CHAPTER 5

# Pseudodifferential operators

## Lecture 5: 22 September, 2005

Since it may be a while before I write up the notes from this fifth lecture, I include here my pre-lecture notes

**L5.1. Conormal sections of bundles.** I had planned to go through the definition of $I^m(X,Y)$ again from the beginning to define instead $I^m(X,Y;E)$ where $E$ is a complex vector bundle over $X$. I will do this in the addenda and instead give a direct definition which has the virtue of brevity. Namely

$$I^m(X,Y;E) = I^m(X,Y) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;E). \tag{L5.1}$$

Here we use the fact that $I^m(X,Y)$ is a module over $\mathcal{C}^\infty(X)$ – we can multiply by arbitrary $\mathcal{C}^\infty$ functions – and so is $\mathcal{C}^\infty(X;E)$, the space of smooth sections of the vector bundle $E$. What *precisely* does (L5.1) mean? It means that we define an element of $I^m(X,Y;E)$ as an equivalence class of finite sums of pairs (written multiplicatively)

$$v = [\sum_i u_i e_i],\ e_i \in \mathcal{C}^\infty(X;E),\ u_i \in I^m(X,Y) \tag{L5.2}$$

where the equivalence relation is generated by $\mathcal{C}^\infty(X)$-linearity, i.e.

$$\sum_i u_i e_i \sim \sum_j u'_j e'_j \text{ if } e_i = \sum_j a_{ij} e'_j \text{ and } u'_j = \sum_i a_{ij} u_i,\ a_{ij} \in \mathcal{C}^\infty(X). \tag{L5.3}$$

Then $I^m(X,Y;E)$ is itself a $\mathcal{C}^\infty(X)$-module and if an element, $u$, has support in an open set over which $E$ is trivial then for any smooth local basis, $e_i$ of $E$, [1]

$$u = \sum_i u_i e_i,\ u_i \in I^m(X,Y). \tag{L5.4}$$

The definition above can be used for the space of distributional sections, that is

$$\mathcal{C}^{-\infty}(X;E) = \mathcal{C}^{-\infty}(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;E) \tag{L5.5}$$

so $I^m(X,Y;E) \subset \mathcal{C}^{-\infty}(X;E)$ and this tensor product definition is equivalent to the duality definition

$$\mathcal{C}^{-\infty}(X;E) = (\mathcal{C}^\infty(X;E^* \otimes \Omega_X))'. \tag{L5.6}$$

It follows that there are natural injections, as there should be

$$\mathcal{C}^\infty(X;E) \hookrightarrow I^m(X,Y;E) \hookrightarrow \mathcal{C}^{-\infty}(X;E). \tag{L5.7}$$

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[1] Instead this can also be used as the basis of the definition.

**L5.2. Integration.** Suppose that $Z$ is compact then integration of distibutions is well-defined provided they are valued in the density bundle of $Z$, for any *vector space* $E$ (not a vector bundle, it has to be globally trivialized)

$$\int_Z : \mathcal{C}^{-\infty}(Z; E \otimes \Omega_Z) \longrightarrow E. \tag{L5.8}$$

Of course this means we can integrate $I^m(Z, Y; E \otimes \Omega_Z)$ under the same conditions.

**L5.3. Restriction.** Now suppose that $Z \subset X$ is an embedded submanifold which is transversal to $Y$, meaning that

$$\forall\ p \in Y,\ T_pY + T_pZ = T_pX. \tag{L5.9}$$

Then, the restriction map for smooth sections $\mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(Z; E\big|_Z)$ extends to conormal sections

$$\big|_Z : I^m(X, Y; E) \longrightarrow I^{m+\frac14 \operatorname{codim} Z}(Z, Y \cap Z; E\big|_Z),\ \sigma_{m+\frac14 \operatorname{codim} Z}(u\big|_Z) = \sigma_m(u)\big|_Z. \tag{L5.10}$$

To prove this, we can use the restriction map from $\mathcal{C}^\infty(X; E)$ to handle any element supported away from $Y$. So to define $u\big|_Z$ for $u \in I^m(X, Y; E)$ we can suppose that $u$ is supported in any preassigned neighbourhood of $Y$. In particular we can assume it is supported in the range of some normal fibration of $Y$.

Now, what does the transversality mean? Fix a point $p \in Y$ then let $Z = \{y_1, \dots, y_k = 0\}$ be given by the vanishing of local defining functions and let $Y = \{t_1, \dots, t_p = 0\}$ be similarly given in terms of local defining b functions. Then (L5.9) means that the differentials of these functions are independent at $p$, so they can be completed to a local coordinate system based at $p$, by adding $s_1, \dots, s_{n-p-k}$ where necessarily $k \le n - p$. Thus the $y$ and $s$ together give local coordinates on $Y$ near $p$. These coordinates give a normal fibration of $Y$ near $p$ – we may identify the normal bundle with the fibres $(y, s)$ =const near $Y$ (and near $p$.) Now, cover $Y$ by such local coordinate systems and normal fibrations and take a finite partition of unity subordinate to this cover. Using this to decompose $u \in I^m(X, Y)$ we see that each piece, $u_i$, is of the form

$$u_i \in \mathcal{C}^\infty\left(\mathbb{R}^k; I^{m+\frac{n-p-k}{4}}(\mathbb{R}^{n-p-k} \times \mathbb{R}^p, Y \cap Z; E)\right) \tag{L5.11}$$

with compact support near the origin in all variables. The first variables here are the $y$'s and $Y \cap Z = \mathbb{R}^p \times \{0\}$. Thus, restriction to $y = 0$, which is to say $Z$, gives a map as in (L5.10) locally. It is clearly consistent[2] under changes of coordinates and so we get (L5.10) with the computation of the symbol also immediate from (L5.11).

**L5.4. Push-forward.** Let $\phi : X \longrightarrow B$ be a fibration (or if you prefer, for present purposes it is enough to take the projection off a product, i.e. $X = B \times Z$). Suppose that this fibration is transversal to the embedded submanifold $Y \subset X$, meaning that for all $p \in Y$,

$$T_pY + T_p(\phi^{-1}(\phi(p))) = T_pX, \tag{L5.12}$$

which is just the condition that each fibre is transversal to $Y$. Then fibre integration gives a linear map

$$\phi_* : I^m(X, Y; \phi^*E \otimes \Omega_X) \longrightarrow \mathcal{C}^\infty(B; E \otimes \Omega_B) \tag{L5.13}$$

[2] See problem X

for any smooth vector bundle $E$ over $B$.

First recall that this is true in the case $m = -\infty$, i.e.

$$\phi_* : \mathcal{C}^\infty(X; \phi^* E \otimes \Omega_X) \longrightarrow \mathcal{C}^\infty(B; E \otimes \Omega_B). \tag{L5.14}$$

Namely, near a point $b \in B$ we can reduce $\phi$ to projection for the product $U \times Z$ to $U$, where $U$ is a neighbourhood of $b \in B$. The density bundles behave well under products, so $\Omega_X = \Omega_U \otimes \Omega_Z$. Then (L5.14) is just locally in $B$ the formula

$$\phi_*(u) = \left(\int_Z u(b,z)\nu(z)\right)\nu(b). \tag{L5.15}$$

In each fibre, i.e. for fixed $b$, $u(b,z)$ is a smooth map in $z$ into the vector space $E_b$, the fibre of the bundle at $b$. Now, to get (L5.13) we just replace the integral in (L5.15) by the integral in (L5.8) after restricting to each fibre using (L5.10) and the result is smooth as claimed in (L5.13).

**L5.5. Pseudodifferential operators.** As already noted, we define the space of pseudodifferential operators, 'acting between' sections of two vector bundles $E$ and $F$ over $X$ to be

$$\Psi^m(X; E, F) = I^m(X^2, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R). \tag{L5.16}$$

Here $\Omega_R = \pi_R^* \Omega_X$ is the pull-back of the density bundle from the right factor of $X$, $\pi_R(x, y) = y$, and $\mathrm{Hom}(E, F)$ is the 'big' homomorphism bundle. Thus $\mathrm{Hom}(E, F)$ is a vector bundle over $X^2$ with fibre at $(x, y)$ the space $\hom(E_y, F_x)$ of linear maps from the fibre, $E_y$, of $E$ at $y \in X$ to the fibre, $F_x$, of $F$ at $x \in X$. Using standard identifications we can think of this bundle as

$$\mathrm{Hom}(E, F) = \pi_L^* F \otimes \pi_R^*(E'). \tag{L5.17}$$

Then the operator associated with (and indeed identified with) the kernel $A \in \Psi^m(X; E, F)$ is

$$(Au)(x) = \int_X A(x,y)u(y)dy,\ \ Au = (\pi_L)_* \left(A \cdot (\pi_R)^* u\right), \tag{L5.18}$$
$$A : \mathcal{C}^\infty(X; E) \longrightarrow \mathcal{C}^\infty(X; F).$$

Here the left 'integral' is formal. The middle expression is supposed to be rigourous and yield the map as shown. Thus, for $u \in \mathcal{C}^\infty(X; E)$ the pull-back to $X^2$ under $\pi_R$ is an element of $\mathcal{C}^\infty(X^2; \pi_R^* E)$. When we multiply it by the kernel $A \in I^m(X^2, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R)$ we get, using (L5.17), an element

$$A \otimes (\pi_R)^* u I^m(X^2, \mathrm{Diag}; \pi_L^* F \otimes (\pi_R)^*(E \otimes E')\Omega_R).$$

Now, we can pair $E$ with $E'$ to get the action of $\hom(E_y, F_x)$ on $E_y$ and hence an element of $I^m(X^2, \mathrm{Diag}; \pi_L^* F \otimes \Omega_R)$. Finally we may apply (L5.13) to get the integral, mapping to $\mathcal{C}^\infty(X; F)$ as expected.

This means that the composite of two pseudodifferential operators acting on appropriate bundles is defined. It is of fundamental importance that the composite is again a pseudodifferential operator,

THEOREM 2. *On any compact manifold, $X$, and for any complex vector bundles, $E$, $F$ and $G$*

$$\Psi^m(X; F, G) \circ \Psi^{m'}(X; E, F) \subset \Psi^{m+m'}(X; E, G). \tag{L5.19}$$

I will prove this after discussing the use of pseudodifferential operators to partially invert elliptic operators.

We also need to see what has happened to the symbol of our conormal distributions in this case. Namely the symbol map simplifies to give a short exact sequence

$$\Psi^{m-1}(X;E,F) \longrightarrow \Psi^m(X;E,F) \longrightarrow \mathcal{C}^\infty(S^*X;N_m \otimes \hom(E,F)). \tag{L5.20}$$

So, the density terms have disappeared, the manifold carrying the symbol has become the cosphere bundle of $X$, $S^*_xX = (T^*_xX \setminus 0)/\mathbb{R}^+$ and the bundle has become the usual homomorphism bundle, over $X$, lifted to $S^*X$.

**L5.6. Action of differential operators.** For the moment we can easily see that differential operators are special cases of pseudodifferential operators and more generally the restricted composition theorem

(L5.21)
$$\mathrm{Diff}^k(X;F,G) \circ \Psi^m(X;E,F) \subset \Psi^{k+m}(X;E,G),\ \sigma_{k+m}(PA) = \sigma_k(P) \circ \sigma_m(A)$$

is easy to deduce. This is enough for our application to Hodge theory.

[Needs proof]

## 5+. Addenda to Lecture 5

**5+.1. The euler class.**