Lectures on Microlocal Analysis
18.157, Spring 2022
Versions

V1 30 November, 2021: L1 and compactification
V2 31 January, 2022: L2 started
V2b 4 February, 2022: Some corrections from Paige
V3 7 February, 2022: L3
V3a 8 February, 2022: L3 reorganized, correction from Zach.
V3b 9 February, 2022: Asymptotic completeness added to L3
V3c 9 February, 2022: Finished L3
V4 14 February, 2022: Problems 1 included, part of L5
V5 16 February, 2022: L6
V5a 16 February, 2022: More of L6
v7 19 February, 2022: Most of L7.
Outline and Practicalities

[Revised: 24 January, 2022.]
In this course I hope to cover four (types of) theorems which involve microlocal analysis and in particular the theory of pseudodifferential operators. Namely

(1) Hörmander’s theorem on the propagation of singularities
(2) Weyl’s law for the distribution of eigenvalues
(3) The Atiyah-Singer index theorem and K-theory
(4) Hodge theory and boundaries

As a first step I will proceed to discuss the algebras of pseudodifferential operators on Euclidean space and on a compact manifold and then similar algebras (and related modules) on manifolds with boundary and for fibrations and more

\[
\Psi^\ast(\mathbb{R}^n), \Psi^\ast(M), \Psi^\ast_\ast(M)
\]

where the upper star is an order and the lower star is some sort of structural information.

To me the four results listed above are fundamental, and I like them! The first two are relatively closely related and both give realization of the ‘semiclassical limit’, the interplay between the non-commutative theory of (pseudo-)differential operators and the more familiar behaviour of analysis of functions. The latter two are more global but both involve the essential invertibility of (pseudo-)differential operators.

Let me briefly indicate what these theorems are about.

Hörmander’s theorem on the propagation of singularities is a precise version, and massive generalization, of ‘Huyghen’s Principle’. The latter describes the spreading of the singular edge of solutions of the wave equation. The precise version is one of the consequences of ‘microlocalization’, transferring analysis from ‘space’ to ‘phase space’ interpreted concretely as a manifold and its cotangent bundle respectively.

Weyl’s asymptotic formula describes, at ‘high energy’, the number of eigenvalues of a self-adjoint elliptic operator, on a compact manifold, in terms of the volume inside the energy surface in the cotangent bundle. The original theorem was actually about the eigenvalues of the Dirichlet problem on a domain in \( \mathbb{R}^2 \).

Elliptic (pseudo-)differential operators on a compact manifold are Fredholm – they are invertible modulo finite dimensional null space and complement of the range. The index, the difference of these two dimensions, is a very stable number in the sense that it only depends on the ‘topology’ defined by the leading part of the operator and the theorem gives a formula for it. One classical version of this is the Riemann-Roch theorem for the \( \bar{\partial} \) operator on (line bundles over) a compact Riemann surface. This already requires some effort to understand! There is a one-dimensional real version of the theorem, due to Toeplitz, which states that the index of an elliptic Toeplitz operator on the circle (the projection onto the Hardy space, consisting of the functions smooth on and holomorphic on the interior of the disk, of multiplication by a non-vanishing smooth function) is equal to (minus) the winding number of the function.

You probably do know the Hodge theorem for a compact manifold without boundary as the identification of the deRham cohomology with the space of harmonic forms. For non-compact manifolds there is no simple generalization, rather
there are many corresponding to structures ‘at infinity’ (meaning near the boundary).

Clearly, each of these theorems could easily expand to take the whole semester. Still I hope to show how they can be approached using pseudodifferential operators and ‘quantization’. In fact an alternative title for this course might be ‘Smooth quantization’. So most of the time will be devoted to preparing the background material, specifically pseudodifferential operators on $\mathbb{R}^n$, pseudodifferential operators on a manifold, families of pseudodifferential operators and then rings of pseudodifferential operators quantizing a Lie algebroid.

I plan to give 26 one-hour lectures in the 9:30-10:30 slot on Tuesdays and Thursdays and leave 20 minutes for questions and discussions (even short presentations by students); if there is sufficient interest I will organize another ‘discussion’ time, perhaps on Wednesdays in the afternoon. There will be notes for each topic (the precise correspondence to the individual lectures will depend on various things), which will include topics I will not have time to cover and will certainly include further references – to books, lecture notes and papers. With any luck at least some of the lectures at should appear on my webpage before the beginning of the semester.

Problem sets: There will be approximately 5, every two weeks. Grading may be by discussion with me.

Grades: Graduate students are expected to participate actively. That is what ‘A’ means to me. By this I mean that I expect people to attend lectures and to ask questions. For undergraduates this course might be heavy lifting, it is for me, so please talk to me by early in the semester at the latest. We can discuss what you should expect. There are no exams.

Prerequisites: I will assume familiarity with manifolds and distributions, essentially as in 18.155 but plan to review pretty much everything.

Why don’t I just follow a book or my earlier lecture notes? This probably reflects some personal failing and general dissatisfaction with how things are done! I find it difficult to think through things without seeing some other way of approaching them. If it is not to your taste, I am sorry but that is the way it is. I may not get to all the results listed above, but I expect to at least get to the point where they are all within reach and that is really what I want to do – try to put these results in a general context that maybe encourages them to be exploited (i.e. applied) and extended.

In the interim, feel free to contact me with questions or comments.

1. Lecture 1

The main aim of this course is to describe various algebras of pseudodifferential operators. Let me start with a traditional ‘crypto-historical’ description of the ‘standard’ algebra of pseudodifferential operators on \( \mathbb{R}^n \). I recall notation for functions below, but let’s assume you know about the spaces of smooth functions on Euclidean and the various subspaces of compactly supported, Schwartz and functions with all derivatives bounded.

\[
C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C_c^\infty(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)
\]

maybe including their topologies and duals.

For any multiindex \( \alpha \in \mathbb{N}_0^n \), \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) being the non-negative integers, the corresponding iterated partial derivative acts on each of these spaces

\[
|\alpha| = \alpha_1 + \ldots \alpha_n
\]

where the normalizing power of \( i \) is inserted to help with notation for the Fourier transform.

These generate the commutative ring of differential operators with constant coefficients with general element

\[
p(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha, \quad c_\alpha \in \mathbb{C}.
\]

This is a filtered ring which is isomorphic to the ring of polynomials in \( n \) variables.

Similarly, each of the spaces in (1.1) is a ring, so multiplication of functions is defined. Combining these we consider linear partial differential operators which are given by sums

\[
P(x, D)u = \sum_{|\alpha| \leq m} p_\alpha(x)D^\alpha u.
\]

In each case, when the coefficients are in one of the spaces (1.1), we get an operator – a continuous linear map – on the corresponding space.

Whilst this is probably very familiar, and the operator product is given explicitly by Leibniz’ formula, it is very significant that these form a ring (and algebra) with product

\[
P(x, D)Q(x, D) = \sum_{\gamma \leq \alpha, \beta} p_\alpha(x)(D_x^\gamma q_\beta(x))D^\alpha+\beta-\gamma, \quad Q(x, D) = \sum_{|\beta| \leq m'} q_\beta(x)D^\beta.
\]
It is worth thinking a little more about what is going on here. First note that (1.4) is not as ‘natural’ as (1.3) in so far as we have chosen to write the ‘coefficients’, the function $p_\alpha(x)$ on the left. This is true in (1.3) as well but there the constants commute with the differentiation operators. Of course this is reflected in the fact that the product (1.3) is not commutative.

Now, let’s concentrate on the Schwartz space. For this we have the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n), \quad \mathcal{F} u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx.$$  

It is a linear isomorphism. We know that

$$u \in \mathcal{S}(\mathbb{R}^n) \implies \mathcal{F}(D^n u)(\xi) = \xi^n \hat{u}(\xi).$$

The Fourier transform conjugates differentiation to multiplication. Of course a monomial such as $\xi^n$ is not in the Schartz space, but it does define an operator on it by multiplication.

So the inverse Fourier transform, $u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \hat{u}(\xi)$, allows us to write

$$D^n u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^n \hat{u}(\xi) d\xi.$$  

A linear partial differential operator, (1.4), is given by a finite sum so we can combine (1.8) with (1.9) and write

$$Pu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} p(x,\xi) \hat{u}(\xi) d\xi, \quad p(x,\xi) = \sum_{|\alpha| \leq m} p_\alpha(x) \xi^\alpha d\xi.$$  

Since $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$, the integral converges absolutely. If we just assume that the coefficients are in $C^\infty(\mathbb{R}^n)$ then the integral converges uniformly on compact subsets in $x \in \mathbb{R}^n$, with all its formal derivatives in $x$ because of the obvious estimates

$$|D_\xi^\gamma p(x,\xi)| \leq C_{K,\gamma}(1 + |\xi|)^m, \quad x \in K \subseteq \mathbb{R}^n, \quad \xi \in \mathbb{R}^n.$$  

We can actually define the ‘standard’ space of pseudodifferential operators of order $m \in \mathbb{R}$ by considering those functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n_\xi)$ which satisfy the symbol estimates

$$|D_\xi^\beta D_\xi^\gamma a(x,\xi)| \leq C_{\beta,\gamma}(1 + |\xi|)^{m-|\beta|}, \quad \forall \gamma, \beta \in \mathbb{N}^n_\xi.$$  

Notice that $p$ in (1.9) satisfies these estimates for an integer $m$ if the coefficients are in the space

$$C^\infty_\gamma(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n); \sup |D_\xi^\gamma f(x)| < \infty \forall \gamma \},$$  

consisting of the smooth functions with all derivatives bounded.

The space of functions satisfying estimates (1.11) is often written $S^{m}_{1,0}$ as part of a more general class of spaces $S^{m}_{\rho,\delta}$ where the exponent $m - |\beta|$ is replaced by $m - \rho|\beta| + \delta|\alpha|$. I will make this notation more precise below, and will probably not talk about the general $\rho, \delta$ space - in fact there are many variants of such estimates (see for instance (1.12)) and we will already have enough things to think about.

It follows directly that if $a \in S^{m}_{1,0}$, in the sense that all the estimates hold, then the direct generalization of (1.11),

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x,\xi) \hat{u}(\xi) d\xi \implies a : \mathcal{S}(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n).$$

In fact much more is true:
Theorem 1. * The space of operators, $\Psi^m_{1,0}(\mathbb{R}^n)$ defined by symbols, $a$, satisfying (157.11) act on $S(\mathbb{R}^n)$ and these form a filtered $*$-closed ($*$ for adjoint here) ring

\[
\Psi^m_{1,0}(\mathbb{R}^n) \circ \Psi^m_{1,0}(\mathbb{R}^n) \subset \Psi^{m+m'}_{1,0}(\mathbb{R}^n), \ \forall \ m, m' \in \mathbb{R}.
\]

This is the main content of the first chapter of [157], see also [Taylor]. Probably the first place this result appeared in this form is [157].

The * in the header of the theorem is to indicate that I will not prove it immediately but a full proof, and more, will follow later. It is not that it is so hard to prove such a result, it is rather that I prefer to approach it from a position of strength, so somewhat indirectly, in the sense that I want to give a good deal of background before proving it.

[Narrowed parts of a lecture are things I don’t expect to have time to cover.]

Still it is important to see what is straightforward to prove and what may require some more thought. First let’s make sure we do have (157.11).

Proof of (157.11): If $u \in S(\mathbb{R}^n)$ then the product

\[
a(x,\xi) \hat{u}(\xi) \in S_{1,0}^{-\infty} = \bigcap_{M \in \mathbb{R}} S^M_{1,0} \tag{1.15}
\]

meaning that the estimates in (157.11) hold for all $m$. Indeed this is just the product rule for differentiation. Written out fully in terms of Leibniz’ formula

\[
D_\xi^\beta D_x^\gamma (a(x,\xi) \hat{u}(\xi)) = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D_\xi^\gamma D_x^\alpha a(x,\xi) \cdot D_\xi^{\beta-\gamma} \hat{u} \tag{1.16}
\]

Then one can apply the more obvious fact the product is rapidly decaying in $\xi$:

\[
S^m_{1,0} \cdot S(\mathbb{R}^n_\xi) \subset S^{m-k}_{1,0} \ \forall \ k \in \mathbb{R}. \tag{1.17}
\]

The integral (1.13) is therefore convergent. Again, if you like to be precise, you can see that

\[
S^m_{1,0} \subset C^0_\infty(\mathbb{R}^n; L^1(\mathbb{R}^n)), \ m < -n \tag{1.18}
\]

since $(1 + |\xi|)^{-n-\epsilon} \in L^1(\mathbb{R}^n)$ if $\epsilon > 0$. Now we can use standard properties of Lebesgue (or improper Riemann) integrals to see that $Au \in C^0_\infty(\mathbb{R}^n)$ is a bounded continuous function and the same holds for all derivatives giving (1.13). □

Now, I want to check a couple of other statements, weaker than Theorem 1. First the stronger mapping property that

\[
A : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n). \tag{1.19}
\]

This is a matter of getting ‘decay’. Namely we need to show that for any monomial and any derivative

\[
x^j D_x^\alpha Au \in C^0_\infty(\mathbb{R}^n). \tag{1.20}
\]

We can approach this one step at a time, asking just about $x_j Au$. Note that we can certainly multiply by $x_j$ but the operator $x_j A$ is not in general in $\Psi^m_{1,0}(\mathbb{R}^n)$ (for any $m$) since $x_j a(x,\xi)$ is not...
bounded as $|x_j| \to \infty$ even for fixed $\xi$. However the integral in (1.13) still converges rapidly in $\xi$ for $x$ in compact sets if we replace $a$ by $x_j a$ so

$$x_j A : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

for instance.

**Lemma 1.** In the sense of operators (1.21)

$$[x_j, A] = x_j A - Ax_j \in \Psi_{1,0}^{-1}(\mathbb{R}^n).$$

**Proof.** We use ‘integration by parts’. Consider the operator $Ax_j$. The Fourier transform of $x_j u$, $u \in \mathcal{S}(\mathbb{R}^n)$ is $i\partial_\xi \hat{u}$ so

$$Ax_j u = (2\pi)^{-n} \int a(x, \xi) e^{ix \cdot \xi} i\partial_\xi \hat{u}(\xi) d\xi = x_j A(x, D)u + b_j(x, D)u,$$

$$b_j(x, \xi) = -i\partial_\xi a(x, \xi) \in \Psi_{1,0}^{-1}(\mathbb{R}^n).$$

The rapid decay of $a(x, \xi) \hat{u}(\xi)$ in $\xi$ means that

$$\int \partial_\xi (a(x, \xi) e^{ix \cdot \xi} \hat{u}(\xi)) d\xi = 0.$$

Proceeding by induction we conclude that

$$x^\gamma A(x, D) = \sum_{\delta \leq \gamma} B_\delta(x, D) x^\delta, \quad B_\delta(x, D) \in \Psi_{1,0}^{-|\gamma|+|\delta|}.$$  

Rather than $\Psi_{1,0}^m(\mathbb{R}^n)$, which will be denoted simply $\Psi^m(\mathbb{R}^n)$ we will be more interested in the smaller space which I will denote $\Psi^m_{cl}(\mathbb{R}^n)$ often called the ring (with the composition property (1.14)) of ‘classical’ pseudodifferential operators where the symbols $a$ have the additional property:

There exists a sequence $a_i \in C^\infty(\mathbb{R}^n \times (\mathbb{R}_\xi^n \setminus \{0\}))$ of homogeneous functions of degree $m-i$ in the $\xi$ variables

$$a_i(x, t\xi) = t^{m-i} a(x, \xi), \quad t > 0, \quad (x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$$

such that for (any) cutoff $\chi \in C^\infty_c(\mathbb{R}^n_\xi)$ with $\chi = 1$ near 0

$$a(x, \xi) - \sum_{i=0}^N (1 - \chi(\xi)) a_i(x, \xi) \in \Psi_{1,0}^{m-N-1}.$$  

These ‘classical’ symbols form a filtered subring $\Psi^m_{cl} \subset \Psi^m = \Psi_{1,0}^m$. The relationship (1.20) is often written

$$a \simeq \sum_i a_i$$

and $a$ is then said to have a complete asymptotic expansion. Such ‘asymptotic summation’ (the existence of $a$ given the $a_i$ is discussed below, it is closely related to E. Borel’s ‘Lemma’ on Taylor series. So there no statement of convergence of the series in (1.20) (although there is one lurking in the background) but you should be able to see that the $a_i$, assuming they exist are determined by the relations (1.20).
Now, when we insert such classical symbols in \((1.13)\) (or if you prefer, restrict to classical symbols) into the definition of pseudodifferential operators then the resulting space of constitutes a filtered subring \(\Psi^m_{cl}(\mathbb{R}^n) \subset \Psi^m(\mathbb{R}^n)\) which for positive integral \(m\) includes the differential operators of order \(m\) discussed above.

These two rings have many important properties but one of the most important is that one can recover the terms \(a_i\) in \((1.26)\) from the operator \(A\) and the leading term defines the principal symbol, \(\sigma_m\), as a map
\[
\sigma_m : \Psi^m_{cl}(\mathbb{R}^n) \to \{a_0 \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \text{ homogeneous of degree } m \text{ in } \xi\},
\]
and this map is surjective, multiplicative and defines a short exact sequence
\[
\Psi^m_{cl}(\mathbb{R}^n) \to \Psi^{m'}_{cl}(\mathbb{R}^n) \to \{a_0 \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \text{ homogeneous of degree } m \text{ in } \xi\}.
\]
Here I have stuck with a cumbersome notation for the homogeneous space which will be refined below. There are similar exact sequences for the large algebra \(\Psi^*(\mathbb{R}^n)\) but the principal symbol lies in a quotient space.

So, we want to prove all these things and a lot more! However, I do not want to go there directly but rather map out the territory a bit first, in particular discussing the ‘symbol spaces’ concretely.

## 2. Manifolds with corners

This might appear to be a serious non-sequitor but I hope you will get used to the idea of these sections on background material and see a bit later why I am proceeding this way.

Both for ‘local’ analysis and the formulation of global results it is very convenient to focus on manifolds with corners as our basic ‘category of spaces’ (which it is as will be made precise later). There are several reasons to introduce these. An immediate one is to understand the symbol spaces and their generalizations. This I will get to next time. This allows me to introduce the spaces of conormal distributions which arise as the Schwartz kernels of the pseudodifferential operators we are interested in. Thinking about the kernels abstractly will allow us to generalize readily later. This involves manifolds with boundary, but then products will get you to manifolds with corners.

So, this is one of the basic settings for the course – analysis on manifolds with corners – but only taken as far as we need for the moment. Let me start with an explicit definition and then explain all the terms used in it. I’m assuming familiarity with the standard definition of a manifold without boundary.

**Definition 1.** A manifold with \(M\) is a metrizable, separable (so second countable) topological space with an open covering giving a (maximal) atlas of \(C^\infty\)-related coordinate patches modeled on \([0, \infty)^n\) and with embedded boundary hypersurfaces.

I will not assume connectedness without explicitly saying so, but the definition then requires all the components to have the same dimension.

So we are given a separable metric space, \(M\), where the ‘metrizable’ means we do not take the actual metric seriously, just the open sets it defines as the unions of open balls. A coordinate patch in such a topological space is a triple \((F, U, V)\)
consisting of a homeomorphism $F : U \rightarrow V$ of an open subset $U \subset M$ onto a (relatively) open subset $V \subset [0, \infty)^n$. So this means there exists an open subset $V' \subset \mathbb{R}^n$ such that the range $V = V' \cap [0, \infty)^n$. The coordinates on the coordinate patch are the pull-backs of the coordinate functions $x_i$ on $\mathbb{R}^n$.

To make clear what ‘$C^\infty$-related’ for two such coordinate patches means, we need to define $C^\infty(V)$ (I will not bother with lower regularity than $C^\infty$):

\[ C^\infty(V) = \{ u : V \rightarrow \mathbb{R} (\text{ or } \mathbb{C}); \exists V' \subset \mathbb{R}^n \text{ open } V = V' \cap [0, \infty)^n, \ u' \in C^\infty(V') \text{ and } u = u'|_V \}. \]

So I am assuming you know about $C^\infty(V')$ for open subsets of $\mathbb{R}^n$.

Now the $C^\infty$-compatibility of two coordinate patches $(F_i, U_i, V_i)$, $i = 1, 2$, as introduced above, means that either $U_1 \cap U_2 = \emptyset$ or else the transition maps

\[ F_{12} = F_1 \circ F_2^{-1} \] from $F_1(U_1 \cap U_2) \rightarrow F_2(U_1 \cap U_2)$ and $F_{21} = F_2 \circ F_1^{-1}$ from $F_2(U_1 \cap U_2) \rightarrow F_1(U_1 \cap U_2)$ are $C^\infty$ in the sense that $F_{12} : C^\infty(F_1(U_1 \cap U_2)) \rightarrow C^\infty(F_2(U_1 \cap U_2))$ and $F_{21} : C^\infty(F_2(U_1 \cap U_2)) \rightarrow C^\infty(F_1(U_1 \cap U_2))$; this is equivalent to saying either pull-back map is an isomorphism. This is also equivalent to saying that the pull-backs of the coordinate functions, under either of the maps $F_i$, restrict to $U_1 \cap U_2$ to be $C^\infty$ functions of the other coordinates.

So now an atlas is a covering by such (pairwise) $C^\infty$-compatible coordinate patches. If some coordinate patches are compatible with all the elements of an atlas then the combined collection is still an atlas – they are necessarily compatible amongst themselves as well. Hence any atlas is contained in a unique maximal atlas – all this is as in the boundaryless case.

If we just stop at this point then $M$ is what I call a tied manifold although there is no general agreement on this. The missing point is the additional condition that ‘boundary hypersurfaces are embedded’. A point in a coordinate patch is a boundary point of codimension $k$ if exactly $k$ of the coordinate functions vanish on it (note that coordinate patches map into $[0, \infty)^n$ so by fiat all coordinates are non-negative – I will actually drop this requirement later but it makes things easier to state initially). By considering the differential of the transition map it follows that the codimension is well-defined at each point, it is independent of the coordinate patch used. This means that $M$ has a stratification, a decomposition into disjoint pieces, based on the codimension

\[ M = M_0 \cup M_1 \cup \cdots \cup M_n \]

where the $M_j$ can be empty (from some $k > 0$ onward). The points of boundary codimension zero are the interior points of the manifold (there is a slight inconsistency between openness of subsets of $[0, \infty)^n$ and this, so the interior there is $(0, \infty)^n$, of course otherwise there would be no point in talking about the interior of a relatively open subset).

Each $M_j$ itself is a manifold without boundary and the closures of the components of the $M_j$ are called the boundary faces of codimension $j$; the set of these boundary faces I will write as $M_j(M)$. In particular the boundary faces of codimension one, the $H_i \in M_1(M)$ are called the boundary hypersurfaces. The ‘boundary hypersurfaces are embedded’ part of the definition is just the statement that the restrictions of the coordinate patches to each $H_i$ given them $C^\infty$-compatible atlases. The point of this functorial, that the boundary hypersurfaces (and in consequence all boundary faces) are themselves manifolds with corners. There are several useful
3. Compactification

Ways to restate this condition but note how it fails for a ‘tear-shaped region’ in the plane.

Pictures needed!

The $C^\infty$ functions on $M$ are those that are $C^\infty$ in each coordinate patch, meaning

$$f \in C^\infty(M) \iff (F^{-1})^*(f|_U) \in C^\infty(V) \text{ for each coordinate patch.}$$

This is equivalent to the same condition for any one compatible atlas.

The direct consequence of the ‘embedded’ requirement is that the boundary hypersurfaces have defining functions:

$$H_i \in \mathcal{M}_1(M) \implies \exists \rho_i \in C^\infty(M), \rho_i \geq 0, H_i = \{\rho_i = 0\},$$

$$d((F^{-1})^*\rho_i)(F(p)) \neq 0 \forall p \in H_i \text{ for all coordinate patches containing } p.$$

This last condition means that for each $p \in H_i$ there is a coordinate patch containing $p$ in which $\rho_i$ is a coordinate function.

If $\tilde{M}$ is a manifold without boundary, i.e. $\tilde{M}_1 = \emptyset$, then $M \subset \tilde{M}$ is an (embedded) submanifold if $M$ has a covering by coordinate patches of $\tilde{M}$ which restrict to give it the structure of a manifold with corners.

**Theorem 2.** For any manifold with corners there exists a manifold without boundary $\tilde{M}$ of the same dimension in which $M$ is embedded as a submanifold; if $M$ is compact then $\tilde{M}$ can be taken to be compact.

Although there is no quite canonical way of constructing such an extension, $\tilde{M}$, all the standard constructions of the tangent, cotangent, form bundles and other bundles associated to the frame bundle, pass over to the case of a manifold with corners in such a way that the restrictions for an extension of this type are canonical

$$TM = T\tilde{M}|_M, T^*M = T^*\tilde{M}|_M \text{ etc.}$$

However, there are important additional structures which arise from the boundary faces as I will discuss later.

So, which work in this degree of generality? Manifolds with corners are the smooth (i.e. $C^\infty$) analogue of smooth algebraic varieties with divisors and they occur for similar reasons. One place manifolds with corners arise is through ‘compactification’.

3. Compactification

Although we will deal with non-compact manifolds, the ones that arise below have some ‘structure at infinity’. One way to describe what this means is through the notion of compactification.

**Definition 2.** A compactification of a manifold $M$ is a compact manifold $\overline{M}$ and a smooth injection $\iota : M \to \overline{M}$ which is a diffeomorphism to a (relatively of course) open dense submanifold.

Here, both $M$ and $\overline{M}$ may have corners. As always when introducing a new notion, we should specify when two compactifications are to be regarded as ‘the same’.
DEFINITION 3. Two compactifications \( \iota_i : M \to \overline{M}_i \) are **equivalent** if there exists a diffeomorphism \( e : \overline{M}_1 \to \overline{M}_2 \) giving a commutative diagramme.

Notice that the equivalence map \( e \) is unique if it exists since it is fixed on an open dense subset by (1.36). We also say that one compactification is **finer** than another if there is a smooth map \( e \) giving a commutative diagramme; again it if it exists it is determined. This defines a partial order on compactification – as we shall see below there can be non-comparable compactifications.

If \( M \) is compact it is a compactification of itself and it is unique in this sense of equivalence.

We might well want more structure for the compactification – for instance if \( M \) is a complex manifold then we might want \( \overline{M} \) to be complex and all maps to be holomorphic. There are important examples from algebraic geometry here. Most relevant at the moment is the projective compactification of a complex vector space \( W \hookrightarrow \mathbb{P}W \) which I mention below but there are much more sophisticated examples to check out. There is the Deligne-Mumford compactification of the Riemann moduli spaces \( \mathcal{M}_{g,n} \) (okay I hear a complaint from someone that the \( \mathcal{M}_{g,n} \) are not quite manifolds, they are orbifolds in general, but take the number of punctures \( n \) large compared to the genus \( g \geq 0 \)). Also there is the deConcini-Procesi ‘wonderful’ compactification of complex adjoint Lie groups \( \mathfrak{sl}(n) \) (here is a real version of this compactification \( \overline{\mathfrak{sl}(n)} \)). Also, compactification of ‘Gravitational Instantons’ (aren’t the Physicists good at inventing names!).

The examples I will consider immediately are more prosaic, namely of a real finite-dimensional vector space \( V \). This is both to illustrate the notion and for later reference. I will discuss

1. The one-point compactification(s) given by a sphere \( \overline{V}^o \).
2. The parabolic compactification given a closed ball \( \overline{V}^p \).
3. The radial compactification also given by a closed ball \( \overline{V} = \overline{V}^R \).

From the notation you can see that I have a preference for the radial compactification – I hope the discussion below shows why. Only the radial compactification is really used subsequently.

These can all be constructed using variants of stereographic projection. So, let’s start with \( V = \mathbb{R}^n \), i.e. choose a basis. We embed \( \mathbb{R}^n \) into \( \mathbb{R}^{n+1} \) as the hyperplane

\[ \mathbb{R}^n \ni x \mapsto (x, 1) \in P \subset \mathbb{R}^{n+1}. \]

In the first case consider the the sphere \( S_o \) of radius \( \frac{1}{2} \) centred at \((0, \frac{1}{2})\) and in the second and third cases take the sphere \( S_R \) of radius 1 centred at the origin. In both these latter cases a point of \( \mathbb{R}^n \) determines a unique line \( L_o(x) \) or \( L_R(x) \) through
the image of \( x \) in \( P \) and the centre of the corresponding sphere then
\[
I_o : \mathbb{R}^n \to S_o, \quad I_o x \text{ is the other point in } S_o \cap L_o(x)
\]
\[
I_R : \mathbb{R}^n \to S_R^+, \quad I_R x \text{ is the other point in } S_R \cap L_1(x) \subset S_R^+ = S_R \cap \{x_{n+1} \geq 0\}
\]
\[
I_p : \mathbb{R}^n \to S_p \subset \mathbb{R}^n, \quad I_p x \text{ is the projection of } L_R x \text{ onto the closed unit ball in } \mathbb{R}^n \times \{x_{n+1} \geq 0\}.
\]
In all three cases the full orthogonal group \( O(n) \), acting on the first factor of \( \mathbb{R}^n \times \mathbb{R} \) satisfies \( I \cdot Ax = AL \cdot x \) for all \( A \in O(n) \), effectively reducing the discussion to the case \( n = 1 \). Explicit formulæ for the maps are easily derived:
\[
I_o x = (\frac{x}{1 + |x|^2}, \frac{1}{1 + |x|^2}) \in S_o
\]
\[
I_R = (\frac{x}{\left(1 + |x|^2\right)^{\frac{1}{2}}}, \frac{1}{\left(1 + |x|^2\right)^{\frac{1}{2}}}) \in S_R^+
\]
\[
I_p x = (\frac{x}{\left(1 + |x|^2\right)^{\frac{1}{2}}} \in \mathbb{R}^n.
\]
Thus, for the radial compactification \( (1 + |x|^2)^{-\frac{1}{2}} \) is a boundary defining function and hence \( |x|^{-1} \), which is a smooth function of it away from \( x = 0 \), is a defining function near the boundary. It follows that
\[
\{ |x| > \epsilon > 0 \} \ni x \mapsto \left(\frac{1}{|x|}, \frac{x}{|x|}\right) \in [0,1) \times S^{n-1}
\]
extends to a smooth product decomposition of \( \mathbb{R}^n \) near the boundary. For the parabolic compactification it follows similarly that
\[
\{ |x| > \epsilon > 0 \} \ni x \mapsto \left(\frac{1}{|x|^2}, \frac{x}{|x|}\right) \in [0,1) \times S^{n-1}
\]
is a product decomposition near the boundary.

It can be seen directly that
\[
I_o(\frac{x}{|x|^2}) = SI_o \text{ where } S : S_o \setminus \{(0,1),(0,0)\} \to S_o \setminus \{(0,1),(0,0)\}, \quad \text{with } S(y,y_n) = (y,-y_n+1)
\]
is equatorial reflection on \( S_o \).

In all cases it is clear either geometrically, or from the formulæ \( \ref{1.39} \), that the action of \( O(n) \) extends smoothly from \( \mathbb{R}^n \) to the compactification. Similarly the scaling action by \( \mathbb{R}^+ \), with generator on \( \mathbb{R}^n \)
\[
\sum_i x_i \frac{\partial}{\partial x_i}
\]
extends smoothly. For the one-point compactification this follows from \( \ref{1.42} \) and in the other two cases
\[
\lim_{|x| \to \infty} \frac{tx}{(1 + t|x|^2)^{\frac{1}{2}}} = \frac{x}{(|x|^2)^{\frac{1}{2}}} \quad \text{and} \quad \lim_{|x| \to \infty} \frac{1}{(1 + t|x|^2)^{\frac{1}{2}}} = 0.
\]
Thus in all cases the action of the conformal group \( O(n) \times \mathbb{R}^+ \) extends smoothly to the compactification.
**Proposition 1.** The action of the general linear group extends smoothly from $\mathbb{R}^n$ to the radial and parabolic compactifications but not to the one-point compactification; the translation action of $\mathbb{R}^n$ extends smoothly to the radial and the one-point compactifications but not to the parabolic compactification and there are smooth surjective maps, which are not diffeomorphisms, giving a commutative diagramme (1.45)

\[
\begin{array}{c}
\text{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n \\
\downarrow \\
\mathbb{S}^{n-1}_R \\
\downarrow I_R \\
\text{O}(n) \rtimes (\mathbb{R}^+ \times \mathbb{R}^n) \\
\downarrow I_\sigma \\
\mathbb{S}^{n-1}_0 \\
\end{array}
\]

\begin{array}{c}
\mathbb{R}^n \\
\downarrow \\
\mathbb{B}^p \\
\end{array}
\]

\[
\begin{array}{c}
\uparrow I_p \\
\end{array}
\]

\[
\text{O}(n) \rtimes (\mathbb{R}^+ \times \mathbb{R}^n) \rightarrow \mathbb{S}^{n-1}_0 \rightarrow \mathbb{R}^n \rightarrow \mathbb{B}^p \rightarrow \text{GL}(n, \mathbb{R}).
\]

**Outline of proof.** That the group actions extend as indicated follows by noting that the Lie algebra of $\text{GL}(n, \mathbb{R})$ consists of vector fields homogenous of degree 0 and similarly the translations are homogeneous of degree $-1$. Similar arguments show that the groups shown are the maximal subgroups of $\text{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n$ which extend to act smoothly on the one-point and parabolic compactifications. □

**Corollary 1.** The one-point compactification is defined for a vector space with conformal-Euclidean structure, the radial compactification is well-defined for an affine space and the parabolic compactification is well-defined for a vector space.

Both the radial and the parabolic compactifications have boundaryless variants, in which the bounding sphere is replaced by an embedded projective space $\mathbb{S}^{n-1}/\pm$ by doubling across the boundary. The apparent advantage of this smaller compactification does not seem to be realized in practice.

**Conjecture 1.** The five compactifications are minimal in their respective categories (i.e. as manifolds with/without boundary) among compactifications with the invariance properties in (1.45).

Although, as noted above, it is the radial compactification which mostly appears below other variants are relevant. In particular none of these compactifications are natural for products – the radial compactification of $V_1 \times V_2$ is not ‘comparable’ to the products of the radial compactifications. Still this relationship is significant and is examined below.
Symbols and conormal distributions at a point

1. Lecture 2

Before tackling the properties of the ring $\Psi^*(\mathbb{R}^n)$ of pseudodifferential operators on $\mathbb{R}^n$ I want to look into the properties of the Schwartz kernels of these operators, so we can get a picture of them. We can ‘guess’ (it is easy to justify) that the Schwartz kernel of an operator $A \in \Psi^m(\mathbb{R}^n)$ defined by a symbol $a$ satisfying (1.11) is

$$A(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x, \xi)e^{i(x-y)\cdot \xi}d\xi. \tag{2.1}$$

Here I use the same letter for the operator and its Schwartz kernel – since the Schwartz kernel theorem (which I will talk a little about later) shows that they determine each other.

We can think of (1.11) in a couple of different ways – in general it is not a convergent integral. We can make a (formal at this stage) linear change of variables on $\mathbb{R}^{2n}$ from $(x, y)$ to $(x, z)$, $z = x - y$ and then

$$A(x, y) = \alpha(x, x - y) \tag{2.2}$$

where

$$\alpha(x, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} a(x, \xi)e^{iz\cdot \xi}d\xi. \tag{2.2}$$

Now the integral is a partial inverse Fourier transform. In fact, since $a$ is smooth in $x$ we can interpret the definition of $\alpha$ in (2.2) as the inverse Fourier transform from $\xi$ to $z$ for each fixed $x$. This in fact is what I will do today. Alternatively one can just check that the partial Fourier transform with respect to a decomposition of Euclidean space into a product behaves ‘correctly’.

So, for the moment, we have dispensed with the ‘coefficients’ and just look at the (commutative) algebra of constant-coefficient pseudodifferential operators where the composition operation is convolution.

Recall the convolution of distributions on $\mathbb{R}^n$. On cannot define the convolution of arbitrary distributions, even arbitrary tempered distributions – this however is an issue of ‘growth’ rather than singularities. In particular the convolution

$$u \ast v \tag{2.3}$$

is defined if either $u$ or $v$ has compact support.

I denote the space of distributions of compact support as

$$C_c^{-\infty}(\mathbb{R}^n). \tag{2.4}$$

So the space of distributions of compact support is actually a commutative ring, since the support of a convolution as in (2.3) satisfies

$$\text{supp}(u \ast v) \subset \text{supp}(u) + \text{supp}(v). \tag{2.5}$$
It is also the case that $S(\mathbb{R}^n)$ is closed under convolution and we know that the Fourier transform satisfies

$$\mathcal{F}(u * v) = \mathcal{F}(u)\mathcal{F}(v), \ u, \ v \in S(\mathbb{R}^n).$$

The ring we are interested in is contained in

$$C_c^{-\infty}(\mathbb{R}^n) + S(\mathbb{R}^n)$$

for which the identity still holds. Note that

$$\mathcal{F}(C_c^{-\infty}(\mathbb{R}^n) + S(\mathbb{R}^n)) \subset C^\infty(\mathbb{R}^n) \cap S'(\mathbb{R}^n).$$

So, we are looking for some interesting spaces of smooth functions on the dual $\mathbb{R}^n$ which are closed under multiplication. You might ask, in view of the identification of the convolution kernels here with the inverse Fourier transforms of symbols, why is there any problem at all? There isn’t a problem for convolution as such because of but recall that the Fourier transform does not ‘behave well’ on say the space $L^\infty(\mathbb{R}^n)$. Of course the Fourier transform maps this to a well-defined linear subspace of the tempered distributions – which includes for instance the delta functions at any point – but it is quite hard, in a certain sense I think impossible, to give a ‘direct’ characterization of the Fourier image of $L^\infty$ and the same is true for our symbols which are modelled on $L^\infty$ in the sense that they are defined by bounds. We will in fact ‘sandwich’ the image between spaces characterized directly (meaning without the Fourier transform) but this still loses information which is rather vital to us!

In the notes related to the first lecture I discussed the radial compactification of a real, finite-dimensional, vector space $V$, to a ball $\overline{V}$. Ignoring all the niceties, for Euclidean space, $\mathbb{R}^n$ with the standard Euclidean norm, we can identify the complement of the origin with the product

$$\mathbb{R}^n \setminus \{0\} \ni x \mapsto (|x|, \frac{x}{|x|}) = (r, \omega) \in (0, \infty) \times S^{n-1}.$$  

The inversion map $r \mapsto 1/r$ is a diffeomorphism of $(0, \infty)$ to itself ‘switching the ends’. This allows us to add the sphere at infinity of $\mathbb{R}^n$ setting

$$\mathbb{R}^n = (\mathbb{R}^n \cup [0, \infty) \times S^{n-1}) / I$$

where

$$I : \mathbb{R}^n \ni x \mapsto (\frac{1}{|x|}, \frac{x}{|x|}) \in (0, \infty) \times S^{n-1}$$

identifies the complement of the origin with the interior of the second part.

Thus $\overline{\mathbb{R}^n}$ is a compact manifold with boundary ‘obtained by introducing inverted polar coordinates near infinity’. The interior is $\mathbb{R}^n$ and the boundary is ‘the sphere at infinity’.

This immediately gives us a ring of functions on $\mathbb{R}^n$, namely

$$C^\infty(\overline{\mathbb{R}^n}) \hookrightarrow C^\infty(\mathbb{R}^n).$$

I can write inclusion here for what is really the restriction from $\overline{\mathbb{R}^n}$ to its interior since this map is injective.

This is the space of ‘classical symbols on $\mathbb{R}^n$ of order zero’ which I would write as

$$S^0_{cl}(\mathbb{R}^n) = C^\infty(\overline{\mathbb{R}^n}).$$
I will approach the issue of characterizing this space precisely on $\mathbb{R}^n$ below.

As a consequence of the discussion of radial compactification in §3, or directly, we can see that the coordinate vector fields on $\mathbb{R}^n$ extend to be smooth on $\mathbb{R}^n$. In fact

**Proposition 2.** The coordinate vector fields on $\mathbb{R}^n$ extend to smooth vector fields on $\mathbb{R}^n$ and span, over $C^\infty(\mathbb{R}^n)$, all the smooth vector fields which are of the form

$$\rho W, \ W \text{ smooth and tangent to the boundary of } \mathbb{R}^n.$$  

Here $\rho \in C^\infty(\mathbb{R}^n)$ vanishes at the boundary.

**Corollary 2.** The space $S^0_{cl}(\mathbb{R}^n)$ consists of smooth functions which satisfy the estimates

$$\sup_{\xi \in \mathbb{R}^n} |(1 + |\xi|)^{|\alpha|} \partial_\xi^\alpha a(\xi)| < \infty \ \forall \ \alpha.$$  

Note that I do not say that this characterizes $S^0_{cl}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$, because it does not.

**Definition 4.** We denote the subspace of $C^\infty(\mathbb{R}^n)$ of functions satisfying all the estimates (2.15) by

$$S^0(\mathbb{R}^n) \supset S^0_{cl}(\mathbb{R}^n).$$

These are the ‘symbols with bounds’ containing the classical symbols.

More generally, consider the function

$$(1 + |x|^2)^{z/2}$$ on $\mathbb{R}^n$, $z \in \mathbb{C}$.  

This is certainly smooth on $\mathbb{R}^n$. It is rather clear that

$$(1 + |x|^2)^{z/2} \in C^\infty(\mathbb{R}^n) \text{ iff } z \in -\mathbb{N}_0.$$  

Indeed, in $x \neq 0$ it can be written

$$t^{-z}(1 + t^2)^{z/2}, \ t = 1/|x|.$$  

This is smooth down to $t = 0$, the boundary of $\mathbb{R}^n$ if and only if $-z$ is a non-negative integer.

We define the space of classical symbols of (complex) order $z$ to be the products

$$S^z_{cl}(\mathbb{R}^n) = (1 + |x|^2)^{z/2} C^\infty(\mathbb{R}^n) = (1 + |x|^2)^{z/2} S^0_{cl}(\mathbb{R}^n).$$

The space of symbols (with bounds) or real order $m$ is similarly defined to be

$$S^m(\mathbb{R}^n) = (1 + |x|^2)^{m/2} S^0(\mathbb{R}^n).$$

Why no complex order in the second case?

**Exercise 1.** Show that in terms of Definition 4

$$(1 + |x|^2)^{is/2} \in S^0(\mathbb{R}^n) \ \forall \ s \in \mathbb{R}.$$  

This in turn implies that

$$S^z_{cl}(\mathbb{R}^n) \subset S^{\text{Re } z}(\mathbb{R}^n) \ \forall \ z \in \mathbb{C}.$$
The space of (Schwartz-) conormal distributions on \( \mathbb{R}^n \), with respect to the origin, of order \( m - n/4 \), is
\[
I^m_{\mathbb{R}}(\mathbb{R}^n) = F^{-1}(S^m(\mathbb{R}^n)).
\]
The corresponding spaces of classical (Schwartz-) conormal distributions at the origin of complex order \( z + n/4 \) are
\[
I^{z + n/4}_{\mathbb{R}}(\mathbb{R}^n) = F^{-1}(S^{z}_{\mathbb{R}}(\mathbb{R}^n)).
\]
So
\[
I^{z}_{\mathbb{R}}(\mathbb{R}^n) \subset S^{z}_{\mathbb{R}}(\mathbb{R}^n).
\]
Why the weird normalization of the order with the \( n/4 \)? This is part of a bigger scheme that I hope will be explained later. It is the standard notion with the \( n \) interpreted as the codimension of the submanifold, here the origin, with respect to which we are defining conormality.

I have made a rather mixed definition of classical and non-classical symbols here. The classical ones defined in terms of the radial compactification and the non-classical ones in terms of estimates on \( \mathbb{R}^n \) more directly, let me try to unravel this.

**THEOREM 3.** If \( u \in I^m_{\mathbb{R}}(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \) then
\[
singsupp(u) \subset \{0\}
\]
\[
(1 - \phi)u \in S(\mathbb{R}^n) \text{ if } \phi \in C^\infty_c(\mathbb{R}^n), \ 0 \notin supp(1 - \phi).
\]
The conditions in \( \ref{2.27} \) do not characterize the conormal distributions.

**PROOF.** By definition a smooth function on \( \mathbb{R}^n \) is a ‘symbol with bounds’ of order \( m \) if it satisfies all the estimates \( \ref{2.15} \). We can reexpress these in the form
\[
\xi^\beta \partial_x^\alpha a = (1 + |\xi|)^m b, \ b \in L^\infty(\mathbb{R}^n) \ \forall \ |\beta| \leq |\alpha|.
\]
\( \square \)

I have made a rather mixed definition of classical and non-classical symbols here. The classical ones defined in terms of the radial compactification and the non-classical ones in terms of estimates on \( \mathbb{R}^n \) more directly, let me try to unravel this.

**LEMMA 2.** The ‘residual symbol spaces’ are
\[
S^{-\infty}(\mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S^m(\mathbb{R}^n) = S(\mathbb{R}^n) = \dot{C}^\infty(\mathbb{R}^n) \subset S^0_{\mathbb{R}}(\mathbb{R}^n) \ \forall \ z \in \mathbb{C}.
\]
Here I am using the notation for any manifold with boundary
\[
\dot{C}^\infty(M) = \{u \in C^\infty(M); u \text{ vanishes to infinite order at } \partial M\}.
\]
So these are the ‘trivial’ symbols in the case of \( \mathbb{R}^n \).

2. Lecture 3

[In the end I covered asymptotic completeness – which is now discussed below – but not cone support/wavefront set and restriction, I will do that a bit later]

Last time I talked about the symbol spaces \( S^m(\mathbb{R}^n) \) and the space of distributions conormal at 0 defined as
\[
I^{m}_{\mathbb{R}}(\mathbb{R}^n; \{0\}) = F^{-1}(S^m(\mathbb{R}^n)).
\]
Since we know that the symbol spaces form a filtered ring under multiplication we deduce a corresponding result for convolution of the conorma spaces

\[ I_S^M(R^n; \{0\}) * I_S^{M'}(R^n; \{0\}) = I_S^{M+M'-\frac{n}{4}}(R^n; \{0\}). \]  

(2.32)

Today I want to talk about some more properties of these conormal distributions before passing onto conormality at a submanifold and the properties of pseudodifferential operators. I hope to discuss this week:

- Topology and density: Propositions
- Integration: Lemma
- Cone support and WF
- Restriction
- Multiplicativity
- Asymptotic completeness
- Diffeomorphism invariance
- Action of \( \Psi^* \)

### 2.1. Topology and asymptotic summation

First the topology on the symbols space \( S^m(R^n) \) is the Fréchet topology given by the norms defining the space

\[ \|a\|_{m,N} = \sup_{\xi \in R^n, |\beta| < N} |(1 + |\xi|)^{-m+|\beta|} \partial_\xi^\beta a(\xi)|, \quad N \in \mathbb{N}_0. \]  

(2.33)

Certainly, \( a \in S^m(R^n) \) if and only if \( a \in C^\infty(R^n) \) and all these norms are finite.

Recall that metric on a countably normed space, such as this, is defined by

\[ d(u,v) = \sum_N 2^{-N} \frac{\|u-v\|_{m,N}}{1 + \|u-v\|_{m,N}}. \]  

(2.34)

**Proposition 3.** The spaces \( S^m(R^n) \) are Fréchet spaces, so complete with respect to the translation-invariant distance (2.34). If it matters to you, they are Montel spaces.

The are not projective limits of Hilbert spaces, which is what the subtlety of density is about.

**Proof.** Convergence with respect to this distance is the same as convergence with respect to each of the norms \( \|\cdot\|_{m,N} \) (without any uniformity in \( N \)). Thus a Cauchy sequence with respect to the metric (2.34) is Cauchy with respect to each of these norms and conversely. So all the derivatives converge locally uniformly and with respect to the distance with the limit in the space. \( \square \)

There is a topology on \( S^\infty(R^n) = \bigcup_m S^m(R^n) \) but I will not worry about this too much for the moment.

So the Fréchet topology on the symbol spaces induces a Fréchet topology on \( I_S^M(R^n, \{0\}) \), since the Fourier transform identifies this with \( S^{M-\frac{n}{4}}(R^n) \). This means we know what a continuous map into the conormal space (and also what a smooth map into it) means.

Now to density. The intersection of the symbol spaces is

\[ S(R^n) = S^{-\infty}(R^n) = \bigcap_m S^m(R^n) = S(R^n). \]

(2.35)
Proposition 4. The ‘residual space’ $S(\mathbb{R}^n)$ is dense in $S^m(\mathbb{R}^n)$ in the topology of $S^{m+\epsilon}(\mathbb{R}^n)$ for any $\epsilon > 0$. More precisely there exist a sequence of ‘regularizing operators’ which are linear maps

$$\Phi_k : S^\infty(\mathbb{R}^n) \to S(\mathbb{R}^n)$$

such that

$$a \in S^m(\mathbb{R}^n) \implies \Phi_k a \text{ is bounded in } S^n(\mathbb{R}^n)$$

and $\Phi_k a \to a$ in the topology of $S^{m+\epsilon}(\mathbb{R}^n)$ for $\epsilon > 0$.

Proof. The $\Phi_k$ can be defined by cut-off. Take $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\phi(\xi) = 1$ in $\{||\xi|| < 1\}$ and set

$$\Phi_k a(\xi) = \phi(\xi/k)a(\xi) \in S(\mathbb{R}^n).$$

The difference

$$\sum_{\gamma} |\partial_\gamma^\beta (1-\phi(\xi/k))| \leq C_\beta(1+Ck)^{\beta}|\beta| < \infty$$

This shows that $1-\phi(\xi/k)$ is bounded with respect to all the seminorms for $S^0(\mathbb{R}^n)$. It follows from (P?) that $\Phi_k a$ is bounded in $S^m(\mathbb{R}^n)$.

The seminorms on $S^s(\mathbb{R}^n)$ on the difference $1-\Phi_k$, which has support in $|\xi| > k$, have an extra factor of $(1+|\xi|)^{-\epsilon}$

$$\sum_{\gamma} |\partial_\gamma^\beta (1-\phi(\xi/k))| \leq C_\beta(1+Ck)^{\beta}|\beta| < \infty$$

where $C_N$ depends on $\phi$ and $N$. Thus $1-\phi(\xi/k) \to 0$ with respect to each seminorm on $S^s(\mathbb{R}^n)$ for $\epsilon > 0$. It follows that

$$\Phi_k a \to a \text{ in } S^s(\mathbb{R}^n) \text{ for } \epsilon > 0.$$

We record the norm estimate which underlies

$$\sum_{\gamma} |\partial_\gamma^\beta (1-\phi(\xi/k))| \leq C_\beta(1+Ck)^{\beta}|\beta| < \infty$$

Lemma 3. For $a \in S^m(\mathbb{R}^n)$ and any $m' > m$

$$\|\text{Id} - \Phi_k a\|_{m',N} \leq C_{N,m,m'}\|a\|_{m,N,k^{m-m'}}$$

where the constant is independent of $a$ and $k$.

2.2. Integration.

Lemma 4. Integration is one of the variables, say the last, gives a continuous linear map

$$\int dx_n : I^m_S(\mathbb{R}^n) \to I^{m-\frac{1}{2}}_S(\mathbb{R}^{n-1}), \quad F\left(\int dx_n\right) = F(u)|_{x_n=0}.$$

Of course we can iterate this, integrating over $k$ variables to get a conormal distribution with order decreased by $k/4$. 
Proof. That the integral is defined follows from (157.111), since integration of both Schwartz functions and distributions of compact support is well-defined. Using the density we can suppose that \( a \in \mathcal{S}(\mathbb{R}^n) \) and then

\[
\int_{\mathbb{R}^n} dx_n F^{-1}(a) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} dx_n e^{ix \cdot \xi} a(\xi) d\xi = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi'} a(\xi', 0) d\xi', \quad x = (x', x_n), \quad \xi = (\xi', \xi_n). \tag{2.46}
\]

by the Fourier inversion formula in one dimension. Thus

\[
\int_{\mathbb{R}^n} dx_n F^{-1}(a) = F^{-1}(a)_{\xi_n=0}, \quad a \in \mathcal{S}(\mathbb{R}^n). \tag{2.47}
\]

Clearly, with \( \Phi_k \) as defined above

\[
\Phi_k(a)_{\xi_n=0} = (\Phi_k a)_{\xi_n=0} \tag{2.48}
\]

so the general case follows. \( \square \)

2.3. Wavefrontset.

The support of a function or distribution on \( \mathbb{R}^n \) is defined by

\[
\text{supp}(u) = \bigcup \{ U \subset \mathbb{R}^n; U \text{ is open and } u = 0 \text{ on } U \}. \tag{2.49}
\]

This is really a notion defined for sheaves (the theory of which I will outline below in case you have not seen it). We define a related notion of symbols but this is only to do with growth at infinity.

If \( V \subset S^{n-1} \) is open then the set

\[
\mathbb{R}^+ V = \{ \xi \in \mathbb{R}^n \setminus 0; \frac{\xi}{|\xi|} \in V \} \tag{2.50}
\]

is what we mean by an open cone – an open subset of \( \mathbb{R}^n \setminus \{0\} \) which is invariant under the radial \( \mathbb{R}^+ \) action. If \( \psi \in C^\infty_c(V) \) and \( \phi \in C^\infty_c(\mathbb{R}) \) is identically equal to 1 near 0 then

\[
\chi(\xi) = (1 - \phi(|\xi|) \psi(\frac{\xi}{|\xi|}) \in S^0(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n). \tag{2.51}
\]

where we define it to be identically zero in \( |\xi| < \epsilon \) where \( \phi(|\xi|) = 1 \) on \( |\xi| < \epsilon \). Thus, \( \phi \) is only there to cut out the singularity the homogeneous function \( \psi(\frac{\xi}{|\xi|}) \) is almost certainly to have at the origin.

Lemma 5. An element of \( C^\infty(\mathbb{R}^n) \) which is homogeneous of complex degree \( z \) vanishes identically unless \( z \in \mathbb{N}_0 \), in which case it is necessarily a polynomial.

Proof. \( \square \)

The product of \( a \in S^m(\mathbb{R}^n) \) and a function \( \chi \) as in (2.31) is always in \( S^m(\mathbb{R}^n) \). However, it might be much smaller.
Definition 6. The cone-support of a symbol $a \in S^m(\mathbb{R}^n)$ is the (relatively) closed subset of $\mathbb{R}^n$

\[ \text{conesupp}(a) = \bigcup \{ \mathbb{R}^+ V; \chi a \in S(\mathbb{R}^n) \forall \psi \in C^\infty_c(V) \} \quad (2.52) \]

In fact the union of the $V$ for which $\psi a \in S(\mathbb{S}^{n-1})$ is still a $V$ for which this holds – i.e. there is a maximal such $V$. Clearly this cone-support is a cone, so the information it contains is the same as the corresponding closed subset of the sphere. It is traditional to think of it as a cone, partly because of the definition, but also because it has a little content as we will see later.

Exercise 2. Show that conesupp$(a) = \emptyset$ iff $a \in S(\mathbb{R}^n) = S^{-\infty}(\mathbb{R}^n)$.

Definition 7. For an element of $I^M_\delta(\mathbb{R}^n; \{0\}) = F^{-1}(S^{M-\frac{n}{4}}(\mathbb{R}^n))$ we set

\[ \text{WF}(u) = \text{conesupp}(a), \ u = F^{-1}(a) \subset T^*_0 \mathbb{R}^m \setminus \{0\} = \mathbb{R}^n \quad (2.53) \]

It matters here that this is the inverse Fourier transform not the Fourier transform, otherwise there is a reflection. The identification of $\mathbb{R}^n \setminus \{0\}$ as the cotangent fibre at 0 on $\mathbb{R}^n$ might appear somewhat arbitrary but is justified by results below on coordinate-invariance. In any case, by definition the wavefront set (that is what WF stands for) of a conormal distribution at the origin in $\mathbb{R}^n$ is a closed cone in $T^*_0 \mathbb{R}^n \setminus \{0\}$.

2.4. Restriction. What is this notion of wavefrontset good for? Notice in (2.45) that integration and restriction are dual under Fourier transform, at least in this special case. In general we cannot expect to restrict a conormal distribution to $x_n = 0$ – for instance this is not reasonable for the delta function at the origin. Dually, we cannot expect to integrate a symbol, it may just be too large at infinity.

Lemma 6. If $a \in S^m(\mathbb{R}^n)$ then

\[ \{ e_n, -e_n \} \cap \text{conesupp}(a) = \emptyset \implies \int_{\mathbb{R}} d\xi_n a(\xi, \xi_n) \in S^{m+1}(\mathbb{R}^{157.125}) \]

(2.54)

Here $e_n = (0, \ldots, 0, 1)$ is the unit vector and this is not in conesupp$(a)$ if and only if the half-line $\mathbb{R}^+ e_n$ does not meet conesupp$(a)$. So you can interpret the condition in (2.54) as saying

\[ \mathbb{R} \cdot e_n \cap \text{conesupp}(a) = \emptyset. \quad (2.55) \]

Proof. The condition on conesupp$(a)$ means that we can find a cut-off function $\psi \in C^\infty(\mathbb{S}^{n-1})$ on the sphere which is non-vanishing at $\pm e_n$ and such that

\[ (1 - \phi|\xi|)\psi(\frac{\xi}{|\xi|}) a \in S(\mathbb{R}^n). \quad (2.56) \]
So this means that in a conic region
\[ \Gamma_\epsilon = \{ \xi \in \mathbb{R}^n; |\xi'| \leq \epsilon |\xi_n| \}, \epsilon > 0 \]

the symbol \( a \) is rapidly decreasing with all its derivatives. In fact we can assume that \( \psi = 1 \) near \( \pm e_n \) and then write
\[ a = a' + \phi(|\xi|) + (1 - \phi(|\xi|))\psi(\frac{\xi}{|\xi|}) \in S^m(\mathbb{R}^n), \quad a = 0 \text{ in } \Gamma_\epsilon, \quad a - a' \in S^m(\mathbb{R}^n) \]

Since integration certainly maps \( S(\mathbb{R}^n) \) into \( S(\mathbb{R}^{n-1}) \) it suffices to consider \( a' \) in place of \( a \) and look at
\[ b(\xi) = \int_{\mathbb{R}} d\xi_n a(\xi', \xi_n). \]

This integral certainly exists since for each \( \xi' \) the integrand is supported in \( |\xi_n| \leq \epsilon^{-1}|\xi'| \). Thus from the leading symbol estimate for \( a \) we see that in \( |\xi'| > 1 \)
\[ |b(\xi')| \leq C \int_{|\xi_n| \leq \epsilon^{-1}|\xi'|} (|\xi'| + |\xi_n|)^m d\xi_n. \]

Now, changing the variable of integration to \( \tau = \xi_n/|\xi'| \) it follows that
\[ |b(\xi')| \leq C \int_{|\tau| \leq \epsilon^{-1}} |\xi'|^{m+1}((1 + |\tau|)^m d\tau \leq C'|\xi'|^m. \]

The same argument applies to all the \( \xi' \) derivatives, so
\[ b \in S^{m+1}(\mathbb{R}^{n-1}). \]

**Corollary 3.** Restriction to the coordinate hyperplane is well-defined as a linear map
\[ |_{x_n=0} : \{ u \in I^M_S(\mathbb{R}^n; \{0\}); \{dx_n, -dx_n\} \cap \text{WF}(u) = \emptyset \} \rightarrow I^{M+\frac{1}{2}}(\mathbb{R}^{n-1}; \{0\}). \]

**2.6. Asymptotic completeness.** The main interest in symbols on \( \mathbb{R}^n \) is their behaviour 'at infinity' (which is the boundary of the radial compactification). This allows for a notion of 'convergence' which corresponds to the 'asymptotic completeness' in the following sense.

**Theorem 4.** If \( a_j \in S^{m_j}(\mathbb{R}^n) \) is a sequence (we think of it as a series) of symbols with \( m_j \rightarrow -\infty \) as \( j \rightarrow \infty \) then there exists a symbol \( a \in S^M(\mathbb{R}^n) \), \( M = \sup m_j \) such that for every \( k \)
\[ \sum_{j \leq k} a_j \in S^{M(k)}(\mathbb{R}^n), \quad M(k) = \sup_{j > k} m_j \]
and \( a \) is determined up to an error in \( S^{-\infty}(\mathbb{R}^n) \) by these conditions.
The relationship \( (2.64) \) between \( a \) and the \( a_j \) is interpreted as ‘a complete asymptotic expansion’ and written
\[
a \sim \sum_j a_j. \tag{2.65}
\]
Note that we are certainly not saying that the series on the right converges in any sense (well people say it converges asymptotically, just meaning the order \( m_j \to -\infty \)).

I have been a little vague here about the range of \( j \), usually one takes \( j \in \mathbb{N}_0 \), so starting off at 0, but this is just a convention.

**Proof.** The ‘uniqueness’ (modulo \( S^{-\infty}(\mathbb{R}^n) \)) is immediate from \( (2.64) \) – given two such ‘asymptotic sums’ \( a \) and \( a' \) the difference satisfies
\[
a' - a = (a' - \sum_{j \leq k} a_j) - (a - \sum_{j \leq k} a_j) \in S^{M(k)}(\mathbb{R}^n) \forall k \implies
\]
\[
a' - a \in S^{-\infty}(\mathbb{R}^n) = S(\mathbb{R}^n). \tag{2.66}
\]

For existence, I will assume, as discussed below, without loss of generality that the \( m_j \) are strictly decreasing, just to simplify notation.

I will use the ‘approximation’ operators \( (\text{Id} - \Phi_l) \) discussed above, where \( \Phi_l \) is multiplication by \( \phi(\xi/l) \) for \( \phi \in C_\infty^c(\mathbb{R}^n) \) equal to 1 near 0. So these are cutoffs near infinity. The \( l \) will vary with \( j \) so we are looking for a sequence of integers
\[
l(j) \to \infty \text{ in } \mathbb{N}. \tag{2.67}
\]

Here is what we want these integers to satisfy – they depend of course on the given sequence \( a_j \).
\[
\sum_{j>k} \|(\text{Id} - \Phi_{l(j)})a_j\|_{m_k,N} < \infty \forall k,N. \tag{2.68}
\]

So this is a countable set of conditions we need to satisfy.

Let’s just examine one of the conditions \( (2.68) \). It makes sense, since the terms are in \( S^{m_j} \subset S^{m_k} \) and \( m_k - m_j > 0 \) for \( j > k \) by assumption. This is often called ‘absolute summability’ of the sequence with respect to the norm. It implies that the series is Cauchy with respect to this norm, and that is what we are after. That is we will ensure that the series
\[
\sum_{j>k} (\text{Id} - \Phi_{l(j)})a_j \text{ is Cauchy with respect to } \| \cdot \|_{m_k,N}. \tag{2.69}
\]

For the moment of course just for one \( N \) and \( k \).

We have, from Lemma \( 157.138 \), an estimate on each of these norms in \( l(j) \)
\[
\|(\text{Id} - \Phi_{l(j)})a_j\|_{m(k),N} \leq C_{N,k}(l(j))^{m_k-m_j} \|a_j\|_{m_k,N} \tag{2.70}
\]
(ultimately because the \( m_j < m_k \)). Here the constant does not depend on \( l(j) \) – the dependence is the power. To make the series converge absolutely it suffices to arrange that
\[
\|(\text{Id} - \Phi_{l(j)})a_j\|_{m(k),N} \leq j^{-2} \tag{2.71}
\]
for instance. In fact convergence is a property of the ‘tail’ of the sequence – the behaviour of any finite number of terms is irrelevant – so it is enough to arrange
from some \( j \) onwards. From (2.70) we see that we can ensure this by choosing \( l(j) \) so that

\[
(2.72) \quad l(j) > L(N, k, j)
\]

where for this \( N \) and \( k \) is some explicit sequence which depends on the norms of the \( a_j \).

Now, this shows we can choose the \( l(j) \) so that any one of the series (labelled by \( k \)) converges absolutely with respect to any one of the norms \( \| \cdot \|_{m, N} \). In fact by a ‘diagonalization’ procedure we can ensure that all the series are Cauchy with respect to all the norms (and hence converge in the corresponding symbol space). To do this, just arrange all the \( (N, k) \) as a sequence, parameterized by \( p \), and demand that (2.72) hold for \( j > p \).

So we can choose the integers \( l(j) \) such that each of the series

\[
(2.73) \quad \sum_{j > k} (\text{Id} - \Phi_{l(j)}) a_j \text{ converges in } S^{m_k}(\mathbb{R}^n)
\]

in the strong sense that it converge absolutely with respect to each of the seminorms.

Now set

\[
(2.74) \quad a = a_0 + \sum_{j \geq 1} (\text{Id} - \Phi_{l(j)}) a_j \in S^{m_0}(\mathbb{R}^n).
\]

This is our asymptotic sum. To check this observe that the difference with a finite sum can be written

\[
(2.75) \quad a - \sum_{j \leq k+1} a_j = - \sum_{j \leq k+1} \Phi_{l(j)} a_j \sum_{j > k+1} (\text{Id} - \Phi_{l(j)}) a_j.
\]

The last sum here is in \( S^{m_{k+1}} \) and the finite sum is actually of compact support in \( S^{-\infty}(\mathbb{R}^n) \). The last term on the left is in the same space, \( S^{m_{k+1}}(\mathbb{R}^n) \) so (2.72) follows.

If we do not have a strictly decreasing sequence of orders, we can rearrange the sequence so that the order is weakly decreasing and then sum up an finite sequences of fixed order. This reduces the problem to the strictly decreasing case and, since we have arranged absolute convergence, we recover (2.64) in general. \( \Box \)

Except that the topology is a little dubious, we have shown that a series with elements in \( S^{m_j}(\mathbb{R}^n)/S^{-\infty}(\mathbb{R}^n) \) ‘always converges’ if the \( m_j \to -\infty \). What this really means is that there exist representatives of the elements in \( S^{m_j}(\mathbb{R}^n) \) such that the series does converge and gives a well-defined limit in \( S^{\sup m_j}(\mathbb{R}^n)/S^{-\infty}(\mathbb{R}^n) \).

3. Lecture 4

As I mentioned last time we will write the ‘full symbol’ map

\[
(2.76) \quad \sigma : l^m_S(\mathbb{R}^n; \{0\}) \to S^{m-\frac{n}{2}}(\mathbb{R}^n)/S^{-\infty}(\mathbb{R}^n)
\]

for the Fourier transform with Schwartz terms ‘dropped’.
3.1. \( I^m_S(\mathbb{R}^n ; \{0\}) \) as a module. The smooth functions of 'slow growth' form a (not very pleasant) linear space which I will denote by \( \mathcal{O}(\mathbb{R}^n) \) which is a space of multipliers on \( \mathcal{S}(\mathbb{R}^n) \). A smooth function is an element \( \psi \in \mathcal{O}(\mathbb{R}^n) \) if for each multiindex \( \alpha \in \mathbb{N}^n_0 \) there exists \( m_\alpha \) such that

\[
(2.77) \quad \sup_{\xi} (1 + |\xi|)^{-m_\alpha} |D_\xi^\alpha \psi(\xi)| < \infty.
\]

Thus multiplication gives a bilinear map

\[
(2.78) \quad \mathcal{O}(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n) \text{ which restricts to } \mathcal{O}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n).
\]

**Proposition 5.** For any \( m \), multiplication defines a map

\[
(2.79) \quad \mathcal{O}(\mathbb{R}^n) \times I^m_S(\mathbb{R}^n ; \{0\}) \rightarrow I^m_S(\mathbb{R}^n ; \{0\}) \text{ with } \sigma(\psi u) \sim \sum_{\alpha \in \mathbb{N}^n_0} \partial_\alpha \psi(0) D_\xi^\alpha \sigma(u).
\]

The sum of the right does determine a unique element of \( S^{m-\frac{n}{4}}/S^{-\infty}(\mathbb{R}^n) \) as we showed last time using asymptotic summation. If we take a representative \( a \in S^{m-\frac{n}{4}}(\mathbb{R}^n) \) – such as the actual Fourier transform of \( u \) – then the terms in the infinite sum are of orders \( m - \frac{n}{4} - |\alpha| \) so the sum 'converges asymptotically'.

**Proof.**
3.2. Action of $Ψ^*$ on $I^*$.

3.3. Diffeomorphism invariance.

4. Problems 1

Due date: The sooner you get in solutions the sooner you will get them returned. I am hoping that you will do them by Feb 26 but am open to discussion – on the problems too of course!

I detected some resistance to the idea of radial compactification of $\mathbb{R}^n$ in class so the main part of the first problem set is to work out some of the details. Quite a bit of this is already in the notes.

0 First recall, for background if nothing else, the basis of projective geometry (which seems to have disappeared as a subject taught to undergraduates not long before I started studying Mathematics). Define complex projective space as a quotient

$$P^n = (\mathbb{C}^{n+1}\{0\})/\mathbb{C}^* = S^{2n+1}/T, \mathbb{C}^* = \mathbb{C}\{0\}, S^{2n+1} = \mathbb{R}^{2n+2}\{0\}\cup = \mathbb{C}^{n+1}\{0\}\cup^.$$ 

Check that this is a complex manifold and that $\mathbb{C}^n$ is identified with an open dense subset by the inclusion

$$C^n \ni z \mapsto (z,1) \in \mathbb{C}^{n+1}\{0\} \hookrightarrow P^n$$

and that the complement of the image may be identified with $P^n - 1$.

(1) Now sort out the real (or more correctly a) real analogue of this. Take the embedding

$$\mathbb{R}^n \ni x \mapsto (x,1) \in \mathbb{R}^n \times [0,\infty) \subset \mathbb{R}^{n+1}$$

and consider the quotient map

$$\iota : \mathbb{R}^n \rightarrow ([\mathbb{R}^n \times [0,\infty) \setminus \{0\}] /R^+ = S^n_+ = \mathbb{R}^n$$

mapping into the upper half-sphere (I hope there is a picture somewhere here); the last equation defines $\mathbb{R}^n$. I take this as the definition of the radial compactification; show that the embedding is given explicitly by

$$\iota(x) = \left( \frac{x}{\sqrt{1 + |x|^2}}, \frac{1}{\sqrt{1 + |x|^2}} \right)$$

and deduce that $t = (1 + |x|^2)^{-\frac{1}{2}} \in C^\infty(S^n_+)$ is a boundary defining function (vanishes only on the boundary and has differential non-zero there).

(2) Derive the Taylor series of $a \in S^0_0(\mathbb{R}^n) = C^\infty(S^n_+)$ (this is the definition of the space of classical symbols of order 0 from lectures) in the form

$$\sum_{k=0}^{\infty} |x|^{-k} a_k(x) \frac{x}{|x|} \in C^\infty(S^n_+), |x| > 2(t \leftrightarrow t < \sqrt{2}).$$

Deduce that Taylor series with remainder gives

$$|a - \sum_{k=0}^{N} |x|^{-k} a_k(x) \frac{x}{|x|}| \leq C_N |x|^{-N-1}.$$

[We want similar estimates for derivatives too].
(3) Introduce projective coordinates on \( \mathbb{R}^n \) given by \( 2n+1 \) coordinate patches on \( S^n \). The first one is \( x \in \mathbb{R}^n \) defining the compactification. Then for each \( k = 1, \ldots, n \) set 

\begin{equation}
D_k^\pm = \{ x \in \mathbb{R}^n; \pm x_k > 0 \}, \quad C_k^\pm = \text{cl}(\iota(D_k^\pm)) \text{ in } S^n
\end{equation}

and show that the diffeomorphisms 

\begin{equation}
P_1.9 \quad C_k^\pm \ni x \mapsto (\frac{1}{x_k}, \pm \frac{x_j}{x_k}) \in (0, \infty) \times \mathbb{R}^{n-1}
\end{equation}

(4) Show that these projective coordinate systems give a coordinate cover of \( \mathbb{R}^n \).

(5) Write out formulas for the images of the vector fields 

\begin{equation}
P_1.10 \quad \partial_{x_j}, \quad x_i \partial_{x_j}
\end{equation}

in these projective coordinate systems (note these span the Lie algebras of the translation group and \( \text{GL}(n, \mathbb{R}) \)).

(6) Show that the \( x_i \partial_{x_j} \) extend to be smooth on \( \overline{\mathbb{R}^n} \) (meaning smooth up to the boundary) and that they are elements of the Lie algebra 

\begin{equation}
P_1.11 \quad \mathcal{V}_b(\mathbb{R}^n) = \{ V \text{ a } C^\infty \text{ vector field tangent to the boundary} \}
\end{equation}

Note that tangency to the boundary means \( Vt = 0 \) at \( t = 0 \).

(7) Show that the images of the \( x_i \partial_{x_j} \) span \( \mathcal{V}_b(\mathbb{R}^n) \) over \( C^\infty(\mathbb{R}^n) \).

(8) Show that the \( \partial_{x_j} \) are also smooth up to the boundary of \( \overline{\mathbb{R}^n} \) and span, over \( C^\infty(\mathbb{R}^n) \) the space 

\begin{equation}
P_1.14 \quad t\mathcal{V}_b(\mathbb{R}^n) = \{ W = tV, \ V \in \mathcal{V}_b(\mathbb{R}^n) \}
\end{equation}

(9) Show that the space \( S^0(\mathbb{R}^n) \) of ‘symbols with bounds’ is identified with the space 

\begin{equation}
P_1.12 \quad \{ u \in L^\infty(\partial \mathbb{R}^n); V_1 \ldots V_N \in L^\infty(\partial \mathbb{R}^n) \ \forall \ V_i \in \mathcal{V}_b(\mathbb{R}^n) \ \forall \ N \},
\end{equation}

[Don’t get hung up on worrying about distributions on a manifold with boundary, we will come back to this later.]

(10) Putting some of these things together show that 

\begin{equation}
P_1.13 \quad S^0_c(\mathbb{R}^n) \subset S^0(\mathbb{R}^n).
\end{equation}

(11) Deduce that an element \( a \in S^0(\mathbb{R}^n) \) is in \( S^0_c(\mathbb{R}^n) \) if and only if 

\begin{equation}
P_1.15 \quad a \sim \sum_k (1 - \phi)(\xi) |\xi|^{-k} a_k(\frac{\xi}{|\xi|})
\end{equation}

where \( \phi \in C^\infty_c(\mathbb{R}^n) \) is equal to one near 0 (to make everything smooth) and the \( a_k \in C^\infty(S^{n-1}) \).
CHAPTER 3

The ring $\Psi^*(\mathbb{R}^n)$

1. Lecture 5

I will start today with the coordinate-invariance of conormal distributions at a point and proceed to discuss the fact that the formal adjoint of a pseudodifferential operator is also a pseudodifferential operator. These might seem to be rather unrelated results, but as we shall see the proofs are closely related.

1.1. Coordinate invariance of $I^m_c(\mathbb{R}^n; \{0\})$. Since we do not want to worry about the global behaviour of diffeomorphisms we will work locally near $0 \in \mathbb{R}^n$. If $\Omega \subset \mathbb{R}^n$ is an open neighbourhood of $0$ set

\[ I^m_c(\Omega; \{0\}) = \{ u \in I^m_c(\mathbb{R}^n; \{0\}); \text{supp}(u) \subseteq \Omega \}. \]

Here of course we are thinking of $\Omega$ as an open subset of $\mathbb{R}^n$ but we can also think of it as a manifold. For the conormal functions to make sense on a manifold we need:

**Proposition 6.** If $F : \Omega \rightarrow \Omega'$ is a diffeomorphism of open neighbourhoods of $0 \in \mathbb{R}^n$ with $F(0) = 0$ then

\[ F^* : I^m_c(\Omega'; \{0\}) \rightarrow I^m_c(\Omega; \{0\}). \]

You should recall that the pull-back of distributions is well-defined under a diffeomorphism (not under a general smooth map). I will remind you of the ‘issues’ arising in the proof of this by duality – namely the need to think about densities – below. Using the density of $C_c^\infty(\Omega)$ in $C_c^{-\infty}(\Omega)$ for any open set $\Omega$ and the fact that $F^*$ extends by continuity I claim that (3.2) already has meaning.

**Proof.** First recall that the ‘full symbol map is still surjective if we restrict supports as in (3.1) since any $u' \in I^m_c(\mathbb{R}^n; \{0\})$ differs from an element of $I^m_c(\Omega; \{0\})$ by an element of $\mathcal{S}(\mathbb{R}^n)$. We will use this in the proof.

First we start with a simple case, when $F \in \text{GL}(n, \mathbb{R})$ is actually an invertible linear map. Then we there is no problem with supports.

**Lemma 7.** Under pull-back by $L \in \text{GL}(n, \mathbb{R})$

\[ L^* : I^m_c(\mathbb{R}^n; \{0\}) \rightarrow I^m_c(\mathbb{R}^n; \{0\}) \]

**Proof.** This corresponds to the fact the Fourier transform behaves ‘well’ under linear change of coordinates. For $u \in \mathcal{S}(\mathbb{R}^n)$ it follows directly that

\[ \mathcal{F}(L^*u)(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(Lx) \, dx = \int_{\mathbb{R}^n} e^{-i(L^{-1}y \cdot \xi) u(y)} |\det L|^{-1} \, dy \]

\[ = \int_{\mathbb{R}^n} e^{-iy \cdot ((L^{-1})^t \xi) u(y)} |\det L|^{-1} \, dy = |\det L|^{-1} \hat{u}(((L^{-1})^t \xi). \]
So the only issue is the constant factor, but this does not affect the fact that \( \mathcal{F}(L^*u) \in S^{m-\frac{n}{2}}(\mathbb{R}^n) \) if \( \hat{u} \in S^{m-\frac{n}{2}}(\mathbb{R}^n) \).

Now, this allows us to simplify the general case in Proposition 3.8. Namely we can write \( F = LG \) where \( L \) is the Jacobian matrix of \( F \) at \( 0 \) and \( G : \Omega \to L^{-1}\Omega' \) still has \( G(0) = 0 \) and now has Jacobian equal to the identity at \( 0 \). Replacing \( G \) by \( F \) again we can therefore assume that

\[
F(x) = x + \sum_{ij} x_i x_j G_{ij}(x) \text{ in } |x| < \epsilon, \quad G_{ij} \in \mathcal{C}^\infty.
\]

There is no problem in shrinking supports to a smaller neighbourhood of \( 0 \) since the conormal distributions are all smooth away from \( 0 \).

We can exploit the triviality of the Jacobian at \( 0 \) by observing that

\[
F_t(x) = x + t \sum_{ij} x_i x_j G_{ij}(x), \quad t \in [0, 1]
\]

is a smooth family of diffeomorphisms of a fixed neighbourhood of \( 0 \) with image containing some fixed neighbourhood of \( 0 \) and with

\[
F_0(x) = x, \quad F_1(x) = F(x).
\]

This allows us to replace the problem by a deformation problem (you might think this is actually harder). However we can now use the variation formula (really just the chain rule) that

\[
\frac{d}{dt}(F_t^* u) = F_t^*(V_t u), \quad V_t = \sum_{k,j,i} x_i x_j a_{i,j,k}(t, x) \partial_{x_k}.
\]

Here \( V_t \) is the \( t \)-dependent vector field which defines \( F_t \) by integration – \( F_t \) is the unique \( 1 \)-parameter family of local diffeomorphisms which satisfies \( F_0 = Id \) and \( F_0 = Id \) (and \( F_0(x) = x \)). It is important here that \( V_t \) vanishes to second order at \( 0 \) (meaning its coefficients vanish quadratically at \( 0 \) of course).

So how does this help us? We need another idea, which I learnt from Jürgen Moser in a rather different context. Namely we can suppose that \( u = u_t \) actually depends smoothly on \( t \) as a parameter (with values in the conormal distributions). Then the variation formula \( (3.8) \) becomes

\[
\frac{d}{dt}(F_t^* u_t) = F_t^*(V_t u_t + \frac{d}{dt} u_t).
\]

Now, the idea is that we try to choose \( u_t \) with \( u_1 = u \) so that \( V_t u_t + \frac{d}{dt} u_t = 0 \). We cannot manage this directly but what we can do is to choose \( u_t \in \mathcal{C}^\infty([0, 1]; I_j^m(\Omega'')) \)

\[
V_t u_t + \frac{d}{dt} u_t \in \mathcal{C}^\infty([0, 1]; C^\infty_\mathcal{C}(\Omega''));
\]

here \( \Omega'' \) is some suitably small open neighbourhood of \( 0 \).

The idea is to solve \( (3.10) \) by successive steps and the crucial point here is that

\[
V_t : I_j^m(\Omega''); \{0\} \to I_j^{m-1}(\Omega'''); \forall \ m \in \mathbb{R}
\]

This follows from Proposition 3.8 and the fact that

\[
\begin{align*}
\times x_i : I_j^m(\Omega''); \{0\} \to I_j^{m-1}(\Omega'''; \{0\}), \quad \partial_{x_k} : I_j^m(\Omega''); \{0\} \to I_j^{m+1}(\Omega'''; \{0\}).
\end{align*}
\]
So, we look for \( u_t \) as a formal, for the moment, sum
\[
(3.13) \quad u_t \sim \sum_j v_j, \ v_j \in \mathcal{C}^\infty([0, 1]; I^{m-j}(\Omega'', \{0\})).
\]

Take
\[
(3.14) \quad v_0 = u, \ v_j = \int_t^1 V_t v_{j-1}, \ j \geq 1.
\]

Then, slightly generalizing the asymptotic summation result (see Problems 2) to include the \('parameter'\ \( t \in [0, 1] \), we can find \( u_t \) satisfying (3.13) and hence (3.10).

However, we do know that \( C^\infty \) is coordinate-invariant so we have proved the Proposition. \( \square \)

The question arises as to what the full symbol of \( F^*u \) might be. The answer is that it is not so simple to write out because of the iteration. We will deduce a few things about this complicated formula below, but for the moment notice that the ‘principal symbol’ is given by a relatively simple formula in terms of the Jacobian.
\[
(3.15) \quad \sigma_m(F^*u)(\xi) = |\det L|^{-1} \sigma_m(u)((L^{-1})^t \xi) \in S^{m-\frac{n}{4}}/S^{m-1-\frac{n}{4}}(\mathbb{R}^n), \ L = D_0 F.
\]

**Lemma 8.** The transformation (3.15) is that of a density on \( T_0^* \mathbb{R}^n \).

**1.2. Left/right invariance.** For differential operators it is conventional to write the coefficients ‘on the left’

\[
(3.16) \quad P = \sum_{|\alpha| \leq m} p_\alpha(x) \partial_x^\alpha, \ p_L(x, \xi) = \sum_{|\alpha| \leq m} p_\alpha(x) \xi^\alpha.
\]

However one can just as well write them on the right

\[
(3.17) \quad P = \sum_{|\alpha| \leq m} D_\alpha^x q_\alpha(x), \ p_R(x, \xi) = \sum_{|\alpha| \leq m} q_\alpha(x) \xi^\alpha.
\]

**Lemma 9.**
\[
(3.18) \quad p_L(x, \xi) = \sum_{\beta \in \mathbb{N}^n_0} \frac{1}{\beta!} \partial_\xi^\beta D_\xi^\beta p_R(x, \xi),
\]
\[
(3.19) \quad p_R(x, \xi) = \sum_{\beta \in \mathbb{N}^n_0} (-1)^{|\beta|} \frac{\beta!}{\beta!} \partial_\xi^\beta D_\xi^\beta p_L(x, \xi).
\]

Here of course only a finite number of terms are non-zero. The formal power series here are those of an exponential so we can write
\[
(3.20) \quad p_L = \exp(D_x \cdot \partial_x)p_R, \ p_R = \exp(-D_x \cdot \partial_x)p_L
\]
to see that one is the inverse of the other.

**Proof.** Leibniz’ formula. \( \square \)

For pseudodifferential operators we can do ‘the same thing’ but it is then not so clear that we get the same space of operators. For a differential operator with coefficients written on the right we see, again using the Fourier inversion formula on \( S \) that the operator is given by the formula
\[
(3.21) \quad F(Pu)(\xi) = \int_{\mathbb{R}^n} e^{-iv \cdot \xi} p_R(y, \xi)u(y)dy, \ u \in S(\mathbb{R}^n).
\]
3. THE RING $\Psi^*(\mathbb{R}^n)$

**Proposition 7.** If $p_R \in C^\infty_0(\mathbb{R}^n_\xi; S^m(\mathbb{R}^n_\xi))$ the operator defined by (3.22) is an element of $\Psi^m(\mathbb{R}^n)$ with left-reduced symbol, $p_L$, given asymptotically by (3.18).

**Proof.** We proceed very much as in the proof above. Namely, the Schwartz kernel of the operator $P$ in (3.20) is

\[ P(x, y) = B(y, x - y), \quad B(y, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} p_R(y, \xi) e^{i(x-y) \cdot \xi} d\xi \]

where we may assume that $p_R$ is of very low order to ensure absolute convergence of the integral (and sort the general case out by continuity). Thus the kernel is given by introducing the coordinates $y$ and $z = x - y$ in $\mathbb{R}^{2n}$ and taking the partial inverse Fourier transform in $z$.

So this is very similar to the original ‘left-reduced’ formula except we have switched $x$ and $y$ as the variable independent of $z = x - y$ on $\mathbb{R}^{2n}$. Using the same idea as above we can consider a 1-parameter family of ‘quantization maps’ including left and right as extreme cases

\[ Q(t)(a_t) = (2\pi)^{-n} \int_{\mathbb{R}^n} a_t(tx + (1-t)y, \xi) e^{i(x-y) \cdot \xi} d\xi, \quad t \in [0, 1] \]

and again allow $a$ to vary smoothly with $t$. The full estimates we are considering on $a$ are therefore

\[ \sup(1 + |\xi|)^{-m+|\beta|} |\partial_\xi^k \partial_\eta^\alpha a(t, x, \xi)| < \infty \] on $[0, 1]_t \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi$.

So again, the claim is that the space of kernels, distributions that is, on $\mathbb{R}^{2n}$ defined by (3.22) is actually independent of $t$.

To see this we compute, as before, the derivative in $t$ and note that it can be written

\[ \frac{d}{dt} Q_t(a_t) = Q_t(i \sum_j \partial_\xi_j \partial_\eta_j a_t + \frac{d}{dt} a_t) \]

where the first term comes from the chain rule and integration by parts since

\[ \frac{d}{dt}(tx + (1-t)y) = x - y \quad \text{and} \quad x - y = -i \partial_\xi i(x - y) \cdot \xi. \]

So, now we want to choose $a_t$ so that

\[ i \sum_j \partial_\xi_j \partial_\eta_j a_t + \frac{d}{dt} a_t \] is of order $-\infty$.

In this case we can solve (3.20) explicitly by taking

\[ a_t(x, \xi) \sim \sum_j \frac{t^j}{j!} (D_x \cdot \partial_\xi)^j a(x, \xi) = \exp(tD_x \cdot \partial_\xi) a \]

in the sense of formal power series at $t = 0$.

If we choose $a_t$ to be an asymptotic sum (uniform in the other variables) as in (3.20) then the ‘error term’ is

\[ \sup(1 + |\xi|)^{-N} |\partial_\xi^k \partial_\eta^\alpha \epsilon_t| < \infty \quad \forall \, k, \alpha, \beta. \]

So we can unload the last step in the proof on the following lemma. \(\Box\)
2. Underlivered Lecture 6

I did not finish the proof of left/right equivalence last time. Let me not start at precisely the place I left off, but instead consider the ‘residual’ operators.

**Lemma 10.** For each $t \in [0, 1]$ the quantization $Q_t$, in (3.22), applied to the residual symbols, which satisfy

$$\sup(1 + |\xi|)^N|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < \infty \forall N, \alpha, \beta$$

(3.28)

which gives the space of kernel of elements of $\Psi^{-\infty}(\mathbb{R}^n)$ are precisely those smooth functions which satisfy

$$\sup(1 + |x - y|)^N|\partial_x^\alpha \partial_y^\beta A(x, y)| < \infty \forall N, \alpha, \beta.$$  

(3.29)

**Proof.** The symbols in (3.28) are exactly the elements of the space $C^\infty(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^n))$. Consider the case $t = 1$, which is the ‘left quantization’ we started with. Since the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$, the space of kernels of elements of $\Psi^{-\infty}(\mathbb{R}^n)$ consists of the functions

$$A(x, y) = B(x, x - y),$$

(3.30)

where $B(x, z)$ is the (partial) inverse Fourier transform

$$B(x, z) = (2\pi)^{-n} \int e^{iz\cdot\xi} a(x, \xi) \Rightarrow B \in C^\infty(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^n)).$$

(3.31)

Thus the space of $B$'s satisfy the same estimates as the symbols

$$\sup_{x, z}(1 + |z|)^N|\partial_x^\alpha \partial_z^\beta B| < \infty.$$  

(3.32)

The general quantization for $t \in [0, 1]$ replaces these kernels by the

$$A_t(x, y) = B(tx + (1 - t)y, z)$$

(3.33)

for the same space of $B$'s. In terms of the $B$'s themselves this corresponds to the change of coordinates $(x, z) \mapsto (X = x - (1 - t)z, z)$. This is invertible and the coordinate vector fields transform to

$$\partial_x = \partial_x - (1 - t)\partial_z, \quad \partial_z = \partial_x$$

(3.34)

from which it is clear that the estimates (3.29) are invariant under such transformations. Thus

$$Q_t(C^\infty(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^n))) = \Psi^{-\infty}(\mathbb{R}^n) \forall t \in [0, 1].$$

(3.35)

This takes care of the residual terms.

So, going back to the proof of Proposition (we are proceeding to construct a family $a_t \in C^\infty([0, 1] \times \mathbb{R}^n; \mathcal{S}^m(\mathbb{R}^n))$ so that

$$\frac{da_t}{dt} + i\partial_\xi \cdot \partial_x a_t \in C^\infty([0, 1] \times \mathbb{R}^n; \mathcal{S}^{-\infty}(\mathbb{R}^n), a_1 \in C^\infty(\mathbb{R}^n; \mathcal{S}^m(\mathbb{R}^n))$$

(3.36)

given. To do this we choose successive family $v_j \in C^\infty([0, 1] \times \mathbb{R}^n; \mathcal{S}^{m-j}(\mathbb{R}^n))$ by

$$v_0 = a, \quad v_j = \frac{(1 - t)^j}{j!}(i\partial_\xi \cdot \partial_x)^j a.$$  

(3.37)

Here I have done the integrals explicitly, so these satisfy

$$\frac{dv_0}{dt} = 0, \quad \frac{dv_j}{dt} + i\partial_\xi \cdot \partial_x v_{j-1} = 0 \ for \ j \geq 1, \ v_0 = a, \ v_j \Big|_{t=1} = 0, \ j \geq 1.$$  

(3.38)
Now we choose \( a_t \) as an asymptotic sum of the \( v_j \)'s. This goes beyond the earlier summation because of the presence of the parameters \( t \in [0,1] \) and \( x \in \mathbb{R}^n \). What we want to do is to ensure that the cutoff series
\[
\sum_{j > k} \Phi_{nk} v_j
\] (3.39)
should converge absolutely with respect to the seminorms of \( C^\infty([0,1] \times \mathbb{R}^n; S^m)(\mathbb{R}^n) \). All the terms have lower order than this. The point is that there are still only a countable number of norms, even though they now involve the supremum over \([0,1] \times \mathbb{R}^n\) as well. So absolute convergence can be ensured for each series (3.39) by choosing the \( n_k \) large enough. Again this only involves a finite number of conditions on each \( n_k \).

Once we choose \( a_t \) to be such an asymptotic sum then we get
\[
\sum_{j \geq k} \Phi_{nk} v_j
\] (3.39)
and following the discussion of the residual terms above, the complete the proof of Proposition 3.27.

2.1. Composition. So, finally we are in a position to prove the multiplicity of (standard) pseudodifferential operators as in Theorem 3.12.

\[ A \in \Psi^m(\mathbb{R}^n), \ B \in \Psi^{m'}(\mathbb{R}^n) \implies A \circ B \in \Psi^{m+m'}(\mathbb{R}^n) \]

and
\[
\sigma(AB) \sim \sum_{\alpha \in \mathbb{N}^n_0} \frac{1}{\alpha!} \partial^\alpha_x \sigma(A) \partial^\alpha_\xi \sigma(B)
\]

where \( \sigma \) is the left-reduced full symbol.

**Proof.** First I suggest the standard proof, which I will not quite follow through. The idea is to write \( A \) in left-reduced form and \( B \) is right-reduced form – now that we know they are equivalent. Thus

\[ Au(x) = (2\pi)^{-n} \int a_L(x,\xi) e^{ix \cdot \xi} \hat{u}(\xi) d\xi, \]

\[ \mathcal{F}(Bv)(\xi) = \int b_R(y,\xi) e^{-iy \cdot \xi} v(y) dy. \]

We can assume that the symbols themselves are of order \(-\infty\) and use density. The composite is then

\[ (AB)v(x) = (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} a_L(x,\xi) b_R(y,\xi) v(y) dy d\xi \]

This is almost what we want, except the ‘amplitude’ in the integral depends explicitly on both \( x \) and \( y \), as well as \( \xi \). So we need to show that the kernel of the composite

\[ K(x,y) = (2\pi)^{-n} \int e^{i(x-y) \cdot \xi} a_L(x,\xi) b_R(y,\xi) d\xi \in \Psi^{m+m}(\mathbb{R}^n). \]

This can be proved by an argument very similar to the left/right reduction. I leave the details to you!

Let’s return for a moment to the spaces of conormal distributions at the origin, \( I^m_S(\mathbb{R}^n; \{0\}) \). We can easily define conormal distributions at another point simply by translation. Thus if \( p \in \mathbb{R}^n \)

\[ I^m_S(\mathbb{R}^n; \{p\}) = \{ u \in \mathcal{S}'(\mathbb{R}^n); u(x + p) = T^*_p u \in I^m_S(\mathbb{R}^n) \}. \]
This is made more convincing by the proof of coordinate invariance. Here $T_q$ is translation by $q \in \mathbb{R}^n$, $T_q x = x + q$.

**Exercise 3.** Define, and formulate (and prove) the coordinate-invariance of the spaces $I^p_c(\Omega, \{p\})$ for $p \in \Omega \subset \mathbb{R}^n$ open.

**Lemma 11.** The Schwartz kernels of elements of $\Psi^m(\mathbb{R}^n)$ may be identified with the space

\[
\mathcal{C}^\infty(\mathbb{R}^n; T^{m+\frac{q}{2}} S(\mathbb{R}^n); \{0\}) \text{ by } A(x, y) \rightarrow A(x - y, y).
\]

**Proof.**

Using earlier results we have another method. What we have shown above, in left/right reduction is that the kernel of an element of $\Psi^m(\mathbb{R}^n)$

If there is a little time left today I want to introduce another algebra of pseudodifferential operators. This is a sign of things to come. I have been rather hard on the ‘coefficient ring’ $C^\infty(\mathbb{R}^n)$ which is involved in the ring $\Psi^m(\mathbb{R}^n)$. What is a ‘nicer’ possibility? The one I have in mind is the symbol space itself.

We can easily introduce the space of ‘symbol-valued symbols’ (in either direction)

\[
S^{m,k}(\mathbb{R}^n; \mathbb{R}^n)
\]

\[
= \{ a \in C^\infty(\mathbb{R}^{2n}_x, x^\xi); \sup(1 + |x|)^{-k+|m|}(1 + |\xi|)^{-m+|\beta|} \partial_x^\alpha \partial_\xi^\beta a(x, \xi) | < \infty, \forall \alpha, \beta \}.
\]

These defining conditions give seminorms. Here there are two orders and differentiation with respect to $x$ lowers the second (but not the first) and differentiation with respect to $\xi$ lowers the first but not the second.

Directly from \[\text{(3.46)}\] we see that

\[
k \leq 0 \implies S^{m,k}(\mathbb{R}^n; \mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n; S^m(\mathbb{R}^n)).
\]

It is also the case that

\[
(1 + |x|^2)^{k/2} \in S^{0,k}(\mathbb{R}^n; \mathbb{R}^n) \text{ and } S^{m,k}(\mathbb{R}^n; \mathbb{R}^n) \cdot S^{m',k'}(\mathbb{R}^n; \mathbb{R}^n) = S^{m+m', k+k'}(\mathbb{R}^n; \mathbb{R}^n).
\]

Combining these two observations we see that

\[
(1 + |x|^2)^{-k/2} S^{m,k}(\mathbb{R}^n; \mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n; S^m(\mathbb{R}^n)).
\]

So, we can quantize these double symbols using left quantization and define

\[
\Psi_{sc}^{m,k}(\mathbb{R}^n) = \{ A : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n); \ A = (1 + |x|^2)^{k/2} Q_L((1 + |x|^2)^{-k/2} a), a \in S^{m,k}(\mathbb{R}^n; \mathbb{R}^n) \}.
\]

This algebra was introduced by Shubin, but I call it the scattering algebra, which is the subscript sc, because it has a direct extension to compact manifolds with boundary.

So, I am getting ahead of myself here:

**Proposition 8.** *The scattering pseudodifferential operators from a double-filtered algebra

\[
\Psi_{sc}^{m,k}(\mathbb{R}^n) \circ \Psi_{sc}^{m', k'}(\mathbb{R}^n) \subset \Psi_{sc}^{m+m', k+k'}(\mathbb{R}^n)
\]
with residual space

\[
\Psi_{sc}^{-\infty,-\infty}(\mathbb{R}^n) = \bigcap_{m,k} \Psi_{sc}^{m,k}(\mathbb{R}^n)
\]

equal to the space of operators with kernels \( A \in \mathcal{S}(\mathbb{R}^{2n}) \).

Maybe you would like to try your hand at proving this! You can easily see why it should be true because Moyal’s formula for the composite of two such operators gives

\[
\sigma_L(AB) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi \sigma_L(A) \cdot D_x \sigma_L(B)
\]

and the individual terms here are in \( \mathcal{S}_{m-|\alpha|,k}(\mathbb{R}^n;\mathbb{R}^n) \cdot \mathcal{S}_{m',k'-|\alpha|}(\mathbb{R}^n;\mathbb{R}^n) \subset \mathcal{S}_{m+m'-|\alpha|,k+k'-|\alpha|}(\mathbb{R}^n;\mathbb{R}^n) \)

which is decreasing in both orders. It takes a little thought to prove that everything ‘works’ correctly; here is an outline of one approach – where I will use Kumano-go’s double symbols.

First go through the left/right reduction argument in this case. For convenience I take \( k \leq 0 \) because we can always recover the general case by multiplying by \( (1 + |x|^2)^{k/2} \). So, we want to choose a 1-parameter family of double symbols, \( a_t \in \mathcal{C}_\infty([0,1]; \mathcal{S}_{m,k}(\mathbb{R}^n;\mathbb{R}^n)) \) so that the identity (3.25) holds in the new sense. Looking at (3.37) we can see that if \( v_0 = a \) is chosen in \( \mathcal{S}_{m,k}(\mathbb{R}^n;\mathbb{R}^n) \) then the \( v_j \in \mathcal{C}_\infty([0,1]; \mathcal{S}_{m+j,k+j}(\mathbb{R}^n;\mathbb{R}^n) \) have both orders decreasing. Going back to the asymptotic summation lemma we now need to do a little more with our cutoffs. So in this case we would consider all the series

\[
\sum_{j>l} (1 - \phi(x/n_j)\phi(\xi/n_j)v_j(t,x,\xi)
\]

where we cut out the region where both \( |x| \) and \( |\xi| \) are less than \( n_j \). So on the support of \( (1 - \phi(x/n_j)\phi(\xi/n_j) \) either \( |x| > n_j \) or \( |\xi| > n_j \), but the symbol lies in the space \( \mathcal{S}_{m-j,k-j}(\mathbb{R}^n;\mathbb{R}^n) \) with \( j > l \). So this is small in the symbol space \( \mathcal{S}_{m-l,k-l}(\mathbb{R}^n;\mathbb{R}^n) \) if we choose \( n_j \) large enough. This means we can make all the series converge by an appropriate choice of the integers \( n_j \) and then the error term and then the error term

\[
\frac{d}{dt} Q_t(a_t) \in \mathcal{S}(\mathbb{R}^{2n})
\]

is a residual operator in the new sense. So we win and we see that \( Q_t(\mathcal{S}_{m,k}(\mathbb{R}^n;\mathbb{R}^n)) = \Psi_{sc}^{m,k}(\mathbb{R}^n) \) for all \( t \in [0,1] \).

Now to Kumano-go’s result. Suppose we ‘overspecify’ the amplitude of the pseudodifferential operator by taking a ‘triple symbol’

\[
b(x, y, \xi) \in \mathcal{S}^{m,k,k}(\mathbb{R}^n;\mathbb{R}^n;\mathbb{R}^n) = \mathcal{S}^k(\mathbb{R}^n_x;\mathcal{S}^k(\mathbb{R}^n_y);\mathcal{S}^m(\mathbb{R}^n))
\]
This means we consider smooth functions on $\mathbb{R}^{3n}$ which satisfy
\[
\sup(1 + |x|)^{-k+\alpha}(1 + |y|)^{-k+\gamma}(1 + |\xi|)^{-m+\beta}|\partial^2_x \partial_y \partial^2_\xi b(x, y, \xi) < \infty.
\]
This defines a countably normed space. Notice that these are symbols ‘separately’ in all the variables, there is no joint decay.

**Proposition 9.** [Kumano-go] The ‘overspecified’ quantization map
\[
Q : b \mapsto \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} b(x, y, \xi) d\xi \in \Psi_{sc}^{m,k}(\mathbb{R}^n).
\]

**Proof.** We can think of the double symbols $a(x, \xi)$ as special cases of the triple symbols which are independent of $y$. Then (3.60) is usual quantization, so the range certainly contains $\Psi_{sc}^{m,k}(\mathbb{R}^n)$. To see that it contains nothing more, we can use the same deformation argument as above and try to construct a family of triple symbols $b_t$ so that the intermediate quantization maps
\[
Q_t : b_t \mapsto \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} b_t(x, tx + (1 - t)y, \xi) d\xi
\]
have derivative a smoothing, here meaning Schwartz, kernel. □
CHAPTER 4

Ellipticity and wavefront set

In the actual Lecture 6 I got a little carried away but let me record here what I tried to cover. So, I am reviewing what we have done, or in some cases partly done, and then expand on it a little.

• Symbols: The spaces $S^m(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ are Fréchet spaces defined by the finiteness of the norms in (157.99). They form a filtered (abelian) ring with identity $1 \in S^0(\mathbb{R}^n)$ and have density (Proposition 157.102) and asymptotic completeness (Theorem 157.140) properties.

When is an element invertible? For $a \in S^m(\mathbb{R}^n)$ to have an inverse in $S^{-m}(\mathbb{R}^n)$ a necessary and sufficient condition is

$$|a(\xi)| \geq \delta (1 + |\xi|)^m \iff a^{-1} \in S^{-m}(\mathbb{R}^n).$$

The necessity of (157.233) follows from the bound

$$|a^{-1}(\xi)| \leq C (1 + |\xi|)^{-m}.$$ Conversely this certainly implies that

$$b(\xi) = \frac{1}{a(\xi)} \in C^\infty(\mathbb{R}^n)$$

and an inductive argument shows that the derivatives are then of the form

$$\partial^\alpha b = \sum_{\gamma \geq \alpha} c_{\gamma} \prod_{i=1}^{\alpha} \partial_\gamma a$$

where each term in the numerator is a constant multiple of a product of $|\alpha|$ terms, each a derivative of $a$, where the sum of the multiindices is $\alpha$. This gives the symbol estimates on $b$.

• We defined $I^m+\frac{m}{2}(\mathbb{R}^n) = \mathcal{F}^{-1}(S^m(\mathbb{R}^n))$ and derived various properties of these conormal distributions at 0.

• For such a Fréchet space we can define $C^\infty_\omega(\mathbb{R}^m; S^m(\mathbb{R}^n))$ as the subspace of $C^\infty(\mathbb{R}^m \times \mathbb{R}^n)$ with all derivatives with respect to the first variables bounded in terms of the seminorms on $S^m(\mathbb{R}^n)$. Then our definition of $\Psi^m(\mathbb{R}^n)$ is in terms of their Schwartz kernels (which we identify with the operators)

$$A \in \Psi^m(\mathbb{R}^n) \implies A(x, x - z) \in C^\infty_\omega(\mathbb{R}^n; I^{m+\frac{m}{2}}(\mathbb{R}^n)).$$

This corresponds to the ‘quantization map’ in terms of the partial Fourier tranform

$$C^\infty_\omega(\mathbb{R}^m; S^m(\mathbb{R}^n)) \ni a(x, \xi) \mapsto Q_L(a) = \mathcal{F}_{\xi}^{-1}(a)(x, x - y) \in \Psi^m(\mathbb{R}^n).$$

• We showed (most of the fact that) that for each $t \in [0, 1]$ the ‘intermediate quantizations’

$$Q_t(a) = \mathcal{F}_{\xi}^{-1}(a)(tx + (1 - t)y, x - y) \in \Psi^m(\mathbb{R}^n)$$
give the same space of operators. For \( t = 1 \) this is ‘left’ quantization and for \( t = 0 \) it is ‘right’ quantization where the kernel is written as

\[
A(x, y) = B\left(\frac{x + y}{2}, x - y\right), \quad B \in C^\infty_\omega(\mathbb{R}^n; I^{m+\frac{\omega}{2}}(\mathbb{R}^n)).
\]

- Note that the case \( t = \frac{1}{2} \) is also of importance. It is called ‘Weyl quantization’ and means writing the kernel as

\[
A(x, y) = B\left(\frac{x + y}{2}, x - y\right), \quad B \in C^\infty_\omega(\mathbb{R}^n; I^{m+\frac{\omega}{2}}(\mathbb{R}^n)).
\]

It has some useful properties.

- The inverses of these quantization maps are the ‘total symbols’

\[
\sigma_L, \sigma_R, \sigma_W : \Psi^m(\mathbb{R}^n) \rightarrow C^\infty_\omega(\mathbb{R}^n; S^m(\mathbb{R}^n)).
\]

- The right and left symbols are related asymptotically by

\[
\sigma_R(A) \sim \sum_{\alpha} \frac{1}{\alpha!} D_\alpha^\alpha \sigma_L(A) = \exp(D_x \cdot \partial_\xi) \sigma_L(A)
\]

where the exponential is to be formally expanded in Taylor series at 0.

**Exercise 4.** Derive a similar asymptotic relationships between \( \sigma_W \) and \( \sigma_L \).

- (Not discussed in lecture) The formal adjoint is defined for any continuous linear operator \( A : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n) \) by duality

\[
A^* : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} (Au) \varphi dx = \int_{\mathbb{R}^n} A^* \varphi dx
\]

(where the distribution pairing is written as an integral). Then

\[
\ast : \Psi^m(\mathbb{R}^n) \rightarrow \Psi^m(\mathbb{R}^n), \quad \sigma_R(A^*) = \sigma_L(A).
\]

- The composition theorem with

\[
\sigma_L(A \circ B) \sim \sum_{\alpha} (\partial_\xi^\alpha \sigma_L(A))(D_\alpha^\alpha \sigma_L(B)).
\]

As suggested in Lecture, check this for differential operators. In fact it is enough to take \( A = D_x^2 \) and \( B = b(x) \) and apply Leibniz’ formula to get

\[
A : H^M(\mathbb{R}^n) \rightarrow H^{M-m}(\mathbb{R}^n).
\]

Proof later. Here \( H^M(\mathbb{R}^n) = \mathcal{F}^{-1}(1 + |\xi|^M) L^2(\mathbb{R}^n) \) is defined as usual as the inverse Fourier transform of the weighted \( L^2 \) spaces on the dual.

### 1. Ellipticity of symbols

In the notes above the notion of ellipticity for elements of \( S^m(\mathbb{R}^n) \) is discussed (although I did not cover this in lectures). We want an extension of this idea to \( C^\infty(\mathbb{R}^N; S^m(\mathbb{R}^n)) \) (I have dropped both the boundedness assumptions on the coefficients and the assumption that \( N = n \) since they are both irrelevant here.

Most importantly a symbol is said to be *elliptic at* \( \bar{x}, \bar{\xi} \in \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}) \) is ‘it is as big as it can be in a cone around this point.

Definition 8. The elliptic set $\text{Ell}(a) \subset \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\})$ of $a \in C^\infty(\mathbb{R}^N; S^m(\mathbb{R}^n))$ consists of those points $\bar{x}, \bar{\xi}$ corresponding to which there exists $\delta > 0$ such that

$$|a(x, \xi)| > \delta|\xi|^m \text{ in } |x| < \delta, \ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\xi|}| < \delta, |\xi| > 1/\delta.$$  

(4.21)

I have engaged in constant-saving here! Clearly the result remains true if $\delta$ is decreased but remains positive. The basic region we are looking at here is a ‘conic neighbourhood’ of $(\bar{x}, \bar{\xi})$ which is then truncated by demanding $|\xi|$ is large as well. So the appearances of $\delta$ can be decreased individually and the estimate remains true. Ellipticity is a ‘local invertibility’ condition on $a$ in the filtered symbol algebra, as shown below. It is a conic set from the definition which only depends on $\xi/|\xi|$ not $\xi$ itself. Thus

$$\text{(4.16)} \quad (\bar{x}, \bar{\xi}) \in \text{Ell}(a) \implies (\bar{x}, t\bar{\xi}) \in \text{Ell}(a), \ t > 0.$$  

(4.25)

It is also clear that $\text{Ell}(a)$ is open for any $a \in C^\infty(\mathbb{R}^N; S^m(\mathbb{R}^n))$ a subset of $\mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\})$ (which is itself open in $\mathbb{R}^{N+n}$. Of course it could be empty, and it certainly is if $a \in C^\infty(\mathbb{R}^N; S^{m'}(\mathbb{R}^n))$, $m' < m$. So one should really write

$$\text{(4.17)} \quad \text{Ell}_m(a) \subset \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}) \text{ defined for } a \in C^\infty(\mathbb{R}^N; S^m(\mathbb{R}^n))$$  

(4.27)

but we treat the ‘$m'$’ as understood from context.

We also give a name to the complement of the elliptic set, it is called the characteristic set of the symbol

$$\text{(4.18)} \quad \text{Char}(a) = (\mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}) \setminus \text{Ell}(a).$$  

(4.28)

It is then a (relatively) closed subset.

We define a third conic set corresponding to the region where the symbol is not locally rapidly decaying with all derivatives as follows

$$\text{(4.19)} \quad \text{conesupp}(a) = \{(\bar{x}, \bar{\xi}) \in \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}); \exists \delta > 0 \text{ and } \phi \in C^\infty(\mathbb{R}^N; S(\mathbb{R}^n))$$

$$\text{with } a = \phi \text{ in } |x - \bar{x}| < \delta, \ |\xi| > \frac{1}{\delta}, \ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\xi|}| < \delta\}^c$$  

(4.29)

It follows that conesupp$(a)$ is relatively closed and that

$$\text{(4.20)} \quad \text{Ell}(a) \subset \text{conesupp}(a)$$  

(4.30)

where this relation is like that between the sets $\{u \neq 0\}$ and supp$(u)$ for a smooth function.

In terms of mutliplication of symbols it is easy to see that

$$\text{Ell}(ab) = \text{Ell}(a) \cap \text{Ell}(b),$$  

(4.21)

$$\text{Char}(ab) = \text{Char}(a) \cup \text{Char}(b),$$  

(4.22)

$$\text{conesupp}(ab) \subset \text{conesupp}(a) \cap \text{conesupp}(b)$$

Note that we can construct symbols which are elliptic at a point $(\bar{x}; \bar{\xi})$ but have cone support in any conic neighbourhood

$$\text{(4.22)} \quad C_\delta = \{(x, \xi) \in \mathbb{R}^N \times (\mathbb{R}^n \setminus \{0\}); |x - \bar{x}| < \delta, \ |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\xi|} < \sigma\}, \ \delta > 0.$$  

(4.23)

Indeed to arrange this we just need to choose a smooth function on the sphere, $\phi$ which is equal to one in $|\frac{\xi}{|\xi|} - \omega| < \delta/2$, where $\omega = \frac{\bar{\xi}}{|\xi|}$, and has support in
\[ |\frac{\xi}{|\xi|} - \omega| < \delta \]
in terms of the distance on the sphere. Similarly choose \( \psi \in \mathcal{C}^\infty(\mathbb{R}^N) \) equal to 1 in \( |x| < \frac{1}{2} \) and supported in \( |x| < 1 \) and a similar cutoff \( \mu \in \mathcal{C}^\infty(\mathbb{R}^n) \) and consider

\[ \chi_\delta(x, \xi) = (1 - \mu(\delta \xi))\psi(\frac{x - \bar{x}}{\delta})\psi(\frac{\xi}{|\xi|}) \in \mathcal{C}_c^\infty(\mathbb{R}^N; \mathcal{S}^0(\mathbb{R}^n)). \]  

\[ \textbf{Lemma 12.} \] The ‘symbolic cutoff’ \( \chi_\delta \) in (4.23) has

\[ \text{Ell}(c_\delta) \supset \{ (x, \xi); |x - \bar{x}| < \delta/2, \frac{|\xi|}{|\xi|} - \frac{\hat{\xi}}{|\hat{\xi}|} < \delta/2 \}, \]

\[ \text{conesupp}(c_\delta) \subset \{ (x, \xi); |x - \bar{x}| < \delta, \frac{|\xi|}{|\xi|} - \frac{\hat{\xi}}{|\hat{\xi}|} < \delta \} \]

(4.25) \text{conesupp}(1 - c_\delta) \cap \{ (x, \xi); |x - \bar{x}| \leq \delta/2, \frac{|\xi|}{|\xi|} - \frac{\hat{\xi}}{|\hat{\xi}|} \leq \delta/2 \} = 0.

\[ \textbf{Proof.} \] Inspection. \[ \square \]

\[ \textbf{Lemma 13.} \] If \( a \in \mathcal{C}^\infty(\mathbb{R}^N; \mathcal{S}^m(\mathbb{R}^n)) \) and \( (\bar{x}, \bar{\xi}) \in \text{Ell}(a) \) then there exists \( b \in \mathcal{C}^\infty(\mathbb{R}^N; \mathcal{S}^{-m}(\mathbb{R}^n)) \) such that

\[ (\bar{x}, \bar{\xi}) \notin \text{conesupp}(ab - 1). \]

\[ \textbf{Proof.} \] Take a symbolic cut-off as in (4.23) and consider

\[ b = \frac{\chi_\delta(x, \xi)}{a}. \]

For \( \delta > 0 \) small enough this is well-defined since by the definition of ellipticity in (4.16), \( a \neq 0 \) on the support of \( \chi_\delta(x, \xi) \); as usual the quotient is extended as zero outside this support. Then (4.27) follows from (4.26) since \( ba = \chi_\delta(x, \xi) \). So it only remains to check that \( b \in \mathcal{C}_c^\infty(\mathbb{R}^N; \mathcal{S}^{-m}(\mathbb{R}^n)) \). Proceeding inductively

\[ \partial_\alpha^\beta \partial_\gamma^\delta b = \frac{g_{\alpha\beta}}{a^{\alpha + 1} \beta + 1}, \quad g_{\alpha\beta} \in \mathcal{C}_c^\infty(\mathbb{R}^N; \mathcal{S}^{(m-1)|\alpha|+1+m|\beta|}(\mathbb{R}^n)). \]

This is certainly true for \( \alpha = \beta = 0 \) and the inductive step follows by differentiating again with respect to either variable. \[ \square \]

It is important to note that

\[ \text{Ell}_m(a + \epsilon) = \text{Ell}_m(a) \quad \text{if} \quad a \in \mathcal{C}^\infty(\mathbb{R}^N; \mathcal{S}^m(\mathbb{R}^n)) \]

\[ \text{and} \quad \epsilon \in \mathcal{C}^\infty(\mathbb{R}^N; \mathcal{S}^{-m-1}(\mathbb{R}^n)), \quad \epsilon > 0. \]

The same is true for \( \text{Char}(a) \) whereas

\[ \text{conesupp}(a + \epsilon) = \text{conesupp}(a) \quad \text{if} \quad \epsilon \in \mathcal{C}^\infty(\mathbb{R}^N; \mathcal{S}^{-\infty}(\mathbb{R}^n)). \]

\[ \textbf{2. Ellipticity of pseudodifferential operators} \]

We now transfer these notions from symbols to pseudodifferential operators.

\[ \textbf{Definition 9.} \] If \( A = Q_L(a) \in \mathcal{P}_m(\mathbb{R}^n), \quad a \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathcal{S}^m(\mathbb{R}^n)) \) we set

\[ \text{Ell}(A) = \text{Ell}_m(A) = \text{Ell}_m(a), \]

\[ \text{Char}(A) = \text{Char}_m(A) = \text{Char}(a), \]

\[ \text{WF}(A) = \text{conesupp}(a). \]
Now consider crolocal invertibility at elliptic points.

a. symbol in terms of the left-reduced symbol subtle, and depends on the formula for the asymptotic expansion of the right-reduced symbol; for the product of operators

\[
\sum_{\alpha} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha a.
\]

If \( a \) is rapidly decreasing in a truncated cone, as \( 4.19 \) then all the terms in \( 4.30 \) are rapidly decaying in the same cone, because of the locality of differential operators. It follows that any asymptotic sum is rapidly decreasing as well. The final part \( 4.35 \) follows similarly.

Perhaps the most important construction associated to these definitions is ‘microlocal invertibility at elliptic points.

**Proposition 10.** If \( A \in \Psi^m(\mathbb{R}^n) \) and \( (\bar{x}, \bar{\xi}) \in \text{Ell}(A) \) there exists \( B \in \Psi^{-m}(\mathbb{R}^n) \) such that

\[
(\bar{x}, \bar{\xi}) \notin \text{WF}'(\text{Id} - AB) \cap \text{WF}'(\text{Id} - BA).
\]

**Proof.** As the notation suggests, we start with \( 4.37 \). The properties of \( b \) mean that

\[
\text{WF}'(B_0) \subset C_\delta = \{ (x, \xi) \in \mathbb{R}^n \xi (\mathbb{R}^n \setminus \{0\}) ; |x - \bar{x}| \leq \delta, \| \frac{\xi}{\xi} \| \leq \delta \}
\]

where we are free to choose \( \delta > 0 \). Then, from the product formula for symbols,

\[
B_0 A = Q_L(c_b) - E, \ E \in \Psi^{-1}(\mathbb{R}^n), \ \text{WF}'(E) \subset C_\delta.
\]

Now, we can almost invert \( \text{Id} - E \) using the Neumann series. That is we can choose

\[
F \in \Psi^{-1}(\mathbb{R}^n), \ F \sim \sum_{k \geq 1} E^k \implies (\text{Id} + F)(\text{Id} - E) = \text{Id} + E'_L, \ (\text{Id} - E)(\text{Id} + F) = \text{Id} + E''_R, \ E'_L, \ E''_R \in \Psi^{-\infty}(\mathbb{R}^n).
\]

Now consider

\[
B = (\text{Id} - F)B_0 \implies BA = (\text{Id} + F)(Q_L(c_b)) - E = \text{Id} + E'_L - (\text{Id} + F)(\text{Id} - (Q_L(c_b))) = \text{Id} + E'',
\]

\[
(\bar{x}, \bar{\xi}) \notin \text{WF}'(E'').
\]
Similarly we can proceed on the right,

\[
AB_0 = Q_L(c_3) - E_R, \quad F_r \sim \sum_{k \geq 1} E_R^k
\]

and see again that \(B_R = B_0(\text{Id} + F_R)\) satisfies

\[
AB_R = \text{Id} + E_R''(\bar{x}, \bar{\xi}) \notin WF(E_R'').
\]

Now, it is a form of the argument which gives the ‘uniqueness of the inverse in a group’ to see that

\[
B_R = BAB_R + S_1 = B_R + S_1 - S_2,
\]

\[S_1 = B(\text{Id} - AB_R), \quad S_2 = (\text{Id} - BA) \text{ so } (\bar{x}, \bar{\xi}) \notin WF(S_i), \quad i = 1, 2.
\]

It follows that \((\bar{x}, \bar{\xi}) \notin WF(B_R - B_R)\) so \(B_R\) also satisfies \((\ref{157.275})\) and \((\ref{157.278})\). \(\square\)

### 3. Wavefront set of a distribution

Now we are in a position to define the wavefront set (or wavefront set) of a distribution on \(\mathbb{R}^n\). First let’s work with compactly supported distributions and then pass to the general case.

**Definition 10.** If \(u \in C_c^{-\infty}(\mathbb{R}^n)\) then

\[
WF(u) = \{(\bar{x}, \bar{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}); \exists A \in \Psi^m(\mathbb{R}^n), \quad Au \in \mathcal{S}(\mathbb{R}^n), \quad (\bar{x}, \bar{\xi}) \in \text{Ell}(A)^c\}^c.
\]

We can characterize the wavefront set of a distribution in a more elementary way.

**Proposition 11.** For \(u \in C_c^{-\infty}(\mathbb{R}^n)\) and \((\bar{x}, \bar{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})\)

\[
(\bar{x}, \bar{\xi}) \notin WF(u) \iff \exists \phi \in C_c^{\infty}(\mathbb{R}^n), \phi(\bar{x}) \neq 0, \psi \in C_c^{\infty}(S^{n-1}), \psi(\frac{\xi}{|\xi|}) \neq 0 \text{ s.t. } |\psi(\frac{\xi}{|\xi|})F(\phi u)| \leq C_N(1 + |\xi|)^{-N} \forall N.
\]

**Proof.** One way is quite easy, the other way depends on the construction of microlocal inverses as above.

First, assume the right side of \((\ref{157.288})\) holds – for some \(\phi\) and \(\psi\) as indicated. Then we can choose another cut-off \(\chi \in C_c^{\infty}(\mathbb{R}^n)\) around zero in \(\mathbb{R}^n\) and conclude that

\[
(1 - \chi(\xi))\psi(\frac{\xi}{|\xi|})F(\phi u)
\]

is also rapidly decreasing. Now

\[
a = (1 - \chi(\xi))\psi(\frac{\xi}{|\xi|})\phi(x) \in C_c^{\infty}(\mathbb{R}^n; S^0(\mathbb{R}^n))
\]

and if \(A = Q_R(a)\) then

\[
F Au = (1 - \chi(\xi))\psi(\frac{\xi}{|\xi|})F(\phi u)
\]
by definition of right quantization. If follows that

\[ Au \in \mathcal{S}(\mathbb{R}^n) \implies (\bar{x}, \bar{\xi}) \notin WF(u) \]

since \( A \) is elliptic at \((\bar{x}, \bar{\xi})\).

For the opposite implication \( \square \)

We can also see that ‘pseudodifferential operators are microlocal’

\textbf{PROPOSITION 12.} For any \( u \in C_\infty^-(\mathbb{R}^n) \) and \( A \in \Psi^m(\mathbb{R}^n) \)

\[ WF(Au) \subset WF'(A) \cap WF(u). \]
Bibliography

