a short exact sequence*

157.473
$$\Psi^{m-1}_{iso}(\mathbb{R}^n) \hookrightarrow \Psi^m_{iso}(\mathbb{R}^n) \overset{\sigma_m}{\twoheadrightarrow} S^m/S^{m-1}(\mathbb{R}^{2n}), \tag{3.66}$$

157.474
$$\sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B) \tag{3.67}$$

for $A \in \Psi^m_{iso}(\mathbb{R}^n)$, $B \in \Psi^{m'}_{iso}(\mathbb{R}^n)$.

PROOF. We know that the $\Psi^{*,*}_{\mathrm{sc}}(\mathbb{R}^n)$ is a (bi-)filtered algebra with products asymptotic to Moyal's formula. Hence, in this sense,

$$A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n),\ B \in \Psi^{m'}_{\mathrm{iso}}(\mathbb{R}^n) \implies AB = Q_L(c)$$

for some $c \in S^k(\mathbb{R}^n; S^k(\mathbb{R}^n))$ and some k where

$$c \sim \sum_\alpha \frac{1}{\alpha!}(D^\alpha_\xi a)(D^\alpha_x b).$$

The the series here has terms in in $S^{k-j,k-j}$. Now since $a \in S^m(\mathbb{R}^{2n})$, $b \in S^{m'}(\mathbb{R}^{2n})$ the terms in the Moyal series are

$$\frac{1}{\alpha!}\partial^\alpha_\xi a \cdot D^\alpha_x b \in S^{m+m'-2|\alpha|}(\mathbb{R}^{2n}).$$

Thus, this series is asymptotic in the isotropic sense. It follows that we can choose an asymptotic sum

$$S^{m+m'}(\mathbb{R}^{2n}) \ni \tilde{c} \sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_\xi a \cdot D^\alpha_x b.$$

Hence, it follows from Lemma 3.6, despite the $\frac{1}{2}$s in the orders, that

$$\tilde{c} \sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_\xi a \cdot D^\alpha_x b$$

in the sense of symbol-valued symbols. From the uniqueness of the asymptotic sum in this sense, and the last part of Lemma 3.6, it follows that

$$\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \cdot \Psi^{m'}_{\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^{m+m'}_{\mathrm{iso}}(\mathbb{R}^n)$$

as claimed. Thus the Moyal product gives an asymptotic formula for the symbol of a product so (157.473 3.66) follows, with (157.474 3.67) essentially being the definition of σ_m.

The closure under passage to adjoints is similar. $\square$

Since these are lectures, one can ask rather impertinent questions such as: Why would one be interested in this? One practical answer is that the algebra is closely related to the harmonic oscillator on $\mathbb{R}^n$.

DEFINITION 3.1 (Harmonic Oscillator). The harmonic oscillator, H, on $\mathbb{R}^n$ is defined as

$$H = \Delta + |x|^2, \text{ where } \Delta = \sum D_{x_i}^2 = -\sum \partial_{x_i}^2.$$

Indeed, $H \in \Psi^2_{\text{iso}}(\mathbb{R}^n)$ is elliptic. I hope that somewhere below I will show how one can use the isotropic algebra to prove Thom isomorphism in K-theory. This is a very special case of the familiar Atiyah-Singer Index Theorem that I want to describe below. The isotropic algebra is also an integral part of the proof, at least for the proof I have in mind.

Now, for elliptic operators, such as H, in the isotropic calculus we can deduce the existence of a parameterix just as is done microlocally in the standard case in the next chapter and globally for manifolds later. It is significant that the error here is compact.

157.475 PROPOSITION 3.5. *If $A \in \Psi^m_{iso}(\mathbb{R}^n)$ is elliptic then it has a two-sided parameterix modulo the ideal $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$.*

PROOF. By definition, $A = Q_L(a)$ where $a \in S^m(\mathbb{R}^{2n})$ is globally elliptic. Thus for some $\delta > 0$

$$|a(x,\xi)| \geq \delta(|x| + |\xi|)^m \text{ in } |x| + |\xi| \geq \frac{1}{\delta},$$

then $b(x,\xi) = \frac{1-\varphi(x,\xi)}{a(x,\xi)} \in S^{-m}(\mathbb{R}^{2n})$ if $\varphi \in C_c^\infty(\mathbb{R}^{2n})$ is equal to 1 on the ball $|x| + |\xi| \leq \frac{1}{\delta}$. $\square$

4. L^2 boundedness

Project 2 P-Bounded

In this project I would like you to go through some 'symbolic arguments', giving L^2 boundedness of pseudodifferential operators.

4.1. Schur's criterion. This is the same Schur as the lemma about irreducibility, hence I just say 'criterion'. This is quite a handy sufficient condition for L^2 boundedness in terms of the Schwartz kernel. It can be generalized to measure spaces (and so manifolds), but for the moment let's think about $\mathbb{R}^n$. Then

Schur PROPOSITION 3.6. *If $A : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a Lebesgue measurable function which satisfies*

P2.2 (3.68) $$\sup_x \int |A(x,y)|dy, \ \sup_y \int |A(x,y)|dx < \infty$$

then the integral operator (say defined initially on $\mathcal{C}_c(\mathbb{R}^n)$)

P2.1 (3.69) $$Au(x) = \int_{\mathbb{R}^n} A(x,y)u(y)dy \textit{ is a bounded operator on } L^2(\mathbb{R}^n).$$

Proof. You might like to look it up, it is basically just a clever use of Schwarz inequality. $\square$

Problem 2.1

Show that if $A \in \Psi^m(\mathbb{R}^n)$ with $m < -n$ then the Schwartz kernel is continuous and satisfies

P2.3
$$\sup_{x,y}(1 + |x - y|)^N |A(x, y)| < \infty \ \forall \ N. \tag{3.70}$$

Deduce that Schur's criterion applies and hence conclude L^2 boundedness.

In fact you can push this argument so that it applies for $m < 0$ but not up to $m = 0$ (think of the identity).

Problem 2.2

For $A \in \Psi^0(\mathbb{R}^n)$ construct $Q \in \Psi^0(\mathbb{R}^n)$ such that

P2.4
$$Q = Q^*,\ Q^2 = C\,\mathrm{Id} - A^*A + E,\ C > 0 \text{ constant},\ E \in \Psi^{-1}(\mathbb{R}^n) \tag{3.71}$$

'Hint': It is enough to choose $C > \sup |a|^2$ where $A = Q_L(a)$. Then show that $q = (C - |a|^2)^{\frac{1}{2}} \in \mathcal{C}^\infty_\infty(\mathbb{R}^n; S^0(\mathbb{R}^n))$ and the set $Q = \frac{1}{2}(Q_L(a)) + Q_L(a)^*$.

Problem 2.3

Now we want to improve the 'error' in (3.71). Show that if $E \in \Psi^{-k}(\mathbb{R}^n)$, $k \geq 1$, and $E^* = E$ where $E = Q_L(e)$ then the choice

P2.5
$$B = Q_L(e/q) + Q_L(e/q)^* \text{ satisfies } (Q - B)^2 = C\,\mathrm{Id} - A^*A - E',\ E' \in \Psi^{-k-1}(\mathbb{R}^n),\ (E')^* = E'. \tag{3.72}$$

Problem 2.4

Using this show that we may 'correct' Q (by adding a lower order term) so that (3.71) holds with $E \in \Psi^{-N}(\mathbb{R}^n)$ for any preassigned N. (Using asymptotic summation this works for $N = -\infty$.

Problem 2.5

Finally deduce L^2 boundedness in the sense that $A \in \Psi^0(\mathbb{R}^n)$ extends by continuity from $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ to a bounded operator on $L^2(\mathbb{R}^n$.

'Hint'. This whole argument is due to Hörmander. It follows from (3.71) that, for $\phi \in \mathcal{S}(\mathbb{R}^n)$, in terms of the L^2 inner product

P2.6
$$0 \leq \langle Q\phi, Q\phi\rangle = \langle Q^2\phi, \phi\rangle = C\|\phi\|^2_{L^2} - \|A\phi\|^2_{L^2} + \langle E\phi, \phi\rangle. \tag{3.73}$$

So, if we know that boundedness of E (which we do) then

P2.7
$$\|A\phi\|_{L^2} \leq (C + C')^{\frac{1}{2}} \|u\|_{L^2}. \tag{3.74}$$

where C' comes from E.

Problem 2.6: Sobolev boundedness

The Sobolev space $H^s(\mathbb{R}^n)$ is defined as consisting of those elements of $\mathcal{S}'(\mathbb{R}^n)$ (becauese we are allowing $s \leq 0$ such that

P2.8
$$(1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n). \tag{3.75}$$

Deduce that the operator $(1 + |D|^2)^t = Q_L((1 + |\xi|^2)^{t/2}) = Q_R((1 + |\xi|^2)^{t/2}) \in \Psi^t(\mathbb{R}^n)$, for any $t \in \mathbb{R}$, is an isomorphism

P2.9
$$(1 + |D|^2)^{t/2} : H^s(\mathbb{R}^n) \longrightarrow H^{s-t}(\mathbb{R}^n). \tag{3.76}$$

From this, L^2 boundedness and the properties of the calculus deduce that

P2.10
$$A \in \Psi^m(\mathbb{R}^n) \Longrightarrow A : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n). \tag{3.77}$$

'Hint': Consider for instance $(1+|D|^2)^{-m+s/2}A(1+|D|^2)^{-s/2}$.

Problem 2.7

For anyone who has read the section on the scattering (Shubin) calculus define the weighted Sobolev spaces

P2.11 (3.78) $$H^{s,t}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); (1+|x|^2)^{t/2}u \in H^s(\mathbb{R}^n).$$

(1) Show that for any real orders

P2.12 (3.79) $$A \in \Psi^{m,k}_{\text{sc}}(\mathbb{R}^n) \Longrightarrow A : H^{s,t}(\mathbb{R}^n) \longrightarrow H^{s-m,t-l}(\mathbb{R}^n).$$

(2) Show that

P2.13 (3.80) $$\mathcal{F} : H^{s,t}(\mathbb{R}^n) \longrightarrow H^{t,s}(\mathbb{R}^n),\ \forall\ s,t.$$

(3) Show that, in contrast to the usual Sobolev spaces, the inclusion $H^{s',t'}(\mathbb{R}^n) \hookrightarrow H^{s,t}(\mathbb{R}^n)$ for $s' > s$, $t' > t$ is *compact*.

Problem 2.7

For anyone who has followed the discussion above of the isotropic calculus.

(1) Define the isotropic Sobolev spaces in terms of the operators

P2.15 (3.81) $$G_s = Q_L((1+|x|^2+|\xi|^2)^{s/2} \in \Psi^s_{\text{iso}}(\mathbb{R}^n)$$

by

P2.16 (3.82) $$H^s_{\text{iso}}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); A_s u \in L^2(\mathbb{R}^n).$$

(2) Show that for any

CHAPTER 4

Ellipticity and wavefront set

In the actual Lecture 6 I got a little carried away but let me record here what I tried to cover. So, I am reviewing what we have done, or in some cases partly done, and then expanding on it a little. L6

- Symbols:- The spaces $S^m(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ are Fréchet spaces defined by the finiteness of the norms in (2.55). They form a filtered (abelian) ring with identity $1 \in S^0(\mathbb{R}^n)$ and have density (Proposition 2.5) and asymptotic completeness (Theorem 2.2) properties.

  When is an element invertible? For $a \in S^m(\mathbb{R}^n)$ to have an inverse in $S^{-m}(\mathbb{R}^n)$ a necessary and sufficient condition is

157.233
$$|a(\xi)| \ge \delta(1+|\xi|)^m \Longleftrightarrow a^{-1} \in S^{-m}(\mathbb{R}^n). \tag{4.1}$$

  The necessity of (4.1) follows from the bound $|a^{-1}(\xi)| \le C(1+|\xi|)^{-m}$. Conversely this certainly implies that

$$b(\xi) = \frac{1}{a(\xi)} \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{4.2}$$
157.234

  and the derivatives are then of the form

$$\partial^\alpha b = \frac{e_\alpha}{a^{|\alpha|+1}} \tag{4.3}$$
157.235

  where e_α is a symbol or order $(m-1)|\alpha|$. This is clear for $|\alpha| = 0$ and follows by induction since taking one more derivative shows that

$$\partial_{\xi_j}\partial^\alpha b = \frac{a\partial_{\xi_j} e_\alpha - (|\alpha|+1)\partial_{\xi_j} a}{a^{|\alpha|+2}} \tag{4.4}$$
157.346

  giving the inductive step. The symbol estimates on b follow.
- We defined $I_{\mathcal{S}}^{m+\frac{n}{4}}(\mathbb{R}^n) = \mathcal{F}^{-1}(S^m(\mathbb{R}^n))$ and derived various properties of these conormal distributions at 0.
- For such a Fréchet space we can define $\mathcal{C}^\infty_\infty(\mathbb{R}^m; S^m(\mathbb{R}^n))$ as the subspace of $\mathcal{C}^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ with all derivatives with respect to the first variables bounded in terms of the seminorms on $S^m(\mathbb{R}^n)$. Then our definiition of $\Psi^m(\mathbb{R}^n)$ is in terms of their Schwartz kernels (which we identify with the operators)

157.236
$$A \in \Psi^m(\mathbb{R}^n) \Longrightarrow A(x, x-z) \in \mathcal{C}^\infty_\infty(\mathbb{R}^n_x; I_{\mathcal{S}}^{m+\frac{n}{4}}(\mathbb{R}^n_z)). \tag{4.5}$$

  This corresponds to the 'quantization map' in terms of the partial Fourier tranform

157.237
$$\mathcal{C}^\infty_\infty(\mathbb{R}^m; S^m(\mathbb{R}^n)) \ni a(x,\xi) \longrightarrow Q_L(a) = \mathcal{F}_\xi^{-1}(a)(x, x-y) \in \Psi^m(\mathbb{R}^n). \tag{4.6}$$

- We showed (most of the fact that) that for each $t \in [0,1]$ the 'intermediate quantizations'

157.238
$$Q_t(a) = \mathcal{F}_\xi^{-1}(a)(tx + (1-t)y, x-y) \in \Psi^m(\mathbb{R}^n) \tag{4.7}$$

give the same space of operators. For $t = 1$ this is 'left' quantization and for $t = 0$ it is 'right' quantization where the kernel is written as

157.239
$$A(x, y) = B(y, x - y),\ B \in \mathcal{C}^\infty_\infty(\mathbb{R}^n_y; I_{\mathcal{S}}^{m+\frac{n}{4}}(\mathbb{R}^n)). \tag{4.8}$$

- Note that the case $t = \frac{1}{2}$ is also of importance. It is called 'Weyl quantization' and means writing the kernel as

157.240
$$A(x, y) = B(\frac{x+y}{2}, x - y),\ B \in \mathcal{C}^\infty_\infty(\mathbb{R}^n_y; I_{\mathcal{S}}^{m+\frac{n}{4}}(\mathbb{R}^n)). \tag{4.9}$$

It has some useful properties.

- The inverses of these quantization maps are the 'total symbols'

157.241
$$\sigma_L,\ \sigma_R,\ \sigma_W : \Psi^m(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty_\infty(\mathbb{R}^n_y; S^m(\mathbb{R}^n)). \tag{4.10}$$

- The right and left symbols are related asymptotically by

157.244
$$\sigma_R(A) \sim \sum_\alpha \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha \sigma_L(A) = \exp(D_x \cdot \partial_\xi)\sigma_L(A) \tag{4.11}$$

where the exponential is to be formally expanded in Taylor series at 0.

157.248 Exercise 7. Derive a similar asymptotic relationships between σ_W and σ_L.

- (Not discussed in lecture) The 'formal' (just meaning non-Hilbert space) adjoint is defined for any continuous linear operator $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ by duality

157.242
$$A^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n),\ \int_{\mathbb{R}^n} (Au)\overline{v}dx = \int_{\mathbb{R}^n} \overline{A^* v}dx \tag{4.12}$$

(where the distribution pairing is written as an integral). Then

157.243
$${}^* : \Psi^m(\mathbb{R}^n) \longrightarrow \Psi^m(\mathbb{R}^n),\ \sigma_R(A^*) = \overline{\sigma_L(A))}. \tag{4.13}$$

- The composition theorem with

157.246
$$\sigma_L(A \circ B) \sim \sum_\alpha (\partial_\xi^\alpha \sigma_L(A))(D_x^\alpha \sigma_L(B)). \tag{4.14}$$

As suggested in Lecture, check this for differential operators. In fact it is enough to take $A = D_x^\gamma$ and $B = b(x)$ and apply Leibniz' formula to get 157.246 (4.14).

- (Also not discussed at all). The elements of $\Psi^m(\mathbb{R}^n)$ define by bounded linear maps for any M on the standard Sobolev spaces

157.247
$$A : H^M(\mathbb{R}^n) \longrightarrow H^{M-m}(\mathbb{R}^n). \tag{4.15}$$

Proof later. Here $H^M(\mathbb{R}^n) = \mathcal{F}^{-1}((1 + |\xi|)^{-M} L^2(\mathbb{R}^n))$ is defined as usual as the inverse Fourier transform of the weighted L^2 spaces on the dual.

1. Ellipticity of symbols

S.EllipticSymbols

In the notes above the notion of ellipticity for elements of $S^m(\mathbb{R}^n)$ is discussed (although I did not cover this in lectures). We want an extension of this idea to $\mathcal{C}^\infty(\mathbb{R}^N; S^m(\mathbb{R}^n))$ (I have dropped both the boundedness assumptions on the coefficents and the assumption that $N = n$ since they are both irrelevant here).

Most importantly a symbol is said to be *elliptic at* $(\bar{x}, \bar{\xi}) \in \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\})$ if 'it is as big as it can be' in a cone around this point.

157.249 Definition 4.1. The elliptic set

$$\mathrm{Ell}(a) = \mathrm{Ell}_m(a) \subset \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}) \text{ of } a \in \mathcal{C}^\infty(\mathbb{R}^N; S^m(\mathbb{R}^n))$$

consists of those points $\bar{x}, \bar{\xi}$ corresponding to which there exists $\delta > 0$ such that

157.250
$$|a(x,\xi)| > \delta|\xi|^m \text{ in } |x| < \delta,\ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| < \delta, |\xi| > 1/\delta. \tag{4.16}$$

I have engaged in constant-saving here! Clearly the result remains true if δ is decreased but remains positive. The basic region we are looking at here is a 'conic neighbourhood' of $(\bar{c}, \bar{\xi})$ which is then truncated by demanding $|\xi|$ is large as well. So the appearances of δ can be decreased individually and the estimate remains true. Ellipticity is a 'local invertibility' condition on a in the filtered symbol algebra, as shown below. It is a conic set from the definition which only depends on $\bar{\xi}/|\bar{\xi}|$ not $\bar{\xi}$ itself. Thus

157.251
$$(\bar{x}, \bar{\xi}) \in \mathrm{Ell}(a) \Longrightarrow (\bar{x}, t\bar{\xi}) \in \mathrm{Ell}(a),\ t > 0. \tag{4.17}$$

It is also clear that $\mathrm{Ell}(a)$ is open for any $a \in \mathcal{C}^\infty(\mathbb{R}^N; S^m(\mathbb{R}^n))$ as a subset of $\mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\})$, which is itself open in $\mathbb{R}^{N+n}$. Of course it could be empty, and it certainly is if $a \in \mathcal{C}^\infty(\mathbb{R}^N; S^{m'}(\mathbb{R}^n))$, $m' < m$. So one should really write

157.253
$$\mathrm{Ell}_m(a) \subset \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}) \text{ defined for } a \in \mathcal{C}^\infty(\mathbb{R}^N; S^m(\mathbb{R}^n)) \tag{4.18}$$

but we treat the 'm' as understood from context.

We also give a name to the complement of the elliptic set, it is called the *characteristic set* of the symbol

157.252
$$\mathrm{Char}(a) = \big(\mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}\big) \setminus \mathrm{Ell}(a). \tag{4.19}$$

It is then a (relatively) closed subset.

We define a third conic set corresponding to the region where the symbol is *not* locally rapidly decaying with all derivatives as follows

157.254
$$\begin{aligned} \mathrm{conesupp}(a) = \big\{(\bar{x}, \bar{\xi}) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}); \exists\ \delta > 0 \text{ and } \phi \in \mathcal{C}^\infty(\mathbb{R}^N; \mathcal{S}(\mathbb{R}^n)) \\ \text{with } a = \phi \text{ in } |x - \bar{x}| < \delta,\ |\xi| > \frac{1}{\delta},\ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| < \delta\big\}^{\complement} \end{aligned} \tag{4.20}$$

It follows that $\mathrm{conesupp}(a)$ is relatively closed and that

157.255
$$\mathrm{Ell}(a) \subset \mathrm{conesupp}(a) \tag{4.21}$$

where this relation is like that between the sets $\{u \neq 0\}$ and $\mathrm{supp}(u)$ for a smooth function.

In terms of mutliplication of symbols it is easy to see that

157.256
$$\begin{gathered} \mathrm{Ell}(ab) = \mathrm{Ell}(a) \cap \mathrm{Ell}(b), \\ \mathrm{Char}(ab) = \mathrm{Char}(a) \cup \mathrm{Char}(b), \\ \mathrm{conesupp}(ab) \subset \mathrm{conesupp}(a) \cap \mathrm{conesupp}(b) \end{gathered} \tag{4.22}$$

Note that we can construct symbols which are elliptic at a point $(\bar{x}; \bar{\xi})$ but have cone support in any conic neighbourhood

157.259
$$C_\delta = \{(x,\xi) \in \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}; |x - \bar{x}| < \delta,\ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| < \sigma\},\ \delta > 0. \tag{4.23}$$

Indeed to arrange this we just need to choose a smooth function on the sphere, ψ_δ which is equal to one in $|\frac{\xi}{|\xi|}-\omega|<\delta/2$, where $\omega=\frac{\bar\xi}{|\bar\xi|}$, is positive on $|\frac{\xi}{|\xi|}-\omega|<\delta/2$ and has support in $|\frac{\xi}{|\xi|}-\omega|\le\delta$ in terms of the distance on the sphere. Similarly choose $\psi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^N)$ positive on $|x|<1$, equal to 1 in $|x|<\frac12$ and supported in $|x|\le1$ and a similar cutoff $\mu\in\mathcal{C}^\infty(\mathbb{R}^n)$ and consider

157.260 (4.24) $$c_\delta(x,\xi)=(1-\mu(\delta\xi))\psi(\frac{x-\bar x}{\delta})\psi_\delta(\frac{\xi}{|\xi|})\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^N;S^0(\mathbb{R}^n)).$$

157.261 LEMMA 4.1. *In terms of* (4.23) *the 'symbolic cutoff'* χ_δ *in* (4.24) *has*

157.262 (4.25) $$\mathrm{Ell}(c_\delta)\supset C_{\delta'},\ \forall\ \delta'<\delta$$

(4.26) $$\mathrm{conesupp}(c_\delta)\subset\overline{C_\delta}\subset C_{\delta''}\ \forall\ \delta''>\delta$$

157.265 (4.27) $$\mathrm{conesupp}(1-c_\delta)\cap C_{\delta'/2}=\emptyset\ \forall\ \delta'<\delta.$$

PROOF. Inspection. □

157.257 LEMMA 4.2. *If* $a\in\mathcal{C}^\infty(\mathbb{R}^N;S^m(\mathbb{R}^n))$ *and* $(\bar x,\bar\xi)\in\mathrm{Ell}(a)$ *then there exists* $b\in\mathcal{C}^\infty(\mathbb{R}^N;S^{-m}(\mathbb{R}^n))$ *such that*

157.258 (4.28) $$(\bar x,\bar\xi)\notin\mathrm{conesupp}(ab-1).$$

PROOF. Take a symbolic cut-off as in (4.24) and consider

157.263 (4.29) $$b=\frac{\chi_\delta(x,\xi)}{a}.$$

For $\delta>0$ small enough this is well-defined since by the definition of ellipticity in (4.16), $a\neq0$ on the support of $\chi_\delta(x,\xi)$; as usual the quotient is extended as zero outside this support. Then (4.28) follows from (4.27), since $ba=\chi_\delta(x,\xi)$. So it only remains to check that $b\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^N;S^{-m}(\mathbb{R}^n))$. Proceeding inductively

157.264 (4.30) $$\partial_x^\alpha\partial_\xi^\beta b=\frac{g_{\alpha,\beta}}{a^{|\alpha|+|\beta|+1}},\ g_{\alpha,\beta}\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^N;S^{(m-1)|\alpha|+m|\beta|}(\mathbb{R}^n)).$$

This is certainly true for $\alpha=\beta=0$ and the inductive step follows by differentiating again with respect to either variable. □

It is important to note that

157.266 (4.31) $$\mathrm{Ell}_m(a+e)=\mathrm{Ell}_m(a)\text{ if }a\in\mathcal{C}^\infty(\mathbb{R}^N;S^m(\mathbb{R}^n))\quad\text{and }e\in\mathcal{C}^\infty(\mathbb{R}^N;S^{m-\epsilon}(\mathbb{R}^n)),\ \epsilon>0.$$

The same is true for $\mathrm{Char}(a)$ whereas

157.267 (4.32) $$\mathrm{conesupp}(a+e)=\mathrm{conesupp}(a)\text{ if }e\in\mathcal{C}^\infty(\mathbb{R}^N;S^{-\infty}(\mathbb{R}^n)).$$

2. Ellipticity of pseudodifferential operators

We now transfer the these notions from symbols to pseudodifferential operators.

157.268 DEFINITION 4.2. If $A=Q_L(a)\in\Psi^m(\mathbb{R}^n)$, $a\in\mathcal{C}^\infty_\infty(\mathbb{R}^n;S^m(\mathbb{R}^n))$ we set

157.269 (4.33) $$\mathrm{Ell}(A)=\mathrm{Ell}_m(A)=\mathrm{Ell}_m(a),$$

157.270 (4.34) $$\mathrm{Char}(A)=\mathrm{Char}_m(A)=\mathrm{Char}(a),$$

157.271 (4.35) $$\mathrm{WF}'(A)=\mathrm{conesupp}(a).$$

Here $\mathrm{WF}'(A)$ is called the 'operator wavefront set' of A for reasons that should become clearer below – it is just a name.

Recall that we defined the principal sybmol of A to be $\sigma_m(A) = [a]$ to be the equivalence class in $\mathcal{C}^\infty_\infty(\mathbb{R}^n; S^m(\mathbb{R}^n))/\mathcal{C}^\infty_\infty(\mathbb{R}^n; S^{m-1}(\mathbb{R}^n))$ and then it is also equal to the equivalence class of the right-reduced symbol.

157.272 LEMMA 4.3. *The elliptic set only depends on $\sigma_m(A)$ and $\mathrm{WF}'(A)$ depends on a modulo symbols of order $-\infty$ and is also equal to the cone-support of the right reduced symbol; for the product of operators*

157.273
$$\mathrm{Ell}_{m+m'}(AB) = \mathrm{Ell}_m(A) \cap \mathrm{Ell}_{m'}(B),\ \mathrm{WF}'(AB) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B). \tag{4.36}$$

PROOF. The first part follows directly from (4.31) and the fact that left- and right-reduced symbols differ by a term of order $m-1$. The last part is a little more subtle, and depends on the formula for the asymptotic expansion of the right-reduced sybmol in terms of the left-reduced symbol a.

157.274
$$\sum_\alpha \frac{1}{\alpha!} D^\alpha_x \partial^\alpha_\xi a. \tag{4.37}$$

If a is rapidly decreasing in a truncated cone, as (4.20) then all the terms in (4.37) are rapidly decaying in the same cone, because of the locality of differential operators. It follows that any asymptotic sum is rapidly decreasing as well. The final part (4.36) follows similarly. □

Perhaps the most important construction associated to these definitions is 'microlocal invertibility' at elliptic points.

157.277 PROPOSITION 4.1. *If $(\bar{x}, \bar{\xi}) \in \mathrm{Ell}(A)$, $A \in \Psi^m(\mathbb{R}^n)$, there exists $B \in \Psi^{-m}(\mathbb{R}^n)$ such that*

157.278
$$(\bar{x}, \bar{\xi}) \notin \mathrm{WF}'(\mathrm{Id} - AB) \cap \mathrm{WF}'(\mathrm{Id} - BA). \tag{4.38}$$

PROOF. As the notation suggests, we start with

157.279
$$B_0 = Q_L(b),\ b \text{ as in Lemma 4.2.} \tag{4.39}$$

The properties of b mean that

157.281
$$\mathrm{WF}'(B_0) \subset C_\delta = \{(x,\xi) \in \mathbb{R}^n \xi(\mathbb{R}^n \setminus \{0\}); |x - \bar{x}| \le \delta,\ \| \frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|} \le \delta\} \tag{4.40}$$

where we are free to choose $\delta > 0$. Then, from the product formula for symbols,

157.280
$$B_0 A = Q_L(c_\delta)) - E,\ E \in \Psi^{-1}(\mathbb{R}^n),\ \mathrm{WF}'(E) \subset C_\delta. \tag{4.41}$$

Now, we can almost invert $\mathrm{Id} - E$ using the Neumann series. That is we can choose

157.282
$$F \in \Psi^{-1}(\mathbb{R}^n),\ F \sim \sum_{k \ge 1} E^k \Longrightarrow \tag{4.42}$$
$$(\mathrm{Id} + F)(\mathrm{Id} - E) = \mathrm{Id} + E'_L,\ (\mathrm{Id} - E)(\mathrm{Id} + F) = \mathrm{Id} + E'_R,\ E'_L,\ E'_R \in \Psi^{-\infty}(\mathbb{R}^n).$$

Define

157.283
$$B = (\mathrm{Id} + F) B_0 \Longrightarrow \tag{4.43}$$
$$BA = (\mathrm{Id} + F)(Q_L(c_\delta) - E) = \mathrm{Id} + E'_L - (\mathrm{Id} + F)(\mathrm{Id} - (Q_L(c_\delta)) = \mathrm{Id} + E'',$$
$$(\bar{x}, \bar{\xi}) \notin \mathrm{WF}'(E'').$$

Similarly we can proceed on the right,

157.284 (4.44) $$AB_0 = Q_L(c_\delta) - E_R,\ F_r \sim \sum_{k\ge 1} E_R^k$$

and see again that $B_R = B_0(\mathrm{Id} + F_R)$ satisfies

157.285 (4.45) $$AB_R = \mathrm{Id} + E_R'',\ (\bar x, \bar\xi) \notin \mathrm{WF}'(E_R'').$$

Now, it is a form of the argument which gives the 'uniqueness of the inverse in a group' to see that

157.286 (4.46) $$\begin{gathered} B = BAB_R + S_1 = B_R + S_1 - S_2, \\ S_1 = B(\mathrm{Id} - AB_R),\ S_2 = (\mathrm{Id} - BA) \text{ so } (\bar x, \bar\xi) \notin \mathrm{WF}'(S_i),\ i = 1, 2. \end{gathered}$$

It follows that $(\bar x, \bar\xi) \notin \mathrm{WF}'(B_L - B_R)$ so B also satisfies (157.285 4.45) and (157.278 4.38). □

157.347 Exercise 8. [Microlocal partition of unity] Suppose that $K \Subset \mathbb{R}^n \times \mathbb{S}^{n-1}$ and $U_a \subset \mathbb{R}^n \times \mathbb{S}^{n-1}$, $a \in A$, is an open cover of K then there exist operators $A_i \in \Psi^0(\mathbb{R}^n)$, $i = 1, \dots, N$ such that, in terms of the cones in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$,

(4.47) $$\mathrm{WF}'(\mathrm{Id} - \sum_i A_i) \cap \mathbb{R}^+ K = \emptyset,\ \mathrm{WF}'(A_i) \subset \mathbb{R}^+ U_{a_i} \text{ for some } a_i \in A.$$

S.WF 3. Wavefront set of a distribution

Now we are in a position to define the wavefront set (or wavefrontset) of a distribution on $\mathbb{R}^n$. First let's work with compactly supported distributions and then pass to the general case.

157.275 Definition 4.3. If $u \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$ then

157.276 (4.48) $$\begin{gathered} \mathrm{WF}(u) = \big\{(\bar x, \bar\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}); \\ \exists\ A \in \Psi^m(\mathbb{R}^n),\ Au \in \mathcal{S}(\mathbb{R}^n),\ (\bar x, \bar\xi) \in \mathrm{Ell}_m(A)\big\}^{\complement}. \end{gathered}$$

We can characterize the wavefront set of a distribution in a more elemenatary way.

157.287 Proposition 4.2. *For* $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *and* $(\bar x, \bar\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$

157.288 (4.49) $$\begin{gathered} (\bar x, \bar\xi) \notin \mathrm{WF}(u) \Longleftrightarrow \\ \exists\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n),\ \phi(\bar x) \neq 0,\ \psi \in \mathcal{C}_c^\infty(\mathbb{S}^{n-1}),\ \psi(\frac{\bar\xi}{|\bar\xi|}) \neq 0 \ s.t. \\ |\psi(\frac{\xi}{|\xi|})\mathcal{F}(\phi u)| \le C_N(1 + |\xi|)^{-N}\ \forall\ N. \end{gathered}$$

Proof. One way is straightforward, the other way depends on the construction of microlocal invserses as above.

First, assume the right side of (157.288 4.49) holds – for some ϕ and ψ as indicated. Then we can choose another cut-off $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$ around zero in $\mathbb{R}^n$ and conclude that

157.289 (4.50) $$(1 - \chi(\xi))\psi(\frac{\xi}{|\xi|})\mathcal{F}(\phi u)$$

is also rapidly decreasing. Now

157.290 (4.51) $$a = (1-\chi(\xi))\psi(\frac{\xi}{|\xi|})\phi(x) \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n; S^0(\mathbb{R}^n))$$

and if $A = Q_R(a)$ then

157.291 (4.52) $$\mathcal{F}Au = (1-\chi(\xi))\psi(\frac{\xi}{|\xi|})\mathcal{F}(\phi u)$$

by definition of right quantization. If follows that

157.292 (4.53) $$Au \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow (\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$$

since A is elliptic at $(\bar{x}, \bar{\xi})$.

Remark. Edited by Paige Dote.

For the opposite implication, we suppose that $u \in C_c^{-\infty}(\mathbb{R}^n)$ and $A \in \Psi^{-m}(\mathbb{R}^n)$ with $(\overline{x}, \overline{\xi}) \in \mathrm{Ell}_m(A)$ are such that $Au \in \mathcal{S}(\mathbb{R}^n)$. Choose $B \in \Psi^{-n}(\mathbb{R}^n)$ as in Proposition 4.1, so $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}'(\mathrm{Id} - BA)$. It follows from the second part of (4.36) that, with c_δ as in (4.24) (so ψ is supported near $\overline{x}$ and ψ near $\overline{\xi}$) and $C_\delta = Q_L(c_\delta)$) for $\delta > 0$ sufficiently small,

$$\mathrm{WF}'(C_\delta) \cap \mathrm{WF}'(\mathrm{Id} - BA) = \emptyset.$$

Then,

$$(C_\delta BA)u = C_\delta u \mod \mathcal{S}(\mathbb{R}^n),$$

and hence, $C_\delta u \in \mathcal{S}(\mathbb{R}^n)$ which implies the condition on the right in (4.37). □

We can also see that 'pseudodifferential operators are microlocal' and combine it with 'microlocal elliptic regularity' which is a partial inverse

157.293 Proposition 4.3. *For any $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and $A \in \Psi^m(\mathbb{R}^n)$*

157.294 (4.54) $$\mathrm{WF}(u) \subset \mathrm{Char}(A) \cup \mathrm{WF}(Au),\ \mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u) \Longrightarrow \mathrm{WF}(u) \cap \mathrm{Ell}(A) = \mathrm{WF}(Au) \cap \mathrm{Ell}(A)$$

We have used the fact that $A \in \Psi^m(\mathbb{R}^n)$ and $\mathrm{WF}'(A) = \emptyset \implies Au \in \mathcal{S}(\mathbb{R}^n)$ if $u \in C_c^{-\infty}(\mathbb{R}^n)$. Note that it is <u>not</u> true that $\mathrm{WF}'(u) = \emptyset$ and $u \in \mathcal{S}'(\mathbb{R}^n) \implies AU \in \mathcal{S}(\mathbb{R}^n)$. This is one of the 'defects' of the algebra $\Psi^*(\mathbb{R}^n)$.

Proof. The first statement is equivalent to saying that

$$(\overline{x}, \overline{\xi}) \in \mathrm{Ell}_m(A),\ (\overline{x}, \overline{\xi}) \notin \mathrm{WF}(Au) \implies (\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u).$$

Again, we use the construction in Proposition ?? to find $B \in \Psi^{-m}(\mathbb{R}^n)$ with $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}'(\mathrm{Id} - BA)$. We can choose the cone-support of the symbol of B such that $B(Au) \in \mathcal{S}(\mathbb{R}^n)$m abd then it follows that $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$. □

So far I have limited the definition of $\mathrm{WF}(u)$ to elements of $\mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$. This is only because $\Psi^m(\mathbb{R}^n)$ does not act on general distributions in $\mathcal{C}^{-\infty}(\mathbb{R}^n)$. To overcome this, we consider a smaller algebra consisting of the properly-supported pseudodifferential operators.

A closed set $S \subset \mathbb{R}^{2n}$ can be considered as a relation between subsets of $\mathbb{R}^n$

$$S \circ U = \pi_L(S \cap \pi_R^{-1}(U))$$

where $\pi_L ; \pi_R : \mathbb{R}^{2n} \to \mathbb{R}^n$ are the two projections

$$\pi_R(x, y) = y \text{ and } \pi_L(x, y) = x.$$

I am thinking here of the support of the kernel of an operator. Thus,

$$x \in S \circ U \iff \exists (x, y) \in S \text{ such that } y \in U.$$

The relation defined by S is said to be *proper* if

$$S \circ K \Subset \mathbb{R}^n \ \forall K \Subset \mathbb{R}^n$$

i.e. it takes compact sets to compact sets.

Definition 4.4 (Properly Supported). A pseudodifferential operator (or indeed any operator) is said to be properly supported if the support of its kernel and the kernel of its adjoint define proper relations.

Lemma 4.4. *The properly supported pseudodifferential operators, denoted $\Psi^m_P(\mathbb{R}^n)$ form a $*$-closed filtered ring defining linear maps on $\mathcal{C}^\infty_c(\mathbb{R}^n)$, $\mathcal{C}^\infty(\mathbb{R}^n)$, $CmIc(\mathbb{R}^n)$, and $\mathcal{C}^{-\infty}(\mathbb{R}^n)$. We further have*

157.494 (4.55)
$$\Psi^m(\mathbb{R}^n) = \Psi^m_p(\mathbb{R}^n) + \Psi^{-\infty}(\mathbb{R}^n).$$

Proof. Let $\varphi \in C^\infty_c(\mathbb{R}^n)$ be a cutoff near 0, say $\varphi(z) = 1$ in $|z| \le 1$. Then,

$$(1 - \varphi(x - y))K(x, y)$$

is the kernel of an element of $\Psi^{-\infty}(\mathbb{R}^n)$ as follows from (??). This proves the equation (1). The relationship of operators and kernels show that

$$Au(x) = \int A(x, y)u(y)dy, \ u \in \mathcal{C}^\infty_c(\mathbb{R}^n$$

vanishes if $A(x, y)u(y)$ vanishes (??). If A has proper support $S = \operatorname{supp}(A)$, then $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp}(\psi) \cap (S \cdot \operatorname{supp}(u)) = \emptyset$ satisfies

$$\psi(x)A(x, y)u(y) = 0 \implies \operatorname{supp}(\psi) \cap \operatorname{supp}(Au) = \emptyset.$$

Thus, $\operatorname{supp}(Au)$ is compact if

$$A : \mathcal{C}^\infty_c(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R}^n).$$

We have assumed the same for the adjoint (and hence the transpose) from which the remaining properties follow by duality. Then,

$$\mathrm{WF}(u) \not\ni (\bar{x}, \bar{\xi}) \text{ if } \exists \ A \in \Psi^m_{\mathrm{p}}(\mathbb{R}^n), \ (\bar{x}, \bar{\xi}) \in \mathrm{Ell}_m(A), \ Au \in \mathcal{C}^\infty(\mathbb{R}^n).$$

It follows that this is consistent with the properties ** defined when $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$. $\square$

One of the important properties if the wavefront set is that it is a refinement of the singular support.

Lemma 4.5. *If $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ then*

$$\bar{x} \notin \operatorname{singsupp}(u) \iff (\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u) \forall \ \bar{\xi} \in \mathbb{R}^n \setminus \{0\}.$$

Proof. In the foward direction, this is immediate, since $\bar{x} \notin \operatorname{singsupp} u$ implies $\exists \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\varphi u \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. As an element of $\Psi^0_{\mathrm{P}}(\mathbb{R}^n)$, this is elliptic at all points $(\bar{x}, \bar{\xi})$ for any $\bar{\xi} \in \mathbb{R}^n \setminus \{0\}$.

For the opposite implication, we need a covering argument. Certainly, for each $\bar{\xi} \ne 0$, by **, there exists $\varphi_{\bar{\xi}} \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\widehat{\varphi_\xi u}$ is rapidly decreasing in a cone around $\overline{\xi}$ with $\varphi_{\overline{\xi}}(\overline{x}) \neq 0$. In fact, we know from Proposition that this remains true if we replace $\varphi_{\overline{\xi}}$ by $\varphi\varphi_{\overline{\xi}}$ for a fixed $\varphi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ around $\overline{\xi}$ with $\varphi(\overline{x}) \neq 0$. The cone of rapid decay does *not* decrease.

Now $\mathbb{S}^{n-1}$ is compact and the cones of rapid decay of the $\widehat{\varphi_{\overline{\xi}} u}$ correspond to an open cone of $\mathbb{S}^{2n-1}$. This has a finite subcover and it follows that we may choose $\varphi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ with $\varphi(\overline{x}) \neq 0$ and $\operatorname{supp}(\varphi) \subset \{\varphi_{\overline{\xi_j}} = 1\}$ for this finite cover. Then $\varphi\varphi_{\overline{\xi_j}} = \varphi$ and $\widehat{\varphi u}$ is rapidly decreasing in all directions of $\mathbb{R}^n$, so $\varphi u \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and hence $\overline{x} \notin \operatorname{singsupp} u$. □

REMARK 3. The scatting and isotropic algebras do have properly supported subalgebras, but the analogue of (157.494 4.55) is not valid in these cases. We will see why later.

Recall that we have insisted that $\mathrm{WF}(u)$ is a closed cone in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. The results above show that this condition in in $\mathbb{R}^{2n}$ is

$$\overline{\mathrm{WF}(u)} = (\operatorname{singsupp}(u) \times \{0\}) \cup \mathrm{WF}(u),$$

which gives a nice picture! L6-end

CHAPTER 5

Propagation of singularities

Microlocal ellipticity, as discussed above, shows us that if $u \in \mathcal{S}'(\mathbb{R}^n)$ and L7
$A \in \Psi^m(\mathbb{R}^n)$ then

157.297 $$\text{(5.1)} \qquad \mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \mathrm{Char}(A).$$

In particular if $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$ then $\mathrm{WF}(u) \subset \mathrm{Char}(A)$.

To see what else we might be able to say, consider a simple case, where $A = D_1$ is differention with respect to the first variable. Letting $p = \xi_1$ be the principal symbol we see that

157.298 $$\text{(5.2)} \qquad \mathrm{Char}(D_1) = \{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}); \xi_1 = 0\}$$

is a smooth hypersurface. Any solution

157.299 $$\text{(5.3)} \qquad D_1 u = 0 \Longrightarrow u(x,x') = v(x')$$

is independent of x_1, meaning that

157.300 $$\text{(5.4)} \qquad u(\phi) = v(\psi),\ \psi(x') = \int \phi(x_1,x')dx_1.$$

We already know that $\mathrm{WF}(u) \subset \{\xi_1 = 0\}$ but more is true. Namely

157.301 $$\text{(5.5)} \qquad \mathrm{WF}(u) = \{(x_1,x',0,\xi'); (x',\xi') \in \mathrm{WF}(v)\}.$$

So the wavefront set of u is a union of lines where x_1 alone varies. These are of course the integral curves of ∂_{x_1} but as a vector field on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and only those inside $\{\xi_1 = 0\}$.

1. Hamiltonian mechanics

As we shall see, something like (157.301 5.5) can be proved much more generally. To do so we use *commutator methods*. The idea here is that we try to get information about solutions of $Pu = 0$ where $P \in \Psi^m(\mathbb{R}^n)$ by taking a commutator with a 'test operator' $B \in \Psi^k(\mathbb{R}^n)$. I will explain this a bit more fully below, but for the moment just recall that

157.302 $$\text{(5.6)} \quad P, \in \Psi^m(\mathbb{R}^n),\ B \in \Psi^k(\mathbb{R}^n) \Longrightarrow [P,B] \in \Psi^{m+k-1}(\mathbb{R}^n) \text{ and}$$
$$\sigma_{m+k-1}([P,B]) = -i\sum_{i=1}^n \left(\frac{\partial p}{\partial \xi_i}\frac{\partial b}{\partial x_i} - \frac{\partial p}{\partial x_i}\frac{\partial b}{\partial \xi_i}\right)$$

Here p and b are really representatives of the principal symbols of P and B and then we are looking at a representative of the symbol of the commutator, all modulo a term of one order lower. In fact we will assume that

157.303 $$\text{(5.7)} \qquad P \in \Psi^m_{\mathrm{cl}}(\mathbb{R}^n),\ B \in \Psi^k_{\mathrm{cl}}(\mathbb{R}^n),\ \sigma_m(P) = p,\ \sigma_k(B) = b$$

where these are now homogeneous functions of orders m and k and we just ignore the singularities at $\xi = 0$, meaing we work on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}$. Then 157.302 (5.6) gives the homogeneous principal symbol of $[P, B]$.

Thw important point is that we can 'recognize' the formula, it is the Poisson bracket of the symbols (ignoring the i). As you will know this comes from the symplectic form on $\mathbb{R}^n \times \mathbb{R}^n = T^*\mathbb{R}^n$

157.304 (5.8)
$$\omega = \sum_i d\xi_i \wedge dx_i$$

which is actually completely independent of coordinates (which we will return to later). Then the *Hamilton vector field* of p is by definition the unique vector field H_p on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ (because p might be singular at $\xi = 0$) satifying

157.305 (5.9)
$$\omega(\cdot, H_p) = dp \Longleftrightarrow H_p = \sum_i \left(\frac{\partial p}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial p}{\partial x_i} \frac{\partial}{\partial \xi_i} \right)$$

and the Poisson bracket of two functions is

157.306 (5.10)
$$\{p, b\} = H_p b = -H_b p.$$

That is,

157.307 (5.11)
$$\sigma_{m+k-1}([P, B]) = -i\{p, b\} = -iH_p b.$$

One of the basic points about Hamiltonian mechanics is that

157.310 (5.12)
$$H_p p = 0$$

which follows from the antisymmetry of ω. This means that

157.311 LEMMA 5.1. *Integral curves of H_p which have a point in $\{p = 0\}$ are contained in $\{p = 0\}$.*

Here of course an integral curve is connected, it is a smooth curve defined on some interval, $I \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \setminus \{0)$ which has tangent vector H_p at each point.

2. Hörmander's Theorem

157.308 THEOREM* 5.1. *If $P \in \Psi^m_{\mathrm{cl}}(\mathbb{R}^n)$ has real principal symbol then for any $u \in \mathcal{S}'(\mathbb{R}^n)$*

157.309 (5.13) $\mathrm{WF}(u) \setminus \mathrm{Char}(p)$

is a union of maximally extended integral curves of H_p in $\mathrm{Char}(P) \setminus \mathrm{WF}(Pu)$.

157.312 EXERCISE 9. Try to see what this says for the flat wave operator

157.313 (5.14)
$$P = D_t^2 - \sum_{j=1}^{b} D_{x_j}^2 \text{ on } \mathbb{R}^{n+1}.$$

Check that the characteristic variety at each point is the cone $\tau^2 = |\xi|^2$ (with the obvious notation) and that the integral curves of H_p within $\mathrm{Char}(P)$ project into $\mathbb{R}^n$ to light rays.

Restated the claim of the Theorem is that a maximally extended integral curve of H_p in the open set $(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \setminus \mathrm{WF}(Pu)$ on which p vanishes either does not meet $\mathrm{WF}(u)$ or lies completely within it.

Proof. We start can by making some elementary reductions before getting to the constructive part of the proof. First note that if p is homogeneous (always meaning in ξ of course) of degree m then H_p is homogeneous of degree $m-1$. If we premultiply P by an elliptic operators, setting $P' = AP$ where $A = (1+|d|^2)^{\frac{-m+1}{2}}$, then

157.315
$$p' = \sigma(P') = (1+|\xi|^2)^{\frac{-m+1}{2}} p,\ H_{p'} = aH_p + pH_a. \tag{5.15}$$

It follows that the integral curves of $h_{p'}$ in $\mathrm{Char}(P') = \mathrm{Char}(P)$ are simply reparameterizations of the integral curves of H_p. Since A is globally elliptic, $\mathrm{WF}(P'u) = \mathrm{WF}(Pu)$ and The statement for P is equivalent to that for P'. Thus we can freely assume that P is of order 1.

A smooth vector field such as H_p has two types of maximally extended integral curves in an open set (of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Namely constant curves valued at points where $H_p = 0$ and through every other point an embedded curve, without stationary points. This dichotomy corresponds to the vanishing of dp. For a constant curve the statement is of course trivial. Thus in fact Hörmander's Theorem tells us nothing about points where

157.316
$$p(\bar{x},\bar{\xi}) = 0,\ dp(\bar{x},\bar{\xi}) = 0. \tag{5.16}$$

There is in fact another trivial case beyond (5.16). Namely a point is *radial* for p (and also P) if

157.317
$$p(\bar{x},\bar{\xi}) = 0,\ dp(\bar{x},\bar{\xi}) = \lambda\bar{\xi}\cdot dc,\ \lambda \in \mathbb{R}. \tag{5.17}$$

Of course (5.16) is the case where $\lambda = 0$. From (5.9) we see that at such a point

157.318
$$H_p = -\lambda\xi\cdot\partial_\xi \tag{5.18}$$

is the radial vector field – hence the name. Since p is homogenous of degree 1 the identiy (5.18) must then hold along the ray $(\bar{x}, \mathbb{R}^+\bar{\xi})$ – which is then the integral curve through $(\bar{x},\bar{\xi})$. Again the statement is trivial for the this integral curve, since $\mathrm{WF}(u)$ is itself radial.

3. The Hamilton vector field

Thus, we are reduced to considering a ray on which p vanishes, maximally extended in $(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \setminus \mathrm{WF}(Pu)$ and through a non-radial point $(\bar{x},\bar{\xi})$. By the local uniqueness of integral curves it must consist of non-radial points. Let the maximally extended integral curve, so defined on an open (possibly infinite) interval be

157.319
$$\chi : I \longrightarrow \{p = 0\} \setminus \mathrm{WF}(Pu),\ \chi(0) = (\bar{x},\bar{\xi}). \tag{5.19}$$

So we need to show is that

157.320
$$\chi(I) \not\subset \mathrm{WF}(u) \Longrightarrow \chi(I) \cap \mathrm{WF}(u) = \emptyset. \tag{5.20}$$

Since $\{t \in I; \chi(t) \in \mathrm{WF}(u)\}$ is closed its complement is a countable union of open intervals. Considering one of these intervals, either it is equal to I of, if not, has at least one endpoint $\bar{t} \in I$ with $\chi(\bar{t}) \in \mathrm{WF}(u)$. Reversing the sign of P does not change the result, so all we have to exclude is that there is some $\bar{t} \in I$ with $\chi(\bar{t}) \in \mathrm{WF}(u)$ but $\chi(t) \notin \mathrm{WF}(u)$ for $t \in (\bar{t}, \bar{t}+\epsilon)$, $\epsilon > 0$. Clearly we can shift the parameterization so that $\bar{t} = 0$.

We are therefore reduced to a (micro-)local statement.

If $(\bar{x}, \bar{\xi}) \in (\mathbb{R}^n \setminus \{0\})$ is a non-radial point for p, $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(Pu)$ and the integral curve of H_p starting at $(\bar{x}, \bar{\xi})$ immediately leaves $\mathrm{WF}(u)$ then $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$.

Q

To prove this we need to look closely at H_p. First we can choose local coordinates near $\bar{x}$ so that $\bar{x} = 0$ $\bar{\xi} = (0, \dots, 0, 1)$. [Despite appearances we will not be using the coordinate-invariance of pseudodifferential operatos here.] By the non-radial assumption $H_p \neq \lambda \partial_{\xi_n}$ since that is the radial direction. So at least one of the other coefficients is non-zero at this point, and hence nearby, it could be any one of the x_i or one of the ξ_j, $j < n$. Denote this special variable ζ, we can solve the initial value problem

157.321 (5.21)
$$\begin{aligned} &H_p y_j = 0, \ j = 1, \dots, 2(n-1), \\ &H_p \Xi = 0, \ \Xi\big|_{\zeta=0} = \xi_n, \\ &H_p \tau = \tau, \ \tau\big|_{\zeta=0} = 0, \end{aligned} \tag{5.21}$$

where the initial conditions on $\{\zeta = 0\}$ for the y_j are all the x_i and ξ_k/ξ_n, $k < n$ except the one that corresponds to ζ (which of course vanishes).

All the solutions exist in an open conic (because H_p is homogeneous of degree 0) neighbourhood of the base point and are homogenous of degree 0, except Ξ which is positive and homogeneous of degree 1. Their differentials are indpendent so they give a coordinate system in terms of which

157.322 (5.22)
$$H_p = \partial_\tau. \tag{5.22}$$

[In fact you can do essentially this with a homogeneous symplectic ('canonical') transformation but we do not need it.]

L-end
L9

4. Construction of symbols

This allows us to construct appropriate classical symbols. What we want is a cut-off near the base point with useful properties, in fact a family of them depending on a parameter $0 < \delta < \delta_0$. The basic function we have in mind is

157.323 (5.23)
$$a_\delta(\tau, y) = \mu(\tau/\delta)\mu((\epsilon/2 - \tau)/\delta)\phi(y/\delta). \tag{5.23}$$

Here $0 \leq \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{2n-2})$ is a typical cut-off, supported in $|\tau| \leq 1$ and strictly positive in $|\tau| < 1$ and $0 \leq \mu \in \mathcal{C}^\infty(\mathbb{R})$ vanishes in $(-\infty, -1)$, is positive on $(-1, \infty)$ and is equal to 1 on $[0, \infty)$. We require

157.362 (5.24)
$$0 < \delta < \epsilon/2. \tag{5.24}$$

Then

157.332 (5.25)
$$\begin{gathered} \mathrm{supp}(a_\delta) = \{(\tau, y); -\delta \leq \tau \leq \epsilon/2 + \delta, \ |y| \leq \delta\}, \\ a_\delta > 0 \text{ on } \mathrm{supp}(a_{\delta'}), \ \delta' > \delta. \end{gathered} \tag{5.25}$$

Then consider

157.324 (5.26)
$$b_\delta(\tau, y) = \mu((\epsilon/2 - \tau)/\delta)^2 \int_{-\delta}^{\tau} (\mu(s/\delta)\phi(y/\delta))^2 ds \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{2n-1}) \tag{5.26}$$

where the missing variable is Ξ. Then

157.325 (5.27)
$$\mathrm{supp}(b_\delta) = [-\delta, \epsilon/2 + \delta] \times \{|y| \leq \delta\} \tag{5.27}$$

and

157.326 (5.28)
$$H_p b_\delta = \partial_\tau b = a_\delta^2 + e_\delta, \ \mathrm{supp}(e_\delta) \subset [\epsilon/2, \epsilon] \times \{|y| \leq \delta\}. \tag{5.28}$$

Here e comes from $\partial_\mu((\epsilon/2-\tau)/\delta)^2$.

Thus b_δ is homogeneous of degree 0 on $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$. We want to consider similar functions homogeneous of degree $2m$, for any m and also 'regularized'. Since $\partial_\tau\Xi=0$ we set

157.327 (5.29) $$b_{\delta,m,N}(\tau,y,\Xi)=(1-\Phi(\Xi))^2\Xi^{2m}\Phi^2(\Xi/N)b_\delta(\tau,y),$$
$$\text{where } 0\le\Phi\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}),\ \Phi(s)=1\text{ in }|s|<\frac12,\ \Phi(s)=0\text{ in }|s|>1\Longrightarrow$$
$$H_pb_{\delta,m,N}=a^2_{\delta,m,N}+e_{\delta,m,N},$$
$$a_{\delta,M,n}=(1-\Phi(\Xi))\Xi^m\Phi(\Xi/N)a_\delta,\ e_{\delta,m,N}=(1-\Phi(\Xi))^2\Xi^{2m}\Phi^2(\Xi/N)e_\delta.$$

In fact we need to go a step further. Namely rather than the vector field H_p what actually arises below is the operator with a zeroth order term

157.328 (5.30) $$H_p+f,\ f\text{ smooth, real-valued and homogenous of degree 0.}$$

Then we replace b_δ by

157.329 (5.31) $$b_\delta(\tau,y,\Xi)=\psi(\tau)^2e^{-q(\tau,y)}\int_{-\delta}^{\tau}e^{q(s,y)}a^2_\delta(s,y)ds$$
$$q(\tau,y)=\int_0^\tau f(s,y)ds.$$

The support properties are unchanged and now

157.330 (5.32) $$(H_p+f)b_{\delta,m,N}(\tau,y,\Xi)=a^2_{\delta,m,N}+e_{\delta,m,N}.$$

5. Proof of regularity

After all this preparation, let me add a few words about L^2 boundedness – which is in Problem set 2 – and uniformity before passing to the actual proof of the statement above..

First uniformity. In the proof below, regularity for Au where $A\in\Psi^m(\mathbb{R}^n)$ is obtained by looking at an approximating sequence $A_N\in\Psi^{-\infty}(\mathbb{R}^n)$ which is bounded in $\Psi^m(\mathbb{R}^n)$ and converges to A in $\Psi^{m+\epsilon}(\mathbb{R}^n)$ for any $\epsilon>0$. This is constructed usual sort of cutoff. Since $\mathrm{WF}'(A_N)=\emptyset$ for finite N we reserve the notation for the *uniform* operator wavefront set defined as previously in terms of the cone-support of the symbol. So in this sense

157.342 (5.33) $$(\bar x,\bar\xi)\notin\mathrm{WF}'(A_*)=Q_L(a_*)\Longleftrightarrow(\bar x,\bar\xi)\notin\mathrm{conesupp}(a_*)\Longleftrightarrow$$
$$c_\delta a_N\text{ is bounded in }\mathcal{C}^\infty_\infty(\mathbb{R}^n;S^{-\infty}(\mathbb{R}^n))\text{ as }N\to\infty.$$

Here c_δ is a conic cut-off as in (157.260 4.24). It follows for instance that for two such families

157.343 (5.34)
$$\mathrm{WF}'(A_*)\cap\mathrm{WF}'(B_*)=\emptyset\Longrightarrow A_*A'_*\psi\in\Psi^{-\infty}(\mathbb{R}^n)\text{ is bounded for }\omega\in\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n).$$

So for instance $A_*A'_*u$ is bounded in $\mathcal{S}(\mathbb{R}^n)$ if $u\in\mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$. Note that we do not necessarily conclude that the product is bounded in $\Psi^{-\infty}(\mathbb{R}^n)$ since we have not assumed any uniformity near infinity is space.

The basic result on L^2 boundedness is that

157.344 (5.35) $$A\in\Psi^m(\mathbb{R}^n):H^M(\mathbb{R}^n)\longrightarrow H^{M-m}(\mathbb{R}^n)\ \forall\ M.$$

In the argument below we need a little more than this. Namely for sequences A_N as above bounded in $\Psi^{2m}(\mathbb{R}^n)$ if $\tilde{A} \in \Psi^m(\mathbb{R}^n)$ is fixed and if

157.345 (5.36)
$$\operatorname{WF}'(A_*) \subset \operatorname{Ell}_m(\tilde{A}) \text{ then } |\langle \psi A_* u, u\rangle_{L^2}| \le C\|\tilde{A}u\|_{L^2} + C\|u\|_{H^{m_0}},$$
$$\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \cap H^{m_0}(\mathbb{R}^n)$$

where the constants may depend on everything except N and u.

157.365 LEMMA 5.2. *If $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and $A_N u$ is bounded in $L^2(\mathbb{R}^n)$ then $Au \in L^2(\mathbb{R}^n)$ and $\|Au\|_{L^2} \le \limsup \|A_N u\|_{L^2}$.*

PROOF. This follows from the fact that $L^2(\mathbb{R}^n)$ is a Hilbert space. Thus the norm boundedness of $A_N u$ implies that it has a weakly convergent subsequence $A_N u \hookrightarrow v$ in L^2. It follows that

157.366 (5.37)
$$\langle Au, \phi\rangle = \lim \langle A_{N_i} u, \phi\rangle = \langle v, \phi\rangle$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ and hence $Au = v$ as a distribution. □

So, to the proof. We are in the setup discussed above, $(\bar{x}, \bar{\xi})$ is a non-radial characteristic point, $(\bar{x}, \bar{\xi}) \notin \operatorname{WF}(Pu)$ and $\exp(tH_p)(\bar{x}, \bar{\xi}) \notin \operatorname{WF}(u)$ for $t \in (0, \epsilon)$ for some $\epsilon > 0$. We can always shrink ϵ. In the local coordinates in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ we choose $0 < \delta_0 < \epsilon/2$ such that

157.367 (5.38)
$$[-\delta_0, \epsilon] \times \{|y| \le \delta_0\} \cap \operatorname{WF}(Pu) = \emptyset,\ [\epsilon/2, \epsilon] \times \{|y| \le \delta_0\} \cap \operatorname{WF}(u) = \emptyset.$$

We proceed to conclude from this that

157.368 (5.39)
$$A_{m,\delta} u \in L^2(\mathbb{R}^n)\ \forall\ \delta < \delta_0,\ \forall\ m \Longrightarrow A_0 u \in \mathcal{C}^\infty(\mathbb{R}^n) \Longrightarrow (\bar{x}, \bar{\xi}) \notin \operatorname{WF}(u)$$

since A_0 is elliptic at $(\bar{x}, \bar{\xi})$.

Set $B_{\delta,m,N} = Q_L(b_{\delta,m,N}(\tau, y, \Xi))$ which is uniformly bounded in $\Psi^{2m}(\mathbb{R}^n)$ and the formal adjoint is $B^*_{\delta,m,N} = B_{\delta,m,N} + S_{\delta,m,N}$ where $S_{\delta,m,N}$ is uniformly bounded in $\Psi^{2m-1}(\mathbb{R}^n)$. Consider the L^2 inner product

157.331 (5.40)
$$\langle Pu, B_{\delta,m,N} u\rangle_{L^2}.$$

This is well-defined since $B_{\delta,m,N}$ is a smoothing operator. As $N \to \infty$ the cone-support of $b_{\delta,m,N}$ is contained in the region in 157.367 (5.38) where Pu has no wavefront set. Thus

157.333 (5.41)
$$B_{\delta,m,N} Pu \longrightarrow B_{\delta,m} Pu \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n).$$

We use similar notation for the other families below.

Then

157.334 (5.42)
$$\begin{aligned} &2 \operatorname{Re}(i\langle Pu, B_{\delta,m,N} u\rangle_{L^2}) \\ &\qquad = i\langle Pu, B_{\delta,m,N} u\rangle_{L^2} - i\langle B_{\delta,m,N} u, Pu\rangle_{L^2} \\ &= \operatorname{Re} i\langle([B_{\delta,m,N}, P] + B_{\delta,m,N} F\rangle_{L^2} + i\langle(S_{\delta,m,N})P)u, u\rangle_{L^2},\ F = (P - P^*) \end{aligned}$$

The last term here converges as $N \to \infty$ for the same reason as 157.333 (5.41). So we conclude that for any m,

157.335 (5.43)
$$i\langle([B_{\delta,m,N}, P] + B_{\delta,m,N} F\rangle_{L^2} \text{ converges as } N \to \infty.$$

Taking f to be the principal symbol of $F = i(P - P^*) \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^n)$ the construction of the family $B_{\delta,m,N}$ means that

157.336

$$(5.44)\quad \begin{gathered} i[B_{\delta,m,N}, P] + B_{\delta,m,N}F = A^2_{\delta,m,N} + E'_{\delta,m,N} + E_{\delta,m,N}, \\ E'_{\delta,m,N} \text{ bounded in } \Psi^{2m-2}(\mathbb{R}^n), \\ \mathrm{WF}'(E') \subset \mathrm{conesupp}(b_\delta),\ \mathrm{WF}'(E) \subset \mathrm{conesupp}(e) \end{gathered}$$

with e the error in (5.32). From the L^2 bounds in (5.42) we deduce that

157.337

$$(5.45)\qquad \|A_{\delta,m,N}u\|_{L^2} \le C|\langle E'_{\delta,m,N}u, u\rangle_{L^2}| + C|\langle E_{\delta,m,N}u, u\rangle_{L^2}|.$$

Such a bound holds uniformly in N and in $0 < \delta < \delta_0 < \frac{1}{2}\epsilon$. By assumption the last term is uniformly bounded since $\mathrm{WF}'(E_{\delta,m,N})$ is (uniformly) contained in a region disjoint from $\mathrm{WF}(u)$.

By L^2 boundedness we know that

157.338

$$(5.46)\qquad |\langle E'_{\delta,m,N}u, u\rangle_{L^2}| \le \|u\|_{H^{m-\frac{1}{2}}}$$

Since u is a distribution and the supports here are compact, $u \in H^{m_0-\frac{1}{2}}(\mathbb{R}^n)$ for some m_0. It follows that the limit

157.339

$$(5.47)\qquad A_{\delta,m_0}u \in L^2(\mathbb{R}^n)$$

using Lemma 5.2.

Now, we can iterate this estimate, half a derivative at a time. The important point is that we know (5.45) for all $\delta < \delta_0$. So at some stage of the iteration we know that (5.47) holds for a given $m - \frac{1}{2}$. However, the cone-supports of the terms A and the lower order error E' are such that

157.340

$$(5.48)\qquad \text{for } \delta < \delta',\ \mathrm{WF}'(E'_{\delta,m,N}) \subset \mathrm{Ell}_m(A_{\delta',m-\frac{1}{2}}).$$

Again from L^2 boundedness and ellipticity (and the fact that the order of $E'_{\delta,m,N}$ is the same as $A^2_{\delta,m-\frac{1}{2}}$) it follows iteratively that

157.341

$$(5.49)\quad \begin{aligned} \|A_{\delta,m,N}u|^2_{L^2} &\le \langle E'_{\delta,m,N}u, u\rangle_{L^2}| + C \\ &\le C_{\delta,\delta'}\|A_{\delta',m-\frac{1}{2}}u\|_{L^2H^{m-/ha}} + C_{\delta,\delta'}\|u\|_{H^{m_0}} + C \end{aligned}$$

giving the inductive step. □

6. A question about the wave equation

One question in lecture was: How much simpler is it to prove this for the wave equation? Who asked this?

If we are talking about the flat wave equation then the discussion can be simplified by shifting the 'smoothing' from the operators to the distribution u. For constant coefficient operators this is straightforward since we can use convolution. In any case smoothing u is an alternative approach but not much different. Apart from that it is a lot simpler. Note however that there are other approaches which can be used here (and more generally). Namely one has in the flat case a forward fundamental solution, and in the general case one can construct forward microlocal parametrices at non-radial points (this requires real work). Then one can shift the discussion to analysing what such operators due to singularities. This amounts to analysing the wavefront set of the (the Schwartz kernel of) the fundamental solution or parametrix. I will likely get around to including something on this as 'the calculus of wavefront sets'.

L9-end

CHAPTER 6

Smoothing operators and K-theory

K-theory L10

I want to take a serious look at smoothing operators. In particular I want to introduce semiclassical (families of) operators in this context – this is another form of 'quantization' closely related to what we have done so far. There are several reasons behind this, the introduction of (topological, complex) K-theory being one. This I will introduce by constructing a smooth classifying space, initially in the odd case but eventually in the even case as well.

I will try to keep the line of reasoning as straight as I can here since it is easy to get distracted by the myriad of possibilities that arise.

1. Hilbert space and operators

Let me recall here some of the basic facts, without proofs, about Hilbert space and bounded operators. Since we do not need the non-separable case, Hilbert space here means a separable, complex, infinite dimensional, Hilbert space, H. One basic fact is that every such space is isometrically isomorphic to $l^2(\mathbb{Z})$.

- The most basic properties of Hilbert space are the Riesz representation thereom, identifying the dual with the conjugate
$$H' = \overline{H}, \qquad \boxed{157.521} \tag{6.1}$$
the existence of orthonormal bases and the convergence of the Fourier-Bessel series.
- Compact sets in Hilbert space are precisely those closed bounded subsets with the additional propery that the Fourier-Bessel series with respect to any (one) orthonormal basis converges uniformly. This amounts to a characterization for $l^2(\mathbb{Z})$.
- A second characterization of compact sets is that they are approximable by finite-dimensional spaces – K has compact closure (is precompact) if it is bounded and for every $\epsilon > 0$ there exists a finite dimensional subspace F such that
$$\sup_{p\in K}\inf_{q\in F} d(p,q) < \epsilon. \qquad \boxed{157.518} \tag{6.2}$$
- The bounded operators form a complete normed, $*$-closed, algebra (the fundamental example of a C^*-algebra), $\mathcal{B}(H)$, with
$$\|A^*\| = \|A\|,\ \|AB\| \le \|A\|\|B\|.$$
- For any closed subspace $R \subset H$
$$H = R \oplus R^\perp$$
defines the unique self-adjoint projection with range R, $P_R : H \longrightarrow R$, $P_R^* = P_R = P_R^2$.
- The open mapping theorem: Any surjective bounded operator is open, meaning they map open sets to open sets.
- Closed graph theorem: A linear map $L : H \longrightarrow H$ with closed graph in $H \times H$ is bounded.
- A bounded bijection has a bounded inverse.

- The resolvent of a bounded operator, B, is a holomorphic map
$$\operatorname{Res}(B) = \mathbb{C} \setminus \operatorname{Spec}(B) \ni z \longmapsto (B-z)^{-1} \in \mathcal{B}(H) \qquad \boxed{\texttt{157.524}} \tag{6.3}$$
defined on the complement of the compact set $\operatorname{Spec}(B) \subset \mathbb{C}$ where $B - z$ is not a bijection.
- The resolvent identity
$$(B-z)^{-1} - (B-\tau)^{-1} = (z-\tau)(B-z)^{-1}(B-z)^{-1},\ z,\tau \in \operatorname{Res}(B), \qquad \boxed{\texttt{157.559}} \tag{6.4}$$
holds.
- If $\chi : \mathbb{S} \longrightarrow \mathbb{C} \setminus \operatorname{Spec}(B)$ is a smooth, simple, positively-oriented curve then
$$P_\chi = \frac{1}{2\pi i} \oint_\chi (B-z)^{-1} dz \qquad \boxed{\texttt{157.558}} \tag{6.5}$$
is a projection commuting with B such that $P_\chi B$ has spectrum in the interior of χ and $(\operatorname{Id} - P_\chi)B$ has spectrum in the exterior.
- The group of invertibles $\operatorname{GL}(H) \subset \mathcal{B}(H)$ is open (but not dense) because of the convergence of the Neumann series
$$(\operatorname{Id} + B)^{-1} = \sum_k (-1)^k B^k,\ \|B\| < 1. \qquad \boxed{\texttt{157.520}} \tag{6.6}$$
- There is a functional calculus for each self-adjoint operator, $A = A^* \in \mathcal{B}(H)$, defining a map of algebras
$$\{f : \operatorname{Spec}(A) \longrightarrow \mathbb{C};\ \text{continuous}\} \longrightarrow \mathcal{B}(H) \text{ s.t. } f(A)g(A) = (fg)(A).$$
- There is a polar decomposition of each bounded operator as a product
$$B = AV \text{ where } A = (B^*B)^{\frac{1}{2}} \text{ and } V \text{ is a partial isometry} \qquad \boxed{\texttt{157.523}} \tag{6.7}$$
- It follows that unitary subgroup $\operatorname{U}(H) \subset \operatorname{GL}(H)$ defined by the identity $B^*B = \operatorname{Id}$ is a deformation retract of $\operatorname{GL}(H)$.
- Kuiper's theorem is the statement that $\operatorname{U}(H)$ is (weakly) contractible in the norm topology (every continuous map from a compact space is homotopic to the map to the identity).
- The compact operators, $\mathcal{K}(H) \subset \mathcal{B}(H)$, consist of the operators mapping bounded to pre-compact sets. They constitute the norm-closure of the finite rank operators (those with finite-dimensional range) and form the only non-trivial closed ideal.
- The Fredholm operators, $\mathcal{F}(H) \subset \mathcal{B}(H)$, are defined by the requirements that they have finite-dimensional null space and closed range of finite codimension. The index
$$\operatorname{ind}(F) = \dim \operatorname{null}(F) - \dim(\operatorname{Ran}(F)^\perp),\ \operatorname{ind} : \mathcal{F}(H) \longrightarrow \mathbb{Z} \qquad \boxed{\texttt{157.526}} \tag{6.8}$$
labels the components.
- A bounded operator B is Fredholm if and only if it has a parametrix modulo compact operators (an inverse in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$) $A \in \mathcal{B}(H)$ such that
$$BA - \operatorname{Id},\ AB - \operatorname{Id} \in \mathcal{K}(H).$$
- The Hilbert-Schmidt ideal, $\operatorname{HS}(H) \subset \mathcal{B}(H)$, is defined by the condition that for any one orthonormal basis
$$\|B\|_{\operatorname{HS}}^2 = \sum_i |\langle Be_i, e_i \rangle|^2 < \infty. \qquad \boxed{\texttt{157.529}} \tag{6.9}$$
This is independent of choice and
$$\|AB\|_{\operatorname{HS}} \le \|A\|_{\operatorname{HS}} \|B\|. \qquad \boxed{\texttt{157.530}} \tag{6.10}$$
- The trace ideal, $\mathcal{T}(H) \subset \mathcal{B}(H)$, is $\operatorname{HS}(H)^2$ – the finite span of products of elements of $\operatorname{HS}(H)$. Any element is the product of two Hilbert-Schmidt operators and
$$\|T\|_{\operatorname{Tr}} = \|(T^*T)^{\frac{1}{4}}\|_{\operatorname{HS}},\ \|AB\|_{\operatorname{Tr}}| \le \|A\|_{\operatorname{Tr}} \|B\|. \qquad \boxed{\texttt{157.535}} \tag{6.11}$$

- The trace functional is

$$\mathrm{Tr} : \mathcal{T}(H) \ni T = \sum_i \langle Te_i, e_i \rangle_H \in \mathbb{C} \qquad \boxed{\texttt{157.531}} \quad (6.12)$$

  for any orthornomal basis and

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \text{ if } A \in \mathcal{T}(H),\ B \in \mathcal{B}(H) \text{ or } A,\ B \in \mathrm{HS}(H). \qquad \boxed{\texttt{157.532}} \quad (6.13)$$

- Any Fredholm operator B has a parametrix A modulo trace class errors and

$$\mathrm{ind}(B) = \mathrm{Tr}(BA - AB). \qquad \boxed{\texttt{157.533}} \quad (6.14)$$

- A Fredholm operator has a unique generalized inverse

$$AB = \mathrm{Id} - P_{\mathrm{null}(B)},\ BA = \mathrm{Id} - P_{\mathrm{Ran}(B)^\perp}. \qquad \boxed{\texttt{157.534}} \quad (6.15)$$

- If $B \in \mathcal{F}(H)$ and $K \in \mathcal{K}(H)$ then $B + K \in \mathcal{F}(H)$ and

$$\mathrm{ind}(B + K) = \mathrm{ind}(B). \qquad \boxed{\texttt{157.536}} \quad (6.16)$$

- On the ring $\mathrm{Id} + \mathcal{T}(H)$ the Fredholm determinant

  $\det : \mathrm{Id} + \mathcal{T}(H) \longrightarrow \mathbb{C}$ satisfies

$$\det\left((\mathrm{Id} + T_1)(\mathrm{Id} + T_2)\right) = \det(\mathrm{Id} + T_1)\det(\mathrm{Id} + T_2). \qquad \boxed{\texttt{157.537}} \quad (6.17)$$

- The group of invertibles in $\mathrm{Id} + \mathcal{T}(H)$ is the inverse image $\det^{-1}(\mathbb{C} \setminus \{0\})$ and if $(-1, 1) \ni s \longrightarrow T(s) \in \mathcal{T}(H)$ is $\mathcal{C}^1$ and $\mathrm{Id} + T(0)$ is invertible then

$$\frac{d}{ds}\det(\mathrm{Id} + T(s))\big|_{s=0} = \det(\mathrm{Id} + T(0))\,\mathrm{Tr}\left((\mathrm{Id} + T(0))^{-1}\frac{dT}{ds}(0)\right). \qquad \boxed{\texttt{157.538}} \quad (6.18)$$

2. Schwartz smoothing algebra

The residual ideal in the standard pseudodifferential algebra $\Psi^{-\infty}(\mathbb{R}^n)$, is the part that cannot be reached by the symbol calculus. As I have muttered all along, it is not a very nice algebra because of the coefficient ring $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$.

Let us instead concentrate on the smaller Schwartz smoothing algebra, $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ – this is actually a two-sided ideal in $\Psi^{-\infty}(\mathbb{R}^n)$. It is also 'very non-commutative' in that it is simple with only the two trivial ideals. Thus

$\boxed{\texttt{157.348}}$ (6.19)
$$\mathcal{S}(\mathbb{R}^{2n}) = \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) \subset \Psi^{-\infty}(\mathbb{R}^n)$$

where the first equality is as a space of kernels. In fact it is the residual part of both the scattering (Shubin) and isotropic algebras in Proposition 3.3 and §3.3. 157.220 C.Psi.Sm isotropic

The product in this algebra is

$\boxed{\texttt{157.349}}$ (6.20)
$$A \circ B(x, y) = \int_{\mathbb{R}^n} A(x, z)B(z, y)dz.$$

In essence, $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ is an infinite-dimensional matrix algebra. To justify this directly we need a 'basis' for $\mathcal{S}(\mathbb{R}^n)$; the standard one is the Hermite basis. This consists of the eigenfunctions for the harmonic oscillator

$\boxed{\texttt{157.350}}$ (6.21)
$$H = \sum_{i=1}^n D^2_{x_i} + |x|^2,\ He_\kappa = (n + 2|\kappa|)e_\kappa,\ \kappa \in \mathbb{N}^n_0.$$

Here the eigenfunctions are given as products of the L^2 normalized eigenfunctions in the case $n = 1$

$\boxed{\texttt{157.351}}$ (6.22)
$$e_\kappa(x) = \prod_{j=1}^n f_{\kappa_j}(x_j),\ (D^2_x + x^2)f_j = (1 + 2j)f_j,\ f_j(x) = h_j(x)\exp(-\frac{1}{2}x^2)$$

where the h_j are the Hermite polynomials.

We need the isomorphism that these provide

157.352 (6.23)
$$\begin{gathered}\mathcal{S}(\mathbb{R}^n) \longrightarrow s(n) = \{c : \mathbb{N}_0^n \longrightarrow \mathbb{C}; \sum_\kappa (1+|\kappa|^2)^N |c_\kappa|^2 < \infty \ \forall \ N\},\\ u \longmapsto \left\{\int_{\mathbb{R}^n} u(x) e_\kappa(x) dx\right\}_\kappa .\end{gathered}$$

This leads to the identification

157.353 (6.24)
$$\begin{aligned}\Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n) \ni A \longrightarrow a = (Ae_{\kappa'}, e_\kappa)\\ \in \Psi_s^{-\infty} = \{a : \mathbb{N}_0^{2n} \longrightarrow \mathbb{C}; \sum_{\kappa,\kappa'} (1+|\kappa|+|\kappa'|)^N |a_{\kappa,\kappa'}| < \infty \ \forall \ N\}\end{aligned}$$

with the composition becoming 'matrix multiplication'. That is, the ring of 'rapidly decreasing infinite matrices'.

157.371 Lemma 6.1. *The algebras $\Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$, for different n, are isomorphic (although not naturally so).*

Proof. Expansion in terms of the Hermite basis reduces $\Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$ to 'matrices' meaning rapidly decreasing maps

157.484 (6.25)
$$a : \mathbb{N}_0^n \times \mathbb{N}_0^n \longrightarrow \mathbb{C}, \ \sum_{\alpha,\beta} (1+|\alpha|+|\beta|)^N |a(\alpha,\beta)|^2 < \infty.$$

The number of α with $|\alpha| \leq N$ is bounded by N^n so rapid decay in the sense of (6.25) is the same as rapid decay in j if $j \longmapsto \alpha_j$ is any ordering in which $|\alpha_j|$ is non-decreasing. This shows that all the algebras are isomorphic to the case $n = 1$. □

157.369 Lemma 6.2. *The elements of $\Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$ act as elements of the trace ideal of compact operators on $L^2(\mathbb{R}^n)$ (or any Sobolev space) and form a 'corner' (not an ideal) in the bounded operators in the sense that if $B \in \mathcal{B}(L^2(\mathbb{R}^n))$*

157.370 (6.26)
$$A_1,\ A_2 \in \Psi^{-\infty}(\mathbb{R}^n) \Longrightarrow A_1 B A_2 \in \Psi^{-\infty}(\mathbb{R}^n).$$

Proof. Schur's criterion (as discussed in Problem set 2) shows that these are bounded operators and the fact that any sequence in $\mathcal{S}(\mathbb{R}^n)$ bounded with respect to the seminorms has a convergent subsequence in $L^2(\mathbb{R}^n)$ shows that they are compact operators.

That these operators are in the trace ideal follows from the fact that the diagonal operator, with respect to the Hermite basis

157.539 (6.27)
$$T_k \ni u \longmapsto \sum_\alpha (1+|\alpha|)^{-k} \langle u, e_\alpha \rangle e_\alpha$$

is in the trace ideal if $(T^*T)^{\frac14}$ is in Hilbert-Schmidt, which follows if

157.540 (6.28)
$$\|(T^*T)^{\frac14}\|_{\mathrm{HS}}^2 = \sum_\alpha (1+|\alpha|)^{-k} < \infty.$$

This holds if $k > n$. If $A \in \Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$ it follows that $AT_k^{-1} \in \Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$ is bounded so $A \in \mathcal{T}(L^2(\mathbb{R}^n))$. □

Why should we be interested in $\Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$? One important reason is topological, arising from the associated group which plays a major rôle in the discussion of K-theory below,

157.354 (6.29) $G_{\mathcal{S}}^{\infty}(\mathbb{R}^n) = \{A \in \Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n); \exists\, B \in \Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n) \text{ with } (\mathrm{Id}+A)\circ(\mathrm{Id}+B) = \mathrm{Id}\}.$

Just as for square matrices, this one-sided inverse condition implies that $\mathrm{Id}+B$ is a 2-sided inverse

157.355
$$(\mathrm{Id}+B)\circ(\mathrm{Id}+A) = \mathrm{Id}\,. \tag{6.30}$$

157.356 LEMMA 6.3. *The group $G_{\mathcal{S}}^{\infty} \subset \Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$ is open and dense in $\Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$ and the union of the subgroups $\mathrm{GL}(N,\mathbb{C})$ of finite $N\times N$ matrices with respect to the Hermite basis is dense.*

157.485 REMARK 4. I habitually write $G_{\mathcal{S}}^{-\infty}(\mathbb{R}^n) \subset \Psi_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$ by removing the identity. This really corresponds to changing the product to $A\circ B = A+B+AB$ now giving a ring structure.

PROOF. I did not go through this in lecture but maybe I should have done so. We know, from Schur's criterion, that the norm as a bounded operator defines a continuous map

$$\Psi^{-\infty}(\mathbb{R}^n) \ni A \longrightarrow \|A\|_{L^2}. \tag{6.31}$$ 157.390

So the elements of a neighbourhood of 0 in $\Psi^{-\infty}(\mathbb{R}^n)$ give invertible elements $\mathrm{Id}+A \in \mathcal{B}(L^2(\mathbb{R}^n))$ by Neumann seris. The inverse being

$$(\mathrm{Id}+A)^{-1} = \mathrm{Id}+B,\ B = \sum_{k\ge 1}(-1)^k A^k. \tag{6.32}$$ 157.391

All the elements in the series are in $\Psi^{-\infty}(\mathbb{R}^n)$ but convergence is in principle only as bounded operators – in fact the Neumann series converges in $\Psi^{-\infty}(\mathbb{R}^n)$, i. e. in $\mathcal{S}(\mathbb{R}^{2n})$. To see this, expand the definition of B to see that

$$B = -A + A^2 + A(\sum_{k\ge 1}(-1)^k A^k)A = -A + A^2 + ABA. \tag{6.33}$$ 157.392

It follows from the corner property that $ABA \in \Psi^{-\infty}(\mathbb{R}^n)$ and that the series for B actually converges in this sense.
Thus, $G^{\infty}(\mathbb{R}^n)$ contains a neighbourhood N of $0 \in \Psi^{-\infty}(\mathbb{R}^n)$ and hence a neighbourhood around any point of $G^{\infty}(\mathbb{R}^n)$. □

3. Odd K-theory

The group $G_{\mathcal{S}}^{-\infty}(\mathbb{R}^n)$ provides an entry point to 'complex K-theory'.

157.357 DEFINITION 6.1. The *odd K-theory,* of a manifold M consists of the smooth homotopy classes of smooth maps

157.358
$$K^1(M) = \{u : M \longrightarrow G_{\mathcal{S}}^{\infty};\ u = \mathrm{Id} \text{ on } M\setminus K \text{ for some } K \Subset M\}/\text{smooth homotopy}. \tag{6.34}$$

A smooth homotopy between two elements u_0 and u_1 is a smooth map

157.359
$$\begin{gathered} v : M\times[0,1] \longrightarrow G_{\mathcal{S}}^{\infty} \text{ with} \\ v = \mathrm{Id} \text{ on } (M\setminus K')\times[0,1] \text{ for some } K' \Subset M, \\ v\big|_{M\times\{0\}} = u_0,\ v\big|_{M\times\{1\}} = u_1. \end{gathered} \tag{6.35}$$

Note that there is no problem in understanding 'smoothness' here since

157.372 (6.36) $\mathcal{C}^\infty(M;\mathcal{S}(\mathbb{R}^m)) = \{u : M\times\mathbb{R}^m \longrightarrow \mathbb{C};\ \text{smooth with all seminorms finite}\}.$

So, the maps in (6.34) actually form a group which can be written

157.486 (6.37)
$$\mathcal{C}^\infty_{\mathrm{c}}(M;G^{-\infty}(\mathbb{R}^n))$$

where as always the identity is 'removed'. Then smooth curves in this group become elements of

157.487 (6.38)
$$\mathcal{C}^\infty_{\mathrm{c}}([0,1]\times M;G^{-\infty}(\mathbb{R}^n))$$

and we can write the odd K-groups as the groups of components

157.488 (6.39)
$$K^1(M) = \pi_0(\mathcal{C}^\infty_{\mathrm{c}}(M;G^{-\infty}(\mathbb{R}^n))).$$

What justifies such a bald definition and how can we start to understand it? It is actually saying that $G^\infty_{\mathcal{S}}$, in any of its variants, is a classifying group for odd K-theory. You might object that K-theory is supposed to be related to vector bundles. It is, as we shall see below, but I assert that (6.34) (and (6.42) below) are rather natural definitions corresponding to the assertion that K-theory is the topology associated to invertible matrices – since (6.34) is a definition which can be reduced to smooth maps into $\mathrm{GL}(N,\mathbb{C})$ (stabilized in N).

I do not want to spend too much time on this, but let me outline some of the things which can be proved relatively easily and give the proofs later, perhaps some of them not in lectures.

157.360 Proposition* 6.1. *The odd K-theory of a manifold is an abelian group with*

157.361 (6.40)
$$K^1(\mathbb{R}^k) = \begin{cases} 0 & k \text{ even} \\ \mathbb{Z} & k \text{ odd}\end{cases}$$

and for any manifold M there are natural isomorphisms

157.387 (6.41)
$$K^1(\mathbb{R}^{2k}\times M) \longrightarrow K^1(M).$$

The fundamental result (6.41) is actually a consequence of (6.40) if you know some homotopy theory. In any case (6.40) is 'Bott periodicity' and (6.41) is periodicity in K-theory. Of course (6.40) is a consequence of (6.41) and the special cases $k = 0$ and $k = 1$. There are lots of competing proofs, none of them really simple as far as I know. I will describe a proof using semiclassical quantization to construct the map (6.41).

The idea is that (6.41) is a prototype for the Atiyah-Singer index theorem for families. It is an odd version, whereas the standard version is in even K-theory, but these are closely related and the index theorems are in fact equivalent.

157.388 Definition 6.2. The even K-groups of a manifold M are the groups

157.389 (6.42)
$$K^0(M) = K^1(\mathbb{R}\times M).$$

S.IGrass

4. The involutive Grassmannian

As you are probably aware, 'K-theory' – usually meaning the even group $K^0(M)$ – for a manifold, is related to vector bundles over the manifold. I have not yet reminded you of these and rather than proceed to do that I will simply *assert* for the moment that vector bundles, like the groups $\mathrm{GL}(N,\mathbb{C})$ can be 'stabilized'. So I will work from the top down and start with the stabilized version.

To be explicit consider the space $L^2(\mathbb{R};\mathbb{C}^2) = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. We need (at least) two copies because I want to have infinite 'upside and downside' options. Now we replace the identity operator considered above for the group $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ by the involutive matrix (meaning the square is the identity)

157.541 (6.43) $$\beta_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Things related to β_∞ are often given a 'super' quantifier but not here.

157.542 Definition 6.3. The (involutive) Grassmannian associated to β_∞ is the space of involutive operators on $L^2(\mathbb{R};\mathbb{C}^2)$ of the form[1]

157.543 (6.44) $$\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2) = \left\{\beta = \beta_\infty + B,\ B \in \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R};\mathbb{C}^2),\ \beta^2 = \mathrm{Id}\right\}.$$

So these are the smoothing perturbations of β_∞ which are involutive.

How is this related to vector bundles? Well, an involutive operator is really a projection (by which I mean what is sometimes called an idempotent) namely

157.544 (6.45) $$P = \frac{1}{2}(\beta + \mathrm{Id}) \Longleftrightarrow P^2 = P,\ \beta = 2P - \mathrm{Id}\,.$$

So the space of involutive perturbations of β_∞ is identified with the space of perturbations

157.545 (6.46) $$P = P_1 + A,\ P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A \in \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R};\mathbb{C}^2),\ P^2 = P.$$

Such a projection is determined by its null space and range. These are the -1 and 1 eigenspaces, respectively, of the corresponding involution β.

157.546 Remark 5. Note that the 'projection' P need not be self-adjoint, so the range need not be orthogonal to the null space. One could, and indeed it is more conventional to do so, work with self-adjoint projections, for which this orthogonality does hold, but I have elected not to do since the resulting 'classifying space' (6.44) 157.543 is cleaner. Two projections with the same range are homotopy through projections.

Consider some points in $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$. To do so, let Q_k be the orthogonal projection on the span of the first $k \in \mathbb{N}$ Hermite functions in $L^2(\mathbb{R}^n)$ (with respect to an order in which $|\alpha|$ is increasing but it really does not matter) and let $Q^{(i)}_k$, for $i = 1, 2$, be the corresponding projections on $L^2(\mathbb{R}^n;\mathbb{C}^2)$ acting on the first and second factors. Thus the $Q^{(i)}_k$ commute with P_1 and $P_2 = \mathrm{Id} - P_1$ with $Q^{(i)}_k$ a subprojection of P_i. Then

157.547 (6.47) $$\beta_k = \begin{cases} 2(P_1 + Q^{(2)}_k) - \mathrm{Id} & \text{for } k > 0 \\ \beta_\infty = 2P_1 - \mathrm{Id} & \text{for } k = 0 \\ 2(P_1 - Q^{(1)}_{-k}) - \mathrm{Id} & \text{for } k < 0 \end{cases}$$

are involutions.

The group $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ is a Fréchet manifold even though I have not defined the meaning – simply because it is an open subset of the Fréchet space $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$. It is much less clear that this Grassmannian is a manifold. As a subset

157.548 (6.48) $$\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^2;\mathbb{C}) \subset \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2) \text{ is closed.}$$

[1] If you are wondering, it is an Upsilon

If it is to be smooth then it should have a decent tangent space, as a subspace of $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$, at each point. So consider a smooth curve

157.549 (6.49) $$(-1,1)\ni t\longmapsto\beta+a(t)\in\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2),\ a(t)\in\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2),\ a(0)=0.$$

Certainly we must have

157.550 (6.50) $$(\beta+a(t))^2=\mathrm{Id}\Longrightarrow\beta\frac{da}{dt}(0)+\frac{da}{dt}(0)\beta=0.$$

We can decompose any $a'=da/dt(0)\in\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ as a sum of four terms, determined by β,

157.551 (6.51) $$a'=Pa'P+(\mathrm{Id}-P)a'P+Pa'(\mathrm{Id}-P)+(\mathrm{Id}-P)a'(\mathrm{Id}-P).$$

The condition in (6.50) reduces to

157.552 (6.52) $$Pa'P=0=(\mathrm{Id}-P)a'(\mathrm{Id}-P)\Longrightarrow a(0)=Pa'(\mathrm{Id}-P)+(\mathrm{Id}-P)a'P.$$

Thus a tangent vector at β must be off-diagonal with respect to the decomposition defined by β.

157.555 Proposition 6.2. *Given $\epsilon>0$ if $a\in\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ is sufficiently close to 0 and $\beta\in\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ the resolvent*

157.556 (6.53) $$(\beta+a-z\,\mathrm{Id})^{-1}\ \text{exists in}\ \{z;|z-1|>\epsilon/2\}\cap\{|z+1|>\epsilon/2\}$$

and the contour integral

157.557 (6.54) $$\beta_a=\frac{1}{2\pi i}\int_{|z-1|=\epsilon}(\beta+a-z\,\mathrm{Id})^{-1}dz-\frac{1}{2\pi i}\int_{|z+1|=\epsilon}(\beta+a-z\,\mathrm{Id})^{-1}dz$$

gives a smooth local retraction to $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ which is a bijection from the intersection with the tangent space in the sense of (6.52).

Proof. The resolvent of β is

R.24 (6.55) $$(\beta-z\,\mathrm{Id})^{-1}=\frac{1}{2}(1-z)^{-1}(\beta+\mathrm{Id})+\frac{1}{2}(z+1)^{-1}(\beta-\mathrm{Id})$$

from which it follows that proved the norm $\|a(0)\|<\delta$ is small $(\beta+a-z\,\mathrm{Id})^{-1}$ is holomorphic outside the two disks in (6.54) where it is a smoothing perturbation of (**??**). Standard arguments from the holomorphic calculus show that it is an involution. So the map from a small L^2 ball in the linear space, given by (6.51), to $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ is well-defined. As a map on bounded operators on L^2 it is differentiable and restricted to the diagonal elements (6.51) it has derivative the identity at $a(0)=0$. Thus the Implict Function Theorem applies and the inverse preserves regularity, showing that the retraction is indeed a local isomorphism from the (putative) tangent space to $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$. □

These retractions show that $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ is a smooth manifold locally modelled on the tangent space at each point.

157.553 Theorem 6.1. *The involutive Grassmannian $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ has components labelled by*

157.554 (6.56) $$\text{R-ind}:\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)\ni\beta\longmapsto\frac{1}{2}\operatorname{Tr}(\beta-\beta_\infty)$$

and on each component $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ acts transitively by conjugation.

The 'relative index' function R-ind clearly takes the value k on the elements β_k in (6.47).

PROOF. The trace is well-defined, since $\beta - \beta_\infty$ is a smoothing operator. Moreover and smooth curve in $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ is locally the image of a curve in the tangent space under (6.54), so we know that R-ind varies smoothly along a curve and has derivative

R.25
$$\frac{1}{2}\operatorname{Tr}(\frac{d\beta}{dt}) = \frac{1}{2}\operatorname{Tr}(a') = 0 \tag{6.57}$$

because of (6.51). Thus R-ind is constant on components of $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$.

For the elements defined in (6.47)

R.26
$$\text{R-ind}(\beta_k) = k. \tag{6.58}$$

Now, consider the action of $g \in G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$. Certainly $g\beta g^{-1}$ is again an involutive perturbation of β_∞ so $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ does act on $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$. We know that $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ is path connected so it must fix components of $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$.

It remains to show that any element of $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ is conjugate to one of the β_k. For a given $\beta \in \Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ with positive projector $P = \frac{1}{2}(\beta + \operatorname{Id})$ let H_P be the range of P acting on $L^2(\mathbb{R};\mathbb{C}^2)$, it is a closed subspace. Then the composite operator

157.510
$$PP_1 : L^2(\mathbb{R}) \oplus \{0\} \longrightarrow H_P \tag{6.59}$$

is Fredholm. This follows from the fact that

$$P_1PP_1 = P_0 + P_1(P - P_1)P_1 = \operatorname{Id} + Q \text{ on } L^2(\mathbb{R}) \oplus \{0\}$$

is a compact (in fact Schwartz-smoothing) perturbation of the identity on the range of P_1. Similarly PP_1P is a compact perturbation of the identity acting on H_P. Thus PP_1 has finite dimensional null space and maps the orthocomplement isomorphically onto a closed subspace of H_P with a finite-dimensional (ortho)-complement. However these two spaces may have different dimensions – that is what $\text{R-ind}(\beta) = \operatorname{Tr}(P - P_1)$ measures as we see below.

If the dimension of the null space of PP_1 in (6.59) is larger than that of the complement of the range then we may replace P_1 by $P_1(\operatorname{Id} - Q_k)$ where k is the difference and Q_k is the finite-rank projector onto the first k Hermite functions. This may not be contained in the null space but now the operator $PP_1(\operatorname{Id} - Q_k)$ has null space and complement of the range of the same dimension. We can then add a finite rank smoothing operator S between these two spaces so that

157.512
$$PP_1(\operatorname{Id} - Q_k) + S = P_1 + S',\ S,\ S' \text{ smoothing} \tag{6.60}$$

which is an isomorphism from the range of $P_1(\operatorname{Id} - Q_k)$ to H_P. Then it follows that

157.513
$$g = P_1 + P_2 + S' \in G^{-\infty}(\mathbb{R};\mathbb{C}^2),\ \tilde{P} = g(P_1(\operatorname{Id} - Q_k)g^{-1} : L^2(\mathbb{R};\mathbb{C}^2) \longrightarrow H_P \tag{6.61}$$

is a projection with the same range as P. The null space $\operatorname{Ran}(\operatorname{Id} - \tilde{P})$ of $\tilde{P}$ is a complement to the range and

157.514
$$(\operatorname{Id} - P) : \operatorname{Ran}(\operatorname{Id} - \tilde{P}) \longrightarrow \operatorname{Ran}(\operatorname{Id} - P) \tag{6.62}$$

is an isomorphism since it has no null space and is surjective. It follows that $\tilde{g}\operatorname{Id} + (\tilde{P} - P) \in G^{-\infty}(\mathbb{R}^2;\mathbb{C}^2)$ conjugates $\tilde{P}$ to P.

If the dimension of the null space of PP_1 is smaller than the range then a similar argument follows with P_1 replaced by $P_1 \oplus Q_k$ where Q_k is the same projection but now on the second factor.

This proves that the involution is conjugate to one of the options in (6.47) and then it follows that the relative trace is either k or $-k$ in the two cases. $\square$

If you are prepared to swallow a bit of homotopy theory you can see why the Grassmannian is relevant here. Consider the component containing $\beta_0 = \beta_\infty$:

157.560
$$\Upsilon^{-\infty}_{0,\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2) = \{\beta \in \Upsilon^{-\infty}_{0,\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2); \operatorname{Tr}(\beta - \beta_\infty) = 0\}. \tag{6.63}$$

The isotropy group of β_∞ is

157.561
$$G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) \times G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) = \{g \in G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2); h\beta_\infty = \beta_\infty g\} \tag{6.64}$$

where the two factors act in the two components of $\mathbb{C}^2$. Thus in fact we have shown that there is a natural isomorphism

157.563
$$\Upsilon^{-\infty}_{0,\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2) \equiv G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)/\left(G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) \times G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)\right). \tag{6.65}$$

These two subgroups commute with each other so we can also write the quotient as

157.562
$$\begin{gathered}\Upsilon^{-\infty}_{0,\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2) \equiv \\ \Big(G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)/\{\mathrm{Id}\} \times G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)\Big)/G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) \times \{\mathrm{Id}\}.\end{gathered} \tag{6.66}$$

The first quotient here is weakly contractible – since $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ can be retracted onto $\{\mathrm{Id}\} \times G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$. This means that this base component is the quotient by the free action of $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ on a contractible space.

157.564 Proposition 6.3. *The base component of* $\Upsilon^{-\infty}_{0,\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$, *is a classifying space for* $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ *and hence*

157.565
$$\pi_j(\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)) = \begin{cases} \mathbb{Z} & j \text{ even} \\ \{0\} & j \text{ odd} \end{cases} \tag{6.67}$$

and $\Upsilon^{-\infty}_{0,\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ *is a classifying space for even K-theory.*

Proof. Not given here but from the long-exact sequence of Serre the homotopy groups can be identified from the identification of the homotopy groups of $G^{-\infty}$ in (6.40). These are proved below, using some of the properties of $\Upsilon^{-\infty}_{0,\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$ – but not (6.67)! $\square$

5. Semiclassical smoothing operators

If $A \in \mathcal{S}(\mathbb{R}^{2n})$ is the kernel of an element of $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ we can, as we did for pseudodifferential operators, introduce

157.373
$$B(x,z) = A(x, x-z) \text{ and } B(x,z) = (2\pi)^{-n}\int_{\mathbb{R}^n} b(x,\xi)e^{-iz\cdot\xi}d\xi. \tag{6.68}$$

157.393 Definition 6.4. A *semiclassical Schwartz family,* $A(\epsilon) \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n)$, is defined by a smooth family

157.377
$$a \in \mathcal{C}^\infty([0,1]_\epsilon;\mathcal{S}(\mathbb{R}^{2n})) \tag{6.69}$$

by setting, for $\epsilon > 0$,

$$A(\epsilon)u(x) = \int_{\mathbb{R}^n} A(\epsilon; x, y)u(y)dy,\ u \in \mathcal{S}(\mathbb{R}^n) \text{ where}$$

157.374 (6.70)
$$A(\epsilon; x, y) = B(\epsilon; x, x - y),\ B(\epsilon; x, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(\epsilon; x, \epsilon\xi)e^{-iz\cdot\xi}d\xi$$
$$= (2\pi\epsilon)^{-n} \int_{\mathbb{R}^n} a(\epsilon; x, \eta)e^{-i\epsilon^{-1}z\cdot\eta}d\eta.$$

For $\epsilon > 0$ this does not do very much but the family certainly becomes singular at $\epsilon = 0$. L10-end

Said another way, an element of $\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n)$ is a family of operators for $\epsilon \in (0, 1]$ L11
defined by

157.378 (6.71)
$$A(\epsilon) = Q_L(a(\epsilon; x, \epsilon\xi)),\ a \in \mathcal{C}^\infty([0, 1]; \mathcal{S}(\mathbb{R}^{2n}_{x,\eta})).$$

The basic result is that these families compose to give an algebra.

157.375 Proposition 6.4. *The operator product on $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$, applied for $\epsilon > 0$ to*

157.376 (6.72)
$$(A_1 \circ_\epsilon A_2)(\epsilon; x, y) = \int_{\mathbb{R}^n} A_1(\epsilon; x, y')A_2(\epsilon, y', y)dy'$$

extends to a smooth product defining the algebra $\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n)$.

Proof. Writing the composite family in terms of the kernels $\epsilon^{-n}B_i(\epsilon, x, \frac{x-y}{\epsilon})$ of the factors shows that

$$A_1 \circ A_2 \text{ has kernel } \epsilon^{-n}D(\epsilon, x, \frac{x - y}{\epsilon}) \text{ where}$$

157.489 (6.73)
$$D(\epsilon, x, t) = \epsilon^{-n} \int_{\mathbb{R}^n} B_1(\epsilon, x, t + \frac{y - z}{\epsilon})B_2(\epsilon, z, \frac{z - y}{\epsilon})dz$$
$$= \int_{\mathbb{R}^n} B_1(\epsilon, x, t - Z)B_2(\epsilon, x + \epsilon Z, Z)dZ,\ Z = \frac{y - z}{\epsilon}.$$

The integrand here is smooth in ϵ with values in the Schwartz functions in the variables x, t, Z – as follows from the fact that

157.490 (6.74)
$$(1 + |x| + |t - Z| + |x + \epsilon Z| + |Z|) \text{ is comparable to } (1 + |x| + |t| + |Z|).$$

Thus the integral is also Schwartz. Expanding in Taylor series at $\epsilon = 0$ gives the Moyal formula below.

Alternatively this follows from our earlier results on $\Psi^0(\mathbb{R}^n)$. Observe that

157.379 (6.75)
$$\epsilon(1 + |\xi|) \le (1 + |\epsilon\xi|),\ \epsilon \in (0, 1)$$

from which it follows that

157.380 (6.76)
$$(0, 1) \ni \epsilon \longrightarrow a(\epsilon, x, \epsilon\xi) \in \mathcal{S}(\mathbb{R}^n; S^0(\mathbb{R}^n))$$

is (uniformly) bounded. Indeed the derivatives satisfy

157.381 (6.77)
$$\partial^\alpha_x\partial^\beta_\xi a(\epsilon; x, \epsilon\xi) = \epsilon^{|\beta|}\partial^\alpha_x\partial^\beta_\eta a(\epsilon; x, \eta)\big|_{\eta=\epsilon\xi} \Longrightarrow$$
$$|\partial^\alpha_x\partial^\beta_\xi a(\epsilon; x, \epsilon\xi)| \le C_{\alpha,\beta}\epsilon^{|\beta|}(1 + |\epsilon\xi|)^{-|\beta|} \le C_{\alpha,\beta}(1 + |\xi|)^{-|\beta|}.$$

So our earlier composition result shows that uniformly for $\epsilon > 0$

157.382 (6.78)
$$A_1(\epsilon) \circ A_2(\epsilon) \text{ is bounded in } \Psi^0(\mathbb{R}^n)$$

Moreover the left-reduced symbol of the composite is given by the asymptotic formula

157.383 (6.79)
$$A_1(\epsilon)\circ A_2(\epsilon) = Q_L(c(\epsilon)),\ c(\epsilon)\sim\sum_\alpha \partial_\xi^\alpha a_1(\epsilon;x,\epsilon\xi)D_x^\alpha a_2(\epsilon;x,\epsilon\xi) = \sum_\alpha \epsilon^{|\alpha|}\partial_\eta^\alpha a_1(\epsilon;x,\eta)D_x^\alpha a_2(\epsilon;x,\eta)\big|_{\eta=\epsilon\xi}.$$

Using Borel's Lemma to sum the Taylor series gives $c\in\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2n}))$ such that

157.384 (6.80)
$$c(\epsilon;x,\eta)-\sum_{|\alpha|\le N}\epsilon^{|\alpha|}\partial_\eta^\alpha a_1(\epsilon;x,\eta)D_x^\alpha a_2(\epsilon;x,\eta)\in\epsilon^{N+1}\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2n}))\ \forall\ N.$$

The claim then is that

157.385 (6.81)
$$E(\epsilon)=Q_L(c(\epsilon;x,\epsilon\xi))-A_1(\epsilon)\circ A_2(\epsilon)\in\mathcal{C}^\infty([0,1];\Psi^{-\infty}(\mathbb{R}^n)) \quad \text{with } \frac{d^kE(\epsilon)}{d\epsilon^k}\Big|_{\epsilon=0}=0\ \forall\ k.$$

□

From either proof we see that again 'Moyal's formula' appears (I think it is more historically legitimate here than before!)

157.397 (6.82)
$$\begin{gathered}A_i\in\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n),\ A_i=Q_L(a_i(\epsilon,x,\epsilon\xi)),\ a_i\in\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2n}))\Longrightarrow\\ A_1\circ A_2=Q_L(c),\ c\in\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2n}),\\ c(\epsilon,x,\eta)\simeq\sum_\alpha\epsilon^{|\alpha|}\partial_\eta^\alpha a_1(\epsilon,x,\eta)D_x^\alpha a_2(\epsilon,x,\eta)\end{gathered}$$

Equality here is in the sense of formal power series (i.e. Taylor series) at $\epsilon=0$. So this formula tells you nothing about what happens for $\epsilon>0$ as is to be expected. However, the symbol of the product is determined by this formula up to terms vanishing to infinte order.

157.398 LEMMA 6.4. *If $a\in\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2n}))$ vanishes to infinite order at $\epsilon=0$ then the semiclassical family*

157.399 (6.83)
$$A(\epsilon)=Q_L(a(\epsilon,x,x\xi))$$
is simply a smooth map $[0,1]\longrightarrow\Psi^{-\infty}(\mathbb{R}^n)$ vanishing to infinite order at $\epsilon=0$.

The analogue of the 'principal symbol map' for pseudodifferential operators is played by the 'semiclassical symbol'

157.394 (6.84)
$$\begin{gathered}A(\epsilon)=Q_L(a(\epsilon,x,x\xi)),\ \epsilon>0\Longrightarrow\\ \sigma_{\mathrm{sl}}:\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n)\ni\longrightarrow a(0,y,\eta)\in\mathcal{S}(\mathbb{R}^{2n}_{x,\eta})\end{gathered}$$

157.395 PROPOSITION 6.5. *The semiclassical symbol map (157.394 6.84) gives a short exact sequence of algebras*

157.396 (6.85)
$$\epsilon\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n)\longrightarrow\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n)\xrightarrow{\sigma_{\mathrm{sl}}}\mathcal{S}(\mathbb{R}^{2n})$$

with the commutative product on $\mathcal{S}(\mathbb{R}^{2n})$.

PROOF. This is just the first term in the Moyal product. □

Make sure you understand what a semiclassical smoothing operator 'looks like'. Its kernel is of the form

157.408
$$A_\epsilon(x,y) = \epsilon^{-n} B(\epsilon, x, \frac{x-y}{\epsilon}),\ B \in \mathcal{C}^\infty([0,1]_\epsilon; \mathcal{S}(\mathbb{R}^{2n}) \tag{6.86}$$

so as $\epsilon \downarrow 0$ the kernel is 'squashed' around the diagonal.

There are semiclassical operators of finite order as well, I hope I will have time to talk a little about them. A semiclassical differential operator on $\mathbb{R}^n$ might be of the form

157.409
$$P_\epsilon = \epsilon^2 \sum_{i=1}^{n} D^2_{x_j} + V(x). \tag{6.87}$$

Notice that if you set $\epsilon = 0$ only the zeroth order term survives. This is not the semiclassical symbol which is instead the 'rescaled' symbol including the lower order term:

157.410
$$\sigma_{\mathrm{sl}}(P_\epsilon) = |\xi|^2 + V(x). \tag{6.88}$$

Operators like (6.87) arose in quantum mechanics where $\epsilon \simeq \hbar$ is the 'coupling constant' relating the frequency of spectral lines to the jump in energy between electron shells which produces them. In practice $\hbar \simeq 1/127$ is small and the idea is to think of it as 'very small' and perturb from $\epsilon = 0$ (I don't like to use $\hbar$ here since it is actually a constant!) Simply setting $\epsilon = 0$ is a bad idea since most of the problem disappears. What is happening here is that the problem is becoming commutative as $\epsilon \downarrow 0$ because of the commutation condition that

157.427a
$$[\epsilon \partial_{x_i}, x_k] = \epsilon \delta_{ik} \tag{6.89}$$

so one is turning on the non-commutative product as ϵ becomes positive.

There is a lot one can do with semiclassical operators – see for instance the book of Zworski [**9**].

6. The group G^{∞}_{sl}

As remarked above we want to consider the semiclassical group analogous to $G^{\infty}_{\mathcal{S}}(\mathbb{R}^n)$. In fact we need to generalize the discussion above by allowing 'smoothing values'. If we consider the Schwartz smoothing operators on $\mathbb{R}^{n'+n}$ then we know that

157.400
$$\mathcal{S}(\mathbb{R}^{2n'+2n}) = \mathcal{S}(\mathbb{R}^{2n'}; \mathcal{S}(\mathbb{R}^{2n})). \tag{6.90}$$

The smoothing operators on $\mathbb{R}^{n'+n}$ can then be considered as 'smoothing operators with values in smoothing operators', in either direction. We can do the same thing for the semiclassical smoothing operators and consider the space

157.401
$$\mathcal{S}(\mathbb{R}^{2n'}; \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n)). \tag{6.91}$$

Then we get a 'stabilized' algebra

157.402
$$\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})). \tag{6.92}$$

For the kernel of an element in this algebra (6.86) is replaced by

157.411
$$A_\epsilon(x,y,z,t) = \epsilon^{-n} B(\epsilon, x, \frac{x-y}{\epsilon}, t, t'),\ B \in \mathcal{C}^\infty([0,1]_\epsilon; \mathcal{S}(\mathbb{R}^{2n'+2n})). \tag{6.93}$$

Observe what the form that the semiclassical symbol takes in this more general case. It is just a symbol with values in the smoothing operators

157.406
$$\sigma_{\mathrm{sl}} : \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})) \longrightarrow \mathcal{S}(\mathbb{R}^{2n}_{x,\xi}; \Psi^{-\infty}(\mathbb{R}^{n'})) \tag{6.94}$$

giving a short exact sequence just like 157.396 (6.85)

157.407
$$\epsilon\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})) \longrightarrow \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})) \longrightarrow \mathcal{S}(\mathbb{R}^{2n}; \Psi^{-\infty}(\mathbb{R}^{n'})). \tag{6.95}$$

So the product formula for the symbol in this case is multiplicative in the variables (x, ξ) but with values in the non-commutative algebra $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$.

157.404 PROPOSITION 6.6. *The semiclassical group*

157.405
$$\begin{aligned} G^{\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^{n'})) = \{&A \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}));\\ &\exists\ B \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})) \text{ with } (\mathrm{Id}+A)(\mathrm{Id}+B) = \mathrm{Id} = (\mathrm{Id}+B)(\mathrm{Id}+A)\} \end{aligned} \tag{6.96}$$

is open in $\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}))$.

PROOF. This is similar too, but a bit more involved than, the proof of Lemma 157.356 6.3. The topology on the semiclassical algebra $\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}(\mathbb{R}^{n'}))$ comes from its identification with $\mathcal{C}^\infty([0,1]; \mathcal{S}(\mathbb{R}^{2n+2n'}))$ through either 157.411 (6.93) or 157.399 (6.83). From the latter it follows that $A(\epsilon)$ defines a bounded family of operators on $L^2(\mathbb{R}^n)$ with a uniform norm bound being a continuous seminorm. So if A lies in a sufficently small neighbourhood of 0 the family, in ϵ, of operators

157.412
$$(\mathrm{Id} - A(\epsilon))^{-1} = \mathrm{Id} + B(\epsilon) \text{ exists for } \epsilon > 0 \tag{6.97}$$

with $B(\epsilon)$ uniformly bounded as $\epsilon \downarrow 0$. So we only need to show that

157.413
$$B(\epsilon) \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})). \tag{6.98}$$

Proceeding 'symbolically' we can see that the model problem, the existence of the inverse

157.414
$$(\mathrm{Id} + \sigma_{\mathrm{sl},\mathcal{S}}(A))^{-1}(y, \eta) = \mathrm{Id} + B_0(x, \eta) \in G^{-\infty}_{\mathcal{S}}(\mathbb{R}^{2n'}), \tag{6.99}$$

has a unique solution, for A in a possibly smaller neighbourhood of 0. This indeed follows from Lemma 157.356 6.3. Once we know that the principal symbol can be inverted, the existence of a formal power series inverse follows from the behaviour of the Moyal product. That is, we can find a sequence of elements $B_k \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}))$ such that for any p

157.415
$$(\mathrm{Id} + A(\epsilon)(\mathrm{Id} + B_0(\epsilon)) + \sum_{k=1}^{p} \epsilon^k B_k(\epsilon) \in \mathrm{Id} + \epsilon^{p+1} \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})). \tag{6.100}$$

Using Borel's lemma again we can sum the series and so find a 'parametrix' $B' \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}))$ such that

157.416
$$(\mathrm{Id} + A(\epsilon)(\mathrm{Id} + B'(\epsilon)) - \mathrm{Id},\ (\mathrm{Id} + B'(\epsilon))(\mathrm{Id} + A(\epsilon)) - \mathrm{Id} \in \dot{\mathcal{C}}^\infty([0,1]; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n+n'})) \tag{6.101}$$

(which is the same as saying a semiclassical error vanishing to all orders at $\epsilon = 0$).

The corner identity as in 157.392 (6.33) then shows that the difference of the inverse and the parametric $B - B'$ is uniformly bounded as a function of $\epsilon \in (0, 1]$ with values in $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n+n'})$. A similar argument for the derivatives with respect to ϵ

shows that this difference is actually smooth and vanishes to infinite order at $\epsilon = 0$ so proving (6.98). $\square$

For the semiclassical group the symbol sequence (6.95) gives the map

157.418 (6.102)
$$\sigma_{\mathrm{sl}} : G^{\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})) \longrightarrow \mathcal{S}(\mathbb{R}^{2n}; G^{\infty}(\mathbb{R}^{n'}))$$

which it is important to note is surjective. In fact we want a stronger lifting property

157.417 Proposition 6.7. *The semiclassical group has the lifting property that for any smooth symbol map $u : M \longrightarrow \mathcal{S}(\mathbb{R}^{2n}; G^{\infty}(\mathbb{R}^{n'}))$ of compact support (reducing to the identity outside a compact set) there is a smooth map, also of compact support $\tilde{u} : M \longrightarrow G^{\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}))$ giving a commutative diagramme*

157.419 (6.103)
$$\begin{tikzcd} G^{\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})) \arrow[r, "\sigma_{\mathrm{sl}}"] & \mathcal{S}(\mathbb{R}^{2n}; G^{\infty}(\mathbb{R}^{n'})) \\ & M. \arrow[u, "u"'] \arrow[ul, "\tilde{u}"] \end{tikzcd}$$

Moreover any two such lifts are smoothly homotopic.

Proof. This is another symbolic argument, just a uniform version of the surjectivity of (6.102). $\square$ RBM:Expand?

This might appear a little arcane! However a consequence of the existence of a lift as in (6.103) is that this semiclassical group serves as a 'bridge' for odd K-theory. Namely a family $u \in \mathcal{C}^{\infty}_{\mathrm{c}}(\mathbb{R}^{2n} \times M; G^{-\infty})$ as in Proposition 6.7 defines an element of the group $K^1(\mathbb{R}^{2n} \times M)$. On the other hand the lifted group, restricted to say $\epsilon = 1$ (or $\epsilon = \frac{1}{2}$ if you prefer) defines an element of $K^1(M)$, since it is valued in $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n+n'})$. Now we are getting closer to Bott periodicity.

157.420 Proposition* 6.8 (Bott periodicity). *The lifting construction in (6.103) defines, for any manifold M and any n, a homomorphism*

157.421 (6.104)
$$K^1(\mathbb{R}^{2n} \times M) \longrightarrow K^1(M)$$

which is an isomorphism.

This is a 'protypical' index map, it is a non-geometric version of Atiyah-Singer which is used in the proof of the general version below.

Once the existence of the map (6.104) is established there are two parts to the proof that it is an isomorphism. First we show that it is surjective, by a computation based on the existence of a 'Bott element' $\beta \in K^0(\mathbb{R}^2)$ which is a generator (so along the way we prove that $K^0(\mathbb{R}^2) = \mathbb{Z}$). This is where the hard work lies – and I may suppress some of it as far as the lectures are concerned (although it will all be in the notes). The second part is a clever idea of Atiyah. This is based on the fact that we can move some of the factors of $\mathbb{R}^2$ in (6.104) into M. Indeed, once we have the map (6.104) for $n = 1$ we can get a similar map by iterating – applying the map

157.422 (6.105)
$$q : K^1(\mathbb{R}^2 \times M) \longrightarrow K^1(M)$$

to $\mathbb{R}^{2n-2} \times M$ instead of M. This leads one to think in terms of the iterated map

$$K^1(\mathbb{R}^4 \times M) \xrightarrow{q} K^1(\mathbb{R}^2 \times M) \xrightarrow{q} K^1(M)$$

The way we are getting (6.104) can be described as 'turning on quantization' by writing $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Looking again at (6) we can think of $\mathbb{R}^4 = \mathbb{R}^2_{x,\xi} \times \mathbb{R}^2_{y,\eta}$ where we first turn on quantization in (x, ξ) for the first map, then turn on quantization in (y, η) for the second map. So we conclude easily enough that this is the same as doing both simultaneously in the sense that we get a commutative diagramme forming the upper right triangle

157.424 (6.106)
$$\begin{tikzcd} K^1(\mathbb{R}^4 \times M) \arrow[r] \arrow[d] \arrow[dr] & K^1(\mathbb{R}^2 \times M) \arrow[d] \\ K^1(\mathbb{R}^2 \times M) \arrow[r] & K^1(M) \end{tikzcd}$$

where the diagonal map is (6.104) for $n = 2$. Atiyah's idea has two parts. The first is that we get a further commutative triangle on the lower left where the roles of the two quantizations are interchanged – the second set of variables are quantized

RBM:Expand or refer first, giving the same final result.

However, we can do a little more, namely we can rotate the variables, smoothly, by looking at $x \cos\theta + y \sin\theta$ and dually on the other variables. This gives us a smooth family of double quantization maps starting at the first and finishing at the second, with the sign reversed. By homotopy invariance these must all give the same result.

The way we use this depends on a good understanding of the lifting map in (6.103) as giving us a right inverse to (6.105) which we can write somewhat mysteriously as

157.425 (6.107)
$$K^1(M) \ni \kappa \longmapsto \beta \otimes \kappa \in K^1(\mathbb{R}^2 \times M)$$

where as above, β is the Bott element of $K^0(\mathbb{R}^2)$.

So, suppose that $\kappa \in K^1(\mathbb{R}^2 \times M)$ is mapped to zero in $K^1(M)$ by (6.105), so along the right side of (6.106). By surjectivity κ comes from $\beta_1 \otimes \kappa \in K^1(\mathbb{R}^4 \times M)$ along the top line of (6.106), so lifting in the first variables. Reversing the quantization as discussed above we see that

157.426 (6.108)
$$\kappa = \pm q_2(\beta_1 \otimes \kappa) = \pm \beta_1 \otimes q_2(\kappa) = 0$$

so $\kappa = 0$ and we have injectivity.

This argument depends on seeing that quantization in one set of variables commutes with the lifting map in a different set of variables which we will see in

L11-end the lifting construction.

L12/13

7. Constructing the Bott element

I am trying to take a minimalist approach here to avoid getting bogged down in K-theory. The aim is to construct the lifting map in (6.103) and think of it as giving (6.108). As already remarked, this is really an exterior product in K-theory

157.427 (6.109)
$$K^0(\mathbb{R}^2) \times K^1(M) \longrightarrow K^1(\mathbb{R}^2 \times M)$$

coming from $\beta \in K^0(\mathbb{R}^2)$ which is a generator.

The standard model for even K-theory over a manifold, X, is in terms of pairs of (compactly supported) complex vector bundles. Such a vector bundle can always be realized as a family of projections in $N \times N$ matrices for sufficiently large N so as a smooth map

157.515 (6.110) $$P : X \longrightarrow M(N, \mathbb{C}),\ P(x)^2 = P(x).$$

For the Bott element in $K^0(\mathbb{R}^2)$ this projection arises in 2×2 matrices and it is convenient (but by no means necesary) to view the projection as coming from a family of involutions. As we have seen above, if β is an involutive matrix (or operator) meaning $\beta^2 = \mathrm{Id}$, then

157.516 (6.111) $P = \frac{1}{2}(\beta + \mathrm{Id})$ is a projection and conversely $\beta = P - (\mathrm{Id} - P) = 2P - \mathrm{Id}$.

The explicit 2×2 family of matrices we will to consider satisfies

157.428 (6.112) $$\beta : \mathbb{R}^2 \longrightarrow M(2, \mathbb{C}),\ \beta^2 = \mathrm{Id},\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P_1 - P_2 \text{ near } \infty.$$

The 'triviality near infinity' is analogous to the group G^∞.

Explicitly in this sense the Bott element (although how it defines an element of $K^0(\mathbb{R}^2)$ is not yet explained) is

157.430 (6.113) $$\beta(x, \xi) = \begin{pmatrix} \cos(\chi(r)) & e^{i\theta} \sin(\chi(r)) \\ e^{-i\theta} \sin(\chi(r)) & -\cos(\chi(r)) \end{pmatrix}.$$

Here $(x, \xi) = r(\cos\theta, \sin\theta)$ are polar coordinates, $\chi \in \mathcal{C}^\infty([0, \infty))$ is constant near 0 and ∞ and decreases monotonically from π to 0. Thus β is constant near 0 and hence smooth and is equal to β_∞ near ∞.

An easy computation shows

157.431 (6.114) $$\beta^2 = \mathrm{Id},\ \beta = 2\Pi(x, \xi) - \mathrm{Id},$$
$$\Pi(x, \xi) = \begin{pmatrix} \cos^2(\frac{1}{2}\chi(r)) & e^{i\theta} \sin(\frac{1}{2}\chi(r)) \cos(\frac{1}{2}\chi(r)) \\ e^{-i\theta} \sin(\frac{1}{2}\chi(r)) \cos(\frac{1}{2}\chi(r)) & \sin^2(\frac{1}{2}\chi(r)) \end{pmatrix}.$$

Thus the range of the projection $\Pi(x, \xi)$ is the span of

157.432 (6.115) $$\begin{pmatrix} \cos(\frac{1}{2}\chi(r)) \\ e^{-i\theta} \sin(\frac{1}{2}\chi(r)) \end{pmatrix}.$$

This is a complex line bundle which comes from a non-trivial line bundle on $\mathbb{S}^2$ as the one-point compactification of $\mathbb{R}^2$.

8. Quantization of the Bott element

The way we will make use of the involution (157.430 6.113) is by quantizing it to a semiclassical family of involutions.

Since $b_0 = \beta - \beta_\infty$ is a compactly supported smooth function with values in 2×2 complex matrices we can apply semiclassical quantization to find

157.433 (6.116) $$B_0 \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}; \mathbb{C}^2) \text{ with } \sigma_{\mathrm{sl}}(B_0) = b_0 = \beta - \beta_\infty$$

So, B_0 is a 2×2 matrix of semiclassical smoothing operators on $\mathbb{R}$. As with the identity we take the semiclassical quantization of the constant matrix $\beta_\infty = P_1 - P_2$ to be itself.

157.434 Proposition 6.9. *There exists $B \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R}; \mathbb{C}^2)$ satisfying*

157.435 (6.117) $$\sigma_{\mathrm{sl}}(B) = b_0,\ (\beta_\infty + B)^2 = \mathrm{Id}.$$

PROOF. First make any choice B_0 as in (6.116). Then, from the semiclassical symbol calculus the principal semiclassical symbol of the square satisfies

157.436
$$\sigma_{\mathrm{sl}}((\beta_\infty + B_0)^2 - \mathrm{Id}) = \sigma_{\mathrm{sl}}(\beta_\infty b_o - b_o\beta_\infty + b_0^2) = 0. \tag{6.118}$$

Thus in fact

157.437
$$(\beta_\infty + B_0)^2 = \mathrm{Id} + \epsilon E_1,\ E_1 \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R};\mathbb{C}^2). \tag{6.119}$$

Now, we proceed, by induction, to construct successive corrections B_k 'as usual' (cf. Project 2) so that

157.438
$$(\beta_\infty + B_0 + \sum_{k=1}^{p} \epsilon^k B_k)^2 = \mathrm{Id} + \epsilon^{k+1} E_{k+1},\ E_{k+1} \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R};\mathbb{C}^2). \tag{6.120}$$

Adding $\epsilon^{p+1}B_{k+1}$, where B_{k+1} has semiclassical symbol b_{k+1} what we need to arrange in the inductive step is

157.439
$$\begin{aligned}(\beta_\infty + B)B_{k+1} + B_{k+1}(\beta_\infty + B) = E_{p+1} \quad \mathrm{mod}\ \epsilon\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R};\mathbb{C}^2)\\ \Longleftrightarrow \beta b_{k+1} + b_{k+1}\beta = e_{k+1} = \sigma_{\mathrm{sl}}(E_{k+1}).\end{aligned} \tag{6.121}$$

This cannot be solved for an arbitrary e_{k+1}. However, it follows from the definition, (6.120), of the error at the previous stage of construction, that

157.440
$$(\beta_\infty + B)E_{k+1} - E_{k+1}(\beta_\infty + B) \in \epsilon\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R};\mathbb{C}^2) \Longrightarrow \beta e_{k+1} = e_{k+1}\beta. \tag{6.122}$$

This means that

157.493
$$e_{k+1} = Pe_{k+1}P + (\mathrm{Id} - P)e_{k+1}(\mathrm{Id} - P) \tag{6.123}$$

is 'diagonal' with respect to the two projections. Thus

157.441
$$b_{k+1} = \frac{1}{2}\left(Pe_{k+1}P - (\mathrm{Id} - P)e_{k+1}(\mathrm{Id} - P)\right) \tag{6.124}$$

solves the inductive condition in (6.121).

Asymptotically summing the resulting series in ϵ^k we find a semiclassical operator B' with the correct principal symbol such that

157.442
$$(\beta_\infty + B')^2 - \mathrm{Id} \in \epsilon^\infty\Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R};\mathbb{C}^2) \tag{6.125}$$

is a smooth family of smoothing operators vanishing to infinite order at $\epsilon = 0$. It follows that as a family of bounded operators on $L^2(\mathbb{R};\mathbb{C}^2)$ the spectrum is, uniformly for $0 < \epsilon < \epsilon_0$ (for suitable small ϵ_0), concentrated very near to ± 1. We can then use a contour integral (the functional calculus for bounded operators on a Hilbert space) to correct the quantization a little further, setting

157.443
$$\beta_\infty + B = B_+ - B_-,\ B_\pm = \pm\frac{1}{2\pi i}\oint_{C^\pm}(\beta_\infty + B' - z\,\mathrm{Id})^{-1}dz. \tag{6.126}$$

Here $C^\pm$ are circular contours around ± 1. Now an argument with resolvents shows that the $B_\pm$ are commuting projections and

157.444
$$(\beta_\infty + B_+ - B_-)^2 = \mathrm{Id} \tag{6.127}$$

where $B_\pm$ are defined only for $0 < \epsilon < \epsilon_0$ but $B = B_+ - B_-$ is equal to a semiclassical family on $0 < \epsilon < \frac{1}{2}\epsilon_0$ (simply by rescaling the parameter ϵ to stretch the interval $[\frac{1}{2}\epsilon_0, \frac{3}{4}\epsilon]$ to $[\frac{1}{2}\epsilon, 1]$). The difference $B - B'$ is a family of smoothing operators vanishing to infinite order at $\epsilon = 0$.

This completes the proof of Lemma 6.116. $\square$

9. Simplifying the quantized Bott operator

RBM:Relate to §6.4

So this was hard work, but we need to do a little more to see what we have really produced and refine it further. Choose an L^2-normalized element $p \in \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ and let Q also denote the orthogonal projection onto $(0, e_0)$ where e_0 is the ground state of the harmonic oscillator (it is just a convenient rank one projection).

157.477 Proposition 6.10. *There is a semiclassical family $B \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R};\mathbb{C}^2)$, quantizing β as in (*157.435 *6.117), such that*

$$\beta_0 + B\big|_{\epsilon=1} = (\Pi_1 + Q) - (P_2 - Q). \tag{6.128}$$

Thus the effect of quantization is to move a one-dimensional space from the negative to the positive eigenspace. This really is the fundamental 'index theorem' from this point of view.

Proof. We first show that any (single, not semiclassical family) involution of the form

157.517
$$\beta_0 + S,\ S \in \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R};\mathbb{C}^2), (\beta_0 + S)^2 = \mathrm{Id} \tag{6.129}$$

can be deformed to a model. This argument does not depend on the particular β_∞ nor on the dimension involved.

Where does the 'one-dimensional' come from. We have constructed P and hence a projection

157.478
$$P = \frac{1}{2}(\beta_\infty + B) + \mathrm{Id} = \Pi_1 + S,\ S \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R};\mathbb{C}^2) \Longrightarrow P^2 = P. \tag{6.130}$$

The range of $P(\epsilon)$ for each $\epsilon > 0$ is a closed subspace of $L^2(\mathbb{R}^2;\mathbb{C}^2)$

So the idea is that for $\epsilon > 0$ this is a projection which differs from Π_1 by a smoothing, hence trace classe, term. The *relative index* of these two projections is therefore well-defined

157.479
$$R(\epsilon) = \mathrm{Tr}(P(\epsilon) - \Pi_1) \in \mathcal{C}^\infty((0,1]). \tag{6.131}$$

In fact, this is constant and necessarily an integer which we then proceed to compute.

The constancy follows from the properties of the trace functional and projections. Differentiating

157.480
$$\frac{dR}{d\epsilon} = \mathrm{Tr}(\frac{dB}{d\epsilon}) = \mathrm{Tr}(\frac{dB}{d\epsilon}) = 0 \text{ since} \\ \frac{dB}{d\epsilon} = P\frac{dB}{d\epsilon} + \frac{dB}{d\epsilon}P \Longrightarrow \frac{dB}{d\epsilon} = P\frac{dB}{d\epsilon}(\mathrm{Id} - P) + (\mathrm{Id} - P)\frac{dB}{d\epsilon}P. \tag{6.132}$$

Note that this is the same identity as used in the construction of B.

Having shown that R is constant we compute it in terms of the semiclassical limit.

157.481 Lemma 6.5. *For a semiclassical family of smoothing operators $B \in \Psi^{-\infty}_{\mathrm{sl},\mathcal{S}}(\mathbb{R};\mathbb{C}^N)$, $\epsilon\,\mathrm{Tr}(B_\epsilon) \in \mathcal{C}^\infty([0,1])$ and if the kernel is*

157.483
$$B = Q_L(b(\epsilon, x, \epsilon\xi)) \tag{6.133}$$

then

157.482
$$\mathrm{Tr}(B_\epsilon) = \epsilon^{-1}\int_{\mathbb{R}^2} \mathrm{tr}(b(0)) + \int_{\mathbb{R}^2} \mathrm{tr}(\frac{\partial b}{\partial\epsilon}(0)) + O(\epsilon). \tag{6.134}$$

Proof. ◻

◻

10. Surjectivity of the periodicity map

Now we have a Bott element and its quantization. Note that I defined $K^0(\mathbb{R}^2)$ to be $K^1(\mathbb{R}^3)$ in Definition 157.388 6.2 so to get an element there we need to do just a little more. Take a smooth function on $\mathbb{R}$ with 'winding number one' such as

157.445 $$\exp(i\psi(s)),\ \psi\in\mathcal{C}^\infty(\mathbb{R};\mathbb{R}),\ \psi=0 \text{ in } s<-R,\ \psi=2\pi \text{ in } s>R. \tag{6.135}$$

Now consider the family of operators

157.446 $$\mathbb{R}\ni s\longrightarrow b\,(\exp(i\psi(s))b_+ + \exp(-i\psi(s)b_-)\,. \tag{6.136}$$

Both factors here are semiclassical families, the second depending on s, (with L12/13-end values in 2×2 matrices) which is the identity for $|s|>R$ and

11. Manifolds with boundary

We have defined the K-theory of any manifold, including a manifold with boundary, M. In this case there are three natural K-groups, well six including even/odd groups. Namely

157.497 $$\begin{gathered} K^1(M)=\mathcal{C}^\infty_{\mathrm{c}}(M;G^{-\infty})/\text{homotopy}\\ K^1(M;\partial M)=\mathcal{C}^\infty_{\mathrm{c}}(M\setminus\partial M;G^{-\infty})/\text{homotopy}\\ K^1(\partial M)=\mathcal{C}^\infty_{\mathrm{c}}(\partial M;G^{-\infty})/\text{homotopy}. \end{gathered}\tag{6.137}$$

According to our definition the even groups are defined as the odd groups for $\mathbb{R}\times M$.

These are related by a 'six-term sequence'

157.498 (6.138)
$$\begin{tikzcd} K^0(M;\partial M) \arrow[r] & K^0(M) \arrow[r] & K^0(\partial M) \arrow[d] \\ K^1(\partial M) \arrow[u] & K^1(M) \arrow[l] & K^1(M;\partial M) \arrow[l] \end{tikzcd}$$

The horizontal maps here are straightforward. Namely there is an inclusion map

$$\mathcal{C}^\infty_{\mathrm{c}}(M\setminus\partial M;G^{-\infty})\hookrightarrow\mathcal{C}^\infty_{\mathrm{c}}(M;G^{-\infty})$$

since the former are maps which are equal to the identity outside a compact set and so can be extended (as the identity) up to the boundary. Similarly there is a restriction map

$$\mathcal{C}^\infty_{\mathrm{c}}(M;G^{-\infty})\xrightarrow{|_{\partial M}}\mathcal{C}^\infty_{\mathrm{c}}(\partial M;G^{-\infty})$$

and both these maps 'descend' through homotopies.

The end maps, the connecting homomorphisms, are as usual a little less obvious. On the right we start from the group

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}\times\partial M;G^{\infty}).$$

By a very simple version of the collar neighbourhood theorem a neighbourhood of the boundary in M is diffeomorphic to a product

$$[0,1)\times\partial M\hookrightarrow M.$$

We can take the radial compactification of $\mathbb{R}$ to $[0,1]$ and thereby convert the map (157.503 11) into a map

$$\mathbb{R} \times \partial M \longrightarrow M.$$

The definition in terms of rapid decay then means that there is a pull-back map

157.505 $$\mathcal{S}(\mathbb{R}; \mathcal{C}^\infty(\partial M; G^{-\infty})) \hookrightarrow \mathcal{C}^\infty_{\mathrm{c}}(M; G^{-\infty}) \tag{6.139}$$

which descends to give the connection homorphism on the right.

This leaves the connecting homomorphism on the left. This is where Bott periodicity comes in. We are starting from (the components of) $\mathcal{C}^\infty_{\mathrm{c}}(M; G^{-\infty})$ on the lower left. However, by Bott perioidicty – Theorem **??** – we can instead start from $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^2 \times M; G^{-\infty})$. Now, separating the factors of $\mathbb{R}^2$ into $\mathbb{R} \times \mathbb{R}$ we can apply the argument of the preceeding paragraph, giving the connecting homomorphism on the right, to $\mathbb{R} \times M$ instead of M, and this gives the map on the left. There is an orientation issue here but since it amounts to a sign the choise one makes does not affect the main result, which is:

157.499 Theorem 6.2. *The six-term sequence (157.498 6.138) is exact for any manifold with boundary.*

12. Chern character-building

Project 3 P-Chern

I have laid rather heavy emphasis on the group(s) $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ in the definition of (complex, topological) K-theory. Indeed by fiat I have declared this to be a classifying group for odd K-theory. Here I want you to sort out the map to deRham cohomology leading to the Atiyah-Hirzebruch isomorphism.

In the first part I want you to explain, step by step, the meaning of the 'odd Chern character'

$$\mathrm{Ch}(g) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2\pi i)^k (2k+1)!} \operatorname{Tr}\left((g^{-1}dg)^{2k+1}\right)$$

This is a formal sum of forms in odd degree.

(1) Recall that $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ is an open subset of $\mathcal{S}(\mathbb{R}^{2n})$ and it is 'an infinite matrix group'. Use this to give a clear meaning to the Maurier-Cartan form $g^{-1}dg$ here.

Hint: Perhaps avoid going into a full discussion of Fréchet manifolds! As an open subset of a Fréchet space the tangent space at each point g is $\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$. Any element $a \in \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ here defines a curve in the group in the obvious way as $g + ta$ and the identification with the tangent space (with elements equivalence classes of curves) is written 'dg' – meaning I think that $d(g+ta)/dt$ is the tangent vector at g. Then g^{-1} acting on the left on the group maps g to the origin. This is a concrete group in the sense that this gives a linear map

$$g^{-1} : \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n).$$

The push-forward on the tangent spaces would be denoted g^{-1}_* but since the map is linear we can drop the $*$ and then

$$g^{-1}dg : \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$$

is the natural map from the tangent space at g to the tangent space at Id (which is the Lie algebra).

(2) Then the formal product $(g^{-1}dg)^j$ is supposed to be the j-fold exterior product, with composiition thrown in, so it is a j-multilinear, totally antisymmetric map from j copies of the tangent space at g to the tangent space at Id .

(3) This explains (P3.14 12) in the sense that the trace functional results in each term being a $(2k+1)$-multilinear function on the tangent space at each point, which is what a form should be.

(4) The trace property $\mathrm{Tr}([a,b]) = 0$ shows that

$$\mathrm{Tr}((g^{-1}dg)^{2k}) = 0$$

which explains why there are no even terms.

(5) The deRhan differential is easy to define on 1-forms, even with values in an infinite dimensional space, show that

$$d(g^{-1}dg) = -g^{-1}dg \wedge g^{-1}dg$$

in an appropriate sense.

(6) Conclude that the terms in $\mathrm{Ch}(g)$ are all closed.

(7) Suppose $[0,1] \ni t \longmapsto G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ is a smooth curve, show that

$$\frac{d}{dt}\,\mathrm{Ch}(g_t) = d\,\mathrm{Et}(g)$$

where Et is the 'Eta' or Chern-Simons form

$$\mathrm{Et}(g) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}k!}{(2\pi i)^k(2k)!}\,\mathrm{Tr}\left(g^{-1}\frac{dg}{dt}(g^{-1}dg)^{2k}\right)$$

Hint: Make sense of the formula

$$\frac{d}{dt}g^{-1}dg = -g^{-1}\frac{dg}{dt}g^{-1}dg + g^{-1}d\frac{dg}{dt}.$$

(8) Now, suppose M is a compact manifold and $\kappa : M \longrightarrow G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$ is a smooth map. Show that the pull-backed form is closed and so defines a map

$$\mathcal{C}^\infty(M; G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)) \ni \kappa \longmapsto \kappa^*\,\mathrm{Ch} \in H^{\mathrm{odd}}_{\mathrm{dR}}(M).$$

(9) Conclude that this map descends to a map

$$\mathrm{Ch} : K^1(M) \longrightarrow H^{\mathrm{odd}}_{\mathrm{dR}}(M) \text{ the odd Chern character.}$$

(10) Contemplate why this might induce the Atiyah-Hirzebruch isomorphism

$$K^1(M) \otimes \mathbb{C} \longrightarrow H^{\mathrm{odd}}_{\mathrm{dR}}(M).$$

(So this means torsion is killed in the K-group. This does not quite work for general non-compact manifolds).

(11) Now, if you have the energy, do the even version! Start from (P3.20 7) and show that on the group $\mathcal{C}^\infty(\mathbb{R}; G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n))$, which is used above to define even K-theory,

$$\mathrm{Ch}_{\mathrm{ev}} = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}k!}{(2\pi i)^k(2k)!}\,\mathrm{Tr}\left(g^{-1}\frac{dg}{dt}(g^{-1}dg)^{2k}\right) dt$$

defines a closed form in each even degree. [I am not sure I have the normalizing constants correct, but here it does not matter.]

(12) Conclude, proceeding as above, that this defines a map

$$K^0(M)\otimes\mathbb{C}\longrightarrow H^{\text{ev}}_{\text{dR}}(M).$$

13. Isotropic K-pop

Project 4 $\boxed{\texttt{P-Iso}}$

P3b.0 Recall to isotropic pseudodifferential algebra, for simplicity of order 0, on $\mathbb{R}^{2n}$ is defined by quantization of the classical symbols joointly in the variables (x,ξ) :

$$\Psi^0_{\text{iso}}(\mathbb{R}^n)=Q_L(\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}).$$

The product defines a smooth bilinear map

$$\Psi^0_{\text{iso}}(\mathbb{R}^n)\times\Psi^0_{\text{iso}}(\mathbb{R}^n)\longrightarrow\Psi^0_{\text{iso}}(\mathbb{R}^n).$$

We can then define $\mathcal{C}^\infty(M;\Psi^0_{\text{iso}}(\mathbb{R}^n)$ for any manifold M and in particular

$$\mathcal{S}(\mathbb{R}^{2n'};\Psi^0_{\text{iso}}(\mathbb{R}^n)).$$

Now, show that this allows us to define the 'stabilized algebra'

$$\Psi^0_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$$

by using the product of Schwartz smoothing operators and then the product ($\overset{\text{P3.3}}{13}$) to define

$$\Psi^0_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\times\Psi^0_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longleftrightarrow\mathcal{S}(\mathbb{R}^{2n'};\Psi^0_{\text{iso}}(\mathbb{R}^n)\times\Psi^0_{\text{iso}}(\mathbb{R}^n))\longrightarrow\Psi^0_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

This is an associative algebra of bounded operators on $L^2(\mathbb{R}^{n+n'})$ which maps $\mathcal{S}(\mathbb{R}^{n+n'})$ to itself.

(1) One reason that this algebra is interestimg is that it has a principal symbol map

$$\sigma_0:\Psi^0_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\mathcal{C}^\infty(\mathbb{S}^{2n-1};\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$$

which is multiplicative and gives a short exact sequence

$$\Psi^{-1}_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\Psi^0_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\longrightarrow\mathcal{C}^\infty(\mathbb{S}^{2n-1};\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

(2) We need to massage this a little more by identifying

$$\mathbb{R}^{2n-1}\longrightarrow\mathbb{S}^{2n-1}$$

as the 1-point compactification – choose your favourite point (mine is the South Pole). Anyway, this allows us to map

$$\mathcal{S}(\mathbb{R}^{2n-1};\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^{n'})\hookrightarrow\mathcal{C}^\infty(\mathbb{S}^{2n-1};\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$$

with image being the subspace of functions vanishing to infinite order at the point at infinity.

(3) Denote by $\dot\Psi^0_{\text{iso}}(\mathbb{R}^n;\Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})$ which is the inverse image under σ_{iso} of $\mathcal{S}(\mathbb{R}^{2n-1};\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^{n'})$

(4) This is an algebra without identity, so add Id to get a ring which we can denote

$$\mathrm{Id} + \dot{\Psi}^0_{\mathrm{iso}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

Check that we now a multiplicative exact sequence

$$\mathrm{Id} + \Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}) \longrightarrow \mathrm{Id} + \dot{\Psi}^0_{\mathrm{iso}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}) \longrightarrow \mathrm{Id} + \mathcal{S}(\mathbb{R}^{2n-1}; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

(5) Finally we can finish the setup by defining

$$\dot{E}(\mathbb{R}^n; \Psi^{-\infty}(\mathbb{R}^{2n'})) = \left\{A \in \mathrm{Id} + \dot{\Psi}^0_{\mathrm{iso}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}; \sigma_{\mathrm{iso}}(A) \in \mathcal{C}^\infty(\mathbb{R}^{2n-1}; G^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'})\right\}'$$

and now there is a multliplicative exact sequence

$$\mathrm{Id} + \Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}) \longrightarrow \dot{E}_{\mathrm{iso}}(\mathbb{R}^n; \Psi^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}) \longrightarrow \mathcal{S}(\mathbb{R}^{2n-1}; G^{-\infty}_{\mathcal{S}}(\mathbb{R}^{n'}).$$

14. Bott and Clifford

Project 5 P-BC

In lectures I wrote down an explicit family of involutive matrices on $\mathbb{R}^2$ representing the 'Bott' element (in even K-theory) and showed that the semiclassical quantization of this family is a 1-dimensional shift from the constant family. Here I ask you to do this for $\mathbb{R}^{2n}$. Initially I assumed you knew about the structure of the complex Clifford algebra, I have now added a brief derivation in case you do not.

(1) Deconstruct the matrix I simply wrote down in lectures

$$\mu = \begin{pmatrix} \cos(\chi(r)) & e^{i\theta}\sin(\chi(r)) \\ e^{-i\theta}\sin(\chi(r)) & -\cos(\chi(r)) \end{pmatrix}.$$

First write it out as

$$\mu = \cos(\chi(r))Z + \sin(\chi(r))\left(\cos\theta E_1 + \sin\theta E_2\right).$$

Show that these 2×2 matrices satisfy

$$E_i^2 = Z^2 = \mathrm{Id},\ E_1E_2 + E_2E_1 = ZE_1 + E_1Z = ZE_2 + E_2Z = 0.$$

(2) Recall the complexified Clifford algebra on a real Euclidean vector space, V, of even dimension, $2n$, defined as the quotient of the infinte tensor algebra

$$\mathcal{T}(V) = \mathbb{C} \oplus V_{\mathbb{C}} \oplus (V_{\mathbb{C}} \otimes V_{\mathbb{C}}) \oplus \cdots = \sum_k V_{\mathbb{C}}^{\otimes k},\ V_{\mathbb{C}} = V \otimes \mathbb{C}$$

by the two-sided ideal (under tensor product) generated by the elements

$$\mathcal{I}(V) \ni \xi\otimes\eta + \eta\otimes\xi - 2\langle\xi,\eta\rangle,\ \xi,\ \eta \in V.$$

Here $\langle,\rangle$ is the Euclidean inner product. Thus,

$$\mathrm{Cl}(V) = \mathcal{T}(V)/\mathcal{I}(V).$$

Remind yourself that this has dimension 2^{2n}, the same as the exterior algebra. You should also note that it is isomorphic to the $2^n \times 2^n$ matrix algebra.

The Clifford algebra is closely related to the exterior algebra, but on $\mathbb{C}^n$ rather than $\mathbb{R}^{2n}$. So we are passing to the standard complex structure on $\mathbb{R}^{2n}$ in which a basis, over the complex numbers, is

$$f_i = e_i + ie_{i+n},\ i = 1, \ldots, n.$$

Show that $\mathrm{Cl}(\mathbb{R}^{2n})$, the complexified Clifford algebra (not the Clifford algebra on the complexification!) acts on $\lambda^*\mathbb{C}^n$ through the formula we saw for the Hodge-Dirac operator

$$\mathrm{cl}(e_k + ie_{k+n})f^\alpha = i\sqrt{2}f_k \wedge f^\alpha,\ \mathrm{cl}(e_k - ie_{k+n})f^\alpha = -i\sqrt{2}\iota(f_k)f^\alpha,\ k \le n$$

where the constants are for length normalization an here α is a strictly increasing sequence in $\{1, \dots, n\}$. This specifies the action of all basis elements and we see that

$$\begin{aligned}\mathrm{cl}(e_k + ie_{k+n})\,\mathrm{cl}(e_k - ie_{k+n}) + \mathrm{cl}(e_k - ie_{k+n})\,\mathrm{cl}(e_k + ie_{k+n}) = 2\,\mathrm{Id},\\ \mathrm{cl}(e_k + ie_{k+n})^2 = 0 = \mathrm{cl}(e_k - ie_{k+n})^2\\ \Longrightarrow \mathrm{cl}(e_k)\,\mathrm{cl}(e_{k'}) + \mathrm{cl}(e_{k'})\,\mathrm{cl}(e_k) = 2\delta_{kk'}\,\mathrm{Id},\ k, k' = 1, \dots, 2n.\end{aligned}$$

(3) If e_i is an (oriented) orthonormal basis of V, let E_i be the corresponding elements in the Clifford algebra so

$$E_iE_j + E_jE_i = \delta_{ij}\,\mathrm{Id}\,.$$

Then show that the Clifford algebra has a 'maximal element'

$$Z = i^{n(2n-1)}E_1E_2\dots E_{2n} \Longrightarrow Z^2 = \mathrm{Id},\ ZE_i + E_iZ = 0.$$

(4) Okay, now observe that (1) is now

P3.35
$$\mu(\zeta) = \cos(\chi(|\zeta|)Z + \sin(\chi(r))(\hat{\zeta} \cdot E_*) \tag{6.140}$$

where $E_* = (E_1, \dots, E_{2n})$ is thought of as a vector with values in matrices and $\cdot$ is the inner product.

(5) Now observe that the definition (6.140) extends to $V = \mathbb{R}^{2n}$ to give a smooth family of involutive matrices. Here, as before $\chi(r)$ is decreasing from π near 0 to 0 near ∞.

(6) Review the proof in the notes to check that there is a semiclassical family of idempotents qunatizing μ (so now to smoothing operators on $\mathbb{R}^n$).

(7) For a bonus, show that the quantization is again has relative index 1 (assuming I got the signs right which would be a pleasant accident). You might like to do this by quantizing in two variables repeatedly, so working by induction over n.

CHAPTER 7

Operators on manifolds

C.OpMa L14

In this second half of the course I will, finally, get to the discussion of analysis on manifolds. First what I hope is a reminder of the invariant description of kernels of operators.

1. Functions and densities

The basic object defined on a smooth manifold M is the algebra of smooth functions $\mathcal{C}^\infty(M)$. Since we will want to include non-compact manifolds we need to also consider the subspace of functions of compact support

157.566
$$\mathcal{C}^\infty_{\mathrm{c}}(M) \subset \mathcal{C}^\infty(M). \tag{7.1}$$

In general there is no good analogue of schwartz functions on a non-compact manifold but on a manifold with corners the space, $\dot{\mathcal{C}}^\infty(M)$, of functions vanishing to infinite order at the boundary satisfies

157.567
$$\begin{array}{ccc} \dot{\mathcal{C}}^\infty(M) & \subset & \mathcal{C}^\infty(M) \\ \cup & & \cup \\ \dot{\mathcal{C}}^\infty_c(M) & \subset & \mathcal{C}^\infty_{\mathrm{c}}(M) \end{array} \tag{7.2}$$

and does correspond to the Schwartz space at least in the sense that

157.568
$$\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n}) = \mathcal{S}(\mathbb{R}^n). \tag{7.3}$$

Recall the change-of-variable formula for the integral of functions on Euclidean space. If $F : \Omega' \longrightarrow \Omega$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ and $f \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ then

157.569
$$\int_{\Omega'} F^* f |\det \frac{\partial F}{\partial x}| dx = \int_\Omega f(y) dy. \tag{7.4}$$

So there is no invariant integral of functions on a manifold.

To integrate we need something that transforms with the absolute value of the determinant of the Jacobian of the diffeomorphism that appears in (157.569 7.4). This is provided by *densities*. Recall that on an n-manifold the maximal degree, $n-$, forms at a point, $\Lambda^n_m M$ for $m \in M$, may be defined as the linear space of totally antisymmetric multilinear maps

157.570
$$T_m M \times \ n \text{ factors } \ \times T_m M \longrightarrow \mathbb{C}. \tag{7.5}$$

Equivalently if we let $\lambda^n T_m M$ denote the totally antisymmetric part of the n-fold tensor product of $T_m M$ with itself (so $\Lambda^n_m M = \lambda^n T^*_m M$) then $\Lambda^n_m M$ is the space of linear maps

157.571
$$\mu : \lambda^n T_m M \longrightarrow \mathbb{C}. \tag{7.6}$$

The closely related space of t-densities, for $t \in \mathbb{R}$, is

157.572 (7.7) $$\Omega^t_m M = \{\nu : \lambda^n T_m M \setminus \{0\} \longrightarrow \mathbb{C}; \nu(sv) = |s|^t \nu(v),\ \forall\, t \in \mathbb{R} \setminus \{0\}\}.$$

So these are t-absolutely homogeneous maps. That this is a vector space depends on the one-dimensionality of $\lambda^n T_m M$. Just as for the form bundles these one-dimensional vector spaces form a smooth bundle over M

157.573 (7.8) $$\Omega^t M \longrightarrow M.$$

The particularly important case $t = 1$ is just denoted ΩM.

So if $\mu \in \mathcal{C}^\infty(M; \Lambda^n M)$ is a smooth n-form on an n-manifold then

157.574 (7.9) $$|\mu|^t \in \mathcal{C}^0(M; \Omega^t) \text{ is a continuous density.}$$

A non-vanishing n-form on M exists if and only if M is orientable and then the t-density (7.9) becomes smooth. Note that positivity of a density is well-defined because of the transformation law in (7.7).

157.575 Lemma 7.1. *The density bundles are always trivial, i.e. there exists a smooth positive section of* $\Omega^t M$.

Proof. In local coordinates the 'Lebesgue section' $|dx|^t$ is smooth and positive so summing over a partition of unity gives a global smooth positive section. □

157.576 Proposition 7.1. *There is a well-defined integral*

157.577 (7.10) $$\int : \mathcal{C}^\infty_c(M; \Omega) \longrightarrow \mathbb{C}$$

for any manifold.

Of course this can be extended to locally integrable (but compactly supported) sections of ΩM.

157.578 Lemma 7.2. *There are natural isomorphisms*

157.579 (7.11) $$\Omega^t M \otimes \Omega^s M = \Omega^{s+t} M,\ \Omega^t(M \times N) = \pi^*_M \Omega^t M \otimes \pi^*_N \Omega^t N.$$

So one can define an integral on $\mathcal{C}^\infty_c(M)$ by choosing a density (probably positive) $\nu \in \mathcal{C}^\infty(M; \Omega)$ and using

157.580 (7.12) $$\int_M f\nu \text{ for } f \in \mathcal{C}^\infty_c(M).$$

It is just that there is, in general, no natural choice of ν. On a Riemannian manifold there is an associated, smooth positive, Riemann density ν_g defined as $|\alpha_1 \wedge \dots \alpha_n|$ for any orthonormal basis α_i of $T^*_m M$.

2. Distributions

The existence of the integral on densities leads to consistent pairings

157.581 (7.13)
$$\begin{tikzcd}
\mathcal{C}^\infty_c(M) \times \mathcal{C}^\infty(M;\Omega) \arrow[dr] & \\
\mathcal{C}^\infty_c(M) \times \mathcal{C}^\infty_c(M;\Omega) \ni (f,g) \arrow[u, hook] \arrow[d, hook] \arrow[r] & \int fg \in \mathbb{C} \\
\mathcal{C}^\infty(M) \times \mathcal{C}^\infty_c(M;\Omega) \arrow[ur] &
\end{tikzcd}$$

Each of the spaces of smooth sections of Ω has a topology coming from the isomorphism with the corresponding space of functions given by the choice of a smooth positive section of Ω (and the fact that multiplication by a positive smooth element $f \in \mathcal{C}^\infty(M)$ is an isomorphism on both $\mathcal{C}^\infty_{\mathrm{c}}(M)$ and $\mathcal{C}^\infty(M)$). This means that we can define the spaces of distributions by

157.582 (7.14)
$$\begin{aligned} \mathcal{C}^{-\infty}_{\mathrm{c}}(M) &= \{u : \mathcal{C}^\infty(M;\Omega) \longrightarrow \mathbb{C} \text{ linear and continuous}\} \\ \mathcal{C}^{-\infty}(M) &= \{u : \mathcal{C}^\infty_{\mathrm{c}}(M;\Omega) \longrightarrow \mathbb{C} \text{ linear and continuous}\}. \end{aligned}$$

This is arranged so that these are 'generalized functions' with natural inclusions

157.583 (7.15)
$$\mathcal{C}^\infty_{\mathrm{c}}(M) \hookrightarrow \mathcal{C}^{-\infty}_{\mathrm{c}}(M),\ \mathcal{C}^\infty(M) \hookrightarrow \mathcal{C}^{-\infty}(M).$$

3. Vector bundles

In fact we want to generalize this in several respects. First for sections of a vector bundle. I am already assuming you know what a vector bundle is. Just for completness sake let me remind you that a vector bundle (either real or complex) over a manifold M is another manifold V (often called the total space of the vector bundle) with a surjective smooth map

157.584 (7.16)
$$\pi : V \longrightarrow M$$

such that each fibre $V_m = \pi^{-1}(m) \subset V$ has a linear space structure which is 'smooth and locally trivial'. This means that each point $m \in M$ has a neighbourhood O_m for which there is a diffeomorphism giving a commutative diagramme

157.585 (7.17)
$$\begin{tikzcd} \pi^{-1}(O_m) \arrow[rr, "F"] \arrow[dr, "\pi"'] & & O_m \times \mathbb{K}^N \arrow[dl, "\pi_1"] \\ & O_m & \end{tikzcd}$$

with F linear on each fibre. Here $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$ in the real or complex case respectively. One can always assume that the O_m are coordinate patches on M and then the maps (157.585 7.17) give a special atlas on V – the transition conditions are automatic.

There are sections of any vector bundle just as for functions (and densities) $\mathcal{C}^\infty_{\mathrm{c}}(M;V) \hookrightarrow \mathcal{C}^\infty(M;V)$ corresponding to smooth maps $u : M \longrightarrow V$ with $\pi \circ u = \mathrm{Id}_M$.

The dual bundle, V', of a vector bundle V is defined, as a set, as the union of the duals of the fibres

157.586 (7.18)
$$V' = \bigsqcup_{m \in m} V'_m.$$

It has a smooth structure coming from the local trivializations (157.585 7.17) by replacing the map $F_m : V_m \longrightarrow \mathbb{K}^N$ by $(F^t)^{-1} : V'_m \longrightarrow \mathbb{K}^N$.

There is then a pointwise pairing

157.587 (7.19)
$$\mathcal{C}^\infty(M;V) \times \mathcal{C}^\infty(M;V') \longrightarrow \mathcal{C}^\infty(M)$$

and this leads to pairings analogous to (7.13):

$$\int \langle f, g\rangle_m,\ f \in \mathcal{C}^\infty_{\mathrm{c}}(M; V),\ g \in \mathcal{C}^\infty(M; V' \otimes \Omega),\ \text{or}$$
$$f \in \mathcal{C}^\infty(M; V),\ g \in \mathcal{C}^\infty_{\mathrm{c}}(M; V' \otimes \Omega).$$

Then, noting that there are again appropriate topologies, the spaces of distributional sections are defined as the spaces of continuous linear maps

157.589
$$\begin{gathered}\mathcal{C}^{-\infty}_{\mathrm{c}}(M; V) = \{u : \mathcal{C}^\infty(M; V' \otimes \Omega) \longrightarrow \mathbb{C};\ \text{linear and continuous}\}\\ \mathcal{C}^{-\infty}(M; V) = \{u : \mathcal{C}^\infty_{\mathrm{c}}(M; V' \otimes \Omega) \longrightarrow \mathbb{C};\ \text{linear and continuous}\}.\end{gathered}\tag{7.20}$$

4. Operators and kernels

The operators we will consider on a manifold are, at worst, continuous linear maps

157.590
$$A : \mathcal{C}^\infty_{\mathrm{c}}(M; V) \longrightarrow \mathcal{C}^{-\infty}(M; W)\tag{7.21}$$

taking smooth sections of one vector bundle to distributional sections of another. In fact we generally will not encounter anything quite this general. One of the important features of distributions is that these operators are again distributions.

157.591 Theorem 7.1 (Schwartz' kernel). *There is a 1-1 correpondence between continuous linear maps (7.21) and distributions*

157.592
$$K_A \in \mathcal{C}^{-\infty}(M^2; \pi_L^* W \otimes \pi_R^*(V' \otimes \Omega)).\tag{7.22}$$

So I am assuming you see how to define the tensor product of two bundles and the pull-back of a bundle under a smooth map. Here π_L and π_R are the projections from M^2 to M as the left and right factors.

Proof. None given, since I do not really use this result below. Still I should at least specify what I mean by the continuity which is in terms of the weak topology on the image. This is really the first part of the proof. Namely we can replace A, in (7.22), by a bilinear map

157.593
$$\mathcal{C}^\infty_{\mathrm{c}}(M; W' \otimes \Omega) \times \mathcal{C}^\infty_{\mathrm{c}}(M; V) \in (f, g) \longmapsto A(g)(f)\tag{7.23}$$

from which A can be recovered. Continuity of A actually means separate continuity of this bilinear form. The theorem is that such a bilinear form extends uniquely to a distribution K_A under the (bilinear) inclusion

157.594
$$\mathcal{C}^\infty_{\mathrm{c}}(M; W' \otimes \Omega) \times \mathcal{C}^\infty_{\mathrm{c}}(M; V) \hookrightarrow \mathcal{C}^\infty_{\mathrm{c}}(M^2; \pi_L^*(W' \otimes \Omega) \otimes \pi_R^* V)\tag{7.24}$$

given by pulling back under the projections.

In this sense the Schwartz kernel theorem is an infinite-dimensional generalization of the identifications for finite-dimensional vector space

157.595
$$\{L : V \longrightarrow W;\ \text{linear}\} \longleftrightarrow \{B_L : W' \times V \longrightarrow \mathbb{C};\ \text{bilinear}\} \longleftrightarrow W \otimes V'.\tag{7.25}$$

The proof is not actually so hard. □

So, the way we will use this is by specifying operators in terms of their kernels,

157.596
$$(Av)(w) = K_A(w \otimes v).\tag{7.26}$$

Note the reversal of order which makes so that pairing corresponds to proximity.

157.602 REMARK 6. I will generally identify operators and their kernels, using the same letter to denote both.

If M is not compact we also need to think again about the 'calculus of supports'.

5. Smoothing operators

Suppose M is a compact manifold without boundary, so all the support annoyances are absent.

157.597 DEFINITION 7.1. The space of smoothing operators between sections of two bundles, V, W over a compact manifold, M, is defined by the space of kernels

157.598 (7.27) $$\Psi^{-\infty}(M;V,W)=\mathcal{C}^\infty(M^2;\pi_L^*W\otimes\pi_R^*(V'\otimes\Omega)).$$

When $W=V$ we abbreviate the notation to $\Psi^{-\infty}(M;V)$.

157.599 REMARK 7. Smoothing operators are characterized (for M compact) by the two conditions that

157.600 (7.28) $$\begin{gathered}A:\mathcal{C}^{-\infty}(M;V)\longrightarrow\mathcal{C}^\infty(M;W)\text{ and}\\ A^t:\mathcal{C}^{-\infty}(M;W'\otimes\Omega)\longrightarrow\mathcal{C}^\infty(M;V'\otimes\Omega).\end{gathered}$$

For M compact it is again the case that the smoothing operators on a fixed bundle V – so with $W=V$ – form an algebra. The composition written in terms of kernels is

157.601 (7.29) $$A\circ B=\int_M A(\cdot,m)B(m,\cdot).$$

Make sure you understand why the integral here makes sense and gives again a smoothing operator.

157.603 PROPOSITION* 7.2. *The group of invertibles in the ring* $\mathrm{Id}+\Psi^{-\infty}(M;V)$ *for a vector bundle over a compact manifold is isomorphic (but not naturally so) to* $G^{-\infty}_{\mathcal{S}}(\mathbb{R}^n)$.

To prove this we need a decent basis for $L^2(M;V)$ – we will get this as the eigenbasis for any self-adjoint elliptic pseudodifferential operator acting on V.

157.604 EXERCISE 10. Give an appropriate definition for the space $\Upsilon^{-\infty}(M;V\otimes\mathbb{C}^2)$ for any complex vector bundle, V, over a compact (connected) manifold M and check that it is isomorphic to $\Upsilon^{-\infty}_{\mathcal{S}}(\mathbb{R}^n;\mathbb{C}^2)$. L14-end

6. Cornormal distributions at the zero section L15

Let W be a real vector bundle (any complex vector bundle has an underlying real bundle) over a manifold S. We proceed to introduce the space of conormal distributions on W, the total space of the vector bundle, relative to S appearing as the zero section of W. We will want this to be a module over $\mathcal{S}(W)=\dot{\mathcal{C}}^\infty(\overline{W})$ and the elements should be singular only at the zero section. In the case that $W=S\times\mathbb{R}^N$ we know what we want, namely

157.605 (7.30) $$I^m_{\mathcal{S}}(S\times\mathbb{R}^N;S\times\{0\})=\mathcal{C}^\infty(S;I^{m-\frac{s}{4}}_{\mathcal{S}}(\mathbb{R}^N)).$$

Here $s=\dim S$ and the only the shift in the order is questionable. We already considered such a space (with restricted coefficients) in case that $S=\mathbb{R}^n$ and

$N = n$ in defining the kernels of pseudodifferential operators on $\mathbb{R}^n$. Note that what this says about the symbol involved is that

157.606 (7.31) $$I^m_S(S\times\mathbb{R}^N; S\times\{0\}) \Longrightarrow u(s) = \mathcal{F}^{-1}(a(s,\xi)),\ a\in S^{m-\frac{s}{4}+\frac{N}{4}}(\mathbb{R}^N)$$

so it is consistent with our earlier notation.

One thing we can see immediately is that we can restrict such distributions to a submanifold $Y\subset S$, since this just restricts the smooth map

157.617 (7.32) $$\big|_Y : I^m_S(S\times\mathbb{R}^N; S\times\{0\}) \longrightarrow I^{m+\operatorname{codim}Y}_S(S\times\mathbb{R}^N; S\times\{0\}),\ Y\longrightarrow S.$$

157.618 REMARK 8. In sofar as I can remember the normalization, this is one way to do it. First, restricting to a submanifold of S increases the (apparent) order by $1/4$ for each dimension, i,e, by $\operatorname{codim}(Y,S)/4$ as indicated. Secondly when 'base dimension' $s=\dim S$ and 'fibre dimension' are equal the order is equal to the order of the symbol.

Why is the order so normalized you might well ask? The ultimate reason is the appearnce of 'Lagrangian deistribution' which do not have a strict separation of base and fibre dimension. Together with the fact that we want a pseudodifferential operator to have the same order as its symbol (so that it is consistent with the order of differential operators) that determines the normalization. The normalization is further complicated (to those like me who are numerically challenged) by the fact that there are three dimensions as work here. As well as s and n there is the dimension of W as a manifold, $n=s+N$. So, N is the codimension of the submanifold S (the zero section) in W which is of dimension n. It follows that the relation ship between the orderI^M and $a\in S^m$ in (7.31) is

157.619 (7.33) $$u\in I^M(W,S) \Longleftrightarrow a\in S^{M'},\ M'=M+\frac{n}{4}-\frac{N}{2}.$$

Let's hope I am right. (See Problems 5?).

We also know what happens if we subject such a conormal distribution to a linear transformation on the fibres, even if it depends on the point in S :

157.607 (7.34)
$$A\in\mathcal{C}^\infty(S;\mathbb{R}^N),\ u\in I^m_S(S\times\mathbb{R}^N; S\times\{0\}) \Longrightarrow u(s,A(s,\cdot))\in I^m_S(S\times\mathbb{R}^N; S\times\{0\})$$
$$u=\mathcal{F}^{-1}(a(s,\xi)) \Longrightarrow u(s,A(s,y)) = \mathcal{F}^{-1}(a(s,((A^{-1})^t\xi)|\det A(s)|).$$

157.608 DEFINITION 7.2. If W is a real vector bundle of rank N over S, a manifold of dimension s then we define

157.609 (7.35) $$I^m_S(W;0_W)\subset\mathcal{C}^{-\infty}(W)$$

to consist of those distributions $u\in\mathcal{C}^{-\infty}(W)$ such that if $\phi\in\mathcal{C}^\infty_c(S)$ is supported in an open set O over which W is trivial, $\tau : W\big|_O \equiv O\times\mathbb{R}^N$ then

157.610 (7.36) $$\pi^*\phi u = \tau^*(v_O),\ v_O\in I^m_S(O\times\mathbb{R}^N; O\times\{0\}).$$

We can see from (7.34) that the symbol of such a distribution, defined locally, will transform as a density on the dual bundle. So let's make sure that this is well-defined.

157.613 LEMMA 7.3. *Suppose U is a real vector bundle over a manifold S and E is a complex vector bundle over S then the space of symbols on (the fibres of) U with values in E is well-defined by reference to local trivializations.*

PROOF. □

157.611 PROPOSITION 7.3. *Under fibre-wise inverse Fourier transform the space of Schwartz conormal distributions with respect to the zero section of a vector bundle is globally isomorphic to a space of symbolic densities*

157.612 (7.37)
$$I^m_{\mathcal{S}}(W;0_W)\in\mathcal{F}^{-1}(\mathcal{C}^\infty(S;S^{m+\frac{\dim W}{4}-\frac{\dim_{\text{fib}} W}{2}}(W';\Omega W'))).$$

PROOF. $\square$

157.620 PROPOSITION 7.4. *If $W\longrightarrow S$ is a real vector bundle, $U\longrightarrow S$ is a complex vector bundle and*

157.621 (7.38)
$$F:D_2\longrightarrow D_1$$

is a diffeomrophism between open neigbhourhoods of the zero section of W which maps the zero identically to itself, then

157.622 (7.39)
$$F^*:\{u\in I^m(W,O_W,U);\operatorname{supp}(u)\Subset D_1\}\longrightarrow\{v\in I^m(W,O_W,U);\operatorname{supp}(u)\Subset D_2\}.$$

PROOF. Since conormal distributions are smooth away from the zero section we may shrink the domain D_1 with D_2 replaced by the image and the result remains unchanged.

We also know that the conormal space is a module over $\mathcal{C}^\infty(S)$ so we can use a partition of unity in S to localize to an open sets over each of which W is trivial. So we may assume $W=S\times\mathbb{R}^N$.

To see how much remains to be proved, consider the following factorization of F – in a possibly smaller domain

157.623 (7.40)
$$F=L\circ\tilde{F}.$$

Here G is the fibre-preserving diffeomorphism constructed from the differential of F as follows. At each point $s\in S$ the tangent space to W is the direct sum

157.624 (7.41)
$$T_sW=T_SS\oplus\mathbb{R}^N$$

where the second part is the tangent space of the fibre. Since S is fixed pointwise by F the differential maps is the identity on T_sS and so decomposes into the sum of two linear maps

157.625 (7.42)
$$F_*\mathbb{R}^N\ni w\longmapsto(L'w,Lw)\in T_sS\oplus\mathbb{R}^n.$$

So L is a well-defined family of linear maps on $\mathbb{R}^N$. We already know the invariance under such maps so it suffices to consider he remainder term $\tilde{F}$.

This defines the first, fibre-linear, factor in (157.623 7.40), and hence the second. By construction this has differential at $s\in S$ which is the sum of the identiy and the 'off-diagonal' term mapping T_sW_s to T_sS. This means that the map itself is of the form, in any locall trivialization

157.626 (7.43)
$$\tilde{F}(s,v)=(s,w)+v\cdot h(s)+E(s,v),\ E(s,v)=O(|v^2),\ v\in\mathbb{R}^N.$$

Then we can deform $\tilde{F}$ to the identity through the family of diffeormorphisms

157.627 (7.44)
$$\tilde{F}(s,v)=(s,w)+t(v\cdot h(s)+E(s,v)),\ t\in[0,1].$$

Now we can use the 'homotopy method' again. Very close to S this 1-parameter family of diffeomorphism is generated by a 1-parameter family of vector fields

157.628 (7.45)
$$\frac{d}{dt}F_t^*u=F_t^*(V_tu),\ V_t=\text{ vanishes at } S \text{ and } V_tv_j=O(||v|)^2.$$

Now we replace u by a smooth family u_t were we want to arrange that

157.629 (7.46)
$$u_1 = u,\ \frac{du_t}{dt} + V_t u_t \in \mathcal{C}^\infty.$$

□

7. Conormal distributions at a submanifold

8. Pseudodifferential operators

L15-end PseudoM

L16 We now have completed the definition of the basic space of sections of a complex vector bundle conormal with respect to a closed embedded submanifold. Moreover we have shown that there is a well-defined (principal) symbol map

157.630 (7.47)
$$\sigma : I^m(M, S; V) \longrightarrow S^{m'}(N^*S; V \otimes \Omega(N**S)),\ m' = m + \frac{1}{4}\dim M - \frac{1}{2}\operatorname{codim} S.$$

Here $\Omega(N^*S)$ is a trivial real line bundle over S with fibre the space of densities on the fibres of N^*S. We know that this captures the leading part of the singularity of the elements so that

157.631 (7.48)
$$I^{m-1}(M, S; V) \hookrightarrow I^m(M, S; V) \twoheadrightarrow S^{m'}(N^*S; V \otimes \Omega(N**S))$$

is a short exact sequence. In the process of examining the coordinate-invariance the error terms were all local at the symbolic level. This means the two sets

157.632 (7.49)
$$\mathrm{WF}(u) \subset N^*S \setminus 0_S$$

defined as the complement of the larges open cone where where the local full symbol is rapidly decaying is invariantly defined.

We use this space to define the space of pseudodifferential operators acting between any two complex bundle vector bundles over a manifold in terms of their kernels

157.633 (7.50)
$$\Psi^m(M; V, W) = I^m(M^2, \mathrm{Diag}; \pi_L^* W \otimes \pi_R^*(V' \otimes \Omega).$$

L16-end

L17

9. Elliptic operators

EllipticM

Now we turn to the construction of a parametrix for a globally elliptic operator $A \in \Psi^m(M; V, W)$. Thus the principal symbol of A has a representative $a \in S^m(T^*M; \pi^* \hom(V, W))$ which is elliptic. This means there is an inverse modulo a compactly supported error,

157.634 (7.51)
$$b \in S^{-m}(T^*M; \pi^* \hom(W, V)),$$
$$ba - \mathrm{Id} \in \mathcal{C}^\infty_c(T^*M; \hom(V)),\ ab - \mathrm{Id} \in \mathcal{C}^\infty_c(T^*M; \hom(W)).$$

157.635 Proposition 7.5. *An elliptic element $A \in \Psi^m(M; V, W)$ has a parametrix $B \in \Psi^{-m}(M; W, V)$ modulo smoothing operators,*

157.636 (7.52)
$$BA - \mathrm{Id} = E \in \Psi^{-\infty}(M; V),\ AB - \mathrm{Id} = E' \in \Psi^{-\infty}(M; W)$$

and any two such parametrices differ by an element of $\Psi^{-\infty}(M; W, V)$.

PROOF. The surjectivity of the symbol map

157.637
$$\Psi^{-m}(M;W,V) \longrightarrow S^{-m}(T^*M;W,V) \tag{7.53}$$

allows us to choose an element $B_0 \in \Psi^{-m}(M;W,V)$ with $\sigma_{-m}(B_0) = b$ as in (7.51). It follows from the multiplicativity and exactness of the symbol sequence that

157.638
$$B_0A - \mathrm{Id} = E_1 \in \Psi^{-1}(M;V) \text{ and } AB_0 - \mathrm{Id} = E_1' \in \Psi^{-1}(M;W). \tag{7.54}$$

Now we proceed by induction to find a sequence $B_j \in \Psi^{-m-j}(M;W,V)$, such that

157.639
$$\begin{gathered}(\sum_{j=0}^{k} B_j)A - \mathrm{Id} = E_{k+1} \in \Psi^{-k-1}(M;V) \text{ and} \\ A(\sum_{j=0}^{k} B_j) - \mathrm{Id} = E_{k+1}' \in \Psi^{-k-1}(M;W).\end{gathered} \tag{7.55}$$

In the inductive step for $B_{k+1} \in \Psi^{-m-k-1}(M;W,V)$ to satisfy the first condition in (7.55) we must have

157.640
$$\begin{gathered}B_{k+1}A - E_{k+1} = E_{k+2} \in \Psi^{-k-2}(M;V) \Longrightarrow \\ \sigma_{-m-k-1} = \sigma_{-k-1}(E_{k+1})b \mod S^{-m-k-2}(T^*;\pi^*\hom(W,V)).\end{gathered} \tag{7.56}$$

So not only is this possible but the choice is unique up to a addition of a term in $\Psi^{-m-k-2}(M;W,V)$. Applying A on the left in the first equation in (7.55) and on the right in the second it follows that

157.641
$$AE_{k+1} = E_{k+1}'A. \tag{7.57}$$

So in fact

157.642
$$a\sigma_{-k-1}(E_{k+1}) = a\sigma_{-k-1}(E_{k+1}') \Longrightarrow \sigma_{-k-1}(E_{k+1})b = \sigma_{-k-1}(E_{k+1}')b \tag{7.58}$$

modulo terms of order $-m-k-2$. Thus in fact $AB_{k+1} - E_{k+1}' = E_{k+2}' \in \Psi^{-k-2}(M;W)$ so B_{k+1} satisfies both conditions required in the inductive step.

The asymptotic completeness of the pseudodifferential spaces means that we can choose

157.643
$$B \sim \sum_{j\ge 0} B_j \in \Psi^{-m}(M;W,V) \tag{7.59}$$

which is then a parametrix in the sense of (7.52).

The uniqueness follows from the existence since if B' is any left parametrix then

157.644
$$\begin{gathered}B'A - \mathrm{Id} \in \Psi^{-\infty}(M;V) \Longrightarrow (B'-B)A \in \Psi^{-\infty}(M;V) \Longrightarrow \\ B'-B = (B'-B)(AB-E') = ((B'-B)A)B - (B'-B)E' \in \Psi^{-\infty}(M;W,V).\end{gathered} \tag{7.60}$$

□

The boundedness properties on Sobolev spaces now show that an elliptic operator of order m is Fredholm as an operator

157.645
$$A : H^s(M;V) \longrightarrow H^{s-m}(M;W). \tag{7.61}$$

Recall that even acting on distributions

157.648
$$A : \mathcal{C}^{-\infty}(M;V) \longrightarrow \mathcal{C}^{-\infty}(M;W) \tag{7.62}$$

the null space of A is a finite dimensional subspace of $\mathcal{C}^\infty(M;V)$ since its elements satisfyb

157.649
$$Au = 0 \Longrightarrow u = Eu,\ E \in \Psi^{-\infty}(M;V). \tag{7.63}$$

If we choose hermitian inner products on V and W and a smooth density ν on M then the formal adjoint of A is the unique element

157.650
$$A^* \in \Psi^m(M;W,V) \text{ satisfying } \int_M \langle Au, v\rangle_W \nu = \int_M \langle u, A^*v\rangle_V \nu. \tag{7.64}$$

The uniqueness of A^* follows from this condition.

To see existence, obvserve that the inner products give (antilinear) identifications of V with V' and W with W' and define an adjoint isomorphism of bundles

157.651
$$\pi_L^* W \otimes \pi_R^* V' \longrightarrow \pi_L^* W' \otimes \pi_R^* V. \tag{7.65}$$

Then the Schwartz kernel is the image of the kernel

$$A = D\pi_R^*\nu,\ D \in I^m(M^2, \mathrm{Diag}; \pi_L^* W \otimes \pi_R^* V')$$

under this isomorphism and reversal of the variables so

$$A^*(x,y) = (B(y,x)^* \pi_R^* \nu.$$

It is straightforward to see that this operator satisfies (7.64).

157.646 Lemma 7.4. *The index of an elliptic element $A \in \Psi^m(M;V;W)$ as an operator* (7.61) *is*

157.647
$$\mathrm{ind}(A) = \dim(A) - \dim(A^*) \tag{7.66}$$

where $A^* \in \Psi^m(M;W,V)$ is the formal adjoint of A with respect to any choice of hermitian inner products on the bundles V, W and smooth density on M.

Proof. From the defining property of A^* it follows that if A is considered as an operator

157.652
$$A : H^m(M;V) \longrightarrow L^2(M;W) \tag{7.67}$$

then the orthocomplement of the range is $(A*)$ where

$$A^* : L^2(M;W) \longrightarrow H^{-m}(M,V).$$

This gives (7.66) for the index. Changing the Sobolev order to $A : H^s(M;V) \longrightarrow H^{s-m}(M;W)$ does not change the null space and changes the range either by replacing it by its intersection with $H^{s-m}(M;W)$ if $s > m$ or by the closure if $s < m$. In either case (A^*) remains a complementary space (but not orthogonal) since the (extension of) the L^2 pairing still vanishes. □

157.663 Lemma 7.5. *If $A \in \mathrm{Ell}_m(M;V,W)$ then there exists $A' \in \Psi^{-\infty}(M;V,W)$ such that*

157.664
$$\begin{gathered} \textit{if } \mathrm{ind}(A) > 0,\ A + A' \textit{ is surjective} \\ \textit{if } \mathrm{ind}(A) = 0,\ A + A' \textit{ is bijective} \\ \textit{if } \mathrm{ind}(A) < 0,\ A + A' \textit{ is injective.} \end{gathered} \tag{7.68}$$

Proof. Suppose $\operatorname{ind}(A) \le 0$. By Lemma [157.646] 7.4 $\dim(A) \le \dim(A^*)$. Thus we can choose a smoothing operator $A' : (A) \longrightarrow (A^*)$ which is injective. Then any element of the null space of $A+A'$, satisfies $Au = -A'u$. These lie in complementary spaces so both must vanish and hence $u \in (A)$ and $A'u = 0$ so $u = 0$. If $\operatorname{ind}(A) = 0$ it follows that $(A^*) \subset \operatorname{Ran}(A + A')$ so $A + A'$ is a bijection. If $\operatorname{ind}(A) \ge 0$ thus argument applies to A^*. $\square$

157.655 Lemma 7.6. *For any real order m and any bundle V there is an elliptic and invertible operator $Q_m \in \Psi^m(M;V)$ with diagonal, positive principal symbol.*

Proof. Take a Riemann metric on m and consider the symbol $|\xi|^m \operatorname{Id}$ valued in $\hom(V)$. Clearly this is elliptic. For the case $m = 0$ we can of course take the identity. Suppose $m < 0$ then choose an operator $L_{m/2} \in \Psi^{m/2}(M;V)$ with symbol $|\xi|^{m/2} \operatorname{Id}$. It is a compact operator on $L^2(M;V)$. Taking an hermitian inner product on V and a positive smooth density on M gives $L^2(M'V)$ an explicity Hilbert inner product. Then the adjoint $L^*_{m/2} \in \Psi^{m/2}(M;V)$ has the same principal symbol so $Q'_m = L^*_{m/2}L_{m/2} \in \Psi^m(M;V)$ is selfadjoint with principal symbol $|\xi|^m \operatorname{Id}$. It is elliptic of order 0 with $(Q'_m) \subset \mathcal{C}^\infty(M;V)$ finite dimensional. Choosing a positive definite matrix on (Q'_m) and adding it to Q'_m gives an invertible element of $\Psi^m(M;V)$ with the same principal symbol. For $m >$ take Q_{-m}. $\square$

The existence of such an element shows that it really suffices to consider operators of order 0 in the discussion of the index since if $A \in \Psi^m(M;V,W)$ is elliptic then so is $A_0 = AQ_{-m} \in \Psi^0(M;V,W)$ and

157.656 (7.69)
$$\sigma_0(A_0) = \sigma_m(A)||\xi|^{-m}.$$

Corresponding to the stability of the index of Fredholm operators on a fixed Hilbert space it also follows that

157.654 (7.70)
$$\operatorname{ind}(A+E) = \operatorname{ind}(A) \text{ if } A \in \Psi^m(M;V,W) \text{ is elliptic and } E \in \Psi^{m'}(M;V,W),\ m' < m.$$

Indeed, the index of $(A+E)Q_{-m} = AQ_{-m} + EQ_{-m}$ is the same as that of $A+E$ and EQ_{-m} is compact from $L^2(M;V)$ to $L^2(M;W)$ so the index is the same as that of AQ_{-m} and hence A.

Said a different way, we have proved that

157.657 Lemma 7.7. *The index of and elliptic element $A \in \Psi^m(M;V,W)$ is determined by any reprentative of*

157.658 (7.71)
$$\sigma_m(A)|\xi|^{-m} \in S^0(T^*M;\pi^*\hom(V,W)).$$

and so defines a map

157.659 (7.72)
$$\operatorname{ind} : \operatorname{Ell}(V,W) = \{a : S^0(T^*M;\pi^*\hom(V,W)); \text{ with an inverse modulo lower order terms}\} \longrightarrow \mathbb{Z}$$

for any bundles V and W.

Of course the bundles certainly have to have the same rank before such an elliptic symbol exists.

We can think of the pseudodifferential algebra (modules really) as define the index map ([157.659] 7.72) through 'quantization of the symbol'.

157.660 LEMMA 7.8. *If $A_1 \in \Psi^m(M;V,W)$ and $A_2 \in \Psi^{m'}(M;W,U)$ are two elliptic operators between bundles then the composite is elliptic and*

157.653
$$\operatorname{ind}(A_2 \circ A_1) = \operatorname{ind}(A_2) + \operatorname{ind}(A_1). \tag{7.73}$$

PROOF. Using Lemma 7.6 We can replace A_1 by AQ_{-m} and A_2 by $Q_{-m'}A_2$ and reduce to the case that $m = m' = 0$. From Lemma 7.4 we see that $\operatorname{ind}(A^*) = -\operatorname{ind}(A)$. So if A_1 and A_2 have opposite signs we can assume that $\operatorname{ind}(A_1) \geq 0$ and $\operatorname{ind}(A_2) \leq 0$ or else pass to the adjoint, which reverses the order. If A_1 and A_2 have the same sign then we can assume that both are non-negative. So it suffices to consider the case $\operatorname{ind}(A_1) \geq 0$. Then, adding a finite rank surjective smoothing operator mapping the null space of A onto a complement to its range we can assume that A_1 is surjective. Then the range of $A_2 \circ A_1$ is the range of A_2 and its null space is inverse image of the null space of A_2 under A_1, which has dimension $\dim((A_1)) + \dim((A_2)$. Thus (7.73) holds. □

157.661 COROLLARY 4. *The index map (7.72) is additive under the product*

157.662
$$\operatorname{Ell}(W,U) \circ \operatorname{Ell}(V,W) \longrightarrow \operatorname{Ell}(V,U) \tag{7.74}$$

(when these spaces are non-empty).

10. Self-adjoint operators

Although I will concentrate on the index theorem in these lectures it is even more important to discuss the spectral theory of self-adjoint ellipitc operators on a compact manifold. Indeed I am assuming that you have seen this before.

P1.16 THEOREM 7.2. *An elliptic pseudodifferential operator acting on sections of a vector bundle V over a compact manifold without boundary, M , $A \in \Psi^m(M;V)$, which is symmetric, with respect to a density on M and a fibrewise Hermitian inner product on V, is self-adjoint and, if $m \neq 0$, has eigenfunctions forming a complete orthonormal basis of $L^2(M;V)$.*

PROOF. Symmetry here means 'formal self-adjointness'

P1.17 (7.75)
$$\langle Av, w\rangle_{L^2} = \int_M \langle Av, w\rangle_V \nu = \int_M \langle Av, w\rangle_V \nu = \langle v, Aw\rangle_{L^2} \ \forall\ v, w \in \mathcal{C}^\infty(M;V).$$

If $m \leq 0$ it follows from the L^2 boundedness of A that this identity extends by continuity to all $v, w \in L^2(M;V)$ and so A is a bounded self-adjoint operator.

If $m > 0$ then we know that $A : H^m(M;V) \longrightarrow L^2(M;V)$ is continuous. Then A is a closed unbounded operator with domain $H^m(M;V) \subset L^2(M;V)$ and elliptic regularity shows us that the adjoint of this operator, which by definition has domain

P1.18 (7.76)
$$\operatorname{Dom}(A^*) = \{w \in L^2(M;V); \mathcal{C}^\infty(M;V) \ni u \longrightarrow \langle Au, w\rangle_{L^2} \text{ is bounded on } L^2\}$$

is A again as an unbounded operator.

If $m < 0$ then A is a compact operator and the theory of compact self-adjoint operators applies to give the complete orthonormal basis. If $m > 0$ we know that A has a unique generalized inverse determined by the L^2 structure and it follows that $B \in \Psi^{-m}(M;V)$ is also self-adjoint. The non-zero eigenvalues of A are the inverses of the non-zero eigenvalues of B and the null spaces are, essentiallt by definition, the same. □

One of the things that I set out to do was to prove that such an elliptic operator has ‘Weyl asymptotics’. L17-end

11. Fibre bundles

L18

The definition of homotopy of elliptic pseudodifferential operators involves a ‘family’ in the sense of a smooth map $[0,1] \ni t \longmapsto A(t) \in \Psi^m(M;V,W)$. However we really want to allow the bundles V and W to ‘vary’ as well. In making this precise we may as well pass to the notion of a family of pseudodifferential operators on the fibres of a *fibre bundle*.

157.665 Definition 7.3. A fibre bundle is a smooth surjective map between manifolds $\phi : X \longrightarrow Y$ which is locally a product in the sense that there is a manifold Z (the model fibre) such that each $y \in Y$ has a neighbourhood $U \subset Y$ with a diffeomorphism giving a commutative diagramme

157.667 (7.77)
$$\begin{tikzcd} \phi^{-1}(U) \arrow[rr, "F"] \arrow[dr, "\phi"'] & & Z \times U \arrow[dl, "\pi_U"] \\ & U. & \end{tikzcd}$$

So this is just like a vector bundle except the fibres are manifolds, not vector spaces, and correspondingly the ‘local trivializations’ F are fibre-preserving diffeomorphisms. Of course the most obvious case is a product

157.668 (7.78)
$$X = Z \times Y$$

which is globally trivial as a fibre bundle. You are probably familiar with the Hopf fibration

157.669 (7.79)
$$\mathbb{S}^{2n+1} \longrightarrow \mathbb{S}^{2n}$$

given by thinking of $\mathbb{S}^{2n+1}$ as the unit sphere in $\mathbb{C}^{n+1}$ and then taking the fibres to be given by the multiplicative action of the circle $\mathbb{S} \subset \mathbb{C}$.

The ‘fibre above $y \in Y$’, meaning $\phi^{-1}(Y)$, is diffeomorphic to the model fibre Z and is often written Z_y. There is in general no natural choice of the trivialization map F in (157.667 7.77) so no natural diffeomorphism between Z and Z_y. A fibration is how we are to interpret the notion of a ‘smooth family of manifolds’.

157.670 Proposition 7.6. *If $\phi : X \longrightarrow Y$ is a surjective smooth map between compact manifolds then it defines a fibre bundle if and only if it is a submersion, i.e. $F_* : T_xX \longrightarrow T_{F(x)}Y$ is surjective for each $x \in X$.*

The compactness, or at least some additional condition, is needed here. The surjectivity is less of an issue, since if ϕ is a submersion between compact manifolds its image if open and closed and hence is some union of components of Y – assuming we are not requiring manifolds to be connected!

Proof. This is a form of the Implicit Function Theorem. For a submersion, there is a local version of (157.667 7.77) near each point of X. Namely, the surjectivity of the differential ϕ_* is equivalent to the injectivity of $\phi^* : T_{F(p)}Y \longrightarrow T^*_pX$. So if y_i are local coordinates in $U \subset Y$ near $F(p)$ then the functions $\phi^* y_i$ defined on $\phi^{-1}(U)$ have independent differentials. So they can be completed to a coordinate system, $\phi^* y_i$, z_k on X in some neighbourhood of p. Then the fibres localy are the surfaces $\phi^* y_i$ =const. This needs to be globalized along $\phi^{-1}(\phi(p))$.

In each of the coordinate patches constructed along the 'base fibre' $Z(p) = \phi^{-1}(\phi(p))$, the fibre containing p, the choice of local coordinates gives vector fields, just the $V_i^{(k)} = \partial_{y_i}$ in the local coordinates label by k, which satisfy $\phi_*(V_i^{(k)}) = \partial_{y_i}$ are the coordinate vector fields in the base. The $V_i^{(k)}$ in different coordinate patches will be different but if we take a partition of unity r_k on X near $Z(p)$ and subordinate to the cover then the $V_i = \rho_k V_i^{(k)}$ satisfy $\phi_*(V_i) = \partial_{y_i}$. The compactness of $Z(p)$ means that this is only a finite sum and so the V_i are defined in an open set $\phi^{-1}(U)$ for some neighbourhood $p \in U$ in Y. The V_i need not commute but we can simply integrate successively with respect them starting at $Z(p)$. So for each $q \in Z(p)$ integrating V_1 gives a smooth surface in Y containing $Z(p)$ and projecting under ϕ onto the y_1 access in U. Then integrating from this surface gives a submanifold of one dimension higher projecting onto the y_1, y_2 coordinate plane in U, where this may need to be shrunk at each step. After $\dim Y$ steps we have a smooth map from a neighbourhood of the form $\phi^{-1}(U')$ of $Z(p)$ in Y to $Z(p)$ which together with the projection to U' gives a diffeomorphism F as required in (157.667 7.77). $\square$

157.672 Proposition 7.7. *If $V \longrightarrow X$ is a vector bundle over the total space of a fibre bundle then there are local trivializations of the bundle as in (157.667 7.77) over which there are bundle isomorphisms*

157.673 (7.80)
$$\begin{tikzcd}
V\big|_{\phi^{-1}(U)} \arrow[r, "\tau"] \arrow[d] & V_Z \times U \arrow[d] \\
\phi^{-1}(U) \arrow[r, "F"] \arrow[dr, "\phi"'] & Z \times U \arrow[d, "\pi_U"] \\
 & U
\end{tikzcd}$$

where V_Z is a fixed bundle over the model fibre.

Notice that there is some constructive confusion between the total space of a vector bundle and the vector bundle itself (which includes the projection to the base).

Proof. $\square$

Now, the kernel of a pseudodifferential operator on Z acting from sections of one bundle to another is a distribution on Z^2. So we need to consider the product of each fibre with itself.

157.671 Proposition 7.8. *If $\phi : X \longrightarrow Y$ is a fibre bundle then $X^2 \ni (x, x') \longrightarrow (\phi(x), \phi(x')) \in Y^2$ is a fibre bundle as is the fibre product, which is the closed embedded submanifold*

157.674 (7.81)
$$X^{[2]} = \{(x, x') \in X^2; \phi(x) = \phi(x')\}$$

and the diagonal is the closed embedded submanifold

157.675 (7.82)
$$\mathrm{Diag} = \{(x, x) \in X^2; x \in X\}.$$

Clearly the diagonal is diffeomorphic to X.

Proof. $\square$

12. Families of pseudodifferential operators

To define the space $\Psi^m_\phi(X;V,W)$ of operators

157.676 (7.83)
$$A : \mathcal{C}^\infty(X;V) \longrightarrow \mathcal{C}^\infty(X;W)$$

for two vector bundles over the total space of a fibre bundle we have two clear options. Use Proposition 7.80 locally in the base and then patch things together or start globally and see what happens. As you have seen I prefer to take the latter course, at least for the moment!

157.677 Definition 7.4. The space of fibrewise pseudodifferential operators for a fibre bundle $\phi : X \longrightarrow Y$ between sections of bundles V and W over X is identified with the space of conormal distributions

157.678 (7.84)
$$\Psi^m_\phi(X;V,W) = I^{m-\frac14\dim Y}(X^{[2]}, \mathrm{Diag}; \pi_L^*W \otimes \pi_R^*(V' \otimes \Omega_{\mathrm{fib}}).$$

Here π_L, $\pi_R : X^{[2]} \longrightarrow Y$ are the restrictions of the two projections from X^2 and Ω is the bundle over Y of densities on the fibres of ϕ.

A more standard notion for this space would be $\Psi^m(X/Y;V,W)$ where there is no space X/Y but it is supposed to suggest a family acting on the fibres of X. I will not use that notation because it does not really fit with various generalizations below.

So, we need to see that this definition does give a linear space of operators as in (7.83) and then that they form an algebra. Again we can do this by localization or we can think globally. What we want to do is make sense of the formula

157.679 (7.85)
$$Av(x) = \int_{\mathrm{fib}} A(x,x')v(x'),\ v \in \mathcal{C}^\infty(X;V).$$

We can certainly pull v back to $X^{[2]}$ under π_R to give the a section of π_R^*V over $X^{[2]}$ which is independent of x – constant on the fibres of π_L. Then the product with the kernel makes sense and using the pairing of V' and V gives a section of $\pi_L^*W \otimes \pi_R^*\Omega_{\mathrm{fib}}$. This is singular only at the diagonal and the projection π_L is transversal to the diagonal. So, we can see that the integral is well-defined and gives a smooth section of W over X.

The tangent bundle of X has a subbundle, $T_{\mathrm{fib}}X \hookrightarrow TX$ consisting of the vectors tangent to the fibe at each point. For each $y \in Y$ the restriction if this fibre tangent bundle is therefore just the tangent bundle of the fibre

157.683 (7.86)
$$T_\phi X\big|_{Z_y} = TZ_y.$$

The dual bundle to $T_\phi X$ which I will denote $T^*_\phi X$ is then the collection of the fibre cotangent bundles

157.684 (7.87)
$$T^*_\phi X\big|_{Z_y} = T^*Z_y.$$

The adjustment in the order in (7.84) is so that:

157.680 Proposition 7.9. *There is a well-defined principal symbol map giving a short exact sequence*

157.681 (7.88)
$$\Psi^{m-1}_\phi(X;V,W) \hookrightarrow \Psi^m_\phi(X;V,W) \xrightarrow{\sigma_m} S^m(T^*_\phi X; \pi^* \hom(V,W))$$

consistent with the single operator case in the sense that for each $y \in Y$ there is a restriction map

157.682 (7.89) $$\Psi^m_\phi(X;V,W) \ni A \longmapsto A(y) \in \Psi^m_\phi(Z_y;V_y,W_y),\ \sigma_m(A(y)) = \sigma_m(A)\big|_{T^*Z_y}.$$

Returning to the discussion of the index of an elliptic operator observe that

157.685 Proposition 7.10. *The 'numerical' index of an elliptic family of pseudodifferential operators* $\operatorname{ind}(A(y))$ *is constant.*

For vector bundles over a manifold, let us denote by $\operatorname{iso}(V,W) \subset \hom(V,W)$ bundle with fibre the invertible linear maps between fibres at each point. For two pairs of bundles V_i, W_i $i = 1,2$ there is a direct sum operation

157.688 (7.90) $$\operatorname{iso}(V_1,W_1) \times \operatorname{iso}(V_2,W_2) \longrightarrow \operatorname{iso}(V_1 \oplus V_2, W_1 \oplus W_2).$$

157.686 Proposition 7.11. *Any elliptic operator is homotopic to a classical elliptic operator and hence for any compact manifold and vector bundle the index induces a map*

157.687 (7.91) $$\operatorname{ind} : \mathcal{C}^\infty(S^*M;\pi^*(\operatorname{iso}(V,W)) \longrightarrow \mathbb{Z}$$

which is additive under direct sums as in (157.688 7.90) *and under products giving commutative diagrammes*

157.689 (7.92) $$\begin{tikzcd} \mathcal{C}^\infty(S^*M;\pi^*\operatorname{iso}(V_1,W_1)) \times \mathcal{C}^\infty(S^*M;\pi^*\operatorname{iso}(V_2,W_2)) \arrow[r, "\operatorname{ind}\times\operatorname{ind}"] \arrow[d, "\oplus"] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathcal{C}^\infty(S^*M;\pi^*\operatorname{iso}(V_1\oplus V_2,W_1\oplus W_2)) \arrow[r, "\operatorname{ind}"] & \mathbb{Z}. \end{tikzcd}$$

(7.93) $$\begin{tikzcd} \mathcal{C}^\infty(S^*M;W,U) \times \mathcal{C}^\infty(S^*M;V,W) \arrow[r, "\operatorname{ind}\times\operatorname{ind}"] \arrow[d, "\circ"] & \mathbb{Z}\times\mathbb{Z} \arrow[d, "+"] \\ \mathcal{C}^\infty(S^*M;V,U) \arrow[r, "\operatorname{ind}"] & \mathbb{Z}. \end{tikzcd}$$

Now, we need to relate this to K-theory. Consider all of the triple (V,W,a) where V and W are complex vector bundles over M and a is an isomorphism between them , which we can write as $a \in \mathcal{C}^\infty(S^*M;\pi^*\operatorname{iso}(V,W))$. We impose three relations on this biggish set of data.

157.690 (7.94)
$$\text{Bundle isomorphism invariance: } (V_1,W_1,a) \simeq (V_2,W_2,bae^{-1}) \\ \text{if } b \in \mathcal{C}^\infty(M;W_1,W_2),\ e \in \mathcal{C}^\infty(M;V_1,V_2)$$

157.692 (7.95)
$$\text{Homotopy invariance: } (V_1,W_1,a_1) \simeq (V_0,W_0,a_0) \text{ if } \exists \\ h \in \mathcal{C}^\infty(S^*M \times [0,1];\operatorname{iso}(V,W)) \text{ where } V,W \text{ are bundles over } M \times [0,1] \\ \text{with } V\big|_{M\times\{0\}} = V_0,\ V\big|_{M\times\{1\}} = V_1,\ W\big|_{M\times\{0\}} = W_0,\ W\big|_{M\times\{1\}} = W_1$$

157.691 (7.96) $$\text{Stability: } (V \oplus U, W \oplus U, a \oplus \operatorname{Id}_U) \simeq (V,W,a) \text{ for any bundle } U \longrightarrow M.$$

Each of these is an equivalence relation – for the first and last this is straightforward, for homotopy equivalence one needs to do a littl work in concatenating two homotopies to get a smooth one. The trick is to replace an initial homotopy by its pull back under a smooth bijective map $[0,1] \longrightarrow [0.1]$ which is constant to infinite order at both ends. Then following one of these with another gives smooth objects over $[0,2]$ which can be reparameterized to a homotopy.

157.693 Theorem 7.3. * *The collection of the* $\mathcal{C}^\infty(S^*M;\pi^*\operatorname{iso}(V,W))$ *subject to the equivalence relation combining* (7.94), (7.95) *and* (7.96) *is an abelin group under the additive operation* (7.90) *which is naturally isomorphic to* $K^0(T^*M)$ *and to which the (numerical) index map descends giving the push-forward map*

157.694
$$K^0(T^*M) \xrightarrow{\pi_!} K^0(\{pt\}) = \mathbb{Z}. \tag{7.97}$$

Most of proof later. The identification of $K^0(T^*M)$ with the equivalence classes of the data in $\mathcal{C}^\infty(S^*M;\pi^*\operatorname{iso}(V,W))$ uses the results in Chapter **??** and is discussed below, as is the definition of the 'wrong-way map' $\pi_!$. The existence of this map, uses the notion of the K-orientation of T^*M. However, the fact that the index map does descend through the equivalence relations (7.94), (7.95) and (7.96) follows directly from the preceding discussion. □

So my basic claim is that it is 'better' to prove the more general families index theorem. In the more general families case the plan is to pass through *semiclassical* quantization since this has better functorial properties (whilst being equivalent to the pseudodifferential qunatization we have been employing).

13. Families index theorem

The 'numerical index' that we have been discussing exactly characterizes the invertibility properties of elliptic operators, up to smoothing perturbation as seen in Lemma 7.5. For an elliptic family we can ask the stronger question:- When is there a perturbation, $A' \in \Psi^{-\infty}_\phi(X;V,W)$ for an elliptic family $A \in \Psi^m_\phi(X;V,W)$ such that $A + A'$ is invertible? Clearly the vanishing of the numerical index is necessary, and this implies the existence of such a perturbation locally near each point in the base, but it is not sufficient to imply the existence of a global perturbation working at every point. The obstruction is precisely the families index which, in line with the discussion above, we will find as a map

157.695
$$\operatorname{ind}_\phi : \mathcal{C}^\infty(S^*_\phi X;\pi^*\operatorname{iso}(V,W)) \longrightarrow K^0(Y). \tag{7.98}$$

To construct such a map we look for a 'good' pertubation and parametrix.

157.696 Proposition 7.12. *If* $A \in \Psi^m_\phi(X;V,W)$ *is an elliptic family then there exists* $A' \in \Psi^{-\infty}_\phi(X;V,W)$ *and a parametrix* $B \in \Psi^{-m}_\phi(X;W,V)$ *such that*

157.697
$$B(A+A') = \operatorname{Id} - p_1,\ (A+A')B = \operatorname{Id} - p_2 \tag{7.99}$$

where $p_1 \in \Psi^{-\infty}_\phi(X;V)$ *and* $p_2 \in \Psi^{-\infty}_\phi(X;W)$ *are finite rank projections (idempotents).*

We show below that the formal difference $p_1 \ominus p_2$ defines an element of $K^0(Y)$ which is defined independent of choices and this gives the map (7.98). In fact this defines a map

157.698
$$\operatorname{ind}_\phi = (\phi\pi)_! : K^0(T^*_\phi X) \longrightarrow K^0(Y) \tag{7.100}$$

the identification if which is the Atiyah-Singer theorem in K-theory.

Proof. □

L18-end
L19

14. Spin and Dirac

Most of the geometric examples of elliptic operators, and families of them, are first order differential operators. Perhaps the most basic is the 'Hodge-Dirac' operator.

If M is a compact manifold (it need not be oriented) the form bundles $\Lambda^k M$ are well-defined real vector bundles over M but we will allow complex coefficients as well. Then the deRham operator is a well-defined first order differential operator

157.702 (7.101)
$$d : \mathcal{C}^\infty(M;\Lambda^k) \longrightarrow \mathcal{C}^\infty(M;\Lambda^{k+1})\ d^2 = 0.$$

By convention, for $k < 0$ or $k > \dim M$, the form bundles are trivial, just the zero vector space at each point. The deRham differential is not an elliptic operator, indeed it symbol is

157.703 (7.102)
$$\sigma_1(d)(x,\xi) = i\xi\wedge \in S^1/S^0(T^*M;\hom(\Lambda^k,\Lambda^{k+1}),\ (x,\xi) \in T^*M.$$

If we choose a Riemann metric on M we get, by definition, a fibrewise Euclidean inner product on TM and hence the dual inner product on T^*M and by standard constructions inner products on all the $\Lambda^k M$. Moreover we also get a well-defined Riemannin density (not a volume form unless M is oriented) which I will write as dg. This data determines a formal adjoint of d,

157.704 (7.103)
$$\delta = d^* : \mathcal{C}^\infty(M;\Lambda^{k+1}) \longrightarrow \mathcal{C}^\infty(M;\Lambda^k)\ \forall\ k,\ \delta^2 = 0.$$

Then the Hodge-Dirac operator is

157.705 (7.104)
$$\eth = d + \delta \in \mathrm{Ell}_1(M;\Lambda^*)\ \sigma_1(\eth)(x,\xi) = i(\xi\wedge - \iota(\xi))$$

where $\iota(\xi)$ is contraction with the metrically dual tangent vector to the cotangent vector ξ.

157.706 THEOREM 7.4 (Hodge). *If M is compact without boundary, the null space of $\eth$ is naturally isomorphic to the deRham cohomology of M.*

PROOF. The operator $\eth$ is elliptic since

157.707 (7.105)
$$\sigma_1(\eth)^2 = \xi\wedge\iota(\xi) + \iota(\xi)\xi\wedge = |\xi|^2 \,\mathrm{Id}\,.$$

Thus $\eth^2 = \Delta$ (the Laplacian, or Laplace-Beltrami operator) has scalar principal symbol which is invertible where $\xi \neq 0$. As an elliptic, formally self-adjoint operator we know that it is self-adjoint and the range has an orthogonal decomposition which restricts to smooth sections to

157.708 (7.106)
$$\mathcal{C}^\infty(M;\Lambda^*) = \mathrm{null}(\eth) \oplus \eth\left(\mathcal{C}^\infty(M;\Lambda^*)\right).$$

This is the Hodge decomposition.

Note that if $h \in \mathrm{null}(\eth) \subset \mathcal{C}^\infty(M;\Lambda^*)$ then

157.711 (7.107)
$$\int_M \langle dh, dh\rangle = \int_M \langle dh, -\delta h\rangle = \int_M \langle d^2 h, h\rangle = 0$$

so $dh = \delta h = 0$.

Applying (157.708 7.106) to a closed form $u \in \mathcal{C}^\infty(M;\Lambda^*)$ gives

157.709 (7.108)
$$u = h \oplus (dv + \delta v),\ v \in \mathcal{C}^\infty(M;\Lambda^*),\ dh = \delta h = 0.$$

This defines a surjective map

157.710 (7.109)
$$\{u \in \mathcal{C}^\infty(M;\Lambda^*); du = 0\} \ni u \longmapsto h \in \mathrm{null}(\eth).$$

since h is mapped to itself.

Taking the inner product of (7.108) with δv shows that $\delta v = 0$ (since u is closed) and hence that

157.712 $$u = h + dv. \tag{7.110}$$

If $u = dw$ then taking the inner product with u shows that u is in the null space of (7.109). Conversely if u is in the null space $u = dv$. Thus the map (7.109) descends to an isomorphism

157.713 $$H^*_{\mathrm{dR}}(M;\mathbb{R}) \longrightarrow \operatorname{null}(\eth). \tag{7.111}$$

$\square$

157.714 Corollary 5. *The restriction of the Hodge-Dirac operator to*

$$\eth^+ : \mathcal{C}^\infty(M\Lambda^{\mathrm{ev}}) \longrightarrow \mathcal{C}^\infty(M;\Lambda^{\mathrm{odd}})$$

has index the Euler characteristic of M

157.715 $$\operatorname{ind}(\eth^+) = \sum_{k=0}^{n}(-1)^k \dim H^k_{\mathrm{dR}}(M). \tag{7.112}$$

L19-end

On a Riemann manifold a *Clifford module* is a complex vector bundle $W \longrightarrow M$ with Hermitian metric together with a norm-preserving bundle map

157.716 $$\mathrm{cl} : T^*M \longrightarrow \hom(W) \tag{7.113}$$

satisfying the condition

157.717 $$\mathrm{cl}(\xi)\,\mathrm{cl}(\eta) + \mathrm{cl}(\eta)\,\mathrm{cl}(\xi) = 2\langle \xi, \eta\rangle \,\mathrm{Id} \text{ in } \hom(W). \tag{7.114}$$

157.718 Lemma 7.9. *A Clifford module has a unitary connection ∇ satisfying*

157.719 $$\nabla_v \,\mathrm{cl}(\xi) w = \mathrm{cl}(\nabla_c i)w + \mathrm{cl}(\xi)\nabla_v w, \quad \forall\ v \in \mathcal{C}^\infty(M;TM),\ \xi \in \mathcal{C}^\infty(M;T^*M),\ w \in \mathcal{C}^\infty(M;W) \tag{7.115}$$

where $\nabla_v \xi$ is the action of the Levi-Civita connection on the cotangent bundle.

Proof. As usual the idea is to construct such a connection locally and then patch using a partition of unity. $\square$

Interpreting the connection as a differential operator $\nabla : \mathcal{C}^\infty(M;W) \longrightarrow \mathcal{C}^\infty(M;T^*\otimes W)$ and the Clifford action as a contraction map $\mathrm{cl} : T^*W \otimes W \longrightarrow W$ the associated Dirac operator is a well-defined differential operator

157.720 $$\eth = \mathrm{cl} \circ \nabla : \mathcal{C}^\infty(M;W) \longrightarrow \mathcal{C}^\infty(M;W). \tag{7.116}$$

So, how do such Clifford modules arise and what is the index of the associated Dirac operator? Let me restrict to the case that the dimension of M is even, $2k$ – the odd-dimensional case is slightly different. Give a Riemann metric on M the complexified Clifford algebra at each point, defined as the quotient of the full complexified tensor algebra of $T^*_m M$ by the ideal generated by the elements as in (7.114), i.e.

157.721 $$\xi \otimes \eta + \eta \otimes \xi - \langle \xi, \eta\rangle \,\mathrm{Id} \tag{7.117}$$

forms the fibre of a bundle

157.722 $$\mathrm{Cl}(M) \longrightarrow M. \tag{7.118}$$

This is an *Azumaya* algebra – not only is it a complex vector bundle by the fibres are algebras and these algebras are isomorphic to the $2^k \times 2^k$ matrix algebra. A local orthonormal basis gives a trivialization of the bundle consistent with these identifications.

A Clifford module is then a vector bundle with a multiplicative bundle map

157.723 (7.119)
$$\mathrm{Cl}(M) \longrightarrow \hom(W).$$

Such modules arise from a Spin structure or a Spin-$\mathbb{C}$ structure on M. Both of these only make sense on an oriented manifold. They have to do with the groups

157.724 (7.120)
$$\mathrm{Spin}-\mathbb{C}(2k) \longrightarrow \mathrm{Spin}(2k) \longrightarrow \mathrm{SO}(2k).$$

Here (for $k > 1$) the spin group is the universal cover of $\mathrm{SO}(2k)$. Since $\pi_1(\mathrm{SO}(2k) = \mathbb{Z}_2$ this is a double cover. One way to construct concretely is using the Clifford algebra on $\mathbb{R}^{2k}$. An element of $\mathrm{SO}(2k)$ acts on the Clifford algebra through its action on $T_0^*\mathbb{R}^{2k} = \mathbb{R}^{2k}$ which preserves the ideal generated by (7.117). Such an algebra-preserving isomorphism of a matrix group is necessarily generated by conjugation so for $O \in \mathrm{SO}(2k)$ there is an element $L_O \in \mathrm{Cl}(\mathbb{R}^{2k})$ such that the action of O is

157.725 (7.121)
$$\mathrm{cl}(O\xi) = L_O \,\mathrm{cl}(\xi) L_O^{-1}.$$

Such an element can be constructed by factorizing O into products of reflections. Then L_O is the product of the clifford actions of the unit normal vectors to the fixed set of the reflection and is determined by up to sign. These L_O form the Spin group. It has a reflection action, given by the kernel of $\mathrm{Spin}(2k) \longrightarrow \mathrm{SO}(2k)$ and the Spin-$\mathbb{C}$ group is

157.726 (7.122)
$$\mathrm{Spin}-\mathbb{C}(2k) = \mathrm{Spin}(2k) \times_{\mathbb{Z}_2} \mathbb{S}.$$

15. Gerbes?

I do not expect to have time to discuss this in lectures. The idea is that this give a rather systematic approach to Spin and Spin-$\mathbb{C}$ structures on manifolds. Since it is relevant, at least as background, let me briefly recall the classification of real and complex line bundles over a manifold – which is generalized by gerbes.

A real line bundle over a manifold M can be given a smooth family of fibre metrics and so reduced to a principal $\mathbb{Z}_2$ bundle the 'sphere' in the line at each point

157.728 (7.123)
$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r, dash] & \hat{M} \arrow[d] \\ & M. \end{tikzcd}$$

Thus, $\hat{M}$ is a double cover of M with the action of $\mathbb{Z}_2$ being to interchage the points in each fibre.

Such a principal bundle is classified by $H^1(M;\mathbb{Z}_2)$. This is most easily seen in terms of Čech cohomology. To be brief about this, any open cover of a manifold has a refinement to a 'good' open cover – one in which all the open sets and all non-trivial finite intersections of them are contractible. A covering by small (with radius below the injectivity radius) Riemannian balls satifies this.

So, one can find such a good open cover over each element of which, U_i, the $\mathbb{Z}_2$ bundle has a section. Then over each intersection $U_{ij} = U_i \cap U_j$ the relation between these sections gives a map

157.729 (7.124) $$\chi_{ij} : U_{ij} \longrightarrow \mathbb{Z}_2.$$

This is a Čech cocycle, since over triple intersections

157.730 (7.125) $$\chi_{ij}\chi_{jk}\chi_{ki} = 1.$$

It follows (this is Čech theory) that this determines a cohomology classe

157.731 (7.126) $$\chi \in \check{H}(M; \mathbb{Z}_2).$$

The vanishing of this cohomology class is equivalent to the exactness of cocylcle (because the cover is good), meaning the existence of smooth (i.e. continuous) maps

157.732 (7.127) $$\eta_i : U_i \longrightarrow \mathbb{Z}_2 \text{ s.t. } \chi_{ij} = \eth_i \eta_j^{-1} \text{ on } U_{ij}.$$

Such a collection of map allows the original sections to be 'corrected' to a global section – implying the $\mathbb{Z}_2$ cover, and hence line bundle, is trivial. Conversely given a cocycle χ_{ij} one can construct a $\mathbb{Z}_2$-principal bundle from it which recover χ.

Similarly for complex line bundles over M are classified by $\check{H}^2(M; \mathbb{Z})$. Again one can choose a metric on the line bundle and so reduce it to a principal-$\mathbb{S}$ bundle

157.733 (7.128)
$$\begin{tikzcd} \mathbb{S} \arrow[r, dash] & S \arrow[d] \\ & M. \end{tikzcd}$$

Now the local trivialization over a good open cover give a map

157.734 (7.129) $$c_{ij} : U_{ij} \longrightarrow \mathbb{S} \text{ s.t. } c_{ij}c_{jk}c_{ki} = 1 \text{ over } U_i \cap U_j \cap U_k.$$

This cocycle yields a cohomology class, the Chern class, $c \in \check{H}^1(M; \mathbb{S})$ which is the same as $\check{H}^2(M; \mathbb{Z})$. The triviality of c implies the existence of a section of (7.128) and conversely. Moreover a prinicpal circle bundle can be constructed from a class $c \in \check{H}^2(M; \mathbb{Z})$ with this as Chern class.

The relationship between real line bundles, and their complexification (obtained by tensoring with $\mathbb{C}$) is the Bocksteim homomorphism

157.735 (7.130) $$H^1(M; \mathbb{Z}_2) \longrightarrow H^2(M; \mathbb{Z}).$$

Gerbes, in particular bundle gerbes, are the next step up from line bundles. I include a brief discussion of 'lifting bundle gerbes', specifically for spin and spinC structures. L20

CHAPTER 8

Semiclassical quantization

1. Blow up

11.4.2022.1 Definition 8.1. If $S \subset M$ is a closed embedded submanifold a blow-up of M along S, also called the blow-up (actually the radial blow-up) of S, is a manifold with boundary $[M;S]$ and smooth surjective map $\beta : [M;S] \longrightarrow M$ (the blow-down map) with the properties

(1) $\beta : [M;S] \setminus \partial[M;S] \longrightarrow M \setminus S$ is a diffeomorphism
(2) $\beta : \partial[M;S] \longrightarrow S$ is a sphere bundle
(3) If $0 \le q \in \mathcal{C}^\infty(M)$ vanishes precisely at S and exactly to second order then $\beta^* q = x^2$ is the square of a boundary defining function $x \in \mathcal{C}^\infty([M;S])$
(4) The Lie algebra of smooth vector fields on M which are tangent to S lift to span, over $\mathcal{C}^\infty([M;S])$, the Lie algebra of vector fields tangent to the boundary.

The 'precise second order vanishing' is the statement that the Hessian of q at each point $s \in S$ is postive definite as a quadratic form $v_i v_j q$ on $N_s S$, the normal bundle to S.

If you recall the notion of compactification from early in the course you will see some similarity. However blow-up is much more functorial.

Why blow up submanigolds? There are several reasons (but it isn't always a good idea!).

11.4.2022.2 Theorem 8.1. *Any closed embedded submanifold has a blow-up and any two are naturally diffeomorphic.*

Recall that the normal bundle to S, $NS = T_S M/TS$ parameterizes the vector fields at S 'pointing into N'. As we shall see the boundary of the blow-up is the corresponding sphere bundle $SNS = (NS \setminus 0_S)/\mathbb{R}^+$. So the definition of $[M;S]$ involves gluing this onto $M \setminus S$ to get a manifold with boundary.

Proof. The existence will use the collar neighbourhood theorem, discussed in § Sect.collar 3.

We start with a simple case, namely $M = W$ is a real vector space and $S = \{0\}$ is the origin. We can get a quadratic function $q = |w|^2$ as the square of the length for some Eucludean norm on W. Then we 'introduce polar coordinates'. These are 'singular coordinates' (whatever that means) but correspond to a smooth map

11.4.2022.3
$$\beta : [0,\infty) \times (W \setminus 0)/\mathbb{R}^+ \longrightarrow [0,\infty) \times \{w \in W; |w| = 1\} \ni (r, \hat{w}) \longmapsto r\hat{w} \in W. \tag{8.1}$$

Here the first map is a diffeomorphism where the metric is used to identify the quotient with the unit sphere. The inverse of β, restricted to $W \setminus 0$, is

11.4.2022.4
$$w \longmapsto (|w|, [w]) \in [0,\infty) \times (W \setminus 0/\mathbb{R}^+). \tag{8.2}$$

So we conclude that under change of norm from $|w|$ to $|w|'$ the identity map on $W\setminus 0$ lifts to the smooth diffeomorphism

11.4.2022.5
$$\beta:[0,\infty)\times(W\setminus 0/\mathbb{R}^+)\ni(x,[w])\longmapsto(\frac{|w|}{|w|'}x,[w]) \tag{8.3}$$

which is the identity on the boundary. So the construction does give the set

11.4.2022.6
$$(W\setminus 0/\mathbb{R}^+)\sqcup(W\setminus 0) \tag{8.4}$$

a unique topology and $\mathcal{C}^\infty$ structure.

Now, we should check that this has the desired properties. The first two conditions are clear since β is clearly a diffeomorphism of $(0,\infty)\times(W\setminus 0)/\mathbb{R}^+$ onto $W\setminus\{0\}$ and the boundary is a sphere. The quadratic function corresponding to the metric used to define (8.3) is $|w|^2$ so the third condition holds for this metric and any other quadratic function is the sum of some other Euclidean metric (its Hessian) and a function vanising to third order at the boundary. The first part is the product of $|w|^2$ with a smooth function on the sphere so it is the square of a defining function for the boundary and the higher order term is smooth (since β is smooth and vanishes to third order at the boundary.

So, it remains to check the fourth property. The vector fields tangent to 0 are those which vanish there and so the are of the form, in any linear coordinates,

157.736
$$\sum_{ij}a_{ij}w_i\partial_{w_j},\ a_{ij}\in\mathcal{C}^\infty(W). \tag{8.5}$$

The coefficient pull back to be smooth, so to show that these lift, i.e. extend smoothly from $x>0$ down to $x=0$, it suffices to show this for the $w_i\partial_{w_j}$. These are homogeneous of degree 0 on W and under β radial scaling becomes $(x,[w])\longrightarrow(tx,[w])$, $t>0$. Thus writtend in terms of the product decomposition (8.1)

157.737
$$w_i\partial_{w_j}=a([w])x\partial_x+V \tag{8.6}$$

where V is a smooth vector field and a is a smooth function on the sphere. Both extend smoothly down to $x=0$. The radial vector field $w\partial_w$ lifts to $x\partial_x$ and the $w_i\partial_{w_j}$ span all vector fields on W away from 0 so the V in (8.6) span all the vector fields on the sphere.

This completes the proof for the blow-up of $0\in W$. It is clear that the linear map reversing one coordinate lifts to be smooth as the reflection in the sphere. Then the smoothness of the lifts of the linear vector fields $w_i\partial_{w_j}$, which form the Lie algebra of $\mathrm{GL}(W)$ shows that the action of this group on $W\setminus\{0\}$ extends to $[W,\{0\}]$. It follows that the action

157.738
$$\mathrm{GL}(W)\times[W,\{0\}]\longrightarrow[W,\{0\}] \tag{8.7}$$

is smooth.

Now we pass to the case of the zero section of a real vector bundle, $U\longrightarrow M$. We define this as the union over the base of the blow-up of the fibres just defined, with blow-down maps

157.739
$$[U,0_U]=\bigsqcup_{m\in M}[U_m,\{0\}]\xrightarrow{\beta}U. \tag{8.8}$$

A local trivialization over an open subset $O\subset M$ gives $\beta^{-1}(O)$ a product trivialization as a fibre bundle and the smoothness of (8.7) means this patches smoothly to make (8.8) into a smooth fibre bundle with smooth map. Similarly the first and third conditions follow directly. The vector fields tangent to 0_U are given in a spanned in a local trivialization by the

157.740 (8.9) $$\partial_{z_i},\ w_i\partial_{w_j}$$

where the z_i are coodinates on M. All these vector fields have smooth lifts in the product decompositon of $[U;0_U]$ and so globally and clearly span the vector fields tangent to the boundary.

Finally then we pass to the general case but this follows from the collar neighbourhood theorem (or working locally if you prefer) which gives a diffeomorphism χ from a neighbourhood Q' of an embedded submanifold $S \subset M$ to a neighbourhood Q of the zero section of its normal bundle. This gives a $\mathcal{C}^\infty$ stucture to

157.741 (8.10) $$[M;S] = (M \setminus S) \cup SNS$$

as a manifold with boundary and by the lifting property it is independent of the choice of χ. □

The discussion above is for a closed embedded submanifold of a manifold without boundary, of course it applies unchanged if M has a boundary, including corners, but S does not meet the boundary. If S does meet the boundary we need to specify the meaning of 'embedded'. What is needed for the existence of a collar neighbourhood theorem is the following condition.

157.742 Definition 8.2. A subset $S \subset M$ of a manifold with corners is a p-submanifold (the 'p-' being for 'product') if at each point of S of codimension k there are 'adapted' local coordinates

157.743 (8.11) $$x_1,\ \dots,\ x_k,\ y_1,\ \dots,\ y_{n-k} \text{ in } O \subset M$$

where the $x_i \ge 0$ are local boundary defining functions and

157.744 (8.12) $$S \cap O = \{x_{l+i} = 0,\ i = 1\dots,k-l,\ y_{j+p} = 0,\ p = 1,\dots,\ n-k-j\}.$$

Here S has codimension $n-l-j$.

157.745 Theorem 8.2. *Any closed p-submanifold of a manifold with corners has a blow-up which is unique up to natural diffeomorphism.*

Proof. Maybe it is best to prove the collar neighbourhood theorem in this context first! □

2. Semiclassical smoothing operators

As usual we have two ways of approaching the definition of semiclassical smoothing operators on a manifold, and more generally on the fibres of a fibration. We can either use the original definition on $\mathbb{R}^n$ and localize or proceed globally. As you can already see, I favour the latter appropach.

First recall the case of Euclidean space. We defined the semiclassical smoothing operators in terms of Schwartz functions $A \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2n}))$, depending on a paramater, with the kernel being

157.699 (8.13) $$A(\epsilon, x, \frac{x-y}{\epsilon})\epsilon^{-n}dy$$

where I have included the measure since we know now that we should!

One reason for introducing blow-up above is that we can understand the kernels directly as smooth sections of a bundle.

157.701 Proposition 8.1. *The space of semiclassical smoothing operators on $\mathbb{R}^n$ have kernels which are smooth sections of a rescaled density bundle over*

157.700
$$[\mathbb{R}^n \times \mathbb{R}^n \times [0,1]; \operatorname{Diag} \times \{0\}] \tag{8.14}$$

L20-end
L21

Proof. $\square$

3. Pull-back and push-forward

This should really have come earlier in the course, but there is some virtue in leaving things until you need them, even if 'just in time' has its drawbacks!

Essentially the definition of smoothness of a map $F : M \longrightarrow N$ between manifolds means that the pull-back map is defined, linear and continuous on functions

157.748
$$F^* : \mathcal{C}^\infty(N) \longrightarrow \mathcal{C}^\infty(M). \tag{8.15}$$

We can add in a vector bundle on the image space to get

157.746
$$F^* : \mathcal{C}^\infty(N; W) \longrightarrow \mathcal{C}^\infty(M; F^*W) \tag{8.16}$$

Indeed, the pulled-back bundle, having fibes

157.747
$$(F^*W)_m = W_{F(m)} \tag{8.17}$$

is defined so that this is true. A trivialization of W over an open set $O \subset N$ induces a trivializaation of F^*W over $F^{-1}(O)$ and then (8.15) applies to the coefficients to give (8.16).

The map (8.16) has a 'formal transpose'

157.749
$$F_* : \mathcal{C}_{\mathrm{c}}^{-\infty}(M; W' \otimes \Omega) \longrightarrow \mathcal{C}_{\mathrm{c}}^{-\infty}(N; W' \otimes \Omega) \tag{8.18}$$

since these are the dual spaces – so

157.750
$$(F_*w)(\phi) = w(F^*\phi),\ \phi \in \mathcal{C}^\infty(N; W),\ w \in \mathcal{C}_{\mathrm{c}}^{-\infty}(M; W' \otimes \Omega). \tag{8.19}$$

We need compactness of the support of w (really a bit less) to make sure that the pairing on the right is defined.

In general the push-foward map does not preserve smoothness. If you consider the constant map $F : M \longrightarrow \{p\} \in N$ then it follows from (8.19) that F_*w has support contained in $\{p\}$. Indeed, if $p \notin \operatorname{supp}(\phi)$ then $\operatorname{supp}(F^*\phi) = 0$. In fact in general it follows that $\phi^*\phi$ is constant on M for this map (with F^*W trivial) so the pairing on the right in (8.19) is just an integral. Clearly this is not always zero, and then F_*w is necessarily singular.

So one cannot expect too much regularity for F_*w even if one assumes that $w \in \mathcal{C}_{\mathrm{c}}^\infty(M; W' \otimes \Omega)$. However, one of the properties of a fibration (the map corresponding to a fibre bundle) is that a version of Fubini's Theorem holds.

157.751 Proposition 8.2. *If $F : M \longrightarrow N$ is a fibre bundle then for any vector bundle $W \longrightarrow N$,*

157.752
$$F_* : \mathcal{C}_c^\infty(M; W' \otimes \Omega) \longrightarrow \mathcal{C}_c^\infty(N; W \otimes \Omega) \tag{8.20}$$

is surjective.

Proof. As I say, Fubini. The regularity of F_*w is a local question on N since we can see from (8.19) that if $\phi \in \mathcal{C}^\infty(N)$ then

157.753
$$\psi F_*(w) = F_*(F^*\psi w). \tag{8.21}$$

So it is enough to assume that w has support in $F^{-1}(O)$ where the fibre bundle is trivial over $O \subset N$ and then use a partition of unity; we can arrange that W is trivial over O as well. Then the diffeomorphism invariance of integration (of densities) means that we really are reduced to the case that $F^{-1}(O) = Z\times)$ and we are integrating over Z. □

Once we know this we can reverse the definition of push-forward and see that for a fibration pull-back extends by continuity to

157.754 (8.22) $$F^* : \mathcal{C}^{-\infty}(N;W) \longrightarrow \mathcal{C}^{-\infty}(M;F^*W) \text{ and is injective.}$$

Really the pull-back is 'constant along the fibres of F'.

We are interested in more refined version of this. In particular notice that if $S \subset N$ is a closed embedded submanifold then for a fibration

157.755 (8.23) $$F^{-1}(S) \subset N \text{ is closed and embedded.}$$

157.756 Proposition 8.3. *Pull back under a fibration, as in (8.22) defines a continuous linear map*

157.757 (8.24) $$F^* : I^m(N,S;W) \longrightarrow I^{m-d/4}(M;F^{-1}(S);F^*W)$$

for any closed embedded submanifold of N of codimension d and any vector bundle W over N.

Proof. □

We can, and should, ask a similar question abovt the push-foward map (8.20). So consider a closed, embedded submanifold $D \subset M$. There are already lots of submanifolds of M as the total space of a fibration, namely the fibres. Recall

157.758 Definition 8.3. Two embedded submanifolds D and Z in a manifold M meet *transversally* if at each point intersection

157.759 (8.25) $$p \in D \cap Z \Longrightarrow T_pD + T_pZ = T_pM \Longleftrightarrow N_p^*D \cap N^*)pZ = \{0\}.$$

157.760 Lemma 8.1. *The transversal intesection of two embedded submanifolds is an embedded submanifold with codimension the sum of the codimenstions.*

In particular two manifolds which do not intersect 'intersect transverally'. For two embedded submanifolds the notation for their intersection $D \pitchfork Z$ means that they intersect transversally.

Proof. By definition near any point of D there are local defining functions w_i which vanish on DW with independent differentials. Similarly there are local defining functions u_j for Z. At a point of intersection the transversality condition means that the w_i and u_j have independent differentials. Since they vanish precisely on the intersection locally, it is an embedded submanifold. □

157.762 Definition 8.4. We say that a smooth map $F : M \longrightarrow N$ is transvesal to an embedded submanifold $D \subset M$ (or that the submanifold is transversal to the map) if the differential $F_* : T_pD \longrightarrow T_{F(p)}N$ is surjective for each $p \in D$.

157.761 Proposition 8.4. *A submanifold $D \subset M$ of the total space of a fibration is transversal to the fibration if and only if it is transversal to each fibre; if D is closed then*

157.763 (8.26) $$F_* : I^m_c(M,D;F^*W \otimes \Omega) \longrightarrow \mathcal{C}^\infty_c(M;W \otimes \Omega).$$

So integrating along the fibres of a fibration which is transvesal to a submanifold 'wipes out' the singularities of conormal distributions.

PROOF. □

4. Abstract product theorem

Now, what I have been building up to here is a result which we can use to prove composition result for operators. The case to bear in mind is one we have already covered. Namely take the triple product M^3 and denote a general point (m_1, m_2, m_3). Then consider the three diagonals,

157.771 (8.27) $$D_1 = \{m_1 = m_2\},\ D_2 = \{m_2 = m_3\},\ S_3 = \{m_1 = m_3\}.$$

Each pair of these intersect transversally in the same 'triple diagonal'

157.772 (8.28) $$T = \{m_1 = m_2 = m_3\}.$$

Now consider the smooth map

157.773 (8.29) $$\phi : M^3 \ni (m_1, m_2, m_3) \longmapsto (m_1, m_3) \in M^2.$$

So

157.774 (8.30) $$D_3 = \phi^{-1}(S),\ S = \mathrm{Diag} = \{(m, m) \in M^2\}.$$

The product formula for pseudodifferential operators involves the multiplication of conormal distributions with respect to D_1 and D_2 and push forward under ϕ.

The first step involves a result which really goes way back to the beginning of the course.

157.765 LEMMA 8.2. *If $D_i \subset M$, $i = 1, 2$ are closed embedded submanifolds which intersect transversally then the product of conormal distributions is well defined*

157.766 (8.31) $$\times : I^*(M, D_1; W_1) \times I^*(M, D_2; W_2) \longrightarrow \mathcal{C}^{-\infty}(M; W_1 \otimes W_2).$$

There is no statement of conormality of the product, because it is not true in general (and with our definitions so far does not make sense which is reassuring)!

157.764 THEOREM 8.3. *If D_1, D_2 are closed embedded transversal submanifolds of the total space of a fibration $F : M \longrightarrow N$ to each of which the fibration restricts to be a diffeomorphism and such that there exists a closed embedded submanifold $S \subset N$ with*

157.767 (8.32) $$\phi^{-1}(S) \pitchfork D_1 = \phi^{-1}(S) \pitchfork D_2 = D_1 \pitchfork D_2$$

and a bundle map

157.769 (8.33) $$h : W_1 \otimes W_2 \longrightarrow \phi^*(W) \otimes \Omega_M$$

then the composite map

157.768 (8.34) $$\phi_*(g\xi) : I^{m_1}_c(M, D_1; W_1) \times I^{m_2}c(M, D_2; W_2) \longrightarrow I^m(N, S; W \otimes W).$$

PROOF. Consider the three submanifolds D_1, D_2 and $D_3 = \phi^{-1}(S)$. By assumption, (157.767 8.32), they intersect transversally in pairs. Since ϕ is assumed to be a diffeomorphism when restricted to D_1 it follows that the restriction of the fibration

157.770 (8.35) $$\phi : D_1 \cap D_2 \longrightarrow S$$

is a diffeomorphism. Now it follows from (8.32) that all three manifolds must have the same codimension – since the intersection has codimension which is equal to the sum of each pair.

To prove (8.34), first note that away from $D_1 \cap D_2$ the image of the product map in (8.31) lies in the sum of $I^*(M; D_1, W_1 \otimes W_2)$ and $I^*(M, D_2; W_1 \otimes W_2)$ so, after composing with the bundle map h, so Proposition 8.4 applies and shows that the push-forward is smooth. Thus, it suffices to consider a small neighbourhood of a point $n \in S \subset N$ and its preimage. We may also assume that the elements of $I^{m_1}_{\mathrm{c}}(M, D_1; W_1)$ and $I^{m_2}_{\mathrm{c}}(M, D_2; W_2)$ are supported near unique preimage n' of n in $D_1 \cap D_2$ (by using a partition of unity and discarding smooth terms).

Now, it is convenient to observe that D_1 and D_2 and the fibration can be brought to simultaneous normal form near such a point $n' \in D_1 \cap D_2$. Let $z = (z', z'')$ be local coordinates in the base, N, near n in which $S = \{z'' = 0\}$. Since ϕ is assumed to be a diffeomorphism when restricted to D_1 it follows that it is a graph over N locally. So we can introduce additional variables y near n' so that (y, z', z'') form a coordinate system and

157.778
$$D_1 = \{(y, z', z''); y = 0\} \text{ near } n'. \tag{8.36}$$

Since D_2 is also a graph over N near n' it takes the form

157.779
$$y_i = Y_i(z', z'') \text{ near } n'. \tag{8.37}$$

The codimension of D_1, the number of y_i, is equal to the codimension of $S \subset N$, i.e. the number of z'' variables.

The assumtion that $D_1 \cap D_2 \subset \phi^{-1}(S)$ means that $Y(z', 0) = 0$ and the transversality of D_1 and D_2 implies that

157.780
$$d_{z''} Y_i(z', z'') \text{ are linearly independent at } (0, 0, 0) \tag{8.38}$$

(since the $d_{z''} Y_i = 0$. These form a square matrix, so the Y_i can be introduced as new variables in the base, in place of the z'' and defining S. This then is the coordinate normal form

157.781
$$D_1 = \{y = 0\},\ D_2 = \{z'' - y = 0\},\ D_3 = \{z'' = 0\}. \tag{8.39}$$

Ignoring the bundles, it follows that the conormal distributions locally take the form

157.782
$$u_1 = (2\pi)^{-d} \int_{\mathbb{R}^d} a(z', z'', \eta) e^{iy\cdot\eta} d\eta,\ u_2 = (2\pi)^{-d} \int_{\mathbb{R}^d} b(z', z'', \eta') e^{i(z''-y)\cdot\eta'} d\eta' \tag{8.40}$$

where we can assume that the symbols a and b are supported near $(z', z'') = 0$. The push-forward of the product is then

157.783
$$\begin{aligned} &\int u_1(z, y) u_2(z, y) \\ &\quad = (2\pi)^{-2d} \int_{\mathbb{R}^{3d}} a(z', z'', \xi + \eta') b(z', z'', \eta') e^{z''\cdot\eta'} e^{iy\cdot\xi} d\eta d\xi dy,\ \xi = \eta - \eta'. \end{aligned} \tag{8.41}$$

Formally at least the ξ, y double integral can be interpreted as a Fourier/inverse Fourier transform which evaluates the integrand at $\xi = 0$ giving

157.784
$$\phi_*(u_1 u_2) = (2\pi)^{-2} \int_{\mathbb{R}^{3d}} a(z', z'', \eta') b(z', z'', \eta') e^{z''\cdot\eta'} d\eta' \in I^*(N, S). \tag{8.42}$$

To justify these last step we use continuity in the symbol topology as usual. $\square$

5. Semiclassical pseudodifferential operators

The semiclassical pseudodifferential algebra quantizes the semiclassical Lie algebroid. First we need to understand what this means.

157.785 Definition 8.5. A Lie algebroid on a manifold N is determined by a real vector bundle, W, over N and a bundle map (the anchor map)

157.786
$$a : W \longrightarrow TN \tag{8.43}$$

with the following additional properties

(1) The space of smooth sections $\mathcal{C}^\infty(N;W)$ is a Lie algebra
(2) The Lie bracket satisfies

157.787
$$a([V_1,V_2]) = [a(V_1),a(V_2)]\ \forall\ V_i \in \mathcal{C}^\infty(N;W),\ i=1,2. \tag{8.44}$$

The space of sections $\mathcal{C}^\infty(N;W)$ is normally called the Lie algebroid.

So of course $\mathcal{V}(N) = \mathcal{C}^\infty(N;TN)$ is a Lie algebroid. In the compact case its 'quantization' is taken to be the space $\Psi^*(N)$ of pseudodifferential operators.

I will leave open for the moment the precise definition of quantization, I may suggest a definition below but it depends rather on how much one wishes to demand. We have met a second example above, namely if $\phi : X \longrightarrow Y$ is a fibration of compact manifolds then

157.788
$$\mathcal{V}_\phi(X) = \{V \in \mathcal{V}(X); V\phi^* f = 0\ \forall\ f \in \mathcal{C}^\infty(Y)\} \tag{8.45}$$

is a Lie algebroid. The quantization is the algebra $\Psi_\phi*^(X)$ of fibre-wise pseudodifferential operators, which is directly involved in the definition of the Atiyah-Singer index above. What is the bundle? If we take local coordinates y_i in the base and extend these to coordinates near a point of X by adding some z_j then the elements of $\mathcal{V}_\phi(X)$ are locally of the form

157.789
$$\sum_j a_j(y,z)\partial_{z_j}. \tag{8.46}$$

So the bundle involved here is precisely the fibre tangent bundle ${}^\phi TX \subset TX$ with the anchor map being the natural inclusion (so of constant rank).

The case of immediate interest is the semiclassicl Lie algebroid which is closely related to $\mathcal{V}_\phi$. Namely, take a manifold M and consider

157.790
$$M[1,\mathrm{sl}] = M \times [0,1]_\epsilon \longrightarrow [0,1]. \tag{8.47}$$

On this space we consider fibre vector fields which in addition vanish at the boundary $\epsilon = 0$. Clearly in local coordinates, z_i, on M this means the smooth vector fields of the form

157.791
$$\sum_i a_i(\epsilon,z)\epsilon\partial_{z_j}. \tag{8.48}$$

These clearly form a Lie algebra, but what is the bundle V? We have to construct it. The form of (8.48) makes it rather clear that we have a local basis with elements

157.792
$$\epsilon\partial_{z_i}. \tag{8.49}$$

So these give a basis for the bundle ${}^{\mathrm{sl}}TM$ (which is a bundle over $M[1,\mathrm{sl}]$ not M despite my notation). You might struggle a bit to think of $\epsilon\partial_{z_i}$ as an 'entity' rather
L21-end than the product of ϵ and ∂_{z_i} but that is precisely what is involved here.
L22

6. Semiclassical index map

7. The Atiyah-Singer index theorem

The setting here is a smooth, compact fibre bundle

157.793 (8.50)
$$\begin{tikzcd} Z \arrow[r, dash] & X \arrow[d, "\phi"] \\ & Y. \end{tikzcd}$$

The compactness of the base is not very critical – we just need to assume that everything is trivial outside a compact set; non-compactness of the fibres is not much worse but see the discussion of Atiyah-Patodi-Singer below.

The basic question then, is given a smooth family of differential operators acting between sections of vector bundles on the fibres of ϕ as a map

157.804 (8.51)
$$A : \mathcal{C}^\infty(M; V) \longrightarrow \mathcal{C}^\infty(M; W)$$

when is it invertible?

This is hard.

Here is a step-by-step outline of the proof of the Atiyah-Singer index theorem (for families, in K-theory) showing what remains to be done – since I do not expect to have enough time to do every thing in detail.

i.data (1) Definition of the index map – this is still a little incomplete. We can think of 'quantization data' as a triple (V, W, a) where V and W are vector bundles over X and

157.794 (8.52)
$$a \in \mathcal{C}^\infty(S^*_\phi X; \pi^* \hom(V, W))$$

is an *invertible* isomorphism between the lifts of V and W from X to the fibrewise cotangent sphere bundle over X. We talked about more general elliptic symbols earlier.

(2) We have not quite finished the proof that the K-group $K^0(T^*_\phi X)$ is identified with the equivalence classes of the data (8.52) under the three relations of bundle isomorphism (over X of V and W), homotopy and stability.

(3) Then we can quantize a to a family of pseudodifferential operators $A \in \Psi^0_\phi(X; V, W)$ – this is the surjectivity of the symbol map. Any two such quantizations are homotopic and there are qunatizations where the null spaces form a vector bundle over Y.

(4) The K-group $K^0(Y)$ can be identified with equivalence classes of pairs of vector bundles over Y with the relations, bundle isomorphism, homotopy and stability (by adding an one bundle to both).

(5) The quantizations have appropriate properties under these maps so that

157.795 (8.53)
$$\operatorname{ind} : K^0(T^*_{\mathrm{fib}} X) \longrightarrow K^0(Y)$$

is well-defined with the index being the difference of the null bundle and a complement to the range for a quantization A (any one for which the null spaces form a bundle).

(6) Now we want to deform the index map (8.53) into the semiclassical calculus. To do this we generalize the data in 1. Namely we consider triples (β_1, β_1, b) where the β_i are involutive families $\overline{T^*_{\mathrm{fib}} X} \longrightarrow \mathrm{GL}(N, \mathbb{C})$ and

$b \in \mathcal{C}^\infty(\overline{T^*_{\mathrm{fib}}X}; \hom(\mathbb{C}^N))$ is such that $b : \operatorname{Ran}(P_1) \longrightarrow \operatorname{Ran}(P_2)$ is an isomorphism over $S^*_{\mathrm{fib}}X$ (the boundary of $\overline{T^*_{\mathrm{fib}}X}$) with $P_i = \frac{1}{2}(\beta_i + \mathrm{Id})$ the positive projections of the involutions. This is the semiclassical data considered above the equivalence classes under homotopy and stability again form the group $K^0_c(T^*_{\mathrm{fib}}X$. Semiclassical quantization results in the same index map (8.53) when the data reduces to that in 1.

i.escl (7) The data where β_2 is constant and $b = \mathrm{Id}$ exhausts $K^8(T^*_{\mathrm{fib}}X)$ so we can get the index map (8.53) from semiclassical smoothing operators – were β_1 is constant near infinity.

(8) Now the main part of the proof of the index theorem is the embedding of ϕ in a trivial fibration $\mathbb{R}^k \times Y$ for some (largish) k :

157.796 (8.54)
$$\begin{tikzcd} Z \arrow[r, dash] & X \arrow[r, hook, "\iota\times\phi"] \arrow[d, "\phi"] & \mathbb{R}^k \times Y \arrow[r, dash] \arrow[dl, "\pi_2"] & \mathbb{R}^k \\ & Y. & & \end{tikzcd}$$

This follows by choosing an embedding $\iota : X \longrightarrow \mathbb{R}^k$ and then into $\mathbb{R}^k \times Y$ by adding the map ϕ.

(9) Now we think of X as a submanifold of $\mathbb{R}^k \times Y$ and contemplate its normal bundle – so an open neighbourhood of the image. Each fibre Z_y of X is embedded in $\mathbb{R}^k$ so the full normal bundle is the bundle over X which over Z_y is $NZ_y \subset \mathbb{R}^k \times \{y\}$. So we actually have a 'tower' of fibrations

157.797 (8.55)
$$\begin{tikzcd} \mathbb{R}^d \arrow[r, dash] & NX \arrow[d, "\psi"] \\ Z \arrow[r, dash] & X \arrow[d, "\phi"] \\ & Y \end{tikzcd}$$

(10) The main idea in this proof by Atiyah-Singer (they had another one too, using cobordism) is that we can 'extend' the data and quantization from $T^*_{\mathrm{fib}}X$ to

157.798 (8.56)
$$T^*_{\psi\psi}NX = NX \oplus N^*X \longrightarrow X.$$

The fibres of ψ in (8.55) are non-compact but of course they are real vector spaces. So the fibewise cotangent bundle in (8.56) has fibres the sum of a vector space and its dual.

(11) So, we can find a family of involutions on $T^*_{\psi\psi}NX$ which are costant outside a compact set and quantize (semiclassically) to have index a trivial one-dimensional bundle over X.

(12) Then we take what is essentially the tensor product of this 'Bott element' (well, better to say a 'Thom' element) with the original family over X to be a family for $\phi \circ \psi : NX \longrightarrow Y$ which quantizes to the have the same image as a given element 7. We do this by quantizing in two steps.

(13) Now, we can arrange the support of the Thom element to be very close to the zero section and thereby move it, using a collar map, to $\mathbb{R}^k \times Y$

as a family. This is ‘excision’ and gives the same index for the extended family.

(14) The index map from $\mathbb{R}^k \times Y$ to Y is an isomorphism – again this is explicit Bott periodicity.

(15) Finally then we have an extended construction of the index map giving a commutative diagram

157.799 (8.57)
$$\begin{tikzcd} K^0(T^*_\phi X) \arrow[r, "\otimes\tau"] \arrow[dr, "\mathrm{ind}"'] & K^0(T^*_{\phi\psi}(NX)) \arrow[r, "\mathrm{exc}"] \arrow[d, "\mathrm{ind}"] & T^*_{\pi_2}(\mathbb{R}^k \times Y) \arrow[dl, "\mathrm{ind}"] \\ & K^0(Y) & \end{tikzcd}$$

(16) The final step then is to see that this diagram is the definition of the push-forward in K-theory, the ‘Gysin’ map in this context.

8. Index formula

Of course this is not the end of the story, quite apart from the fact that there are a few gaps in the argument – which I try to fill in below. The index in K-theory as above precisely captures the obstruction to an ellipitic family (or semiclassical family) have a smoothing perturbation which makes it invertible. In the case that Y is a point $K^0(Y) = \mathbb{Z}$ and we can look for a formula for the actual ‘numerical index’. In the case of a family we can look for a simpler obstruction to perturbative invertibility, corresponding to the image of the the index in (let’s say deRham) cohomology under the Chern character

157.800 (8.58)
$$\mathrm{Ch} : K^0(Y) \longrightarrow H^{\mathrm{ev}}(X;\mathbb{R}).$$

Either of these is the index formula.

157.801 THEOREM 8.4. *The index of the image in cohomology is*

157.802 (8.59)
$$\mathrm{Ch} \circ \mathrm{ind} = \phi_*(\mathrm{Ch}([(V,W,a]) \wedge \mathrm{Td}) = \int \mathrm{Ch}([(V,W,a]) \wedge \mathrm{Td}$$

where the push-forward in cohomology if realized as integration of a (compactly supported) form over the fibres of $T^*\mathrm{fib}X$.

The extra factor is the Todd class, which we can see from the proof above should be

157.803 (8.60)
$$\mathrm{Td} = \mathrm{Ch}(\tau)$$

appropriately interpreted.

9. The Dirac case

L22-end
L23
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L24

CHAPTER 9

Manifolds with boundary

Cmb

In these last three lectures I want to go through another example of quantization, leading to an algebra of pseudodifferential operators. In fact this is better thought of as 'microlocalization' of a Lie algebroid. In this case we consider a compact manifold with boundary M. We have already come across the Lie algebroid of smooth vector fields on M which are tangent to the boundary. You might like to check what happens on a manifold with corners but for the monment I will stick with codimension one.

Set

157.805 (9.1) $$\mathcal{V}_{\mathrm{b}}(M) = \{V \in \mathcal{C}^\infty(M; TM);\ \text{V is tangent to the boundary}\}.$$

There is always a boundary defining function $x \in \mathcal{C}^\infty(M)$ and the tangency condition just requires

157.806 (9.2) $$V \in \mathcal{V}_{\mathrm{b}}(M) \Longleftrightarrow Vx \in x\mathcal{C}^\infty(M).$$

So these are the vector fields which map the ideal of functions vanishing at the boundary (which is a primitive ideal generated by x) into itself. There is a strong 'naturality' case for the consideration of $\mathcal{V}_{\mathrm{b}}(M)$ since it is the Lie algebra of the group of diffeomorphism of M.

If we take local coordinates near a boundary point with y_i coordinates on the boundary then locally

157.807 (9.3) $$V \in \mathcal{V}_{\mathrm{b}}(M) \Longleftrightarrow V = a(x,y)x\partial_x + \sum_i b_i(x,y)\partial_{y_i}$$

for arbitrary smooth coefficients. This means that there is a vector bundle ${}^{\mathrm{b}}TM$ over M with sections precisely these vector fields

157.808 (9.4) $$\mathcal{V}_{\mathrm{b}}(M) = \mathcal{C}^\infty(M; {}^{\mathrm{b}}TM).$$

Let's think a little about the structure of the vector bundle ${}^{\mathrm{b}}TM$, since I am asserting it is, in context, the appropriate replacement for the 'ordinary' tangent bundle TM (to which is it isomorphic – just not naturally so). Over the interior of M there is not much to say since these two bundles are naturally isomorphic. Since the elements of $\mathcal{V}_{\mathrm{b}}(M)$ *are* smoth vector fields there is a completely natural smooth vector bundle map (the anchor map of the Lie algebroid $\mathcal{V}_{\mathrm{b}}(M)$)

157.899 (9.5) $${}^{\mathrm{b}}TM \longrightarrow TM.$$

Over the boundary this has corank 1 – there is a 1-dimensional null space since the vector field (in local coordinates) $x\partial_x$ vanishes in the ordinary sense at the boundary. So there is a 1-dimensional subbundle

157.900 (9.6) $${}^{\mathrm{b}}N\partial M \subset {}^{\mathrm{b}}T_{\partial M}M.$$

In fact this is a *canonically trivial* subbundle. The element $x\partial_x$ is actually (at a boundary point) defined independently of coordinates. Indeed it satisfies

157.901
$$(x\partial_x)\rho = \rho + O(\rho^2) \tag{9.7}$$

for any defining function ρ. This just reflects the fact that any other defining function $\rho = a(y)x + O(x^2)$.

In this behaviour ${}^{\mathrm{b}}T_{\partial M}M$ is 'reversed' from $T_{\partial M}.M$ The latter has $T\partial M$ as a subbundle, with the quotient being $N\partial M$, the normal bundle. The former has a (trivial) subbundle with *quotient* naturally $T\partial M$.

Let's go a little further with this analysis of tangency. The primitive ideal of functions vanishing at the boundary, I_∂, generated by x, leads to a Lie ideal

157.902
$$I_\partial \cdot \mathcal{V}_{\mathrm{b}}(M) \subset \mathcal{V}_{\mathrm{b}}(M),\ [I_\partial \cdot \mathcal{V}_{\mathrm{b}}(M), \mathcal{V}_{\mathrm{b}}(M)] \subset I_\partial \cdot \mathcal{V}_{\mathrm{b}}(M). \tag{9.8}$$

This means that the space of sections of ${}^{\mathrm{b}}T_{\partial M}M$ as a bundle over ∂M is itself a Lie algebra. This is clear enough in local coordinates. So there is actually a Lie algebra map

157.903
$$\mathcal{V}_{\mathrm{b}}(M) \longrightarrow \mathcal{C}^\infty(\partial M; {}^{\mathrm{b}}TM). \tag{9.9}$$

I have perhaps not emphasized enough uthat the Lie algebra structure of $\mathcal{V}(M)$ in the boundaryless case, or $\mathcal{V}_{\mathrm{b}}(M)$ here, is what leads to the properties of the differential operators, in particular that the leading part defines the (polynomial) symbol map. This is fair warning that we should expect something similar at the boundary for our, yet to be defined, b-pseudodifferential operators. It will be the 'indicial operator' and arises precisely because (157.903 9.9) is a map of Lie algebras. We can even guess it should take values in the pseudodifferential operators on the boundary but with 'an extra parameter'.

So what we want to find is an algebra of operators, say on $\mathcal{C}^\infty(M)$, which include the vector fields $\mathcal{V}_{\mathrm{b}}(M)$ and multliplication by $\mathcal{C}^\infty(M)$ and which away from the boundary should reduce to ordinary pseudodifferential operators. One can approach this as for the semiclassical calculus. Writing, informally, a pseudodifferential operator in terms of symbols we can try to replace a symbol

157.809
$$a(x,y,\xi,\eta) \text{ by } a(x,y,x\xi,\eta) \tag{9.10}$$

where (ξ,η) are the dual variables to (x,y). This does work but there are significant issues involved.

Proceeding formally we can plug such a symbol into the inverse Fourier transform and then change variables as for the semiclassical calculus (ignoring isses of domains and convergence)

157.867
$$\int a(x,y,x\xi,\eta)e^{i(x-x')\xi+i(y-y')}d\xi d\eta \\ = \int a(x,y,x\xi,\eta)e^{i(\frac{x-x'}{x})\tau+i(y-y')}\frac{d\tau}{x}d\eta,\ \tau = x\xi. \tag{9.11}$$

So, from this point of view what we need to do is to find a space on which $(x-x')/x$ is smooth, at least where it is finite. This we can do by an appropriate blow-up.

1. The b-generalized products

The approach I have taken, from the beginning, to pseudodifferential operators is to try to define them directly as spaces of conormal kernels. I have mentioned in passing the problem that immediately arises on a manifold with boundary, that the diagonal in M^2 – which is where we expect the singularities to be – does not meet the boundaries transversally. Namely there is an obvious dependence relation between the two defining functions x on the left and x' on the right and one of the defining functions for the diagonal, $x - x'$, at the intersection $x = x' = 0$, i.e. the corner. There is no problem with the tangential variables.

The geometric solution to this conundrum is to do as we did for the semiclassical caluclus and blow up the offending submanifold, in this case the corner. Thus we define

157.810
$$\beta : M[2,\mathrm{b}] = [M^2; (\partial M)^2] \longrightarrow M^2. \tag{9.12}$$

by blowing up the corner $x = x' = 0$.

There is quite a lot to get used to in this new space! The result of the blow up of the corner is that the new manifold has a new boundary hypersurfaces separatig two corners to the single corner before blow up (assuming the boundary is connected, which I am doing implicitly here; nothing really bad happens if there are several components). We know that the blow-up can be defined in terms of polar coordinates, in this case since only the variables x and x' are involved,

157.811
$$(x, x') = r(\cos\theta, \sin\theta),\ r \ge 0,\ \theta \in [0, \frac{\pi}{2}]. \tag{9.13}$$

The new boundary hpersurface, here $r = 0$, is the 'front face' denoted

157.812
$$\mathrm{ff}(M[2,\mathrm{b}]) = I \times \partial M \times \partial M,\ I_\theta = [0, \frac{\pi}{2}]. \tag{9.14}$$

We have also seen that we can cover a neighbourhood of the front face of a blow-up by projective coordinates. In fact here this can be done with just one coordinate system as far as x and x' are concerned – of course we also need coordinates in the two copies of ∂M – because the one variable $x + x'$ dominates both x and x' over the manifold. So the two functions

157.813
$$x + x' \text{ and } \mu = \frac{x - x'}{x + x'} \in [-1, 1] \tag{9.15}$$

together with tangential coordinates cover the front face. The two functions $1 + \mu$ and $1 - \mu$ are defining functions for the lifts of the 'old' boundaries $x = 0$ and $x' = 0$ (the lift here means the closure of the inverse image of the complement of the centre, $(\partial M)^2$, of blow up). Then the lifted diagonal is locally

157.814
$$\mathrm{Diag}_{\mathrm{b}} = \{\mu = 0\} \times \mathrm{Diag}_{\partial M} \text{ near } \mathrm{ff}\,. \tag{9.16}$$

It follows that now it *is* transversal to the boundary which it only meets in ff .

In fact the coordinates (9.15) are sometimes a bit awkward and it is simpler to use the more obvious projective coordinates

157.815
$$x' \text{ and } s = \frac{x}{x'} \text{ or } x \text{ and } t = \frac{x'}{x} = 1/s \tag{9.17}$$

valid respectively away from the lifts of $\{x' = 0\}$ and $\{x = 0\}$. In particular either of these simpler systems is valid near the diagonal.

The idea of this blow-up is to introduce a space of kernels which is much easier to describe there than directly on M^2. Note that one version of the Schwartz kernel theorem states that, on a manifold with corners, continuous linear operators

157.816
$$A : \dot{\mathcal{C}}^\infty(M) \longrightarrow \mathcal{C}^{-\infty}(M) = (\dot{\mathcal{C}}^\infty(M;\Omega))' \tag{9.18}$$

are identified with distributions

157.817
$$A \in \mathcal{C}^{-\infty}(M^2; \pi_R^*\Omega). \tag{9.19}$$

The image space in (9.18) is the space of *extendible distributions* as is the space of kernels on M^2 (apart from the density factor). These distributions are defined on any compact manifold with corners X as the dual of $\dot{\mathcal{C}}^\infty(X;\Omega)$ (they are the analogue of tempered distributions on $\mathbb{R}^n$ with which they are identified for the radial compactification).

157.818 Lemma 9.1. *The blow-down map gives an isomorphism*

157.820
$$\beta^* : \dot{\mathcal{C}}^\infty(M^2) \longrightarrow \dot{\mathcal{C}}^\infty(M[2,\mathrm{b}]) \tag{9.20}$$

and in consequence the Schwartz kernel theorem also identifies the space of continuous linear operators (9.18) *with*

157.819
$$A \in \mathcal{C}^{-\infty}(M[2,\mathrm{b}]; \pi_R^*\Omega) \tag{9.21}$$

The point is that we are not actually changing the space of extendible distributions by passing from M^2 to $M[2,\mathrm{b}]$, what is changing is the space of smooth functions (which is getting bigger) and the space of conormal distributions with respect to the diagonal (which is a pain to define on M^2).

Now, before going on let's check that the passage to $M[2,\mathrm{b}]$ does 'resolve' the Lie algebroid $\mathcal{V}_\mathrm{b}(M)$ in an appropriate sense. The vector fields in $\mathcal{V}_\mathrm{b}(M)$ acting on the left (or the right) factor of M in M^2 are tangent to the corner $x = 0 = x'$ since they annihilate x' and satisfy (9.2). Thus they lift to be smooth on $M[2,\mathrm{b}]$. In terms of (9.3) and the local coordinates (9.17), the ∂_{y_i} lift unchanged whereas

157.821
$$x\partial_x = s\partial_s. \tag{9.22}$$

This may not seem like much of an improvement! However, the lifted diagonal is at $s = 1$ so this vector field does not vanish there and we see:

157.822 Lemma 9.2. *The elements of* $\mathcal{V}_\mathrm{b}(M)$ *lifted to* $M[2,\mathrm{b}]$ *from the left (or right) factor of* M *in* M^2 *are transversal to the lifted diagonal* $\mathrm{Diag}_\mathrm{b} \subset M[2,\mathrm{b}]$ *and so the normal bundle to this submanifold is identified with* ${}^\mathrm{b}TM$.

This is a minimal requirement for 'resolution' of $\mathcal{V}_\mathrm{b}(M)$.

As well as the 'stretched double space' there are similar replacements for the higher products M^k. I will invoke the stretched triple space below. let me continue to assume that the boundary of M is connected – if it has more than one component you should proceed component by component, not thinking of interaction between the components which are 'far apart'.

We can see that the boundary faces of M^k consist of products where in each factor we have either ∂M or M. Now arrange these in order of increasing dimension – starting at $(\partial M)^k$ – and then blow them up, one after another

$$M[k,\mathrm{b}] = [M^k; (\partial M)^k; \mathcal{M}_{k-1}(M^k), \dots, \mathcal{M}_2(M^k)] \quad \boxed{157.868} \tag{9.23}$$

where $\mathcal{M}_p(M^k))$ is the collection of boundary faces of codimension p. We stop at $p = 2$ since boundary faces of codimension 1 are hypersurfaces and blowing them up does nothing. Now, what (9.23) really means is that we blow up the successive *lifts* of the boundary faces. To see that (9.23) is well-defined we need to note first that

157.869 Lemma 9.3. *Under blow up of a boundary face the lift of the other boundary faces are boundary faces.*

157.870 Lemma 9.4. *The sequence of blow ups in (9.23) is well-defined since after the blow up of $\mathcal{M}_p(M^k)$ the lifts of the elements of $\mathcal{M}_{p-1}$ are disjoint.*

This means that the order at each step in (9.23) is immaterial.

157.871 Proposition 9.1. *All the projections $M^k \longrightarrow M^j$, for $j < k$, lift to be smooth maps and simple/x b-fibrations.*

Here a b-fibration is a natural extension of the notion of a fibration to the category of manifolds with corners (it is the analogue of a Lefschetz map in algebraic geometry if that helps!) Rather than discuss these in detail here let me just say they are smooth surjective maps which near a point of the domain, of codimension l take the form in appropriately chosen coordinates in domain and range

$$F(x_1,\dots,x_l,y_1,\dots,y_m)) = (x^{\alpha_1},\dots,x^{\alpha_k},y_1,\dots,y_q),\\ \text{where the } x^{\alpha_i} \text{ are monomials in the } x_j\\ \text{with no common factor.} \tag{9.24}$$

Thus each x_j can appear as a positive power in at most one of the x^{α_i}. In Lemma (9.1) the each x_j at most once and as a single power which is the meaning of 'simple'. For such maps the α_i can be identified with disjoint subsets of $\{1,\dots,l\}$.

The importance of b-fibrations is that they have some of the properties of fibration – to which they reduce in the absence of boundaries.

157.873 Proposition 9.2. *Under a simple b-fibration $F :: M \longrightarrow N$ between compact manifolds with corners*

$$F_* : \{u \in \mathcal{C}^\infty(M\Omega_{\rm b}; u \equiv 0 \ \text{at}\ \mathcal{M}_2(M)\} \longrightarrow \mathcal{C}^\infty(M;\Omega_{\rm b}) \tag{9.25}$$

In fact much more is true than (9.25) for a simple b-fibration. Namely even if we don't assume the vanishing of the Taylor series at the corners as in (9.25), ?$\mathcal{C}^\infty(M;\Omega_{\rm b}$ pushes forward into

$$\sum_{\text{finite}} (\log x)^\beta \mathcal{C}^\infty(M;\Omega_{\rm b}). \tag{9.26}$$

So the push-forward is smooth except for powers of logs of the defining functions. These powers come from the behaviour at the corners of M, as follows from (9.25), and can be described much more precisely.

In particular, Proposition 9.2 must apply to the projection maps

$$\pi_{L,R}\beta : M[2,\mathrm{b}] \longrightarrow M. \qquad \boxed{157.876} \qquad (9.27)$$

Here it is essy to check Proposition 9.1 by hand. Away from ff the map is just locally one of the projection from M^2 to M for which (9.24) certainly holds. Near the front face we have one of the two coordinate systems

$$s = \frac{x}{x'},\ x',\ y,\ y' \text{ or } t = \frac{x'}{x},\ x,\ y,\ y'. \qquad \boxed{157.877} \qquad (9.28)$$

The left projection, to (x, y) therefore becomes either

$$(s, x', y, y') \longmapsto (x = sx', y) \text{ or } (x, y) \qquad \boxed{157.878} \qquad (9.29)$$

depending on the point in ff both of which satisfy (9.24).

It follows directly from (9.24) that a simple b-fibration is locally a fibration in the interior and also at boundary points of codimension one. This allows us to deduce

157.879 Lemma 9.5. *For a compact manifold with corners*

$$(\pi_L)_* : \Psi^m_{\mathrm{b}}(M) \longrightarrow \mathcal{C}^\infty(M). \qquad \boxed{157.880} \qquad (9.30)$$

This gives a direct proof of the mapping property (8.51).

2. Conormality at the boundary

Cmb.con

3. The b-calculus

The most basic operator in (9.18) is the identity operator. We know that in local coordinates the kernel of this is the Dirac 'function' at the diagonal

157.823 (9.31)
$$\delta(x-x')\delta(y-y').$$

This is, as it must be, a distributional section of the right density bundle

157.824 (9.32)
$$\mathrm{Id} \in I^0(M^2; \pi_R^*\Omega).$$

To see that this makes invariant sense observe that it must be possible to pair the delta 'function' with an element of $\mathcal{C}^\infty(M^2;\pi_L^*\Omega)$ since

$$\Omega(M^2) = \pi_L^*\Omega \otimes \pi_R^*\Omega.$$

Say using a partition of unity to localize, we need to be able to make sense of the distributional pairing, written formally as an integral

157.826 (9.33)
$$\int_{M^2} \delta(x-x')\delta(y-y')\psi(x,y,x',y')|dxdy| = \int_M \psi(x,y,x,y)|dxdy|$$

which is indeed invariantly defined – so this is what (9.32) actually means.

Now, what happens when we look at the lift of this kernel to $M[2,\mathrm{b}]$ as an extendible distribution – these form a subspace of the distributions on the interior. It is certainly still supported at the diagonal, so it is only a question of what it looks like in the new coordinates say (9.17). The homogeneity of delta means it becomes

157.827 (9.34)
$$\delta(x-x')\delta(y-y') = (x')^{-1}\delta(s-1)\delta(y-y').$$

We can absorb the extra singular factor of x' into the measure to see that

157.828 (9.35)
$$\mathrm{Id} \in I^0(M[2,\mathrm{b}], \mathrm{Diag}_{\mathrm{b}}; \pi_R^*\Omega_{\mathrm{b}})$$

where Ω_{b} is the density bundle coming from ${}^{\mathrm{b}}TM$ so in fact in a natural way it has a basis near the boundary

157.829 (9.36)
$$|\frac{dx}{x}dy| \Longrightarrow \Omega_{\mathrm{b}} = x^{-1}\Omega.$$

This leads us to the definition of b-pseudodifferential operators through their kernels.

157.830 Definition 9.1 ('Small' b-calculus). The space of b-pseudodifferential operators on a compact manifold with boundary, acting between sections of vector bundles V and W is

157.831 (9.37)
$$\Psi_{\mathrm{b}}^m(M;V,W) = \{A \in I^m(M[2,\mathrm{b}], \mathrm{Diag}_{\mathrm{b}}; \pi_L^*W \otimes \pi_R^*(V' \otimes \Omega_{\mathrm{b}})); A \equiv 0 \text{ at both lifted boundaries}\}.$$

I have not actually defined the conormal space here but it is exactly the restriction of the usual conormal space if one extends across the boundary. These distributions are smooth away from the diagonal so (9.37) makes sense since these 'old' boundaries do not meet the diagonal. Locally such a kernel, with the b-density removed, just looks like

$$A(x,y,s,y-y') \text{ smooth in } (x,y) \text{ and conormal at } s=1,\ y-y'=0.$$

The symbol map for conormal distributions gives us

157.833 (9.38)

$$\sigma_m : \Psi_{\mathrm{b}}^m(M;V,W) \longrightarrow (S^m/S^{m-1})(N^*\operatorname{Diag}_{\mathrm{b}};\pi^*(\hom(V,W)\otimes\Omega_{\mathrm{b}})\otimes\Omega(N^*)).$$

Recall that here $N^*\operatorname{Diag}$ is the dual of the normal bundle, so we see from Lemma 9.2 that this is identified with ${}^{\mathrm{b}}T^*M$ (replacing T^*M in the boundaryless case). The last book-keeping bundle in (9.38) is therefore $\Omega({}^{\mathrm{b}}T^*M)$ which is the dual of Ω_{b} so these two factor cancel and the symbol map is as simple as we could hope giving a short exact sequence

157.834 (9.39)

$$\Psi_{\mathrm{b}}^{m-1}(M;V,W) \hookrightarrow \Psi_{\mathrm{b}}^m(M;V,W) \xrightarrow{\sigma_m} (S^m/S^{m-1})({}^{\mathrm{b}}T^*M;\pi^*\hom(V,W)).$$

157.835 Theorem 9.1. *The b-pseudodifferential operators, $\Psi_{\mathrm{b}}^m(M;V,W)$, define continuous linear maps*

157.836 (9.40)
$$A : \mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^\infty(M;W)$$

*and form modules over the filtered *-closed algebra $\Psi_{\mathrm{b}}^m(M;V)$ for which the symbol sequence (9.39) is multiplicative.*

Proof. First let us check that the b-psuedodifferential operators defined by (9.37) do indeed define operators (9.40). This is not quite obvious, but notice that the space of kernels in (9.37) is a module over $\mathcal{C}^\infty(M[2,\mathrm{b}])$ so we can localize as we wish using a partition of unity. In particular we can work in (relatively) open subsets of M over which the bundles are trivial and so we are free to ignore them and assume that $V = W = \mathbb{C}$; this simplifies the notation. If we look at a pair of neighbourhoods which do not meet the boundary then we are in the interior case where we know (9.40). Similarly if one of the open sets does not meet the boundary then the operator is again locally an interior pseudodifferential operator plus a smoothing operator vanishing rapidly at the boundary from which (9.40) follows.

So we can localize to a product of neighbourhoods of points in the boundary – although the two open sets need not meet the diagonal. One thing that is easy to see is then is that

157.844 (9.41)
$$A : \dot{\mathcal{C}}^\infty(M;V) \longrightarrow \dot{\mathcal{C}}^\infty(M;W).$$

Indeed (localized) the action on $u \in \dot{\mathcal{C}}^\infty(U)$ where $U \subset M$ is a coordinate neighbourhood of a boundary point is by pushing forward the product

157.845 (9.42)
$$(\pi_L)_*(A\cdot\pi_R^*u)$$

By assumption, u vanishes to infinite order at $x' = 0$ (the boundary of the right factor) so π_R^*u vanishes to infinite order at the preimage, which includes ff. It follows that the product also vanishes to infinite order at ff and then we are dealing again with an ordinary pseudodifferential operator on M^2.

Now, it actually follows from this that

157.846 (9.43)
$$A : \mathcal{C}^{-\infty}(M;V) \longrightarrow \mathcal{C}^{-\infty}(M;W)$$

so we know that Au is defined if $u \in \mathcal{C}^\infty(M;V)$ and we just need to show that it is smooth up to the boundary. The argument giving (9.41) is included to show that in this case Au 'at the boundary' should only depend on u 'at the boundary'.

To make this precise we can again localize as above and only the terms where x and x' are near 0 are not clearly $\mathcal{C}^\infty$. Using the density properties of conormal distributions and of smooth functions we can assume more about A and then use continuity. Namely we can suppose that the kernel A is continuous (we could assume smoothness) and has support actually disjoint from the two 'old boundaries'. Then the coordiantes $t = x'/x$, x y and y' are admissible over the support after localization

157.847 (9.44)
$$A = A(x, y, t, y - y')|\frac{dx'}{x'}dy'|.$$

The action of U on u is then the integral

157.848 (9.45)
$$Au = \int_{x' \leq \epsilon} A(x, y, t, y - y')u(x', y')|\frac{dx'}{x'}dy'|$$

where u has compact support down to $x' = 0$. This integral certainly exists for $x > 0$ since then the integrand has compact supprt. We can change the variable of integration from x' to t and see that

157.849 (9.46)
$$Au = \int_{[0,\infty)} A(x, y, t, y - y')u(x/t, y')|\frac{dt}{t}dy'|.$$

Now the integral exists (by fiat the support in t is in $[C, 1/C]$ for some finite C) and the integrand is smooth in x and y, so the result is $\mathcal{C}^\infty$ and we have (9.40) under these assumptions on A. In fact we can pass to unrestricted A as in (9.37) since these kernels vanish rapidly at $t = 0$ and $t = \infty$ and the only singularity is conormal at $t = 1$ – across which we are integrating. Thus (9.40) follows in general and as a bonus we see that, as anticipated above

157.850 Corollary 6. *Restriction to the boundary in (9.40) gives*

157.851 (9.47)
$$\begin{gathered} Au\big|_{\partial M} = (A_\partial)u\big|_{\partial M},\ A_\partial \in \Psi^m(\partial M; V, W), \\ A = \tilde{A}(x, y, t, y - y')|\frac{dx'}{x'}dy'| \Longrightarrow A_\partial = \int_{\mathbb{R}^+} A(0, t, y, y - y')dt|dy'| \end{gathered}$$

so the kernel of A_∂ is the integral over the fibres of $\beta : \mathrm{ff} \longrightarrow (\partial M)^2$ of the kernel of A.

We still need to prove that the product of two b-pseudodifferential operators is b-pseudodifferential. □

Thus everything is very much as in the boundaryless case except that we have much more structure at the boundary. You might like to reflect on the similarity to the behaviour of the semiclassical calculus here.

For a moment return to the Lie algebra $\mathcal{V}_{\mathrm{b}}(M)$ – these of course define elements of $\Psi^1_{\mathrm{b}}(M)$. Certainly they satisfy (9.40) but also

157.837 (9.48)
$$(Vu)\big|_{\partial M} = (V\big|_{\partial M})u\big|_{\partial M},\ u \in \mathcal{C}^\infty(M).$$

They are 'localized at the boundary.' However we know that the analogous statement for the semiclassical calculus holds but misses important structure at the boundary. Much the same happens here.

The Collar Neighbourhood Theorem reminds us that a neighbourhood of a submanifold looks like a neighbourhood of the zero section of its normal bundle. For the boundary this translates to mean that a model for M near the boundary is the inward-point half of the normal bundle

157.838 (9.49)
$$N^+\partial M = \{v \in T_{\partial M}; vx \geq 0\}/T\partial M.$$

The choice of a boundary defining function gives a positive section and hence trivialization

157.839
$$N^+\partial M \longleftrightarrow [0,\infty)_{dx} \times \partial M \tag{9.50}$$

where dx defines a linear functional on the fibres of $T_{\partial M}M$ which vanishes on the subbundle $T\partial M$.

Let's pass to the radial compactification of $\overline{N^+\partial M} = I \times \partial M$ where I is a closed interval, thought of as the radial compactification of $[0,\infty)$ – there is no natural trivialization but any choice of boundary defining function provides one through (9.50). The fibre $\mathbb{R}^+$ action extends smoothly to the compactification (fixing the two boundaries $\{0\} \times \partial M$ and $\{\infty\} \times \partial M$).

157.840 Proposition 9.3. *There is a natural multiplicative map, the 'indicial map' to $\mathbb{R}^+$-invariant operators on the normal bundle, giving a short exact sequence*

157.841
$$x\Psi^m_{\mathrm{b}}(M;V,W) \hookrightarrow \Psi^m_{\mathrm{b}}(M;V,W) \xrightarrow{I} \Psi^m_{\mathbb{R}^+}(\overline{N^+\partial M};V_{\partial M},W_{\partial M}). \tag{9.51}$$

Proof. Let's choose a boundary defining function, rather than try to do things invariantly – in the end nothing will depend on this choice. Then the model at the boundary is

157.842
$$\overline{N^+\partial M} = I \times \partial M,\ I = [0,\infty]_x. \tag{9.52}$$

The space on which the kernels for the b-pseudodifferential operators are defined is therefore

157.843
$$I[2,\mathrm{b}] = [I^2, \{0\}\times\{0\}, \{\infty\}\times\{\infty\}]. \tag{9.53}$$

If you consider the $\mathbb{R}^+$ action on both factors starting at a point in the interior of $I[2,\mathrm{b}]$ you will see that it is an open interval but the closure is smooth up to both the front faces. An $\mathbb{R}^+$-invariant operator, as on the right of (9.51) corresponds to a kernel which is constant under this action. So in fact it is determined by its restriction to either of the front faces. Nothing much happens at the other boundaries since everything is required to vanish to infinite order there.

So, the invariant operators are determined uniquely by their restrictions to the front face ff_0 over $x = 0$. However this face for the model space is precisely the same as the face $\mathrm{ff}(M[2,\mathrm{b}])$ – canonically diffeomorphic to it (independent of the choice of defining function). So the indicial map in (9.51) is restriction of the kernel to the front face, and then its null space consists of kernels that vanish there. You might object that $x + x'$, not $x = (x + x')/(1 + s)$ is the defining function for the front face, but the kernel vanish to infinite order where $s \to \infty$ so we can just as well divide by x.

This does not explain the multiplicativity of the sequence but it follows that the null space in (9.51) is pretty clearly an ideal, so the quotient is an algebra – it is a question of what the product is!

To approach this reconsider Corollary 6. We have already noted that the space of kernels in (9.37) is a module over $\mathcal{C}^\infty(M[2,\mathrm{b}])$ but more is true because of the assumption of rapid vanishing at the boundary hypersurfaces other than ff . The quotient of defining functions from the left and right, x/x', is smooth except at one these two hypersurfaces and the rapid vanishing of the kernels there quashes the singularity from $x' = 0$. In fact the same is true for any power of this quotient, which is to say that

157.852 (9.54) $$(x/x')^{iz} \text{ is a multiplier on } \Psi_{\mathrm{b}}^m(M;V,W) \text{ for all } z\in\mathbb{C}.$$

157.853 LEMMA 9.6. *Conjugation generates an entire holomorphic family*

157.854 (9.55) $$\mathbb{C}\ni z\longmapsto x^{iz}Ax^{-iz}$$

of automorphisms of $\Psi_{\mathrm{b}}^m(M;V,W)$.

It follows (even without knowing the multiplicative property for the operators) that

157.855 (9.56) $$(AB)_\partial = A_\partial B_\partial$$

and we see from (9.47) that

157.856 (9.57) $$(x^{iz}Ax^{-iz})_\partial = \int_{\mathbb{R}^+} A(0,t,y,y-y')t^{-iz}dt|dy'|$$

is an entire family of pseudodifferential operators on ∂M.

The integral in (9.57) is the Mellin transform of the kernel of A restricted to $\mathrm{ff}(M[2,\mathrm{b}])$. This is the (inverse) Fourier transform with respect to the variable $-\log t\in\mathbb{R}$. In terms of $\log t$ the kernel is conormal at $0\times\{y=y'\}$ and decreases faster than any exponential at $\pm\infty$. It follows that this *indicial family* determines and is determined by the image in (9.51). In fact the multiplicativity of the map I then follows from (9.56) and (9.57), with the latter being a convolution representation of the $\mathbb{R}^+$-invariant operators on $\overline{N^+\partial M}$. □

We also want to analyse 'L^2 boundedness' of b-pseudodifferential operators.

To conform to the general 'b-yoga' we should replace 'ordinary L^2 – meaning computed with respect to a non-vanishing smooth density on a compact manifold with boundary – with $L^2_{\mathrm{b}}(M)$ computed with respect to a non-vanishing b-density. Since the latter is just x^{-1} times the former, we see that

157.882 (9.58) $$L^2(M)=x^{-\frac12}L^2_{\mathrm{b}}(M).$$

These spaces are well-defined for sections of vector bundles.

157.883 PROPOSITION 9.4. *Elements of* $\Psi^0_{\mathrm{b}}(M;V,W)$ *are bounded operators*

157.884 (9.59) $$x^sL^2_{\mathrm{b}}(M;V)\longrightarrow x^sL^2_{\mathrm{b}}(M;W)\ \forall\ s\in\mathbb{R}.$$

PROOF. The case of general s in (9.59) follows from the case $s=0$ in view of Lemma 9.6.

For $s=0$ we note that if we divide $A\in\Psi^0_{\mathrm{b}}(M;V,W)$, using a cut-off, into a part supported very near $\mathrm{ff}(M[2,\mathrm{b}])$ and a part supported away from this boundary hypersurface then the boundedness of regular pseudodifferential operators shows the boundedness of the second part. We can further localize and reduce to the case that $V=W=\mathbb{C}$.

The argument for boundedness in Problems2, using the symbol sequence can be applied almost verbatim here to show that boundedness follows if we can show the boundedness of the 'residual term' $\Psi_{\mathrm{b}}^{-\infty}(M)$.

Now, consider the $\mathbb{R}^+$-invariant calculus on $[0,\infty]\times\partial M$. We can characterize the space $L^2_{\mathrm{b}}([0,\infty]\times\partial M)$ in terms of the Mellin transform applied globally in the first variable x – since this reduces to the Fourier transform on $\log x \in \mathbb{R}$. Thus in fact

157.885 (9.60)
$$v(x,y) \longrightarrow v_M(z,y) = \int_{\mathbb{R}} v(x,y) x^{iz} \frac{dx}{x}$$

extends from $\dot{\mathcal{C}}^\infty([0,\infty] \times \partial M)$ to an isomorphism

157.886 (9.61)
$$L^2_{\mathrm{b}}([0,\infty] \times \partial M) \longrightarrow L^2(\mathbb{R}_z \times \partial M),\ z \in \mathbb{R}.$$

The fact that the transformed operator in (9.57) is a family of pseudodifferential operators in $\Psi^0(\partial M)$ which is bounded as a function of $z \in \mathbb{R}$ shows that it is bounded on the image of (9.61) so in fact the $\mathbb{R}^+$-invariant operators are bounded on the space on the left in (9.61). For the invariant operators we can again localize near and away from the boundary and deduce that the part localized near the boundary is bounded on $L^2_{\mathrm{b}}([0,\infty] \times \partial M)$.

In view of the exact sequence (9.51) it suffices to consider elements of $x\Psi^{-\infty}_{\mathrm{b}}(M)$ (using the preceding argument). The extra vanishing at ff shows that this follows directly from Schur's criterion. □

Of course we really want boundedness on Sobolev spaces, but the ones we want here are the $x^s H^m_{\mathrm{b}}(M)$ which we need to define. If we work in a fixed product decomposition near the boundary we can use the Mellin isomorphism (9.60) as we would for Euclidean space and define

157.887 (9.62)
$$H^m_{\mathrm{b}}([0,\infty) \times \partial M) = \{v \in \mathcal{C}^{-\infty}([0,\infty) \times \partial M); v_M \in L^1_{\mathrm{loc}}(\mathbb{R}_z; H^m(\partial M), \\ (1+|z|)^m v_M(z) \in L^2(\mathbb{R} \times \partial M)\},\ m \geq 0$$

L24-end For $m < 0$ we can use duality

L25

4. Metrics and boundaries

There are several intersting classes of metrics on a compact manifold with boundary.

> The most 'obvious' one I do not have time to talk about. This is the case of a metric smooth, and non-degenerate up to the boundary. It is rather a standard result, not too hard to see, that the distance from the boundary is, at least near the boundary, a smooth defining function and so can be extended to be smooth and positive in the interior. Then the metric takes a particular form in terms of the product decomposition near the boundary given by flow along the normal geodesic
>
> $$g = dx^2 + h + xh',\ h \text{ a metric on } \partial M. \quad 157.857 \quad (9.63)$$
>
> Here h' is actually an x-dependent family of symmetric tensor on ∂M but in any case can be taken to be a smooth symmetric 2-tensor on M near the boundary.
>
> Near the boundary there is then a corresponding decomposition of the form bundle on M
>
> $$\Lambda^* M = \Lambda^* \partial M \oplus (dx \wedge \Lambda^* \partial M) \quad 157.858 \quad (9.64)$$

in which it is reduced to two copies of the forms on ∂M. The Hodge-Dirac operator becomes a 2×2 matrix of operators

$$\eth = d + \delta = \begin{pmatrix} \eth_{\partial M} & -\partial_x \\ \partial_x & -\eth_{\partial M} \end{pmatrix} \qquad \boxed{\texttt{157.859}} \tag{9.65}$$

where ∂_x is acting on the coefficients in ([157.858]9.64).

Certainly

$$\eth : \mathcal{C}^\infty(M;\Lambda^*) \longrightarrow \mathcal{C}^\infty(M;\Lambda^*) \qquad \boxed{\texttt{157.860}} \tag{9.66}$$

but is not symmetric on this domain (with respect to the inner products on forms and density induced by g.) There are two standard boundary conditions which lead to symmetry, namely the vanishing of one or other of the two summands in ([157.858]9.64). More formally

$$\begin{aligned} \mathrm{Dom}_{\mathrm{Abs}} &= \{u \in \mathcal{C}^\infty(M;\Lambda^*); i^*_{\partial M}(\iota_\nu u) = 0\} \\ \mathrm{Dom}_{\mathrm{Rel}} &= \{u \in \mathcal{C}^\infty(M;\Lambda^*); i^*_{\partial M} u = 0\} \end{aligned} \qquad \boxed{\texttt{157.861}} \tag{9.67}$$

where $\nu = \partial_x$ is the Riemannian normal vector field at the boundary.

Then

$$\eth : \mathrm{Dom}_{\mathrm{Abs/Rel}} \longrightarrow \mathcal{C}^\infty(M;\Lambda^*) \text{ are Fredholm} \qquad \boxed{\texttt{157.862}} \tag{9.68}$$

and lead to Hodge decompositions as in the boundaryless case. From this one deduces the two Hodge theorems

$$\begin{aligned} \mathrm{null}(\eth) \cap \mathrm{Dom}_{\mathrm{Abs}} &\equiv H^*_{\mathrm{dR}}(M), \\ \mathrm{null}(\eth) \cap \mathrm{Dom}_{\mathrm{Rel}} &\equiv H^*_{\mathrm{dR}}(M,\partial M). \end{aligned} \qquad \boxed{\texttt{157.863}} \tag{9.69}$$

These two deRham theories can be identified as the cohomologies of the 'absolute' and 'relative' sequences

$$\begin{aligned} d &: \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M;\Lambda^1) \longrightarrow \cdots \longrightarrow \mathcal{C}^\infty(M;\Lambda^{\dim M}) \\ d &: \dot{\mathcal{C}}^\infty(M) \longrightarrow \dot{\mathcal{C}}^\infty(M;\Lambda^1) \longrightarrow \cdots \longrightarrow \dot{\mathcal{C}}^\infty(M;\Lambda^{\dim M}). \end{aligned} \qquad \boxed{\texttt{157.864}} \tag{9.70}$$

The main work here is to prove ([157.862]9.68) and discuss the corresponding Hodge decompositions – which ultimately are very much as in the boudaryless case. This can be done using the 'edge' calculus, or Boutet de Monvel's 'transmission' calculus. I probably will not have the time/energy to include these.

There is a long exact sequence relating the two cohomology theories and the cohomology of the boundary that we will encounter below

$$\dots \longrightarrow H^{k-1}(\partial M) \longrightarrow H^k(M,\partial M) \longrightarrow H^k(M) \longrightarrow \qquad \boxed{\texttt{157.866}} \tag{9.71}$$

5. Hodge theorems

Rather than the 'regular metrics' as in ([157.857]9.63) I want to consider two classes of metrics which are known in the geometric literature as 'cylindrical end' and 'asymptotically locally Euclidean' metrics. In fact the precise definition of these terms is a bit vague, so instead I will use the following notation.

157.865 Definition 9.2. A *b-metric* on a compact manifold with boundary is a fibre metric on ${}^{\mathrm{b}}TM$ which near the boundary is of the form

157.881 (9.72)
$$g_{\mathrm{b}} = \frac{dx^2}{x^2} + h + xh'$$

on some product decomposition, where h is a Riemann metric in ∂M and h' is a smooth quadratic form on ${}^{\mathrm{b}}TM$.

A *scattering metric* is then of the form

157.888 (9.73)
$$g_{\mathrm{sc}} = x^{-2}g_{\mathrm{b}} = \frac{dx^2}{x^4} + \frac{h}{x^2} + x^{-1}h'$$

near the boundary.

Since a product decomposition near the boundary always exists, every compact manifold with boundary has a metric of either type.

Note in particular that an example of (9.73) is a Euclidean metric on a real vector space, written in terms of the radial compactification.

Now, the idea is that we are supposed to think of these metrics 'categorically'. To analyse the Hodge-Dirac operator, we decompose the form bundle on M, near the boundary, in terms of the product decomposition. So in the case of (9.72)

157.889 (9.74)
$$\Lambda^*_{\mathrm{b}}M = \Lambda^*\partial M \oplus \frac{dx}{x} \wedge \Lambda^*\partial M \text{ near } \partial M.$$

Then

157.890 (9.75)
$$\eth_{\mathrm{b}} = d + \delta_{\mathrm{b}} = \begin{pmatrix} \eth_\partial & -x\partial_x \\ x\partial_x & -\eth_\partial \end{pmatrix} + xD$$

sign change comes from having to move past the dx/x factor. The 'error' term here is a b-differential operator. Thus

157.891 (9.76)
$$\eth_{\mathrm{b}} \in \Psi^1_{\mathrm{b}}(M; \Lambda^*_{\mathrm{b}}M),\ I(\eth_{\mathrm{b}}) = \begin{pmatrix} \eth_\partial & -x\partial_x \\ x\partial_x & -\eth_\partial \end{pmatrix}.$$

If you do not put the dx/x in (9.74), but use (9.64) instead you will *not* get a b-differential operator.

There is an analogue of the Hodge isomorphism here.

157.892 Theorem 9.2. *For a b-metric (9.72) on a compact manifold with boundary there is a natural isomorphism*

157.893 (9.77)
$$\{u \in L^2_{\mathrm{b}}(M; \Lambda^k_{\mathrm{b}}M); \eth u = 0\} \longrightarrow \mathrm{Im}\left(H^k_{\mathrm{dR}}(M, \partial M) \longrightarrow H^k_{\mathrm{dR}}(M)\right).$$

For a scattering metric we can proceed in a similar fashion. We 'rescale' the form bundle according to the forms in (9.73) – which are the ones that pair smoothly with the vector fields in $x\mathcal{V}_{\mathrm{b}}(M)$ – and see that

157.894 (9.78)
$$\Lambda^k_{\mathrm{sc}}M = x^{-k}\Lambda^k\partial M \oplus \frac{dx}{x^2} \wedge x^{-k+1}\Lambda^{k-1}\partial M \text{ near } \partial M.$$

A short calculation shows that in terms of this decomposition

157.895 (9.79)
$$\eth_{\mathrm{sc}} = d + \delta_{\mathrm{sc}} = \begin{pmatrix} x\eth_\partial & -x^2\partial_x - x(k+d) \\ x^2\partial_x - xk & -x\eth_\partial \end{pmatrix} + x^2 D \text{ on } \Lambda^k_{\mathrm{sc}}M,\ d = \dim \partial M$$

where D is again a *b-differential operator.*

Now you may see why I have not needed to develop the theory of 'scattering pseudodifferential operators' to handle this case since in fact

157.896 (9.80) $$\eth_{\mathrm{sc}} = xR,\ R \in \Psi^1_{\mathrm{b}}(M;\Lambda^* M),\ I(R) = \begin{pmatrix} \eth_\partial & -x\partial_x - (k-d-1) \\ x\partial_x - k & -\eth_\partial \end{pmatrix}.$$

This means that it is b-pseudodifferential operators which are relevant for the null space of $\eth_{\mathrm{sc}}$. It is a different matter if you wish to discuss the spectral theory of this operator (which I feel you should want to do)– which is indeed really scattering theory.

157.897 Theorem 9.3. *For a scattering metric (9.73) on a compact manifold with boundary of dimension n there is a natural isomorphism*

157.898 (9.81) $$\{u \in x^{\frac12 n} L^2_{\mathrm{b}}(M;\Lambda^k_{\mathrm{b}} M); \eth_{\mathrm{sc}} u = 0\} \longrightarrow \begin{cases} H^k(M,\partial M) & k < \frac12 n \\ \operatorname{Im}\left(H^{\frac12 n}_{\mathrm{dR}}(M,\partial M) \longrightarrow H^{\frac12 n}_{\mathrm{b}}(M)\right) & k = \frac12 n \\ H^k(M) & k > \frac12 n. \end{cases}$$

The L^2 space in (9.81) is the metric L^2 space for g_{sc}. Of course the 'middle dimensional case' can only occur if n is even.

Try it out for the Euclidean metric on $M = \overline{\mathbb{R}^n}$. It follows that there is no L^2 null space at all! This corresponds to the fact that here are no L^2 harmonic forms on $\mathbb{R}^n$ – their coefficients would be harmonic functions which would mean they decay at infinity.

Let's think about a strategy for proving Theorem 9.2. First we need to get some way to approach the deRham cohomology in this setting.

157.904 Proposition 9.5. *For $\epsilon > 0$ the cohomology of the deRham complex*

157.905 (9.82) $$x^\epsilon H^\infty_{\mathrm{b}}(M) \xrightarrow{d} x^\epsilon H^\infty_{\mathrm{b}}(M;\Lambda^1) \xrightarrow{d} \dots \xrightarrow{d} x^\epsilon H^{-\infty}(M;\Lambda^n_{\mathrm{b}})$$

is naturally isomorphic to $H^*_{\mathrm{dR}}(M,\partial M)$ and the cohomology of

157.906 (9.83) $$x^{-\epsilon} H^\infty_{\mathrm{b}}(M) \xrightarrow{d} x^{-\epsilon} H^\infty_{\mathrm{b}}(M;\Lambda^1) \xrightarrow{d} \dots \xrightarrow{d} x^{-\epsilon} H^{-\infty}(M;\Lambda^n_{\mathrm{b}})$$

is naturally isomorphic to $H^*_{\mathrm{dR}}(M)$.

The main step is the parametrix construction giving some 'elliptic regularity' and a Hodge decomposition.

157.907 Proposition 9.6. *On a compact manifold with boundary and for $\epsilon > 0$ small enough the Hodge-Dirac operator for a b-metric has* $\mathrm{null}(\eth)_{L^2_{\mathrm{b}}} \subset x^\epsilon H^\infty_{\mathrm{b}}(M;\Lambda^*_{\mathrm{b}})$ *and is Fredholm as an operator on $x^{-\epsilon} H^\infty_{\mathrm{b}}(M;\Lambda^*_{\mathrm{b}})$ satisfying*

157.909 (9.84) $$x^{-\epsilon} H^\infty_{\mathrm{b}}(M;\Lambda^*_{\mathrm{b}}) = \mathrm{null}(\eth)_{L^2_{\mathrm{b}}} \oplus d\left(x^{-\epsilon} H^\infty_{\mathrm{b}}(M;\Lambda^*_{\mathrm{b}})\right) \oplus \delta\left(x^{-\epsilon} H^\infty_{\mathrm{b}}(M;\Lambda^*_{\mathrm{b}})\right).$$

L25-end

L26

6. Ellipticity and parametrices

How do we prove say Proposition 9.6? We try to construct a parametrix as a b-pseudodifferential operator; as we shall see this does not quite work; we shall soon see why. An extension of the bounded result above is that

157.910 Lemma 9.7. *An element of $\Psi^0_{\mathrm{b}}(M;V)$ is compact as an operator on $L^2_{\mathrm{b}}(M;V)$ if and only if both its principal symbol and its normal operator vanish.*

It follows immediately that the same condition is necessary and sufficient for compactness on any of the weighted spaces $x^sL^2_{\mathrm{b}}(M;V)$.

Proof. The important point is the sufficiency, well the necessity is important but only to know! □

To prove that $\eth_{\mathrm{b}}$ is Fredholm as a map $x^{-\epsilon}H^1(M;\Lambda^*_{\mathrm{b}}) \longrightarrow x^{-\epsilon}H^1(M;\Lambda^*_{\mathrm{b}})$ we will want to construct a right parametrix modulo compact operators on $L^2_{\mathrm{b}}(M;\Lambda^*_{\mathrm{b}})$. If there *were* to be an element $B \in \Psi^{-1}_{\mathrm{b}}(M:\Lambda^*_{\mathrm{b}})$ satisfying

157.911 (9.85)
$$\eth_{\mathrm{b}}B = \mathrm{Id} - E,\ \ E \in x\Psi^{-1}(M;\Lambda^*_{\mathrm{b}})$$

with compact remainder E then we would need to have

157.912 (9.86)
$$\sigma_1(\eth_{\mathrm{b}})\sigma_{-1}(B) = \mathrm{Id},\ \ I(\eth_{\mathrm{b}})I(B) = \mathrm{Id}\,.$$

The first, symbolic, statement or 'division problem' is just ellipticity and is straighforward. The second is an issue, and is almost never possible to satisfy *within the class of operators* $\Psi^{-1}_{\mathrm{b}}(M;\Lambda^*_{\mathrm{b}})$.

What is the problem? Well we know that the indicial operator is equivalent, in terms of information, to the indicial family and in this case we can see from (9.76) what it is:

157.913 (9.87)
$$\hat{I}(\eth_{\mathrm{b}})(z) = \begin{pmatrix} \eth_\partial & -iz \\ iz & -\eth_\partial \end{pmatrix}.$$

We are asking that the inverse of this entire family of pseudodifferential operators on ∂M exist for all $z \in \mathbb{C}$. This is totally unreasonable!

What we can see is that there are values of z for which this operator has null space. In fact we can see exactly what they are. Suppose we have an eigenvector for $\eth_\partial$ (which we do!)

157.914 (9.88)
$$\eth_\partial u = \lambda u \Longrightarrow \hat{I}(i\lambda)\begin{pmatrix} u \\ -u \end{pmatrix} = 0,\ \ \hat{I}(-i\lambda)\begin{pmatrix} u \\ u \end{pmatrix} = 0$$

Then there are points of non-invertibility at

157.915 (9.89)
$$i\,\mathrm{Spec}(\eth_\partial) \subset \mathbb{C}.$$

157.916 Lemma 9.8. *The indicial family $\hat{I}(\eth_{\mathrm{b}})$ is invertible for $z \in \mathbb{C} \setminus i\,\mathrm{Spec}(\eth_\partial)$ and defines a meromorphic family*

157.917 (9.90)
$$\hat{I}(\eth_{\mathrm{b}})^{-1} : \mathbb{C} \setminus i\,\mathrm{Spec}(\eth_\partial) \longrightarrow \Psi^{-1}(\partial M;\Lambda^*_{\mathrm{b}}\big|_{\partial M})$$

with poles of order 1 at (9.89) with finite residues the orthogonal projections onto the mull spaces of $\hat{I}$.

157.918 Proposition 9.7. *For any $w \in \mathbb{R}$ the operator $I(\eth_{\mathrm{b}})$ has an $\mathbb{R}^+$-invariant inverse with kernel*

157.919 (9.91)
$$K(s,y,y') \in \mathrm{ff}(M[2,\mathrm{b}];\Lambda^*_{\mathrm{b}} \otimes \pi^*_R\Omega_{\mathrm{b}})$$

which lies in $s^{w-\epsilon}H^\infty$ near $s=0$ and $t^{-w+\epsilon}H^\infty$ near $t=0=1/s$ for $\epsilon>0$ sufficiently small.

We are mainly interested in the case $w = 0$.

In fact we can be much more specific about the behaviour of this kernel which has expansions as discussed in §9.2. There is a significant difference between the cases that $w \in \operatorname{Spec}(\eth_\partial)$ or not. We are actually interested in the case that w is an eigenvalue, in particular 0, of the Hodge-Dirac operator on the boundary.

By elliptic regularity we know that any solution of

157.920
$$I(\eth_{\mathrm{b}})K = \delta(s-1)\delta(y-y') \tag{9.92}$$

must be smooth away from $\operatorname{Diag}_{\mathrm{b}}$ where the delta function is supported. The solution given by the Proposition is unique. It has an expansion in the sense of §9.2 at the boundaries of $\operatorname{ff}(M[2,\mathrm{b}])$ determined by the spectrum of $\eth_\partial$

157.921
$$\begin{gathered} K \sim \sum_{\lambda_i \ge w} s^{\lambda_i} a_i(y,y') \text{ as } s \downarrow 0 \\ K \sim \sum_{\lambda_i > -w} t^{-\lambda_i} a_i'(y,y') \text{ as } s \downarrow 0. \end{gathered} \tag{9.93}$$

Proof. We are really working with the spectral theory of $\eth_\partial$ here. First notice that any solution of (9.92) has a conormal singularity at the 'diagonal' appearing on the right. Morevover, using the symbol map, we can construct a parameterix satisfying

157.922
$$I(\eth_{\mathrm{b}})K_0 = \delta(s-1)\delta(y-y') - E(s,y,y'),\ E \in \mathcal{C}^\infty_{\mathrm{c}}(\operatorname{ff}(M[2,\mathrm{b}]). \tag{9.94}$$

So it remains to 'solve away' the error term E which has support in the interior of ff .

We 'know' (I hope) that the expansion of a smooth function such as E (valued here in 2×2 matrices acting on $\Lambda^*\partial M$) in the y variable in terms of the eigenbasis of $\eth_{\partial M}$ converges rapidly. So on each eigenspace, with eigenvalue $\lambda_i \in \operatorname{Spec}(\eth_{\partial M})$ we wish to solve

157.924
$$\begin{pmatrix} \lambda_i & -s\partial_s \\ s\partial_s & -\lambda_i \end{pmatrix} K_i'(s,y') = E_i(s,y'). \tag{9.95}$$

This ordinary differential, and $\mathbb{R}^+$-invariant, equation has a unique solution which vanishes near $s = 0$ and any two solutions differ by an element of the 2-dimensional null space which is spanned by

$$x^{\lambda_i}\begin{pmatrix} u \\ -u \end{pmatrix} \text{ and } x^{-\lambda_i}\begin{pmatrix} u \\ u \end{pmatrix}$$

where u is the eigenvector.

As $s = 1/t \to \infty$ the chosen solution is in the null space. Thus we can arrange (9.93) for this one term by adding the approriate element of the null space. Summing over the eigenexpansion gives rapid convergence and hence we do in fact find a unique solution to (9.94) with the desired behaviour, (9.93), with respect to a given $w \in \mathbb{R}$. There can only be equality there if the wright w is equal to one of the eigenvalues but since these form a discrete set the kernel satisfies

157.926
$$K \in x^{w-\epsilon}H^\infty_{\mathrm{b}} \text{ near } \{s = 0\} \text{ and } K \in x^{-w+\epsilon}H^\infty_{\mathrm{b}} \text{ near } \{t = 0\}. \tag{9.96}$$

$\square$

The idea of course is that this kernel is to be the restriction to $\mathrm{ff}(M[2,\mathrm{b}])$ of the parametrix for $\eth_{\mathrm{b}}$. Now it should be clear why we cannot get a parametrix in the space $\Psi_{\mathrm{b}}^{-1}(M;\Lambda_{\mathrm{b}}^*)$ – because the solution we have found to (9.86) does *not* decay rapidly at the boundaries of the front face as required by Definition 9.1.

157.928 DEFINITION 9.3. Given a pair of index sets $\mathcal{E}=(\mathcal{E}_L,\mathcal{E}_R)$ we define

157.929
$$\begin{aligned}\Psi_{\mathrm{b}}^{-\infty,\mathcal{E}}(M;V,W)=&\mathcal{A}^{0,\mathcal{E}_L,\mathcal{E}_R}(M[2,\mathrm{b}];\pi_L^*W\otimes\pi_R^*(V'\otimes\Omega_{\mathrm{b}}))\\&+\mathcal{A}^{\mathcal{E}_L,\mathcal{E}_R}(M^2;\pi_L^*W\otimes\pi_R^*(V'\otimes\Omega_{\mathrm{b}})).\end{aligned}\tag{9.97}$$

Here the first space consists of $\mathcal{C}^\infty$ sections away from the left and right boundaries of $M[2,\mathrm{b}]$ which are conormal in the sense of §2 with index sets $\mathcal{E}_L$ and $\mathcal{E}_R$ at the left and right boundaries.

The second term will not appear below but needs to be there if we want to capture the structure of the generalized inverse.

In fact these operators form a module over the operators in Definition 9.1 and we write

157.930
$$\Psi_{\mathrm{b}}^{m,\mathcal{E}}(M;V,W)=\Psi_{\mathrm{b}}^{m}(M;V,W)+\Psi_{\mathrm{b}}^{-\infty,\mathcal{E}}(M;V,W).\tag{9.98}$$

157.931 LEMMA 9.9. *The elements of* $\Psi_{\mathrm{b}}^{m,\mathcal{E}}(M;V,W)$ *define bounded operators from* $x^{w_1}H_{\mathrm{b}}^s(M;V)$ *to* $x^{w_2}H_{\mathrm{b}}^{s-m}(M;W)$ *for any* $s\in\mathbb{R}$ *and* $w_1\ge w_2$ *provided* $w_2<\inf\operatorname{Re}\mathcal{E}_L$ *and* $w_1>-\inf\operatorname{Re}\mathcal{E}_R$.

7. Hodge theorem for b-metrics

Using the parametrix constructed above we proceed to prove Theorem 9.2. First we need to finish the proof of Proposition 9.6 and so (9.84). The parametrix construced in Proposition 9.7 for $w=0$ shows that

157.943
$$\eth_{\mathrm{b}}:x^{-\epsilon}H_{\mathrm{b}}^{\infty}(M;\Lambda^*)\longrightarrow x^{-\epsilon}H_{\mathrm{b}}^{\infty}(M;\Lambda^*)\tag{9.99}$$

is Fredholm. A complement to the range, namely the annihilator with respect to the metric pairing, is the null space of

157.944
$$\eth_{\mathrm{b}}:x^{\epsilon}H_{\mathrm{b}}^{-\infty}(M:\Lambda^*)\longrightarrow x^{\epsilon}H_{\mathrm{b}}^{-\infty}(M:\Lambda^*).\tag{9.100}$$

In fact we already know that the null space lies in $x^{\epsilon}H_{\mathrm{b}}^{\infty}(M:\Lambda^*)$. In fact this is elliptic regularity at the level of the 'small calculus' as in (9.85) – or bettwer we know that the error can be arranged to be in $\Psi_{\mathrm{b}}^{-\infty}(M;\Lambda^*)$. This is not compact but does map

157.945
$$\Psi_{\mathrm{b}}^{-\infty}(M:\Lambda^*):x^{\epsilon}H_{\mathrm{b}}^{-\infty}(M:\Lambda^*)\longrightarrow x^{\epsilon}H_{\mathrm{b}}^{\infty}(M:\Lambda^*).\tag{9.101}$$

We are really interested in the Hodge cohomology, the null space of $\eth_{\mathrm{b}}$ on the metric space $L_g^2(M;\Lambda^*)=L_{\mathrm{b}}^2(M;\Lambda^*)$. The same elliptic regularity shows that this the same as the null space of $\eth_{\mathrm{b}}$ on $H_{\mathrm{b}}^{\infty}(M;\Lambda^*)$ and so contained in the null space of (9.99) – so finite dimensional. Now, the form of the parametrix shows the second 'boundary' part of elliptic regularity, namely that the null space of (9.99) consists of elements with expansions, i.e. is contained in $\mathcal{A}^{\mathcal{E}}(M;\Lambda^*)$ where the index set is as in (9.89). So we conclude that

157.946
$$\{u\in x^{-\epsilon}H_{\mathrm{b}}^{-\infty}(M;\Lambda^*);\eth_{\mathrm{b}}u=0\}\Longrightarrow u-u_0\in x^{\epsilon}H_{\mathrm{b}}^{\infty}(M;\Lambda^*)\tag{9.102}$$

where u_0 is smooth up to the boundary, corresponding to $0 \in \operatorname{Spec}(\eth_\partial)$. However, $u_0 \notin L^2_{\mathrm{b}}(M : \Lambda^*)$ – the integral is logarithmically divergent – so

157.947 (9.103) $\{u \in L^2(M : \Lambda^*); \eth_{\mathrm{b}} u = 0\} \subset x^\epsilon H^\infty_{\mathrm{b}}(M; \Lambda^*)$ is the null space of (9.100).

So now we have our Hodge decomposition, initially in the form we have derived from (9.99)

157.948
$$x^{-\epsilon} H^\infty_{\mathrm{b}}(M; \Lambda^*) = (d+\delta) x^{-\epsilon} H^\infty_{\mathrm{b}}(M; \Lambda^*) \oplus \{u \in L^2(M : \Lambda^*); \eth_{\mathrm{b}} u = 0\}. \tag{9.104}$$

The Hodge summand here consists of the closed and coclosed eleemnts of $x^\epsilon H^\infty_{\mathrm{b}}(M; \Lambda^*)$ since the positive order of decay is enough to justify the integration by parts argument

157.949
$$0 = \langle (d+\delta)u, du\rangle = \|du\|^2_{L^2}. \tag{9.105}$$

So, assuming we accept Proposition 9.5, we already see that the Hodge cohomology, given by (9.103), is mapped into $H^*_{\mathrm{dR}}(M; \partial M)$. The usual proof of injectivity works. In fact the extra decay means that RBM:Proof of Proposition 9.5 needed

157.950
$$v \in x^{-\epsilon} H^\infty_{\mathrm{b}}(M; \Lambda^*), dv = 0 \Longrightarrow \langle v, u \Longrightarrow_{L^2} = 0. \tag{9.106}$$

This map is therefore injective to

157.951
$$\begin{aligned} \{u \in x^\epsilon H^\infty_{\mathrm{b}}(M; \Lambda^*\}/ \left(x^\epsilon H^\infty_{\mathrm{b}}(M; \Lambda^*) \cap dx^{-\epsilon} H^\infty_{\mathrm{b}}(M; \Lambda^*)\right) \\ = \operatorname{Im}\left(H^k_{\mathrm{dR}}(M, \partial M) \longrightarrow H^k_{\mathrm{dR}}(M)\right) \end{aligned} \tag{9.107}$$

so we have the existence and injectivity of the map (9.77). The Hodge decomposition (9.104), applied to a closed form in $x\epsilon H^\infty_{\mathrm{b}}(M : \Lambda^*)$ gives a two-sided inverse.

Thus Theorem 9.2 is proved.

8. Hodge theorem for scattering metrics

To prove Theorem 9.3 we proceed very much as for a b-metric, obviously with some changes.

First, the L^2 space with respect to a scattering metric is

157.940
$$x^{n/2} L^2_{\mathrm{b}}(M; \Lambda^*) \tag{9.108}$$

since the Riemannian density is a positive multiple of $x^{-n}\nu_{\mathrm{b}}$. Of course the L^2 space uses the inner product from g_{sc} on the form bundles, which gives a positive-definite inner product on the rescaled bundle Λ^*_{sc} in (9.78). The Fredholm properties of $\eth_{\mathrm{sc}}$ are determined by those of $I(R)$ in (9.80).

So we apply Proposition 9.7 for $w = n/2$ to the elliptic b-differential operator $x^{-1}\eth_{\mathrm{sc}}$ and conclude that

157.952
$$\eth_{\mathrm{sc}} : x^{n/2-1-\epsilon} H^\infty_{\mathrm{b}}(M; \Lambda^*_{\mathrm{sc}}) \longrightarrow x^{n/2-\epsilon} H^\infty_{\mathrm{b}}(M; \Lambda^*_{\mathrm{sc}}) \tag{9.109}$$

is Fredholm. Just as in the b-case, the Hodge cohomology

157.953
$$H^*_{\mathrm{sc}}(M) = \{u \in L^2_g(M; \Lambda^*); \eth_{\mathrm{sc}} u = 0\} \subset x^{n/2+\epsilon} H^\infty_{\mathrm{b}}(M; \Lambda^*_{\mathrm{sc}}) \tag{9.110}$$

is a complement, to the range in (9.109).

We need to analyse the indicial roots of $R = x^{-1}\eth_{\mathrm{sc}}$, which is to say the singular values of

157.954
$$I(R) = \begin{pmatrix} \eth_\partial & -x\partial_x - (N+n) \\ x\partial_x + N & -\eth_\partial \end{pmatrix} \tag{9.111}$$

where N is the number operator on boundary forms. Again these are generated by the spectrum of $ð_\partial$ but the relationship is more complicated than in the b-case above.

Each harmonic k-form, γ, on the boundary, in the null space of $ð_\partial$, generates a two-dimensional space on which $I(R)$ acts, spanned by

157.955 (9.112)
$$x^{-k}\gamma,\ \frac{dx}{x^2}\wedge x^{-k}\gamma.$$

These each correspond to an solution of $I(R)v=0$, namely

157.956 (9.113)
$$x^k\frac{\gamma}{x^k},\ \frac{dx}{x^2}\wedge x^{n/2-k}\frac{\gamma}{x^k}.$$

The first can be the leading term of a square-integrable form only if $k>n/2$ and the second only if $k<n/2$.

To claify square-integrability we need to analyze the other indicial roots. An eigenform for $ð_\partial$ with non-zero eigenvector corresponds to a pair of forms, a coclosed $(k-1)$-form h_{k-1} and a closed k-form u_k with $du_{k-1}=\lambda u_k$ and $\delta u_k=\lambda e_{k-1}$. These generate a 4-dimensional bundle invariant under $I(R)$ spanned by

157.957 (9.114)
$$\frac{u_{k-1}}{x^{k-1}},\ \frac{dx}{x^2}\wedge\frac{u_k}{x^k},\ \frac{u_k}{x^k}\ \text{and}\ \frac{dx}{x^2}\frac{u_{k-1}}{x^{k-1}}.$$

Since the 2-dimensional space spanned by the first two elements maps into that spanned by the second two and conversely, the null space of $I(R)$ is conatained in these two subspaces.

In the two cases the indicial roots correspond to a null vector of

157.958 (9.115)
$$\begin{pmatrix}\lambda & -iz+(n-k-1))\\ iz-k+1 & -\lambda\end{pmatrix}\ \text{and}\ \begin{pmatrix}\lambda & -iz+(n-k)\\ iz-k & -\lambda\end{pmatrix}$$

which occur when

157.959 (9.116)
$$(iz-k+1)(iz-n+k+1)=\lambda^2,\ (iz-k)(iz+(n-k)=\longrightarrow^2\Longrightarrow$$
$$(iz)^2-(n-2)iz-(n-k-1)(k-1)-\lambda^2=0,\ (iz)^2+(n-2k+1)iz)-k(n-k+1)-\lambda^2=0\Longrightarrow$$
$$iz=\frac{(n-2)}{2}\pm\frac{1}{2}\sqrt{(n+1-2(k-1))^2-4(n-k)(k-1)+4\lambda^2},\ iz=x\frac{(n+1)}{2}\pm\frac{1}{2}\sqrt{(n+1)^2+4\lambda^2}$$

157.941 LEMMA 9.10. *The indicial family $\hat{I}(R)(z)$ acting on $\Lambda^k\partial M\oplus\Lambda^{k-1}\pm M$ is invertible except for*

157.942 (9.117)
$$z=\frac{d+1}{2}\pm\sqrt{\lambda_{j,k}^2+(\frac{d+1}{2}-k)^2}.$$

where the $\lambda_{j,k}^2$ are the eigenvalues of the Laplacian on ∂M acting on closed k-forms.

9. Atiyah-Patodi-Singer index theorem

157.932 PROPOSITION 9.8. *Any element $A\in\Psi_{\mathrm{b}}^m(M;V,W)$ which is elliptic, i.e. $\sigma(A)$ is invertible, is Fredholm as a map*

157.933 (9.118)
$$x^wH_{\mathrm{b}}^s(M;V)\longrightarrow x^wH_{\mathrm{b}}^{s-m}(M;W)$$

for $w\in\mathbb{R}\setminus S$ where $S\subset\mathbb{R}$ is discrete.

Proof. The main point here is that ellipticity alone does imply that the indicial operator $I(A)$ is invertible as an $\mathbb{R}^+$-convolution operator such a set of weights. The main part of this is that the indicial family

157.934 (9.119)
$$\hat{I}(A) : \mathbb{C} \longrightarrow \Psi^m(\partial M; V, W)$$

which is entire, has a meromorphic inverse – so with a discrete set of poles

157.935 (9.120)
$$\text{b-Spec}(A) \subset \mathbb{C} \text{ s.t. } z_j \in \text{b-Spec}(A),\ |z_j| \to \infty \Longrightarrow |\operatorname{Re} z_j| \to \infty.$$

So this means there are only finiteley many poles in any strip $\operatorname{Re} z| < R$.

This in turn is a form of 'analytic Fredholm theory'. For a holomorphic family of elliptic operators, such as we have here, defined on a connected open set, the inverse is meromorphic there is one point at which the operator is invertible. In this case the existence of such a point, and the bound on the set of poles in (9.120) is a consequence of

157.936 Lemma 9.11. *The indicial family $I(A)(t+i\tau)$ for $t \in \mathbb{R}$ is a semiclassical family down to $\epsilon = \pm 1/\tau$ as $\tau \to \pm\infty$ in $\mathbb{R}$.*

Proof. □

Then the parametrix construction above generalizes to yield the Fredholm propery (9.118) for

157.937 (9.121)
$$w \notin \operatorname{Re} \text{b-Spec}(A).$$

□

So the Fredholm condition for (9.118) for w in an open set with discrete complement. The index is necessarily constant on the open sets but changes as w crosses an end-point. In fact there is a multiplicity function corresponding to the algebraic multiplicity of the residues at the poles of $\hat{I}(A)$,

157.938 (9.122)
$$\text{rank} : \text{b-Spec}(A) \longrightarrow \mathbb{Z}$$

and the change of the index in passing from one interval to the next is the sum of the multiplicity of the points in b-Spec(A) with real part corresponding to the end-point. It is elementary to see that the index is a decreasing function of w.

For Dirac operators Atiyah, Patodi and Singer ([?]) gave a formula for the index which applies in this case. This is extensively discussed, from the present point of view, in [?]. You might ask, is there a formula for the index in this general b-pseudodifferential case? The answer of course is yes!

If you have survived this far, you would certainly be tempted to ask: Is there a families index theorem? There is, at least there is a families index formula for the Dirac case in the literature and this can be extended to the case of families of Fredholm b-pseudodifferential operators. However, unlike the case of one operator discussed above, for an elliptic family of b-pseudodifferentia operators on the fibres of a fibre bundle (so of course the fibres are compact manifolds with boundary) there is an obstruction to this forming a Fredholm family.

157.939 Proposition 9.9. *The family of indicial operators of an elliptic family of b-pseudodifferential operators on the fibres of a compact fibre bundle define an index class in K^1 the vanishing of which is a necessary and sufficient condition for the existence of a Fredholm family with the same symbol.*

10. Spectral and scattering theory

11. What else?

L26-end

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