# Lectures on Microlocal Analysis
# 18.157, Spring 2022

**Versions**

(1) 30 November, 2021: L1 and compactification

## Outline and Practicalities

[Revised: 24 January, 2022.]

In this course I hope to cover four (types of) theorems which involve microlocal analysis and in particular the theory of pseudodifferential operators. Namely

(1) Hörmander's theorem on the propagation of singularities
(2) Weyl's law for the distribution of eigenvalues
(3) The Atiyah-Singer index theorem and K-theory
(4) Hodge theory and boundaries

As a first step I will proceed to discuss the algebras of pseudodifferential operators on Euclidean space and on a compact manifold and then similar algebras (and related modules) on manifolds with boundary and for fibration ...)

$$\Psi^*(\mathbb{R}^n),\ \Psi^*(M),\ \Psi^*_*(M) \tag{1}$$

where the upper star is an order and the lower star is some sort of structural information.

To me the four results listed above are fundamental, and I like them! The first two are relatively closely related and both give realization of the 'semiclassical limit', the interplay between the non-commutative theory of (pseudo-)differential operators and the more familiar behaviour of analysis of functions. The latter two are more global but both involve the essential invertibility of (pseudo)differential operators.

Let me briefly indicate what these theorems are about.

Hörmander's theorem on the propagation of singularities is a precise version, and massive generalization, of 'Huyghen's Principle'. The latter describes the spreading of the singular edge of solutions of the wave equation. The precise version is one of the consequences of 'microlocalization', transferring analysis from 'space' to 'phase space' interpreted concretely as a manifold and its cotangent bundle respectively.

Weyl's asymptotic formula describes, at 'high energy', the number of eigenvalues of a self-adjoint elliptic operator, on a compact manifold, in terms of the volume inside the energy surface in the cotangent bundle. The original theorem was actually about the eigenvalues of the Dirichlet problem on a domain in $\mathbb{R}^2$.

Elliptic (pseudo-)differential operators on a compact manifold are Fredholm – they are invertible modulo finite dimensional null space and complement of the range. The index, the difference of these two dimensions, is a very stable number in the sense that it only depends on the 'topology' defined by the leading part of the operator and the theorem gives a formula for it. One classical version of this is the Riemann-Roch theorem for the $\overline{\partial}$ operator on (line bundles over) a compact Riemann surface. This already requires some effort to understand! There is a one-dimensional real version of the theorem, due to Toeplitz, which states that the index of an elliptic Toeplitz operator on the circle (the projection onto the Hardy space, consisting of the functions smooth on and holomorphic on the interior of the disk, of multiplication by a non-vanishing smooth function) is equal to (minus) the winding number of the function.

The Hodge theorem you probably do know for a compact manifold without boundary as the identification of the deRhm cohomology with the space of harmonic forms. For non-compact manifolds there is no simple generalization, rather there are many corresponding to structures at infinity.

Clearly, each of these could easily expand to take the whole semester. Still I hope to show how they can be approached using pseudodifferential operators and 'quantization'. In fact an alternative title for this course might be 'Smooth quantization'. So most of the time will be devoted to preparing the background material, specifically pseudodifferential operators on $\mathbb{R}^n$, pseudodifferential operators on a manifold, families of pseudodifferential operators and then rings of pseudodifferential operators quantizing a Lie algebroid.

I plan to give 26 one-hour lectures in the 9:30-10:30 slot on Tuesdays and Thursdays and leave 20 minutes for questions and discussions (even short presentations by students); if there is sufficient interest I will organize another 'discussion' time, perhaps on Wednesdays in the afternoon. There will be notes for each topic (the precise correspondence to the individual lectures will depend on various things), which will include topics I will not have time to cover and will certainly include further references – to books, lecture notes and papers. With any luck at least some of the lectures at should appear on my webpage before the beginning of the semester.

Problem sets: There will be approximately 5, every two weeks. Grading may be by discussion with me.

Grades: Graduate students are expected to participate actively. That is what 'A' means to me. By this I mean that I expect people to attend lectures and to ask questions. For undergraduates this course might be heavy lifting, it is for me, so please talk to me by early in the semester at the latest. We can discuss what you should expect. There are no exams.

Prerequisites: I will assume familiarity with manifolds and distributions, essentially as in 18.155 but plan to review pretty much everything.

Why don't I just follow a book or my earlier lecture notes? This probably reflects some personal failing and general disatisfaction with how things are done! I find it difficult to think through things without seeing some other way of approaching them. If it is not to your taste, I am sorry but that is the way it is. I may not get to all the results listed above, but I expect to at least get to the point where they are all within reach and that is really what I want to do – try to put these results in a general context that maybe encourages them to be exploited (i.e. applied) and extended.

In the interim, feel free to contact me with questions or comments.

Richard Melrose, 17 November, 2021.

CHAPTER 1

# Pseudodifferential operators, Manifolds and compactification

## 1. Lecture 1

The main aim of this course is to describe various algebras of pseudodifferential operators. Let me start with a traditional 'crypto-historical' description of the 'standard' algebra of pseudodifferential operators on $\mathbb{R}^n$. I recall notation for functions below, but let's assume you know about the spaces of smooth functions on Euclidean and the subspaces of compactly supported, Schwartz and functions with all derivatives bounded

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n) \tag{2}$$

maybe including their topologies and duals.

For any multliindex $\alpha \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ being the non-negative integers, the corresponding iterated partial derivative acts on each of these space

$$u \longmapsto D^\alpha u,\ D^\alpha u(x) = i^{-|\alpha|} \frac{\partial^{\alpha_1}}{\partial x_1} \dots \frac{\partial^{\alpha_n}}{\partial x_n} \dots u(x),\ |\alpha| = \alpha_1 + \dots \alpha_n \tag{3}$$

where the normalizing power of $i$ is inserted to help with notation for the Fourier transform.

These generate the commutative ring of *differential operators with constant coefficients* with general element

$$p(D) = \sum_{|\alpha| \le m} c_\alpha D^\alpha,\ c_\alpha \in \mathbb{C}. \tag{4}$$

This is a filtered ring which is isomorphic to the ring of polynomials in $n$ variables.

Similarly, each of the spaces in (2) is a ring, so multiplication of functions is defined. Combining these we consider *linear partial differential operators* which are given by sums

$$P(x, D)u = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha u. \tag{5}$$

In each case when the coefficients are in one of the spaces (2) we get an operator – a continuous linear map – on the corresponding space.

Whilst this is probably very familiar, and the *operator product* is given explicitly by Leinbiz' formula, it is very significant that these form a ring (and algebra)

$$P(x, D)Q(x, D) = \sum_{\gamma \le \alpha, \beta} p_\alpha(x)(D_x^\gamma q_\beta(x)) D^{\alpha+\beta-\gamma}, Q(x, D) = \sum_{|\beta| \le m'} q_\beta(x) D^\beta. \tag{6}$$

It is worth thinking a little more about what is going on here. First note that (5) is not as 'natural' as (4) in so far as we have chosen to write the 'coefficients', the function $p_\alpha(x)$ on the left. This is true in (4) as well but there the constants commute with the differentiation operators. Of course is reflected in the fact that the product (6) is not commutative.

Now, let's concentrate on the Schwartz space. For this we have the Fourier transform

$$\mathcal{F}:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n),\ \mathcal{F}u(\xi)=\hat{u}(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}dx. \tag{7}$$

It is a linear isomorphism. We know that

$$u\in\mathcal{S}(\mathbb{R}^n)\Longrightarrow\mathcal{F}(D^\alpha u)(\xi)=\xi^\alpha\hat{u}(\xi). \tag{8}$$

The Fourier transform conjugates differentiation to muliplitaction. Of course a monomial such as $\xi^\alpha$ is not in the Schartz space, but it does define an operator on it by multiplication.

So the inverse Fourier transform allows us to write

$$D^\alpha u(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\xi^\alpha\hat{u}(\xi)d\xi. \tag{9}$$

A linear partial differential operator, (5), is given by a finite sum so we can combine (9) with (5) and write

$$Pu(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}p(x,\xi)\hat{u}(\xi)d\xi,\ p(x,\xi)=\sum_{|\alpha|\le m}p_\alpha(x)\xi^\alpha d\xi. \tag{10}$$

Since $\hat{u}\in\mathcal{S}(\mathbb{R}^n)$, the integral converges absolutely. If we just assume that the coefficients are in $\mathcal{C}^\infty(\mathbb{R}^n)$ then the integral converges uniformly on compact subsets in $x\in\mathbb{R}^n$, with all its formal derivatives in $x$ because of the obvious estimates

$$|D_x^\gamma p(x,\xi)|\le C_{K,\gamma}(1+|\xi|)^m,\ x\in K\Subset\mathbb{R}^n,\ \xi\in\mathbb{R}^n. \tag{11}$$

We can actually define the 'standard' space of pseudodifferential operators of order $m\in\mathbb{R}$ by considering those functions $a\in\mathcal{C}^\infty(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)$ which satisfy the *symbol estimates*

$$|D_x^\gamma D_\xi^\beta a(x,\xi)|\le C_{\beta,\gamma}(1+|\xi|)^{m-|\beta},\ \forall\ \gamma,\beta\in\mathbb{N}_0^n. \tag{12}$$

Notice that $p$ in (10) satisfies these estimates for an *integer* $m$ if the coefficients are in the space

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n)=\{f\in\mathcal{C}^\infty(\mathbb{R}^n);\sup|D_x^\gamma f(x)|<\infty\ \forall\ \gamma\} \tag{13}$$

the space of smooth functions with all derivatives bounded.

The space of functions satifying estimates (12) is often written $S^m_{1,0}$ as part of a more general class of spaces $S^m_{\rho,\delta}$ where the exponent $m-|\beta|$ is replaced by $m-\rho|\beta|+\delta|\alpha|$. I will make this notation more precise below, and will probably not talk about the general $\rho$, $\delta$ space – in fact there are many variants of such estimates (see for instance [?]) and we will already have enough things to think about.

It follows directly that if $a\in S^m_{1,0}$, in the sense that all the estimates (12) hold, then the direct generalization of (10),

$$Au(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}a(x,\xi)\hat{u}(\xi)d\xi\Longrightarrow a:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{C}^\infty_\infty(\mathbb{R}^n). \tag{14}$$

In fact much more is true

THEOREM 1. * *The space of operators,* $\Psi^m_{1,0}(\mathbb{R}^n)$ *defined by symbols a satisfying* (12) *act on* $\mathcal{S}(\mathbb{R}^n)$ *and these form a filtered* $*$*-closed (*$*$ *for adjoint here) ring*

$$\Psi^m_{1,0}(\mathbb{R}^n) \circ \Psi^{m'}_{1,0}(\mathbb{R}^n) \subset \Psi^{m+m'}_{1,0}(\mathbb{R}^n),\ \forall\ m, m' \in \mathbb{R}. \tag{15}$$

This is the main content of the first chapter of [**?**], see also [**?**]. Probably the first place this result appeared in this form is [**?**].

The * in the header of the theorem is to indicate that I will not prove it immediately but a full proof will follow later and more. It is not that it iso hard to prove such a result, it is rather that I prefer to approach it from a position of strength, so somewhat indirectly, in the sense that I want to give a good deal of background before proving it.

Still it is important to see what is straightforward to prove and what may require some more thought. First let's make sure we do have (14).

PROOF OF (14). If $u \in \mathcal{S}(\mathbb{R}^n)$ then the product

$$a(x,\xi)\hat{u}(\xi) \in S^{-\infty}_{1,0} = \bigcap_{M \in \mathbb{R}} S^M_{1,0} \tag{16}$$

meaning that the estimates in (12) hold for all $m$. Indeed this is just the product rule for differentiation. Written out fully in terms of Leibniz' formula

$$D^\gamma_x D^\beta_\xi \left(a(x,\xi)\hat{u}(\xi)\right) = \sum_{\gamma \le \beta} \binom{\beta}{\gamma} D^\alpha_x D^\gamma_\xi a(x,\xi) \cdot D^{\beta-\gamma}_\xi \hat{u}. \tag{17}$$

Then one can apply the more obvious fact the product is rapidly decaying in $\xi$ :

$$S^m_{1,0} \cdot \mathcal{S}(\mathbb{R}^n_\xi) \subset S^{m-k}_{1,0}\ \forall\ k \in \mathbb{R}. \tag{18}$$

The integral (14) is therefore convergent. Again it you like to be precise you can see that

$$S^m_{1,0} \subset \mathcal{C}^0_\infty(\mathbb{R}^n; L^1(\mathbb{R}^n_\xi)),\ m < -n \tag{19}$$

since $(1+|\xi|)^{-n-\epsilon} \in L^1(\mathbb{R}^n)$ if $\epsilon > 0$. Now use can use standard properties of Lebesgue (or improper Riemann) integrals to see that $Au \in \mathcal{C}^0_\infty(\mathbb{R}^n)$ is a bounded continuous function and the same holds for all derivatives giving (14). □

Now, I want to check a couple of other statements, weaker than Theorem 1. First the stronger mappting property that

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{20}$$

This is a matter of getting 'decay'. Namely we need to show that for any monomial and any derivative

$$x^\gamma D^\alpha_x Au \in \mathcal{C}^0_\infty(\mathbb{R}^n). \tag{21}$$

We can approach this one step at a time, asking just about $x_j Au$. Note that we can certainly multiply by $x_j$ but the operator $x_j A$ is *not* in general in $\Psi^m_{1,0}(\mathbb{R}^n)$ (for any $m$) since $x_j a(x,\xi)$ is not bounded as $|x_j| \to \infty$ even for fixed $\xi$. Howver the integral in

(14) still converges rapidly in $\xi$ for $x$ in compact sets if we replace $a$ by $x_j a$ so

$$x_j A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \tag{22}$$

for instance.

Lemma 1. *In the sense of operators* (22)

$$[x_j, A] = x_j A - A x_j \in \Psi^{m-1}_{1,0}(\mathbb{R}^n). \tag{23}$$

Proof. We use 'integration by parts'. Consider the operator $Ax_j$. The Fourier transform of $x_j u$, $u \in \mathcal{S}(\mathbb{R}^n)$ is $i\partial_{\xi_j}\hat{u}$ so

$$Ax_j u = (2\pi)^{-n} \int a(x,\xi) e^{ix\cdot\xi} i\partial_{\xi_j}\hat{u}(\xi) d\xi = x_j A(x,D)u + b_j(x,D)u,\ b_j(x,\xi) = -i\partial_{\xi_j} a(x,\xi) \in S^{m-1}_{1,0}.$$

The rapid decay of $a(x,\xi)\hat{u}(\xi)$ in $\xi$ means that

$$\int \partial_{\xi_j}\left(a(x,\xi)e^{ix\cdot\xi}\hat{u}(\xi)\right) d\xi = 0. \tag{24}$$

Proceeding by induction we conclude that

$$x^\gamma A(x,D) = \sum_{\delta\le\gamma} B_\delta(x,D) x^\delta,\ B_\delta(x,D) \in \Psi^{m-|\gamma|+|\phi|}_{1,0}(\mathbb{R}^n).a1 \tag{25}$$

□

In fact rather than $\Psi^m_{1,0}(\mathbb{R}^n)$ I am more interested in the smaller space which I will denote just $\Psi^m(\mathbb{R}^n)$ often called the ring (with the composition property (15)) of 'classical' pseudodifferential operators where the symbols $a$ have the additional property:

There exists a sequence $a_i \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ of homogeneous functions of degree $m - i$ (in the $\xi$ variables)

$$a_i(x, t\xi) = t^{m-i} a(x,\xi),\ t > 0,\ (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \tag{26}$$

such that for (any) cutoff $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n_\xi)$ with $\chi = 1$ near 0

$$a(x,\xi) - \sum_{i=0}^{N} (1 - \chi(\xi)) a_i(x,\xi) \in S^{m-N-1}_{1,0}. \tag{27}$$

These 'classical' symbols form a filtered subring $S^m \subset S^m_{1,0}$. The relationship (27) is often written

$$a \simeq \sum_i a_i \tag{28}$$

and $a$ is then said to have a complete asymptotic expansion. There is no statement of *convergence* in (27) (although there is one lurking in the background) but you should be able to see that the $a_i$, assuming they *exist* are determined by the relations (27).

Now, when we insert the classical symbols in (14) (or if you prefer, restrict to classical symbols) then the space of operators constitutes a filtered subring $\Psi^m(\mathbb{R}^n) \subset \Psi^m_{1,0}(\mathbb{R}^n)$ which for positive integral $m$ includes the differential operators of order $m$ discussed above.

These rings have many important properties but one of the most importan is that one can recover the terms $a_i$ in (27) from the operator $A$ and the leading term defines the principal symbol as a map

$$\Psi^m(\mathbb{R}^n) \xrightarrow{\sigma_m} \{a_0 \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \text{ homogeneous of degree } m \text{ in } \xi\} \tag{29}$$

and this map is surjective, multiplicative and defines a short exact sequence

$$\sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B),\ A \in \Psi^m(\mathbb{R}^n),\ B \in \Psi^{m'}(\mathbb{R}^n)$$

$$\begin{split}\Psi^{m-1}(\mathbb{R}^n) \hookrightarrow \Psi^m(\mathbb{R}^n) \longrightarrow \\ \{a_0 \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \text{ homogeneous of degree } m \text{ in } \xi\}\end{split} \tag{30}$$

Here I have stuck with a cumbersome notation for the homogeneous space which will be refined below.

So, we want to prove all these things and a lot more! However, I do not want to go there directly but rather map out the territory a bit first, in particular discussing the 'symbol spaces' concretely.

## 2. Manifolds with corners

This might appear to be a serious non-sequitor but I hope you will get used to the idea of these sections on background material and see a bit later why I am proceeding this way.

Both for 'local' analysis and the formulation of global results it is very convenient to focus on *manifolds with corners* as our basic 'category of spaces' (which it is as will be made precise later). There are several reasons to introduce these. An immediate one is to understand the symbol spaces and their generalizations. This I will get to next time. This allows me to introduce the spaces of conormal distributions which arise as the Schwartz kernels of the pseudodifferential operators we are interested in. Thinking about the kernels abstractly will allow us to generalize readily later. This involves manifolds with boundary, but then products will get you to manifolds with corners.

So, this is one of the basic settings for the course – analysis on manifolds with corners – but only taken as far as we need for the moment. Let me start with an explicit definition and then explain all the terms used in it. I'm assuming familiarity with the standard definition of a manifold *without boundary.*

Definition 1. A manifold with $M$ is a metrizable, separable (so second countable) topological space with an open covering giving a (maximal) atlas of $\mathcal{C}^\infty$-related coordinate patches modelled on $[0,\infty)^n$ and with embedded boundary hypersurfaces.

I will not assume connectedness without explicitly saying so, but the definition then requires all the components to have the same dimension.

So we are given a separable metric space, $M$, where the 'metrizable' means we do not take the actual metric seriously, just the open sets it defines as the unions of open balls. A coordinate patch in such a topoligical space is a triple $(F, U, V)$ consisting of a homeomorphism $F : U \longrightarrow V$ of an open subset $U \subset M$ onto a (relatively) open subset $V \subset [0,\infty)^n$. So this means there exists an open subset $V' \subset \mathbb{R}^n$ such that the range $V = V' \cap [0,\infty)^n$. The coordinates on the coordinate patch are the pull-backs of the coordinate functions $x_i$ on $\mathbb{R}^n$.

To make clear what '$\mathcal{C}^\infty$-related' for two such coordinate patches means, we need to define $\mathcal{C}^\infty(V)$ (I will not bother with lower regularity than $\mathcal{C}^\infty$):

(31)
$$\mathcal{C}^\infty(V) = \{u : V \longrightarrow \mathbb{R}(\text{ or } \mathbb{C});\ \exists\, V' \subset \mathbb{R}^n \text{ open } V = V' \cap ]0,\infty)^n,\ u' \in \mathcal{C}^\infty(V') \text{ and } u = u'\big|_V\}.$$

So I am assuming you know about $\mathcal{C}^\infty(V')$ for open subsets of $\mathbb{R}^n$.

Now the $\mathcal{C}^\infty$-compatibility of two coordinate patches $(F_i, U_i, V_i)$, $i = 1, 2$, as introduced above, means that either $U_1 \cap U_2 = \emptyset$ or else the transition maps

(32)
$$F_{12} = F_1 \circ F_2^{-1} : F_2(U_1 \cap U_2) \longrightarrow F_1(U_1 \cap U_2) \text{ and } F_{21} = F_2 \circ F_1^{-1} : F_1(U_1 \cap U_2) \longrightarrow F_2(U_1 \cap U_2)$$

are $\mathcal{C}^\infty$ in the sense that $F_{12}^* : \mathcal{C}^\infty(F_1(U_1 \cap U_2)) \longrightarrow \mathcal{C}^\infty(F_2(U_1 \cap U_2))$ and $F_{21}^* : \mathcal{C}^\infty(F_2(U_1 \cap U_2)) \longrightarrow \mathcal{C}^\infty(F_1(U_1 \cap U_2))$; this is equivalent to saying either pull-back map is an isomorphism. This is also equivalent to saying that the pull-backs of the coordinate functions under either of the maps $F_i$ restrict to $U_1 \cap U_2$ to be $\mathcal{C}^\infty$ functions of the other coordinates.

So now an atlas is a covering by such (pairwise) $\mathcal{C}^\infty$-compatible coordinate patches. If some coordinate patches are compatible with all the elements of an atlas then the combined collection is still an atlas – they are necessarily compatible amongst themselves as well. Hence any atlas is contained in a unique maximal atlas – all this is as in the boundaryless case.

If we just stop at this point then $M$ is what I call a *tied manifold* although there is no general agreement on this. The missing point is the additional condition that 'boundary hypersurfaces are embedded'. A point in a coordinate patch is a boundary point of codimension $k$ if exactly $k$ of the coordinate functions vanish on it (note that coordiante patches map into $[0,\infty)^n$ so by fiat all coordinates are non-negative – I will actually drop this requirement later but it makes things easier to state initially). By considering the differential of the transition map it follows that the codimension is well-defined at each point, it is independent of the coordinate patch used). This means that $M$ has a stratification, a decomposition into disjoint pieces, based on the codimension

$$M = M_0 \cup M_1 \cup \cdots \cup M_n \tag{33}$$

where the $M_j$ can be empty (from some $k > 0$ onward). The points of boundary codimension zero are the interior points of the manifold (there is a slight inconsistency between openness of subsets of $[0,\infty)^n$ and this, so the interior there is $(0,\infty)^n$, of course otherwise there would be no point in talking about the interior of a relatively open subset).

Each $M_j$ itself is a manifold without boundary and the *closures* of the components of the $M_j$ are called the boundary faces of codimension $j$; the set of these boundary faces I will write as $\mathcal{M}_j(M)$. In particular the boundary faces of codimension one, the $H_i \in \mathcal{M}_1(M)$ are called the boundary *hypersurfaces.* The 'boundary hypersurfaces are embedded' part of the definition is just the statement that the restrictions of the coordinate patches to each $H_i$ given them $\mathcal{C}^\infty$-compatible atlases. The point of this functorial, that the boundary hypersurfaces (and in consequence all boundary faces) are themselves manifolds with corners. There are several useful ways to restate this condition but note how it fails for a 'tear-shaped region' in the plane.

Pictures needed!

The $\mathcal{C}^\infty$ functions on $M$ are those that are $\mathcal{C}^\infty$ in each coordinate patch

$$f \in \mathcal{C}^\infty(M) \Longleftrightarrow (F^{-1})^*(f\big|_U) \in \mathcal{C}^\infty(V) \text{ for each coordinate patch.} \tag{34}$$

This is equivalent to the same condition for any one compatible atlass.

The direct consequence of the 'embedded' requirement is that the boundary hypersurfaces have defining functions:

$$\begin{gathered} H_i \in \mathcal{M}_1(M) \Longrightarrow \ \exists\ \rho_i \in \mathcal{C}^\infty(M),\ \rho_i \ge 0,\ H_i = \{\rho_i = 0\}, \\ d((F^{-1})^*\rho_i)(F(p)) \ne 0\ \forall\ p \in H_i \text{ for all coordinate patches containing } p. \end{gathered} \tag{35}$$

This last condition means that for each $p \in H_i$ there is a coordinate patch containing $p$ in which $\rho_i$ is a coordinate function.

If $\tilde{M}$ is a manifold without boundary, i.e. $\tilde{M}_1 = \emptyset$, then $M \subset \tilde{M}$ is a(n embedded) submanifold if $M$ has a covering by coordinate patches of $\tilde{M}$ which restrict to give it the structure of a manifold with corners.

Theorem 2. *For any manifold with corners there exists a manifold without boundary $\tilde{M}$ of the same dimension in which $M$ is embedded as a submanifold; if $M$ is compact then $\tilde{M}$ can be taken to be compact.*

Although there is no quite canonical way of constructing such an extension $\tilde{M}$ all the standard constructions of the tangent, cotangnet, form bundles and other bundles associated to the frame bundle, pass over to the case of a manifold with corners in such a way that the restrictions for an extension of this type are canonical

$$TM = T\tilde{M}\big|_M,\ T^*M = T^*\tilde{M}\big|_M \text{ etc.} \tag{36}$$

However, there are important additional structures which arise from the boundary faces as I will discuss later.

So, which work in this degree of generality? Manifolds with corners are the smooth (i.e. $\mathcal{C}^\infty$) analogue of smooth algebraic varieties with divisors and the occur for similar reasons. One such corresponds to the notion of compactification.

## 3. Compactification

Although we will deal with non-compact manifolds, the ones that arise below have some 'structure at infinity'. One way to describe what this means is through the notion of compactification.

Definition 2. A *compactification* of a manifold $M$ is a compact manifold $\overline{M}$ and a smooth injection $\iota : M \longrightarrow \overline{M}$ which is a diffeomorphism to a (relatively of course) open dense submanifold.

Here both $M$ and $\overline{M}$ may have corners. As always we need to specifiy when two compactifications are 'the same'.

DEFINITION 3. Two compactifications $\iota_i : M \longrightarrow \overline{M}_i$ are *equivalent* if there exists a diffeomorphism $e : \overline{M}_1 \longrightarrow \overline{M}_2$ giving a commutative diagramme

$$\begin{tikzcd} & \overline{M}_1 \arrow[dd, "e"] \\ M \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & \\ & \overline{M}_1. \end{tikzcd} \tag{37}$$

Notice that the equivalence map $e$ is unique if it exists since it is fixed on an open dense subset by (37). We also say that one compactification is *finer* than another if there is a smooth map $e$ giving a commutative digaramme; again it if it exists it is determined. This defines a partial order on compactification – as we shall see below there can be non-comparable compactifications.

If $M$ is compact it is a compactification of itself and it is unique in this sense of equivalence..

We might well want more structure for the compactification – for instance if $M$ is a complex manifold then we might want $\overline{M}$ to be complex and all maps to be holomorphic. There are important examples from algebraic geometry here. Most relevant at the moment is the projective compactification of a complex vector space $W \hookrightarrow \mathbb{P}W$ which I mention below but there are much more sophisticated examples to check out. There is the Deligne-Mumford compactification of the Riemann moduli spaces $\mathcal{M}_{g,n}$ (okay I here a complaint from someone that the $\mathcal{M}_{g,n}$ are not quite manifolds, they are orbifolds in general, but take the number of punctures $n$ large compared to the genus $g \geq 0$). Also there is the deConcini-Procesi 'wonderful' compactification of complex adjoint Lie groups [] (if you are interested look also the real version of this in []). Also, compactificaiton of 'Gravitational Instantons' (aren't the Physicists good at inventing names!).

The examples I will consider immediately are more prosaic, namely of a real finite-dimensional vector space $V$. This is both to illustrate the notion and for later reference. I will discuss

(1) The one-point compactification(s) given by a sphere $\overline{V}^o$.
(2) The parabolic compactification given a closed ball $\overline{V}^p$
(3) The radial compactification also given by a closed ball $\overline{V} = \overline{V}^R$.

From the notation you can see that I have a preference for the radial compactification – I hope the discussion below shows why. Only the radial compactification is really used subsequently.

These can all be constructed using variants of stereographic projection. So, let's start with $V = \mathbb{R}^n$, i.e. choose a basis. We embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as the hyperplane

$$\mathbb{R}^n \ni x \longmapsto (x, 1) \in P \subset \mathbb{R}^{n+1}. \tag{38}$$

In the first case consider the the sphere $\mathbb{S}_o$ of radius $\frac{1}{2}$ centred at $(0, \frac{1}{2})$ and in the second and third cases take the sphere $\mathbb{S}_R$ of radius 1 centred at the origin. In both cases a point of $\mathbb{R}^n$ determines a unique line $L_o(x)$ or $L_R(x)$ through the image of

$x$ in $P$ and the centre of the corresponding sphere then

$$
\begin{aligned}
&I_o : \mathbb{R}^n \longrightarrow \mathbb{S}_o,\ I_o x \text{ is the other point in } \mathbb{S}_o \cap L_o(x)\\
&I_R : \mathbb{R}^n \longrightarrow \mathbb{S}_R^+,\\
&\qquad I_R x \text{ is the other point in } \mathbb{S}_R \cap L_1(x) \subset \mathbb{S}_R^+ = \mathbb{S}_R \cap \{x_{n+1} \geq 0\}\\
&I_p : \mathbb{R}^n \longrightarrow \mathbb{B}_p \subset \mathbb{R}^n,\\
&\qquad I_p x \text{ is the projection of } L_R x \text{ onto the closed unit ball in } \mathbb{R}^n \times \{0\}.
\end{aligned}
\tag{39}
$$

In all three cases the full orthogonal group $\mathrm{O}(n)$, acting on the first factor of $\mathbb{R}^n \times \mathbb{R}$ satisfies $I_\bullet Ax = AL_\bullet x$ for all $A \in \mathrm{O}(n)$, effectifely reducing the discussion to the cae $n = 1$. Explicit formulæ for the maps are easily derived:

$$
\begin{aligned}
I_o x &= (\frac{x}{1+|x|^2}, \frac{1}{1+|x|^2}) \in \mathbb{S}_o\\
I_R &= (\frac{x}{(1+|x|^2)^{\frac12}}, \frac{1}{(1+|x|^2)^{\frac12}})) \in \mathbb{S}_R^+\\
I_p x &= \frac{x}{(1+|x|^2)^{\frac12}} \in \mathbb{R}^n.
\end{aligned}
\tag{40}
$$

Thus, for the radial compactification $(1+|x|^2)^{-\frac12}$ is a boundary defining function and hence $|x|^{-1}$, which is a smooth function of it away from $x = 0$, is a defining function near the boundary. It follows that

$$
\{|x| > \epsilon > 0\} \ni x \longmapsto (\frac{1}{|x|}, \frac{x}{|x|}) \in [0,1) \times \mathbb{S}^{n-1}
\tag{41}
$$

extends to a smooth product decomposition of $\overline{\mathbb{R}^n}^R$ near the boundary. For the parabolic compactification it follows similarly that

$$
\{|x| > \epsilon > 0\} \ni x \longmapsto (\frac{1}{|x|^2}, \frac{x}{|x|}) \in [0,1) \times \mathbb{S}^{n-1}
\tag{42}
$$

is a product decomposition near the boundary.

It can be seen directly that

$$
I_o(\frac{x}{|x|^2}) = SI_o \text{ where } S : \mathbb{S}_o \setminus \{(0,1),(0,0)\} \longrightarrow \mathbb{S}_o \setminus \{(0,1),(0,0)\},\quad \text{with } S(y, y_n) = (y, -y_n + 1)
\tag{43}
$$

is equatorial reflection on $\mathbb{S}_o$.

In all cases it is clear either geometrically, or from the forumlæ (40), that the action of $\mathrm{O}(n)$ extends smoothly from $\mathbb{R}^n$ to the compactification. Similarly the scaling action by $\mathbb{R}^+$, with generator on $\mathbb{R}^n$

$$
\sum_i x_i \frac{\partial}{\partial x_i}
\tag{44}
$$

extends smoothly. For the one-point compactification this follows from (43) and in the other two cases

$$
\lim_{|x|\to\infty} \frac{tx}{(1+t|x|^2)^{\frac12}} = \frac{x}{(|x|^2)^{\frac12}} \text{ and } \lim_{|x|\to\infty} \frac{1}{(1+t|x|^2)^{\frac12}} = 0.
\tag{45}
$$

Thus in all cases the action of the conformal group $\mathrm{O}(n) \times \mathbb{R}^+$ extends smoothly to the compactification.

Proposition 1. *The action of the general linear group extends smoothly from $\mathbb{R}^n$ to the radial and parabolic compactifications but not to the one-point compactification; the translation action of $\mathbb{R}^n$ extends smoothly to the radial and the one-point compactifications but not to the parabolic compactification and there are smooth surjective maps, which are not diffeomorphisms, giving a commutative diagramme*
(46)

$$\begin{tikzcd}
 & & \mathrm{GL}(n,R) \ltimes \mathbb{R}^n \arrow[d, dash] & & \\
 & & \mathbb{S}^{n+}_R \arrow[dl] \arrow[dr] & & \\
\mathrm{O}(n) \ltimes (\mathbb{R}^+ \times \mathbb{R}^n) \arrow[r, dash] & \mathbb{S}^n_0 & \mathbb{R}^n \arrow[l, "I_o"'] \arrow[u, "I_R"] \arrow[r, "I_p"] & \mathbb{B}^n_p \arrow[r, dash] & \mathrm{GL}(n,R).
\end{tikzcd}$$

Outline of proof. That the group actions extend as indicated follows by noting that the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$ consists of vector fields homogenous of degree 0 and similarly the translations are homogeneous of degree $-1$. Similar arguments show that the groups shown are the maximal subgroups of $\mathrm{GL}(n,R)\ltimes\mathbb{R}^n$ which extend to act smoothly on the one-point and parabolic compactifications. $\square$

Corollary 1. *The one-point compactification is defined for a vector space with conformal-Euclidean structure, the radial comactification is well-defined for an affine space and the parabolic compactification is well-defined for a vector space.*

Both the radial and the parabolic compactifications have boundaryless variants, in which the boundaing sphere is replaced by an embedded projective space $\mathbb{S}^{n-1}/\pm$ by doubling across the boundary. The apparent advantage of this smaller compactification does not seem to be realized in practice.

Conjecture 1. *The five compactifications are minimal in their respective categories (i.e. as manifolds with/without boundary) among compactifications with the invariance properties in* (46).

Although, as noted above, it is the radial compactification which mostly appears below other variants are relevant. In particular none of these compactifications are natural for products – the radial compactification of $V_1 \times V_2$ is not 'comparable' to the products of the radial compactifications. Still this relationship is significant and is examined below.

CHAPTER 2

# Symbols and conormal distrubutions at a point

## 1. Lecture 2

Before tackling the properties of the ring $\Psi^*(\mathbb{R}^n)$ of pseudodifferential operators on $\mathbb{R}^n$ I want to look into the properties of the Schwartz kernels of these operators, so we can get a picture of them. For a start let me dispense with the 'coefficients' and just look at the (commutative) algebra of constant-coefficient pseudodifferential operators.

Recall the convolution of distributions on $\mathbb{R}^n$. On cannot define the convolution of arbitrary distributions, even arbitary tempered distributions – this however is an issue of 'growth' rather than singularities. In particular the convolution

$$u * v \text{ is defined if either } u \text{ or } v \text{ has compact support.} \tag{47}$$

I denote the space of distributions of compact support as

$$\mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n). \tag{48}$$

So the space of distriutions of compact support is actually a commutative ring, since the support of a convolution as in (47) satiesfies

$$\operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v). \tag{49}$$

It is also the case that $\mathcal{S}(\mathbb{R}^n)$ is closed under convolution and we know that the Fourier transform satisfies

$$\mathcal{F}(u * v) = \mathcal{F}(u)\mathcal{F}(v),\ u,\ v \in \mathcal{S}(\mathbb{R}^n). \tag{50}$$

The ring we are interested in is contained in

$$\mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n) \tag{51}$$

for which the identity (50) still holds. Note that

$$\mathcal{F}(\mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n)) \subset \mathcal{C}^{\infty}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n). \tag{52}$$

So, we are looking for are some interesting space of smooth functions on the dual $\mathbb{R}^n$ which are closed under multiplication.

In the notes related to the first lecture I discussed the radial compactification of a real, finite-dimensional, vector space $V$, to a ball $\overline{V}$. Ignoring all the niceties, for Euclidean space, $\mathbb{R}^n$ with the standard Euclidean norm, we can identify the complement of the origin with the product

$$\mathbb{R}^n \setminus \{0\} \ni x \longmapsto (|x|, \frac{x}{|x|}) = (r, \omega) \in (0, \infty) \times \mathbb{S}^{n-1}. \tag{53}$$

The inversion map $r \longrightarrow 1/r$ is a diffeomorphism of $(0, \infty)$ to itself 'switching the ends'. this allows us to add the sphere at infinity of $\mathbb{R}^n$ setting

$$\overline{\mathbb{R}^n} = \mathbb{R}^n \sqcup [0, \infty) \times \mathbb{R}^{n-1}/I \tag{54}$$

where

$$I : \mathbb{R}^n \setminus \ni x \longmapsto (\frac{1}{|x|}, \frac{x}{|x|}) \in (0, \infty) \times \mathbb{S}^{n-1} \tag{55}$$

identifies the complement of the origin with the interior of the second part.

Thus $\overline{\mathbb{R}^n}$ is a compact manifold with boundary 'obtained by introducing inverted polar coordinates near infinity'. The interior is $\mathbb{R}^{n-1}$ and the boundary is 'the sphere at infinity.

This immediately gives us a ring of functions on $\mathbb{R}^n$, namely

$$\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^n). \tag{56}$$

I can write inclusion here for what is really the restriction from $\overline{\mathbb{R}^n}$ to its interior since this map is injective.

This is the space of 'classical symbols on $\mathbb{R}^n$ of order zero' which I would write as

$$S^0(\mathbb{R}^n) = \mathcal{C}^\infty(\overline{\mathbb{R}^n}). \tag{57}$$

I will approach the issue of characterizing them precisely on $\mathbb{R}^n$ below.

As a consequence of the discussion of radial compactification in § 3, or directly, we can see that the coordinate vector fields on $\mathbb{R}^n$ extend to be smooth on $\overline{\mathbb{R}^n}$. In fact

Proposition 2. *The coordinate vector fields on $\mathbb{R}^n$ extend to smooth vector files on $\overline{\mathbb{R}^n}$ and span, over $\mathcal{C}^\infty(\overline{\mathbb{R}^n}$, all the smooth vector fields which are of the form*

$$\rho W,\ W \text{ smooth and tangent to the boundary of } \overline{\mathbb{R}^n}. \tag{58}$$

*Here $\rho \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}$ vanishes at the boundary.*

Corollary 2. *The space $S^0(\mathbb{R}^n)$ consists of smooth functions which satisfy the estimates*

$$\sup_{\xi \in \mathbb{R}^n} |(1 + |\xi|)^{|\alpha|} \partial_\xi^\alpha a(\xi)| < \infty \ \forall \ \alpha. \tag{59}$$

Note that I do not say that this characterizes $S^0(\mathbb{R}^n) = \mathcal{C}^\infty(\overline{\mathbb{R}^n})$, because it does not.

Definition 4. We denote the subspace of $\mathcal{C}^\infty(\mathbb{R}^n)$ of functions satisfying all the estimates (59) by

$$S^0(\mathbb{R}^n) \supset S^0_{\mathrm{cl}}(\mathbb{R}^n). \tag{60}$$

These are the 'symbols with bounds' containing the classical symbols.

More generally, consider the function

$$(1 + |x|^2)^{z/2} \text{ on } \mathbb{R}^n,\ z \in \mathbb{C}. \tag{61}$$

This is certainly smooth on $\mathbb{R}^n$. It is rather clear that

$$(1 + |x|^2)^{z/2} \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}) \text{ iff } z \in -\mathbb{N}_0. \tag{62}$$

Indeed, in $x \neq 0$ it can be written

$$t^{-z}(1 + t^2)^{z/2},\ t = 1/|x|. \tag{63}$$

This is smooth down to $t = 0$, the boundary of $\overline{\mathbb{R}^n}$ if and only if $-z$ is non-negative integer.

We define the space of classical symbols of (complex) order $z$ to be the products

$$S^z_{\mathrm{cl}}(\mathbb{R}^n) = (1+|x|^2)^{z/2}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) = (1+|x|^2)^{z/2}S^0_{\mathcal{L}}(\mathbb{R}^n). \tag{64}$$

The space of symbols (with bounds) or real order $m$ is similarly defined to be

$$S^m(\mathbb{R}^n) = (1+|x|^2)^{m/2}S^0(\mathbb{R}^n). \tag{65}$$

Why no complex order in the second case?

Exercise 1. Show that in terms of Definition 4

$$(1+|x|^2)^{is/2} \in S^0(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}. \tag{66}$$

This in turn implies that

$$S^z_{\mathrm{cl}}(\mathbb{R}^n) \subset S^{\mathrm{Re}\, z}(\mathbb{R}^n)\ \forall\ z \in \mathbb{C}. \tag{67}$$

Definition 5. The space of (Schwartz-) conormal distributions on $\mathbb{R}^n$, with respect to the origin, of order $m - n/4$, is

$$I^{m-n/4}_{\mathcal{S}}(\mathbb{R}^n) = \mathcal{F}^{-1}\left(S^m(\mathbb{R}^n)\right). \tag{68}$$

The corresponding spaces of *classical* (Schwartz-) conormal distributions at the origin of complex order $z - n/4$ are

$$I^{z-n/4}_{\mathrm{cl},\mathcal{S}}(\mathbb{R}^n) = \mathcal{F}^{-1}\left(S^z_{\mathrm{cl}}(\mathbb{R}^n)\right). \tag{69}$$

So

$$I^z_{\mathrm{cl},\mathcal{S}}(\mathbb{R}^n) \subset I^{\mathrm{Re}\, z}_{\mathcal{S}}(\mathbb{R}^n). \tag{70}$$

Why the weird normalization of the order with the $n/4$? This is part of a bigger scheme that I hope will be explained later. It is the standard notion with the $n$ interpreted as the codimension of the submanifold, here the origin, with respec to which we are defining conormality.

So, apart from the issue with the order these are just the inverse Fourier tranforms of our 'classical symbols'.

Theorem 3. *If* $u \in I^m_{\mathrm{cl},\mathcal{S}}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ *then*

$$\begin{gathered} \operatorname{singsupp}(u) \subset \{0\} \\ (1-\phi)u \in \mathcal{S}(\mathbb{R}^n)\ \text{if}\ \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n),\ 0 \notin \operatorname{supp}(1-\phi). \end{gathered} \tag{71}$$

The conditions in (71) *do not* characterize the conormal distributions.

I have made a rather mixed definition of classical and non-classical symbols here. The classical ones defined in terms of the radial compactification and the non-classical ones in terms of estimates on $\mathbb{R}^n$ more directly, let me try to unravle this.

Lemma 2. *The 'residual symbol spaces' are*

$$S^{-\infty}(\mathbb{R}^n) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n) = \dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n}) \subset S^z_{\mathrm{cl}}(\mathbb{R}^n)\ \forall\ z \in \mathbb{C}. \tag{72}$$

Here I am using the notation for any manifold with boundary

$$\dot{\mathcal{C}}^\infty(M) = \{u \in \mathcal{C}^\infty(M); u \text{ vanishes to infinite order at } \partial M\}. \tag{73}$$