Lectures on Pseudodifferential operators

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Abstract. These are the lectures notes, together with additional material, for the course 18.157, Microlocal Analysis, at MIT in Fall of 2005.

Contents

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Introduction

I plan to take a rather direct and geometric approach to microlocal analysis in these lectures. The initial goal is to define the space of pseudodifferential operators

$$\Psi^m(X; E, F) \ni A : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F). \tag{0.1}$$

Here, m is the 'order' of the pseudodifferential operator, X is the compact manifold on which it is defined and E and F are complex vector bundles over X between the sections of which it acts. Thus, the first few lectures are devoted to the definition, and investigation of the elementary properties, of these operators.

In the approach taken here, this space is defined in terms of another, more general, object

$$\Psi^m(X; E, F) = \mathrm{I}^m(X^2, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R). \tag{0.2}$$

Namely, the space on the right is the space of *conormal distributions* on the compact manifold X^2, with respect to the submanifold Diag, the diagonal, as sections of the bundle $\mathrm{Hom}(E, F) \otimes \Omega_R$, where the precise definition of these bundles is discussed later. So we will proceed to define the right side, but in general for any *embedded* compact submanifold of a compact manifold and any complex vector bundle over the latter

$$\mathcal{C}^\infty(X; E) \hookrightarrow \mathrm{I}^m(X, Y; E). \tag{0.3}$$

Here I have included the fact that smooth sections of the bundle are included in the conormal space, for any order. In fact the elements of $\mathrm{I}^m(X, Y; E)$ are arbitrary smooth sections away from Y, they are singular only at Y and then only in a very special way.

To define the space we use the collar neighbourhood theorem to define a 'normal fibration'. This means identifying a neighbourhood of Y in X with a neighbourhood of the zero section of the normal bundle to Y in X. We denote the latter NY (in which the notation for X does not appear, perhaps it should, say as in $N\{Y; X\}$ but that is a bit heavy-handed) and then, by definition,

$$\mathrm{I}^m(X, Y; E)/\mathcal{C}^\infty(X; E) \longleftrightarrow \mathrm{I}^m_{\mathcal{S}}(NY, O_{NY}; E)/\mathcal{S}(NY, E). \tag{0.4}$$

The space on the top on the right hand side here is almost the same as the one on the left, except that the total space of a vector bundle is not compact, so we need to specify the behaviour of things at infinity and in this case they are required to be 'Schwartz', meaning rapidly decaying with all derivatives; that is what the subscript '$\mathcal{S}$' indicates.

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Finally (going backwards) then we free ourselves of the origin of the bundle NY and replace it by a general real vector bundle W over a compact manifold Y, with E a vector bundle over Y (pulled back to W) and we want to define

$$\mathrm{I}^m_{\mathcal{S}}(W, O_W; E) \xleftarrow{\mathcal{F}^{-1}} \rho^{-m}\mathcal{C}^\infty(\overline{W'}; E \otimes \Omega_{\mathrm{fib}}). \tag{0.5}$$

Here we use the *fibrewise* Fourier transform to identify distributions on W with distributions on the dual bundle, W' (they really have to be fibre-densities accounting for the extra factor Ω_{fib}) and $\rho \in \mathcal{C}^\infty(\overline{W'})$ is a defining function for the boundary of the radial compactification, $\overline{W'}$, which is a compact manifold with boundary made up from W'.

Of course we do all this in the opposite order, which corresponds to the first four lectures. Namely, in the remainder of this first lecture I will first describe various compactifications of a vector space and their invariance under linear transformations, so that the fibrewise compactification of a vector bundle makes sense. In particular the meaning of the notation for the compact manifold $\overline{W'}$ on the right in (0.5) is fixed. Once we have that, and the properties of the Fourier transform are recalled, we can define the left side of (0.5) in terms of the right and discuss the main properties of these spaces. Enough information is needed to show that the identification (0.4) makes sense independent of the choice of the normal fibration which underlies it, which is the main content of Lecture 3. Then (0.2) gives a definition of $\Psi^m(X; E, F)$. Of course we need to discuss more of the properties, in particular the way these act as operators and especially the 'symbolic' and the composition properties; I hope most of this will be done by the end of Lecture 5.

A word is in order about why I have chosen to take this rather sophisticated approach to pseudodifferential operators. The idea is that this approach allows easy generalization. As we shall see below, there are many 'variants' of the space $\Psi^m(X; E, F)$. A large class (namely the 'geometric' ones) of these variants can be readily obtained by changing the compactification of the normal bundle to the diagonal to a different one. Then the same procedure gives a class of operators and, under certain conditions, composition properties can be proved the same way.

Now, my *aim* (this of course is written right at the beginning of the semester) in the rest of the course is to cover the following topics.

(1) Pseudodifferential operators on compact manifolds.
(2) Hodge theorem.
(3) Hörmander's theorem.
(4) Spectral asymptotics for the Laplacian.
(5) Dirac operators
(6) Isotropic algebra
(7) K-theory and classifying spaces.
(8) Chern forms.
(9) Fibrations and product-type operators.
(10) Index theorem.
(11) Eta invariant.
(12) Determinant bundle.
(13) Gerbes.

What will I assume? I hope this is at the level of graduate students with a bit of background. By this I mean I will rather freely use the following

- Differential Geometry:

Manifolds and vector bundles
Forms and deRham theorem
Lie groups
Local symplectic geometry

- Differential Analysis:
  Schwartz distributions
  Fourier transform
  Borel's lemma
  Sobolev spaces on $\mathbb{R}^n$
  Operators on Hilbert space

Lack of knowledge of one or two of these things should not be taken as a bar to preceeding!

In these lecture notes I will limit the 'body' of the notes, forming the first section of each chapter, to the material I think is essential for the main line of the course – and this will be pretty much the content of the lectures. On the other hand I will try to include as addenda to each lecture some more background, various extension and refinements, some indications of directions for further reading, exercises and problems (by this I mean I claim to know the answer to the former but not necessarily to the latter!) I hope to persuade participants in the course to write something up for these addenda.

After a few lectures I will be able to indicate how the present treatment is related to some of the many other treatments of this subject which are available elsewhere.

CHAPTER 1

Compactifications of a vector space

Lecture 1: 8 September, 2005

We need to consider the spaces of functions we will insert in the right hand side of (0.5); such functions are often called 'symbols' or (probably better) 'amplitudes.' To start, consider the case in which the submanifold Y is a point, so its normal bundle is just a vector space. There is nothing special about the dual space to a given vector space, so we just consider an arbitrary, real, finite-dimensional vector space W. This is also a $\mathcal{C}^\infty$ manifold so the space $\mathcal{C}^\infty(W)$ of smooth functions is well defined. However these functions are unconstrained near infinity. To introduce appropriate classes of functions we introduce various compactifications of W. Although these compactifications are introduced here for the specific purpose of describing functions with 'good behaviour at infinity' they have many other uses – some of which will be indicated later.

The general idea of compactification is that if U is a smooth manifold which is not compact then we may be able to find a compact manifold, possibly with boundary or with corners, X, and a smooth injection

$$U \hookrightarrow X \tag{L1.1}$$

which is a diffeomorphism of U onto an open dense subset of X. Since the pull-back map is then injective (a smooth function being determined by its values on a dense set), we may identify $\mathcal{C}^\infty(X)$ as a subset of $\mathcal{C}^\infty(U)$; these functions may be thought of as 'controlled at infinity'.

For a vector space we will define several different compactifications. To do so we start with $\mathbb{R}^n$, define a compactification and then check invariance under choice of the basis which leads to the identification $W \longleftrightarrow \mathbb{R}^n$. If invariance under all general linear transformations does not hold then the compactification depends on some additional structure on W.

L1.1. One-point compactification. The first compactification I will discuss is the 1-point compactification. In fact it will turn out that this is not used for quite a while below, for reasons that will become apparent. However, its relation to the compactifications that we will use is worth understanding and it will eventually reappear in the proof of the Atiyah-Singer index theorem.

One way to define the 1-point compactificaton of $\mathbb{R}^n$ is to use a stereographic projection. Thus we first identify $\mathbb{R}^n$, with variable z, with a hyperplane in $\mathbb{R}^{n+1}$,

$$\mathbb{R}^n \ni z \longmapsto (1, z) \in \mathbb{R}^{n+1}_{z_0,z}. \tag{L1.2}$$

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Now, consider the sphere of radius 1 centred at the origin in $\mathbb{R}^{n+1}$ and draw the line through $(-1,0) \in \mathbb{R} \times \mathbb{R}^n$ and $(1,z)$; it meets the sphere at the point $P(z)$ which we can easily find. Namely, the line is

$$\mathbb{R} \ni t \longmapsto (2t-1, tz) \tag{L1.3}$$

which meets the sphere at the solutions of

$$4t^2 - 4t + 1 + t^2|z|^2 = 1.$$

This has the trivial solution $t = 0$, just the South Pole, and the non-trivial solution $P(z)$ given by

$$t = \frac{4}{4+|z|^2} \Longrightarrow P(z) = \left(\frac{4-|z|^2}{4+|z|^2}, \frac{z}{4+|z|^2}\right). \tag{L1.4}$$

Thus P is a diffeomorphism from $\mathbb{R}^n$ into the complement of the South Pole in the sphere. Indeed, the inverse is given by

$$(Z_0, Z) \longmapsto z = \frac{8Z}{1+Z_0},\ |z|^2 = 4\frac{1-Z_0}{1+Z_0}, \tag{L1.5}$$

which is smooth in $Z_0 > -1$ on the sphere. This formula also shows that the reflection in the equatorial plane, $Z_0 \longmapsto -Z_0$, on the sphere induces the inversion $z \longmapsto z/|z|^2$. So, a smooth function on $\mathbb{R}^n$ is of the form P^*f for $f \in \mathcal{C}^\infty(\mathbb{S}^n)$, if and only if there exists $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $f(z) = g(z/|z|^2)$ outside the origin.

So, what is wrong with the 1-point compactification? For one thing, it does not have enough invariance. Let me use the notation

$$^1\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\mathrm{SP}\} \tag{L1.6}$$

for this set with the $\mathcal{C}^\infty$ structure coming from P, so it is just the sphere. Then P is the inclusion $P : \mathbb{R}^n \longrightarrow {}^1\overline{\mathbb{R}^n}$. Certainly orthogonal transformations lift to this manifold, so there is a commutative diagramme

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{P} & {}^1\overline{\mathbb{R}^n} \\ {\scriptstyle O}\downarrow & & \downarrow{\scriptstyle \tilde{O}} \\ \mathbb{R}^n & \xrightarrow[P]{} & {}^1\overline{\mathbb{R}^n} \end{array}\ ,\ \text{for } O \in \mathrm{O}(n),\ \tilde{O}(Z_0, Z) = (Z_0, OZ). \tag{L1.7}$$

On the other hand, not all elements of $\mathrm{GL}(n,\mathbb{R})$ lift smoothly in this way. To see this, suppose $G \in \mathrm{GL}(n,\mathbb{R})$ lifts in the sense that there is a commutative diagramme of smooth maps as in (L1.7). Then the smoothness of the inversion means that $|Gz|^{-2}$ must be a smooth function of the variables $z/|z|^2$ near infinity. Inverting again, and using the homogeneity of G this means that

$$\frac{|z|^2}{|Gz|^2} \text{ is a smooth function of } z \text{ near } 0. \tag{L1.8}$$

Now, it is well known that this is only the case if $|Gz|^2 = s^2|z|^2$ for some $s > 0$, i.e. if G is *conformal*[1]. So, for instance the scaling in one variable, $z = (z_1, z') \longmapsto (sz_1, z')$, is not conformal, hence does not extend smoothly to the 1-point compactification of $\mathbb{R}^n$ (if $n > 1$!)

[1] Exercise: Check this!

L1.2. Radial compactification. Next we consider the most important compactification in the sequel, the *radial compactification.* We use the same approach as above for the 1-point compactification. So with the same embedding of $\mathbb{R}^n$ as the hyperplane $Z_0 = 1$ in $\mathbb{R}^{n+1}$ as in (L1.2), consider the modifed sterographic projection based on the line through the origin of the unit sphere, rather than the South Pole. The intersection of $[0, 1] \ni t \longrightarrow (t, tz)$ with the unit sphere in $Z_0 > 0$ occurs at $t = (1 + |z|^2)^{-\frac{1}{2}}$. Thus the compactifying map is

$$\text{(L1.9)}\quad R : \mathbb{R}^n \ni z \longrightarrow \Big(\frac{1}{(1+|z|^2)^{\frac{1}{2}}}, \frac{z}{(1+|z|^2)^{\frac{1}{2}}}\Big) \in \mathbb{S}^{n,1} = \{(Z_0, Z); Z_0 \geq 0, Z_0^2 + |Z|^2 = 1\}.$$

It is clearly a diffeomorphism, since the inverse can we written

$$\text{(L1.10)}\qquad z = Z/Z_0 \text{ in } Z_0 > 0.$$

I will denote this radial compactification by $\overline{\mathbb{R}^n} = \mathbb{S}^{n,1}$ with R used to identify the interior with $\mathbb{R}^n$.

Thus the radial compactification embeds $\mathbb{R}^n$ as the open upper half-sphere. This is diffeomorphic to a closed ball and it is tempting to look at the projection on the last n variables in (L1.9) and consider

$$\text{(L1.11)}\qquad Q : \mathbb{R}^n \ni z \longrightarrow \frac{z}{(1+|z|^2)^{\frac{1}{2}}} \in \mathbb{B}^n = \{Z \in \mathbb{R}^n; |Z| \leq 1\}.$$

This is the *quadratic compactification*. It is *not* the same as the radial compactification (L1.9) since the function $Z_0 = (1 + |z|^2)^{-\frac{1}{2}}$ is not smooth on it! Rather $(1 + |z|^2)^{-1}$ is the pull back of a defining function for the boundary of the ball under Q. This corresponds to the fact that the inverse of the projection of the upper half-sphere to the ball has a square-root singularity. When it comes up, and it will, the quadratic compactification will be denoted ${}^q\overline{\mathbb{R}^n}$.

So, returning to the radial compactification, observe as before that orthogonal transformations lift to $\overline{\mathbb{R}^n}$. Indeed the orthogonal transformation can be extended to act on the $\mathbb{R}^n$ factor of $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ and then intertwines with the standard action on $\mathbb{R}^n$ as in (L1.7).

To examine the lift of a general linear transformation we can proceed directly using homogoneity. Subsequently I will proceed more indirectly, by considering the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$. The indirect approach has certain advantages as we shall see below. However, if $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an invertible linear transformation we can see directly that it lifts to a diffeomorphism of the radial compactification $\tilde{G} : \overline{\mathbb{R}^n} \longrightarrow \overline{\mathbb{R}^n}$. This just means showing that the diffeomorphism $R^{-1}GR$ induced on the interior of the upper half sphere by its identification, through R, with $\mathbb{R}^n$, extends smoothly up to the boundary. Notice that a neighbourhood of the boundary of $\mathbb{S}^{n,1}$ can be identified with the product $[0, \frac{1}{2}) \times \mathbb{S}^{n-1}$ using the variables $\frac{1}{|z|}, \frac{z}{|z|}$. Indeed a smooth function on $\mathbb{R}^n$ extends to be smooth on $\mathbb{S}^{n,1}$ under the identification R if and only if it is a smooth function of $|z|^{-1}$ and $\frac{z}{|z|} \in \mathbb{S}^{n-1}$ outside the origin. To see this, just note that

$$\text{(L1.12)}\qquad (1 + |z|^2)^{-\frac{1}{2}} = s(1 + s^2)^{\frac{1}{2}},\ s = |z|^{-1}$$

is a smooth function of $|z|^{-1}$, and conversely. Similarly

$$\frac{z}{(1+|z|^2)^{\frac{1}{2}}} = (1+|z|^{-2})^{-\frac{1}{2}}\frac{z}{|z|}$$

is a smooth function of $z/|z|$ and $1/|z|$ and conversely $z/|z|$ is a smooth function of $z/(1+|z|^2)^{\frac{1}{2}}$ and $(1+|z|^2)^{\frac{1}{2}}$.

Thus the smoothness on the radial compactification is reduced to showing that

$$\frac{1}{|Gz|} \text{ and } \frac{Gz}{|Gz|} \text{ are smooth functions of } \frac{1}{|z|},\ \frac{z}{|z|}$$

up to $1/|z| = 0$. Since G is invertible, $|Gz| > 0$ on the sphere $|z| = 1$, so this the smoothness holds there and by the linearity (hence homogeneity) of G,

$$|Gz| = |z|\left|G(\frac{z}{|z|})\right| \Longrightarrow \frac{1}{|Gz|} = \frac{1}{|z|}\frac{1}{|G(\frac{z}{|z|})|},\ \frac{Gz}{|Gz|} = \frac{G\frac{z}{|z|}}{|G\frac{z}{|z|}|}.$$

This proves the desired smoothness.

Let me show directly that the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$ lifts to the radial compactification, although this could also be shown by checking that the lift $\tilde{G}$ depends smoothly on $G \in \mathrm{GL}(n,\mathbb{R})$. For the standard action on $\mathbb{R}^n$, $\mathfrak{gl}(n,\mathbb{R})$ is represented by 'linear' vector fields with the basis

$$z_i\partial_{z_j},\ i,j = 1,\dots,n. \tag{L1.13}$$

Now we wish to show that $z_i\partial_{z_j}$ lifts to a smooth vector field on $\mathbb{S}^{n,1}$ under the indentification R. Set $s = 1/|z|$ and $\omega = z/|z|$. Then outside the origin

$$z_i\partial_{z_j} = a(s,\omega)\partial_s + V(s,\omega) \tag{L1.14}$$

where $a(s,\omega) \in \mathcal{C}^\infty((0,\infty)\times\mathbb{S}^{n-1})$ and V is a smooth vector field on $\mathbb{S}^{n-1}$ depending smoothly on $s \in (0,1)$. We want to understand what happens as $s \downarrow 0$. However, observe that the linear vector field is constant under the homotheity, $z \to rz$, $0 < r \in \mathbb{R}$. The decomposition (L1.14) is unique and so it must scale in the same way. By the definition of these variables the homotheity becomes $s \to r^{-1}s$, $\omega \to \omega$, so we must have

$$a(s,\omega) = sa(1,\omega),\ V(s,\omega) = V(1,\omega) \Longrightarrow z_i\partial_{z_j} = a(\omega)s\partial_s + V(\omega)$$

where now $a(\omega) \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $V(\omega)$ is a smooth vector field on $\mathbb{S}^{n-1}$. This shows that the linear vector fields lift to be smooth on $\mathbb{S}^{n,1}$ and even that

$$z_i\partial_{z_j} \text{ is tangent to the boundary of } \mathbb{S}^{n,1}. \tag{L1.15}$$

From this we can also deduce that $\mathrm{GL}(n,\mathbb{R})$ lifts to act smoothly under radial compactification. Indeed, in any Lie group a neighbourhood of the identity is given by exponentiation of the Lie algebra. Here exponentiation corresponds to integration of the vector field on $\mathbb{R}^n$, or of its extension to $\mathbb{S}^{n,1}$. So the elements in some neighbourhood of the identity extend smoothly to $\mathbb{S}^{n,1}$. More generally, any element of $\mathrm{GL}(n,\mathbb{R})$ is given by a finite composite of an element of $O(n)$ (an orthogonal transformation, needed if the orientation is reversed) and a finite number of elements of some fixed neighbourhood of the identity. Thus the action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ extends smoothly to $\mathbb{S}^{n,1}$ under R.

It is also the case that translations extend to the radial compactification. Here any translation is obtained by exponentiation of a linear combination of the vector fields ∂_{z_i}. Using the same argument as above, the smoothness near the boundary of $\overline{\mathbb{R}^n}$ can be examined in terms of s,ω and the unique decomposition

$$\partial_{z_i} = a_i(s,\omega)\partial_s + V_i(s,\omega) \tag{L1.16}$$

into a vector field on $(0,\infty)_s \times \mathbb{S}^{n-1}_\omega$. Since ∂_{z_i} is homogeneous of degree -1 under the homotheity $z \to rz$ it follows that

$$a_i(s,\omega) = s^2 a_i(1,\omega),\ V_i(s,\omega) = sV_i(1,\omega)$$

and (L1.16) becomes

$$\partial_{z_i} = s(a_i(\omega)s\partial_s + V_i(\omega)) \tag{L1.17}$$

for a smooth function and a smooth vector field on $\mathbb{S}^{n-1}$. Thus for the translations the generating vector fields actually lift to be Z_0 times a smooth vector field tangent to the boundary of $\mathbb{S}^{n,1}$. This will turn out to be important! In any case the translations also lift to smooth diffeomorphisms of $\mathbb{S}^{n,1}$.

This is our basic compactification of a vector space. Why are we interested in it? One very important reason is that the space $\mathcal{C}^\infty(\overline{W})$ is well-defined and is invariant under the general linear group (and translations). It is given many other names in the literature, typically the 'space of classical symbols of order 0'. More generally we can set

$$S^z_{\mathrm{cl}}(W) = \left\{u \in \mathcal{C}^\infty(W); \rho^{-z}u \in \mathcal{C}^\infty(\overline{W})\right\} \tag{L1.18}$$

where $\rho \in \mathcal{C}^\infty(\overline{W})$ is a boundary defining function. This is the space of 'classical symbols of (possibly complex) order z' on W. I will not use this notation very much because there are all sorts of confusions in the literature.

L1.3. Quadratic compactification. I introduced the quadratic compactification of $\mathbb{R}^n$ in (L1.11) above. Essentially by definition, the canonical map between the interiors (given by identification with $\mathbb{R}^n$) extends to a smooth map from the radial to the quadratic compactification, but not the reverse.

A neighbourhood of infinity in ${}^q\overline{\mathbb{R}^n}$ may be smoothly identified with the product $(0,1) \times \mathbb{S}^{n-1} \ni (t,\omega)$ where $\omega = y/|y| \in \mathbb{S}^{n-1}$ and $t = |y|^{-2}$. Since the generators of the translations satisfy

$$\partial_{y_j} t = -2\frac{y_j}{|y|^4} = -2t^{\frac{3}{2}}\omega_j$$

the translations do *not* lift to be smooth.

The radial vector field is

$$\sum_i y_i \partial_{y_i} = -2t\partial_t,$$

so the homotheity does lift to be smooth, namely it becomes $t \to r^{-2}t$. The homogeniety argument used above for the radial compactification then shows that all general linear transformations lift to be smooth, since

$$z_k \partial_{z_i} = a(\omega)t\partial_t + U_{kj} \tag{L1.19}$$

where the U_{kj} are smooth vector fields on the sphere and a is a smooth function on the sphere.

Thus the quadratic compactification is well-defined for a vector space, since it is preserved under linear transformations, but not for an affine space since it is *not* preserved by translations.

1+. Addenda to Lecture 1

1+.1. Explicit models. It is useful to think of the radial compactification of a vector space, $\overline{W}$, as an explicit set with a $\mathcal{C}^\infty$ structure. By abstract nonsense one can do this from the embedding of $\mathbb{R}^n$ into $\mathbb{S}^{n,1}$, but as I show below there is also a more natural geometric approach.

First let me review in a more sophisticated way the construction of the manifold $\overline{W}$ above. First, for $\mathbb{R}^n$ we have an explicit map into $\mathbb{S}^{n,1}$ such that the action of $\mathrm{GL}(n,\mathbb{R})$ extends smoothly

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "P"] \arrow[d, "G"'] & \mathbb{S}^{n,1} \arrow[d, "\tilde{G}"] \\ \mathbb{R}^n \arrow[r, "P"'] & \mathbb{S}^{n,1} \end{tikzcd}, \ \forall\, G \in \mathrm{GL}(n,\mathbb{R}). \tag{1+.20}$$

Now, to a real vector space, W, of dimension n we can associate the set of all linear ismorphisms to $\mathbb{R}^n$

$$P = \{T : W \longrightarrow \mathbb{R}^n,\ \text{linear and invertible}\}. \tag{1+.21}$$

This is a principal $\mathrm{GL}(n,\mathbb{R})$ space. That is, the action of $\mathrm{GL}(n,\mathbb{R})$

$$\mathrm{GL}(n,\mathbb{R}) \times P \ni (G,T) \longmapsto GT \in P \tag{1+.22}$$

is free and transitive. Then we can 'recover' the original vector space W as the quotient, namely as the vector space associated to the standard action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$

$$\tilde{W} = (P \times \mathbb{R}^n)/\sim,\ (T,v) \sim (GT, Gv)\ \forall\, G \in \mathrm{GL}(n,\mathbb{R}). \tag{1+.23}$$

This is canonically isomorphic to to W with the map being

$$\tilde{W} \ni [(T,v)] \longmapsto T^{-1}v \in W \tag{1+.24}$$

since this does not depend on the representative under (1+.23).

Now, what we have done above is to define the radial compactification $\overline{W}$ as the manifold with boundary associated to P by the action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{S}^{n,1}$

$$\overline{W} = (P \times \mathbb{S}^{n,1})/\sim,\ (T,p) \sim (GT, \tilde{G}p)\ \forall\, G \in \mathrm{GL}(n,\mathbb{R}). \tag{1+.25}$$

This is all very well, but it is a little nicer to have something a little lower-tech in mind. If we consider the action of $G \in \mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ it also induces an action on the associated (projective) sphere. That is, consider the set of half rays through the origin

$$\mathbb{S}^{n-1} = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+,\ \mathbb{R}^+ \times \mathbb{R}^n \ni (s,z) \longmapsto sz \in \mathbb{R}^n. \tag{1+.26}$$

This is a definition of the manifold $\mathbb{S}^{n-1}$. For a general vector space we can similarly define

$$\mathbb{S}W = (W \setminus \{0\})/\mathbb{R}^+,\ \mathbb{R}^+ \times W \ni (s,w) \longmapsto sw \in W. \tag{1+.27}$$

Here is an exercise for you:-

Lemma 1. *The set $W \sqcup \mathbb{S}W$ (the disjoint union) has a unique $\mathcal{C}^\infty$ structure such that the elements of $\mathcal{C}^\infty(W)$ which are homogeneous (for the $\mathbb{R}^+$ action in* (1+.27)*), of non-positive integral degree, outside some compact neighbourhood of the origin, lift to be smooth (with their asymptotic values on $\mathbb{S}W$ for homogeneity* 0 *and* 0 *there for negative homogeneity) and generate the* $\mathcal{C}^\infty$ *structure (i.e. this set of functions contains a coordinate system at each point).*

Thus we can identify $\overline{W} = W \sqcup \mathbb{S}W$ as a set.

Exercise 1. Show that the quadratic compactification of a vector space can be defined as a space associated to the principal $\mathrm{GL}(n, \mathbb{R})$ space P discussed above and also that it is given in a manner similar to Lemma 1 as a different $\mathcal{C}^\infty$ structure on the same set.

Exercise 2. In the case of the 1-point compactification, formulate precisely the notion of a conformal structure on a real vector space and show that the 1-point compactification only depends on it.

1+.2. Inclusions. All three compactificaitons behave well under inclusion of vector spaces – the inclusion extends to a smooth map of the corresponding compactifications (with metric, or at least conformal, consistency required for the one-point compactification).

Proposition 1. *If* $i : V \subset W$ *is a linear subspace of a vector space over the reals then the inclusion map extends by continuity to a smooth map*

$$\begin{aligned} i &: \overline{V} \hookrightarrow \overline{W} \\ i &: {}^q\overline{V} \hookrightarrow {}^q\overline{W}. \end{aligned} \tag{1+.28}$$

Proof. It suffices to check this in a model case $\square$

Exercise 3. In the case of the 1-point compactification, formulate the notion of the conformal structure induced on a subspace $V \subset W$ by the choice of a conformal structure on W and show that provided the compactifications are compatible in this sense then

$$i : {}^1\overline{V} \hookrightarrow {}^1\overline{W} \tag{1+.29}$$

extends to be smooth.

Exercise 4. Show that an injective linear map between vector spaces always extends to a smooth map between the radial or quadratic compactifications. For non-trivial vector spaces (i.e. of positive dimensions) is there ever a map which is not injective yet which has a smooth extension to (one of) these compactifications? Show that there is always a linear map which does not have a smooth extension between either the radial or quadratic compactifications.

1+.3. Relative compactification. If you have done Exercise 4 you will know that the compactifications discussed above do not behave well with respect to projections of vector spaces. The problem is that the points at infinity 'do not know where to go'. For this reason (and others as it turns out) there is more to be done.

Suppose $V \subset W$ is a subspace and we choose a complementary subspace and hence a product decomposition, $W = V \times U$. Take metrics on V and on U and then consider a map analogous to, but more complicated than, (L1.7)

$$\begin{gathered} R_V : W \ni w = (v, u) \longmapsto (t, s, v', u') = \\ \left(\frac{1}{(1+|u|^2)^{\frac12}}, \frac{(1+|u|^2)^{\frac12}}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{v}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{u}{(1+|u|^2)^{\frac12}}\right) \in \mathbb{R}^2 \times W. \end{gathered} \tag{1+.30}$$

On the image, $t^2+|u'|^2=1$, $s^2+|v'|^2=1$. These two 'cylinders' meet transversally (their normals are independent) so the intersection is a smooth manifold. The image of the map lies in

(1+.31)
$$ {}^V\overline{W}=\left\{(t,s,v',u')\in\mathbb{R}^2\times V\times U; t\ge 0,\ s\ge 0,\ t^2+|u'|^2=1,\ s^2+|v'|^2=1\right\}$$

which is a compact manifold with corners in which the image is precisely the dense interior, $s>0$, $t>0$. Of course, in principle this depends on the metrics and the choice of transversal subspace U, but in fact it does not.

LEMMA 2. *All translations on W lift to be smooth on ${}^V\overline{W}$ as do all general linear transformations of W which map V into itself.*

PROOF. The map

$$(t,s,v',u')\longmapsto(v/st,u'/t),\ s,t>0, \tag{1+.32}$$

is a smooth inverse to R_V in (1+.30) so R_V is a diffeomorphism onto this set. Moroever, orthogonal transformations on U and on V left to diffeomorphisms of ${}^V\overline{W}$ since they just act on the variables u' and v'. A general linear transformation of W leaving V fixed can be factored into $G_1\cdot G_2\cdot G_3$ where $G_1\in\mathrm{GL}(V)$ act as the identity on U, G_2 is of the form

$$G_2(v,u)=(v,u+Sv) \tag{1+.33}$$

for a linear map $S:V\longrightarrow U$ and $G_3\in\mathrm{GL}(U)$ acts as the identity on V. Then G_1 is the product of an orthogonal transformation and a finite number of elements of $\mathrm{GL}(V)$ in any preassigned neighbourhood of the identity and similarly for G_3. On the other hand G_2 is connected to the identity by scaling S to 0. Thus, it is enough to show that the Lie algebra lifts to be smooth on ${}^V\overline{W}$, which is to say the vector fields

$$v_i\partial_{v_j},\ u_k\partial_{v_j},\ u_k\partial_{u_l} \tag{1+.34}$$

(which span the linear vector fields on W tangent to V) lift to be smooth.

In the interior, i.e. where $s>0$ and $t>0$ these are certainly smooth. So we consider the three regions near the boundary separately, where

$$\begin{gathered} s\simeq 0,\ t>\epsilon_0>0\\ t\simeq 0,\ s>\epsilon_0>0 \text{ and}\\ t,s\simeq 0.\end{gathered} \tag{1+.35}$$

Arguing as before that near $x=0$ a smooth function of $x(1+x^2)^{-\frac12}$ is just a smooth function of x, we may use as local generating functions (so including coordinates) in these three regions

$$\begin{gathered} \frac{1}{|v|}(\simeq 0),\ \frac{v}{|v|},\ u\\ \frac{1}{|u|}(\simeq 0),\ \frac{v}{|u|},\ \frac{u}{|u|} \text{ and}\\ \frac{1}{|u|}(\simeq 0),\ \frac{|u|}{|v|}(\simeq 0),\ \frac{v}{|v|},\ \frac{u}{|u|}.\end{gathered} \tag{1+.36}$$

This allows us to apply homogeneity arguments much as above but now for the two homogeneities in u and v separately. Note that each of the vector fields in

(1+.34) is homogeneous of non-positive degree in both senses. It follows that all these vector fields lift to be smooth on ${}^V\overline{W}$ proving the Lemma. $\square$

LEMMA 3. *A short exact sequence of linear maps*

$$0 \longrightarrow V \longrightarrow W \longrightarrow W/V \longrightarrow 0 \tag{1+.37}$$

lifts to a sequence of smooth maps

$$\overline{V} \longrightarrow {}^V\overline{W} \longrightarrow \overline{W/V}. \tag{1+.38}$$

PROOF. We just have to do this for the 'model' as in (L1.8). The inclusion is just $v \to (v, 0)$ and

$$R_V(v, 0) = \left(1, \frac{1}{(1+|v|^2)^{\frac{1}{2}}}, \frac{v}{(1+|v|^2)^{\frac{1}{2}}}, 0\right) = (1, P(v), 0) \tag{1+.39}$$

in terms of the map (L1.9). Similarly the map from ${}^V\overline{W}$ to $\overline{U} = \overline{W/V}$ extending the projection $(v, u) \longmapsto u$ is just

$${}^V\overline{W} \ni (t, s, v', u') \longmapsto (t, u') \in \overline{U} = \mathbb{S}^{n,1}. \tag{1+.40}$$

$\square$

COROLLARY 1. *If* $A : V \longrightarrow W$ *is any linear map between real vector spaces* V *and* W, *with null space* $\mathrm{null}(A) \subset V$, *then* A *extends to a smooth map*

$$A : {}^{\mathrm{null}(A)}\overline{V} \longrightarrow \overline{W}. \tag{1+.41}$$

EXERCISE 5. Make sure you can give an elegant proof of this!

EXERCISE 6. Show that the second map in (1+.38) is a fibration. In fact, since the base is contractible (being a ball) it is then necessarily reducible to a product. Thus there exists a diffeomorphism $F : \overline{V} \times \overline{W/V} \longrightarrow {}^V\overline{W}$ such that the composite map

$$\overline{V} \times \overline{W/V} \xrightarrow{F} {}^V\overline{W} \longrightarrow \overline{W/V} \tag{1+.42}$$

is just the projection. However, there is no natural choice of F.

1+.4. Products. One thing we can certainly do is take the product of two vector spaces, $W = V \times U$. Then we can consider the compactification of W given by $\overline{V} \times \overline{U}$. The projection from W to U certainly extends to a smooth map from $\overline{V} \times \overline{U}$ to $\overline{U}$, namely the projection. However we still have the problem of the relationship of $\overline{W}$ to $\overline{V} \times \overline{U}$. The natural map between the interiors, both of which are identified with W, does not extend to a smooth map either way. We are part of the way to overcoming this difficulty with ${}^V\overline{W}$, but this is certainly not 'symmetric' in how it treats V and U so cannot be the full answer.

EXERCISE 7. Define the doubly-relative radial compactification of the product of two vector spaces. Do so by choosing metrics on U and V and then taking the map

$$R_{U,V} : W = (v,u) \longmapsto \Big(\frac{1}{(1+|u|^2)^{\frac12}}, \frac{1}{(1+|v|^2)^{\frac12}}, \frac{(1+|u|^2)^{\frac12}}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{(1+|v|^2)^{\frac12}}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{v}{(1+|v|^2)^{\frac12}}, \frac{(v,u)}{(1+|u|^2+|v|^2)^{\frac12}}, \frac{u}{(1+|u|^2)^{\frac12}}\Big) \in \mathbb{R}^4 \times V \times W \times U \tag{1+.43}$$

and showing it to be a diffeomorphism onto its image. Then check that the closure of the image is a manifold with corners (it has three boundary faces provided U and V have dimension at least 2, more if one of them is one-dimensional). Show that all translations on W lift to be smooth as do all linear transformations of W mapping U and V into themselves (i.e. direct products of linear transformations of U and of V.) Denoting the resulting compactifications by ${}^{U,V}\overline{U \times V}$ show that both the inclusions of U and V extend to be smooth

$$i : \overline{U} \longrightarrow {}^{U,V}\overline{U \times V},\ i : \overline{V} \longrightarrow {}^{U,V}\overline{U \times V}. \tag{1+.44}$$

Show that the identity map extends to be smooth in two different ways

$$\begin{tikzcd} & {}^{U,V}\overline{U \times V} \arrow[dl] \arrow[dr] & \\ \overline{U \times V} & & \overline{U} \times \overline{V}. \end{tikzcd} \tag{1+.45}$$

The non-invertibility of these maps goes some way to explaining the difference between the radial compactification of the product and the product of the radial compactifications. Draw a picture!

1+.5. Blow up. If you go so far as to actually do Exercise 7 you will come to look for a better way of doing such things. Fortunately there is – and it is discussed in more detail starting in the addenda to the second lecture. For the moment, consider the relationship between the 1-point compactfication of $\mathbb{R}^n$ and its radial compactification. We know (or if you prefer have defined things so) that a function is smooth near the point at infinity of the one-point compactification if it is a smooth function of $z/|z|^2$. On the other hand, a function is smooth near the sphere at infinity of $\overline{\mathbb{R}^n}$ if it is a smooth function of $x = 1/|z|$ and $\omega = z/|z| \in \mathbb{S}^{n-1}$ near $x = 0$. Since $z/|z|^2 = x\omega$ we see that smoothness on ${}^1\overline{\mathbb{R}^n}$ implies smoothness on $\overline{\mathbb{R}^n}$. This of course means that the map sending the whole of infinity to the one point is smooth

$$\beta : \overline{\mathbb{R}^n} \longrightarrow {}^1\overline{\mathbb{R}^n},\ \overline{\mathbb{R}^n} = [{}^1\overline{\mathbb{R}^n}, \{\infty\}]. \tag{1+.46}$$

In fact we can see more. Namely, in the coordinates discussed above, the map β is nothing other than the introduction of polar coordinates,

$$z/|z|^2 = Z = x\omega \tag{1+.47}$$

and this is what the final notation in (1+.46) indicates.

Definition 1. A manifold with boundary X (denoted subsequently by $[M, \{p\}]$ is the blow-up at $p \in M$ of a manifold M if there is a smooth map

$$\beta : X \longrightarrow M, \tag{1+.48}$$

which is a diffeomorphism of $X \setminus \partial X$ to $M \setminus \{p\}$, maps ∂X to p and is such that polar coordinates for some local coordinates aound p lift to a diffeomorphism of a neighbourhood of ∂X to $[0,1) \times \mathbb{S}^{n-1}$, $n = \dim M$.

This definition just means that the blow-up of a point is the introduction of polar coordinates. It would not make much sense if it depended on the choice of local coordinates based at (i.e. vanishing at) p in which polar coordinates where introduced.

Exercise 8. Confirm that change of local coordinates based at $0 \in \mathbb{R}^n$ induces a diffeomorphism on $[0,\epsilon)_r \times \mathbb{S}^{n-1}$ for some $\epsilon > 0$. Hint: First do the linear case; for which one can either use the linear invariance of radial compactification above, or model the argument. Then check the case that the Jacobian is the identity directly.

1+.6. Radial and relative compactification. I will show below that the blow up of a closed embedded submanifold of a manifold is always well-defined and reduces locally to the introduction of polar coordinates in the normal variables. The same notation as above, $[M,Y]$ is used for the blown-up manifold in this more general case; it comes equipped with a smooth blow-down map $\beta : [M,Y] \longrightarrow M$. The reason I bring this up here is that the relative compactification introduced above can also be defined through blow-up from the radial compactification. In this case we are blowing up an embedded submanifold of the boundary of a manifold with boundary.

Proposition 2. *If $V \subset W$ is a non-trivial subspace of a real vector space then there is a natural diffeomorphism*

$$ {}^{V}\overline{W} \equiv [\overline{W}, \mathbb{S}V]. \tag{1+.49}$$

Exercise 9. See if you can check this in local coordinates – of course it is a bit tricky since I have not explained what the blow-up map really is.

Exercise 10. See what happens in the 'trivial cases' excluded from Proposition 2, meaning $V = \{0\}$ or $V = W$. Namely show that

$$ {}^{\{0\}}\overline{W} \equiv \overline{W},\ {}^{W}\overline{W} \equiv {}^{q}\overline{W}. \tag{1+.50}$$

A similar discussion applies to the double-relative compactification of a product. Namely, in $\overline{U \times V}$ the two bounding spheres, $\mathbb{S}U$ and $\mathbb{S}V$, of the subspaces are disjoint embedded submanifolds of the boundary. Since they are disjoint the blow-ups of $\mathbb{S}U$ and $\mathbb{S}V$ are completely independent.

Proposition 3. *For any real vector spaces, there is a canonical diffeomorphism*

$$ {}^{U,V}\overline{U \times V} \longrightarrow [\overline{U \times V}, \mathbb{S}U, \mathbb{S}V] \tag{1+.51}$$

We may also blow up embedded submanifolds of boundary faces of manifolds with corners, provided they meet the other boundary faces in a 'product manner'. In particular we can blow up any boundary face.

Proposition 4. *For any real vector spaces, there is a canonical diffeomorphism*

$$ {}^{U,V}\overline{U \times V} \longrightarrow [\overline{U} \times \overline{V}, \mathbb{S}U \times \mathbb{S}V]. \tag{1+.52}$$

Notice that $\mathbb{S}U \times \mathbb{S}V$ is indeed the corner of $\overline{U} \times \overline{V}$, since it is the product of the boundaries.

1+.7. Parabolic compactifications. If that wasn't enough there are actually other compactifications, which are not obtained by blow up of the ones I have already considered. What's more they really do show up in analysis, in particular in complex analysis – about which I will say nothing much in this course (but see [**2**].)