CHAPTER 8

Smoothing operators

Lecture 8: 13 October, 2005

Now I am heading towards the Atiyah-Singer index theorem. Most of the results proved in the process untimately reduce to properties of smoothing operators, so let me review these today.

Recall that the space of smoothing operators on a compact manifold X acting between bundles E and F is identified with smooth sections of the 'big homomorphism bundle' over X^2 :

$$\Psi^{-\infty}(X;E,F) = \mathcal{C}^\infty(X^2;\pi_L^*F\otimes\pi_R^*E'\otimes\pi_R^*\Omega) \tag{L8.1}$$

where we identify $\operatorname{Hom}(E,F) = \pi_L^*F\otimes\pi_R^*E'$. These are bounded operators on L^2 sections as follows directly from the Cauchy-Schwarz inequality

$$\begin{gathered}\Psi^{-\infty}(X;E,F)\ni A: L^2(X;E)\longrightarrow L^2(X;F),\\ Au(x)=\int_X A(x,y)u(y),\ \|Au\|\le\|A\|_{L^2}\|u\|_{L^2}.\end{gathered} \tag{L8.2}$$

This just uses the square-integrability of the kernel.

Lemma 15. *If $A\in\Psi^{-\infty}(X;E)$ (so $E=F$) and its norm as a bounded operator on $L^2(X;E)$ is less than 1 then $(\operatorname{Id}+A)^{-1}=\operatorname{Id}+B$ for $B\in\Psi^{-\infty}(X;E)$.*

Proof. Since $\|A\|<1$ the Neumann series converges as a sequence of bounded operators so

$$B=\sum_{l=1}^{\infty}(-1)^l B^l \tag{L8.3}$$

is bounded on $L^2(X;E)$. As a 2-sided inverse $(\operatorname{Id}+A)(\operatorname{Id}+B)=\operatorname{Id}=(\operatorname{Id}+B)(\operatorname{Id}+A)$ which shows that

$$B=-A+A^2+ABA. \tag{L8.4}$$

From this it follows that $B\in\Psi^{-\infty}(X;E)$ since $ABA\in\Psi^{-\infty}(X;E)$. Indee the A on the right may be considered locally as a smooth map from X into $L^2(X;E)$ and hence remains so after applying B but then applying the second copy of A gives a smooth map into $\mathcal{C}^\infty(X;E)$ so the kernel of the composite is actually smooth on X^2. □

Corollary 2. *For any compact manifold and complex vector bundle E*

$$G^{-\infty}(X;E)=\left\{A\in\Psi^{-\infty}(X;E);(\operatorname{Id}+A)^{-1}=\operatorname{Id}+B,\ B\in\Psi^{-\infty}(X;E)\right\} \tag{L8.5}$$

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is an open subset of $\Psi^{-\infty}(X;E)$ *which is a topological group.*

Proof. For any point $A \in G^{-\infty}(X;E)$ the set $A+B$ such that $\|B\| < 1/\|(\mathrm{Id}+A)^{-1}\|$ is open in $\Psi^{-\infty}(X;E)$ and for such B it follows from the discussion above that $A+B \in G^{-\infty}(X;E)$ since $(\mathrm{Id}+A+B)^{-1} = (\mathrm{Id}+A)^{-1}(\mathrm{Id}+B(\mathrm{Id}+A)^{-1})^{-1}$. Similarly the maps $A \to (\mathrm{Id}+A)^{-1} - \mathrm{Id}$ and $(A,B) \longrightarrow (\mathrm{Id}+A)(\mathrm{Id}+B) - \mathrm{Id}$ are continuous. $\square$

Notice that I insist on $G^{-\infty}(X;E) \subset \Psi^{-\infty}(X;E)$ onto to make such statements easy to say. 'Really' of course you should think of $G^{-\infty}(X;E)$ as something like the invertible bounded operators on $L^2(X;E)$ which are of the form $\mathrm{Id}+A$ with $A \in \Psi^{-\infty}(X;E)$.

In fact, as we shall see later, $G^{-\infty}(X;E) \subset \Psi^{-\infty}(X;E)$ is actually an open dense subset, just like the invertible matrices in all matrices. As a topological algebra it is independant of X and E (provided $\dim X > 0$).

Definition 4. An operator has finite rank if its range is finite dimensional.

We are particularly interested in finite rank smoothing operators.

Lemma 16. *A smoothing operator* $A \in \mathcal{C}^\infty(X;E,F)$ *is of finite rank if and only if there are elements* $f_i \in \mathcal{C}^\infty(X;F)$, $e_i \in \mathcal{C}^\infty(X;E')$ $i = 1,\dots,N$ *and* $\nu \in \mathcal{C}^\infty(X;\Omega)$ *such that*

$$A = \sum_{i=1}^{N} f_i(x) e_i(y) \nu(y). \tag{L8.6}$$

Proof. By definition if $A \in \mathcal{C}^\infty(X;E,F)$ has finite rank, its range must be a finite dimensional subspace of $\mathcal{C}^\infty(X;F)$. Let the f_i be a basis of this space. Thus, we can write $Au = \sum_{i=1}^{N} (A_i u) f_i$ where $A_i : \mathcal{C}^\infty(X;E) \longrightarrow \mathbb{C}$ is continuous. If the f_i are orthonormalized with respect to an hermitian inner product on F and a density on X then $A_i u = \langle Au, f_i \rangle$ so these functionals are given by pairing with the smooth density

$$A_i = \int_X \langle A(x,y), f_i \rangle_F \nu(x) \in \mathcal{C}^\infty(X; E' \otimes \Omega) \tag{L8.7}$$

Dividing by a fixed density $0 < \nu \in \mathcal{C}^\infty(X;\Omega)$ gives $e_i = A_i/\nu \in \mathcal{C}^\infty(X;E')$ and this shows that the kernel can be written in the form (L8.6). $\square$

If we insist that the e_i be independent, or even orthonormalized with respect to some choice of hermitian inner product on E (hence on E') and density on X then the kernel takes the form

$$A = \sum_{i=1}^{N} a_{ij} f_i(x) e_j(y) \nu(y). \tag{L8.8}$$

We may also use the antilinear isomorphism of E' and E in terms of the chosen inner product to think of the e_i as sections of $\mathcal{C}^\infty(X;E)$. Then (L8.8) can be written rather fancifully as

$$A = \sum_{i=1}^{N} a_{ij} f_i(x) \overline{e_j}(y) \nu(y),\ a_{ij} \in \mathbb{C}, \tag{L8.9}$$

where the operation $\overline{e_i}$ is the antilinear isomorphism. Then the action of A is through the inner product

$$Au(x) = \sum_{i,j} a_{ij} \int_X \langle e_i(y), u(y)(y)\rangle f_i(x). \tag{L8.10}$$

If $E = F$ then we can orthonormalize the collection of all the e_i and f_j together and denote the result as e_i. In this case we have embedded A inside the collection of $N \times N$ matrices via (L8.10) which now becomes

$$Au(x) = \sum_{i,j} a_{ij} \int_X \langle e_i(y), u(y)\rangle e_i(x). \tag{L8.11}$$

Notice in fact that these finite rank smoothing operators do form a subalgebra of $\Psi^{-\infty}(X; E)$ which is isomorphic *as an algebra* to $M(N, \mathbb{C})$.

LEMMA 17. *The finite rank operators are dense in* $\Psi^{-\infty}(X; E, F)$.

I will give a rather uninspiring proof of this in which the approximation is done rather brutally. One can give much better approximation schemes, and I will, but first one needs to show that such approximation is possible (since this result is so basic it is actually used in the spectral theory which lies behind the better approximations. . .).

PROOF. In the special case that $X\mathbb{T}^n$ is a torus and $E = \mathbb{C}^k$ and $F = \mathbb{C}^{k'}$ are trivial bundles we can use Fourier series. Let $\nu = |d\theta_1 \dots d\theta_n|$ be the standard density on the torus then and element $A \in \Psi^{-\infty}(\mathbb{T}^n; \mathbb{C}^k, \mathbb{C}^{k'})$ is a $k \times k'$ matrix with entries in $\Psi^{-\infty}(\mathbb{T}^n)$, so acting on functions. The kernel, using the trivialization of the density bundle, is just an element $a \in \mathcal{C}^\infty(\mathbb{T}^{2n})$ which we can therefore expand in Fourier series. Let us write this expansion with the sign reversed in the second variable (in $\mathbb{T}^n$)

$$a(\theta, \theta') = \sum_{I,J} a_{IJ} e^{iJ\cdot\theta} e^{-iJ\cdot\theta'} \tag{L8.12}$$

where the sum is over all I, $J \in \mathbb{Z}^n$ and the coefficients are rapidly decreasing, because of the smoothness of a

$$a_{iJ} = (2\pi)^{-2n} \int_{\mathbb{T}^{2n}} e^{-iI\cdot\theta_i J\cdot\theta'} d\theta d\theta'. \tag{L8.13}$$

Since this double Fourier series converges rapidly the truncated kernels

$$a_N(\theta, \theta') = \sum_{|I|,|J|\le N} a_{IJ} e^{iJ\cdot\theta} e^{-iJ\cdot\theta'} \tag{L8.14}$$

converge to a in the $\mathcal{C}^\infty$ topology. Clearly a_N is a finite rank smoothing operator, so this proves the result in the case of the torus.

In the general case of a compact manifold X and bundles E, F, choose a covering of X by coordinate patches U_i over which both bundles are trivial and a partition of unity of the form ρ_p^2 subordinate to this cover. We may think of each of the U_p as embedded as an open subset of $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, by translating and scaling U_p until it is contained in $(0, 2\pi)^n$. Then we may apply the discussion above to the kernel $\rho_p A(x, y)\rho_q(y)$ which may be interpreted as acting between trivial bundles over the torus. Of course, from (L8.14) the resulting approximating finite rank kernels $a_{N,p,q}$ will not have support in $U_p \times U_q$ when regarded as subsets of $\mathbb{T}^{2n}$.

However $\rho_p(x)a_{N,p,q}\rho_q(y)$ does have such support and is of the form (L8.9) with the e_j being the $e^{iJ\cdot\theta'}\rho_q(y)$ and similarly for the f_i. Thus, summing these finitely many kernels we obtain a sequence of finite rank operators on X converging to A in the $\mathcal{C}^\infty$ topology. □

We need to consider families of operators, so note that this proof of approximation works uniformly on compact sets with the e_i and f_j fixed, i.e. independent of the parameters so only the coefficients in the approximating kernels depend on the parameters.

Now, recall that I have defined the odd K-theory of a compact manifold as

$$K^{-1}(X) = [X, G^{-\infty}(Y;E)] = \pi_0(\mathcal{C}^\infty(X;G^{-\infty}(Y;E)) \tag{L8.15}$$

So this includes the claim that the result is independent of the choice of Y and the bundle E (provided that $\dim Y > 0$). Note that

(L8.16)
$$\mathcal{C}^\infty(X;G^{-\infty}(Y,E)) = \{K \in \mathcal{C}^\infty(X\times Y^2;\operatorname{Hom}(E)\otimes\pi_L^*\Omega);\ \exists\,(\operatorname{Id}+K(x,\cdot))^{-1}\ \forall\, x\in X\}.$$

So the equivalence relation defining $K^{-1}(X)$ is just that $K \equiv K'$ if there exists $\tilde{K} \in \mathcal{C}^\infty(X\times[0,1];G^{-\infty}(Y;E))$ such that $\tilde{K}\big|_{t=0} = K$ and $\tilde{K}\big|_{t=1} = K'$.

The standard definition of odd K-theory is as the stable homotopy classes of (continuous) maps in $\mathrm{GL}(N,\mathbb{C})$. I will not work with this directly, but if you think a little about the proof below that $K^1(X)$ is independent of the choice of Y and E you will see how to show the equivalence of (L8.15) and the standard definition.

Proposition 15. *The groups $G^{-\infty}(Y;E)$ are connected and the set* (L8.15) *for any compact manifold X is independent of the choice of Y and E, so given two choices Y, E and Z, F there is a natural bijection between $[X;G^{-\infty}(Y;E)]$ and $[X, G^{-\infty}(Z,F)]$.*

Proof. That $G^{-\infty}(Y;E)$ is connected follows from the fact that that it is locally connected, so if $a_N \to a$ in $G^{-\infty}(Y;E)$ then for large N, a_N may be connected to a and the fact that $\mathrm{GL}(N;\mathbb{C})$ is connected. Or once can proceed more directly, as discussed below.

Let us choose a fixed 'model', namely $Y = \mathbb{S}$ and $E = \mathbb{C}$. Now, we may embed

$$G(N,\mathbb{C}) \subset G^{-\infty}(\mathbb{S}) \tag{L8.17}$$

by mapping the $N\times N$ matrices to the smoothing operators

$$M(N,\mathbb{C}) \ni a_{kl} \longmapsto A = \sum_{k,l=1,N}(a_{kl}-\delta_{kl})e^{ik\theta}e^{-l\theta'}|d\theta'|. \tag{L8.18}$$

The identity $N\times N$ matrix is subtracted here since we want $\mathrm{GL}(N,\mathbb{C})$ to be embedded as a subgroup of $G^{-\infty}(\mathbb{S})$, which it is for each N.

Given some compact manifold Y and bundle E any smooth map $A : X \ni x \longrightarrow A(x) \in G^{-\infty}(Y;E)$ may be approximated by finite rank operators $A_{(N)}$ as in Lemma 17. Choosing a basis as in (L8.11) we may identify the coefficients $\delta_{kl}+a_{kl}$ with an element of $\mathrm{GL}(N,\mathbb{C})$ and then use (L8.18) to map it to $\tilde{A} : X \longrightarrow G^{-\infty}(\mathbb{S})$. It is important to see that this procedure is well defined at the level of homotopy classes. That is, that the element $[\tilde{A}] \in \pi_0(X;G^{-\infty}(\mathbb{S})]$ is independnet of choices. With the approximations fixed, the procedure only depends on the choice of basis. Since (see the remarks following Lemma 17) the basis is independent of the parameters in X the choice only corresponds to a choice of basis (possibly including redundant elements). Add redundant elements to the basis does not change the family $\tilde{A}$ and changing the basis results in its conjugation by a fixed element of $G^{-\infty}(\mathbb{S})$, replacing $\tilde{A}$ by $B^{-1}\tilde{A}(x)B$, $B \in G^{-\infty}(\mathbb{S})$. Since we know that $G^{-\infty}(Y, E)$ is connected, B may be smoothly connected to the identity, so the conjugated element gives the same homotopy class. All families sufficiently close to a given family are in the same homotopy class so in fact for large enough N the homotopy class of $\tilde{A}$ only depends on the homotopy class of A. Applying the construction to $X \times [0,1]$ shows that homotopic families lift to the same homotopy class, so the map

$$\pi_0(X; G^{-\infty}(Y; E)) \longrightarrow \pi_0(X; G^{-\infty}(\mathbb{S})) \tag{L8.19}$$

is well-defined. An inverse to it can be constructed in essentially the same way, so this is a bijection independent of choices. □

The trace of matrices may be defined as the sum of the diagonal elements

$$\operatorname{tr}(a_{ij}) = \sum_i a_{ii}. \tag{L8.20}$$

It is invariant under change of basis since if $a' = b^{-1}ab$ then

$$\operatorname{tr}(a') = \sum_i (b^{-1}ab)_{ii} = \sum_{i,j,k} b^{-1}_{ij} a_{jk} b_{ki} = \sum_{i,j,k} a_{jk} b_{ki} b^{-1}_{ij} = \operatorname{tr}(a). \tag{L8.21}$$

Thus, $\operatorname{tr} : \hom(V) \longrightarrow \mathbb{C}$ is a well-defined linear map for any vector space V.

If we apply this to the finite rank operators in (L8.11) we find, using the assume orthonormality of the basis, that

$$\sum_i a_{ii} = \sum_i a_{ii} \int_Y \langle e_i(y), e_i(y) \rangle = \int_Y \operatorname{tr}(A(y,y))\nu(y) \tag{L8.22}$$

in terms of the trace on $\hom(E)$ of which $A(y,y) = A\big|_{\mathrm{Diag}}$ is a section. Thus for general smoothing operators we may simply define

$$\operatorname{Tr}(A) = \int_Y \operatorname{tr}(A\big|_{\mathrm{Diag}}). \tag{L8.23}$$

Proposition 16. *The trace functional is a well-defined continuous linear map*

$$\operatorname{Tr} : \Psi^{-\infty}(Y; E) \longrightarrow \mathbb{C} \tag{L8.24}$$

which satisifies

$$\operatorname{Tr}([A, B]) = 0 \ \forall \ A,\ B \in \Psi^{-\infty}(Y; E). \tag{L8.25}$$

Proof. If A, $B\Psi^{-\infty}(Y; E)$ then

$$\operatorname{Tr}(AB) = \operatorname{Tr}(C),\ C(x,z) = \int_Y A(x,y) \cdot B(y,z)$$

where the $\cdot$ refers to composition in the '$Hom(E)$' bundles. Thus in fact

$$\operatorname{Tr}(AB) = \int_Y \operatorname{tr}\left(A(x,y) \cdot B(y,x)\right) = \int_Y \operatorname{tr}\left(B(y,x) \cdot A(x,y)\right) = \operatorname{Tr}(BA) \tag{L8.26}$$

using the same identity for $\hom(E)$. □

Note that it follows from (L8.22) that under approximation by smoothing operators,

$$\operatorname{Tr}(A) = \lim_{N\to\infty} \operatorname{Tr}(A_N). \tag{L8.27}$$

Using this one can show that the determinant extends to smoothing operators in the following sense.

THEOREM 4. *(Fredholm) There is a unique map*

$$\Psi^{-\infty}(Y;E) \ni A \longrightarrow \det(\operatorname{Id} + A) \in \mathbb{C} \tag{L8.28}$$

which is entire and satisfies

$$\begin{gathered} \det(\operatorname{Id} + A)(\operatorname{Id} + B)) = \det(\operatorname{Id} + A)\det(\operatorname{Id} + B) \\ \partial_s \det(\operatorname{Id} + sA)\big|_{s=0} = \operatorname{Tr}(A) \\ A \in G^{-\infty}(Y;E) \Longleftrightarrow A \in \Psi^{-\infty}(Y;E),\ \det(\operatorname{Id} + A) \neq 0. \end{gathered} \tag{L8.29}$$

From this it follows that $\Psi^{-\infty}(Y;E) \subset G^{-\infty}(Y;E)$ is an open *dense* subset. The determinant can be defined on $G^{-\infty}(Y;E)$ by using the connectedness to choose a smooth curve $\gamma_A : [0,1] \longrightarrow G^{-\infty}(Y;E)$ from Id to a given point A and then setting

$$\det(\operatorname{Id} + A) = \exp(\int_0^1 \operatorname{Tr}\left((\operatorname{Id} + \gamma_A(t))^{-1}\frac{d\gamma_A(t)}{dt}\right)dt. \tag{L8.30}$$

Of course it needs to be shown that this is independent of the choice of γ_A, that it extends smoothly to all of $\Psi^{-\infty}(Y;E)$ (as zero on the complement of $G^{-\infty}(Y;E)$ and that it satisfies (L8.29).

8+. Addenda to Lecture 8

There are many other results on smoothing operators which reinforce the sense in which they are 'infinite rank matrices.' Think for instance of the spectrum.

PROPOSITION 17. *If $A \in \Psi^{-\infty}(X;E)$ then*

(8+.31)

$$\operatorname{spec}(A) = \{z \in \mathbb{C} \setminus \{0\}; (z\operatorname{Id} - A) : L^2(X;E) \longrightarrow L^2(X;E) \text{ is not invertible}\}$$

is discrete except (possibly) at $0 \in \mathbb{C}$

and for each $0 \neq z \in \operatorname{spec}(A)$ the associated generalized eigenspace

(8+.32)

$$E(z) = \{u \in \mathcal{C}^\infty(X;E); (z\operatorname{Id} - A)^N u = 0 \text{ for some } N \in \mathbb{N}\} \text{ is finite dimensional.}$$

PROOF. If we could use the Fredholm determinant – although at this stage I have not finished the proof of its properties – then the discreteness would be clear once since certainly

$$\operatorname{spec}(A) \subset \{z \in \mathbb{C}; \det(\operatorname{Id} - \frac{A}{z}) = 0\} \tag{8+.33}$$

and the latter is the set of zeros of a holomorphic function on $\mathbb{C}\setminus\{0\}$. So, we would only need to show that the determinant is not identically zero.

In any case we can proceed more directly, without using the determinant but instead using 'analytic Fredholm theory'. First of all, if we give E an inner product and choose a density on Y then we know that $\|A/z\| = \|A\|/|z|$ so for $|z| > \|A\|$

it follows that $(\mathrm{Id} - \frac{A}{z})^{-1}$ exists. Thus $\operatorname{spec}(A) \subset \{z; |z| \leq \|A\|\}$, meaning that $(A - z\,\mathrm{Id})^{-1}$ is a holomorphic family of bounded operators, and hence map in $G^{-\infty}$ for $\|z\| > \|A\|$. $\square$