CHAPTER 22

# Eta forms

### Lecture 22: 6 December, 2005

The index formula for product-type will involve a 'regularized Chern character' which we interpret as an 'eta' form. To construct these forms, by regularization, we use holomorphic families of pseudodifferential operators. This leads us to a discussion of the residue trace, the regularized trace and the trace-defect formula and then finally to $\eta$-forms.

**L22.1. Trace functional.** For smoothing operators I have already discussed the trace. Namely

$$\mathrm{Tr} : \Psi^{-\infty}(Z; E) \longrightarrow \mathbb{C},\ \mathrm{Tr}(A) = \int_Z \mathrm{tr}_E(A(z, z)) \tag{L22.1}$$

where $\mathrm{tr}_E$ is the trace functional on the fibres of $\hom(E) = \mathrm{Hom}(E)\big|_{\mathrm{Diag}}$. It is straightforward to extend the trace to low order operators, for which the kernel is continuous (and a little more) across the diagonal.

Theorem 12. *The trace functional extends canonically to*

$$\mathrm{Tr} : \Psi^s(Z; E) \longrightarrow \mathbb{C},\ s \in \mathbb{C},\ \mathrm{Re}(s) < -\dim Z. \tag{L22.2}$$

Proof. To see this, and derive a formula for the extended functional, observe that the trace vanishes on any smoothing operator with kernel having support not meeting the diagonal. Since we can decompose and pseudodifferential operators as

$$A = A_1 + A_2,\ A_2 \in \Psi^{-\infty}(X; E),\ \mathrm{supp}(A_2) \cap \mathrm{Diag} = \emptyset \tag{L22.3}$$

we only need to consider the part, $A_1$, of $A$ with support near the diagonal. Directly from our original definition of pseudodifferential operators, this is given as the inverse Fourier transform of a symbol on the cotangent bundle and then transferred to $Z^2$ using a bundle isomorphism (from Hom to hom) covering a normal fibration of the diagonal:

$$A_1 = F^* \mathcal{F}^{-1}(a),\ a \in \rho^{-s} \mathcal{C}^\infty(\overline{T^*Z}; \hom(E)). \tag{L22.4}$$

This is the case even for a smoothing operator, when $a \in \dot{\mathcal{C}}^\infty(\overline{T^*Z}; \hom(E))$ is Schwartz on the fibres of $T^*Z$.

By definition of a normal fibration, the diagonal is carried to the zero section of $TZ$ under $F$. Thus, for a smoothing operator in (L22.4), the trace may be written

$$\mathrm{Tr}(A_1) = \int_{O \subset TM} \mathcal{F}^{-1}(a) = (2\pi)^{-d} \int_{T^*Z} a\omega^d,\ d = \dim Z. \tag{L22.5}$$

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Here I am just using the fact that, fibre by fibre, the value of a function at 0 is the integral of its Fourier transform. We also really need to check that the measures behave correctly in (L22.5) – but that is something I have sloughed over anyway. Here $\omega^d$ is the symplectic measure on $T^*Z$, just the maximal exterior power of the symplectic form. In local coordinates one can check (L22.5) directly. Notice that the (full) symbol $a$ is by no means invariantly defined, but we see from (L22.5) that its integral is, and that it continues to make sense provided $\operatorname{tr}_E(a)$ is integrable, which is just the condition $\operatorname{Re} s < -n$ in (L22.2). □

As I shall show in the addenda, under this condition $A_1$ is indeed trace class so (L22.5) does represent the trace of an operator in the usual sense.

**L22.2. Holomorphic families (of holomorphic order).** For topological vector spaces such as $\Psi^m(Z;E)$, the topoolgy here being very similar to that on $\mathcal{C}^\infty(M)$, there is no difficulty in defining (strongly) holomorphic families, i.e. holomorphic maps from some open set

$$\mathbb{C} \supset \Omega \longrightarrow \Psi^m(Z;E). \tag{L22.6}$$

Namely, this is just a smooth function of the parameter, i.e. and element of $\mathcal{C}^\infty(\Omega;\Psi^m(Z;E))$ which satifies the Cauch-Riemann equtions

$$\overline{\partial}A = (\partial_x + i\partial_y)A = 0 \text{ in } \Omega. \tag{L22.7}$$

We do want to consider such maps, but we need something more. Namely holomorphic families where the order is changing holomorphically as well. These are *not* holomorphic maps into a fixed topological vector space, so we need to be a little careful about their properties. In fact it is probably better to think of them as 'yet-another-variant' of the spaces of pseudodifferential operators. Note that we have defined the space of pseudodifferential operators of complex order, it is the holomorphy that needs to be analyzed.

Definition 9. A map $A : \Omega \longrightarrow \Psi^s(Z;E,F)$ is said to be a holomorphic family of order $\mu : \Omega \longrightarrow \mathbb{C}$, a given holomorphic function on an open set $\Omega \subset \mathbb{C}$, if for any function $\chi \in \mathcal{C}^\infty(Z^2)$ with $\operatorname{Diag} \cap \operatorname{supp}(\chi) = \emptyset$,

$$\chi A : \Omega \longrightarrow \Psi^{-\infty}(Z;E,F) \text{ is holomorphic} \tag{L22.8}$$

(in the usual sense) and for some (any) normal fibration and bundle trivialization and an appropriate cutoff

$$\mathcal{F}(G^*(1-\chi)A) = \rho^{-\mu(s)}a,\ \Omega \ni s \longrightarrow a(s) \in \mathcal{C}^\infty(\overline{T^*Z}; \hom(E,F)). \tag{L22.9}$$

Thus coefficinet $a$ in (L22.9) is itself is holomorphic in the usual sense, as a smooth function on $\Omega \times \overline{T^*Z}$ and only the factor $\rho^{-\mu(s)}$ is 'extraordinary'. Note that changing to another boundary defining function $\rho'$ merely multiplies $a$ by $(\rho/\rho')^{-\mu(s)}$ which is holomorphic in the usual sense, since it is $b^{\mu(s)}$ for a positive smooth function $b$. We are mostly interested in the case $\mu(s) = \pm s$; the case $\mu(s) = m$, constant, is the usual notion of holomorphy.

Of course, it needs to be checked that this definition is independent of the normal fibration and the bundle isomorphism. This however proceed exactly as before so I pass over it without too much comment. The crucial point being that the space of functions $\rho^{-s}\operatorname{Hom}(\Omega \times X)$ for any compact manifold with boundary and any open set $\Omega \subset \mathbb{C}$ is invariant under the action of smooth vector fields on $X$ which are tangent to the boundary. It is also necessary to do the asymptotic summation lemma, not only uniformly in $\Omega$ but holomorphically as well – this is quite straightforward.

Since the proof of the product formula follows from the freedom to change normal fibrations and bundle isomorphisms, it is also straightforward to check that composition makes sense.

Lemma 35. *If $A \in \Psi^{\mu(s)}(Z;E,F)$ and $B \in \Psi^{\nu(s)}(F,G)$ are holomorphic families in an open set $\Omega \subset \mathbb{C}$ then $BA \in \Psi^{\mu(s)+\nu(s)}(Z;E,G)$ is holomorphic.*

Ellipticity of such a family is just pointwise ellipticity and a useful result is a version of 'holomorphic Fredholm theory'.

Lemma 36. *If $A(s) \in \Psi^{m(s)}(X;E,F)$ is an elliptic holomorphic family on a connected open set $\Omega$ such that $A(s_0)^{-1} \in \Psi^{-\mu(s_0)}(Z;F,E)$ exists for one $s_0 \in \Omega$ then $A(s)^{-1} \in \Psi^{-\mu(s)}(Z;F,E)$ exists for $s \in \Omega \setminus D$, with $D$ discrete in $\Omega$, and there is a holomorphic family $B(s) \in \Psi^{-\mu(s)}(Z;F,E)$ and a meromorphic map on $E : \Omega \longrightarrow \Psi^{-\infty}(Z;F,E)$ with poles only at $D$ and of finite rank, such that*

$$A^{-1}(s) = B(s) + E(s),\ \forall\ s \in \Omega \setminus D. \tag{L22.10}$$

The standard examples of such holomorphic families are the complex powers of a positive, self-adjoint, elliptic operator. For instance if $\varDelta$ is the Laplacian on some compact manifold then $(\varDelta+1)^s$ is a holomorphic family of order $2s$. In fact $\varDelta^s$, defined correctly, is itself a holomorphic family of order $2s$. Althouth the residue trace was defined using such complex poweres this is by no means necessary (as was shown originally by Victor [**3**]). Instead the following is enough for our purposes:-

Proposition 47. *For any bundle $E$ on any compact manifold $Z$ there is an entire family (i.e. holomorphic on $\mathbb{C}$) $E(s) \in \Psi^s(Z;E)$ which is everywhere elliptic and satisfies*

$$E(0) = \operatorname{Id}. \tag{L22.11}$$

Using complex powers (or otherwise) one can show that there is such a family which is everywhere invertible as well.

Proof. For any normal fibration and bundle isomorphism, the identity is always represented by the full symbol $\operatorname{Id}_E$ . Thus if we simply choose a boundary defining function $\rho \in \mathcal{C}^\infty(\overline{T^*Z})$ and take the quantization of the symbol $a = \rho^{-s}\operatorname{Id}_E$,

$$E(s) = (1-\chi)F^*\mathcal{F}^{-1}(\rho^{-s}\operatorname{Id}_E) \tag{L22.12}$$

we get such a family. □

**L22.3. Seeley's theorem on the trace.** The important relationship of holomorphic families and the trace functional is given by a theorem of Seeley, originally in the context of zeta functions.

Theorem 13 (Seeley). *For any holomorphic family of order $s$ on a connected open set $\Omega \subset \mathbb{C}$ such that $\Omega' = \Omega \cap \{\operatorname{Re}(s) < -\dim Z\}$ is non-empty and connected,*

$$\operatorname{Tr}(E(s)) : \Omega' \longrightarrow \mathbb{C} \tag{L22.13}$$

*extends to a meromorphic function with at most simple poles at the divisor $-\dim Z + \mathbb{N}$*

$$\operatorname{Tr}(E(s)) : \Omega \setminus \{-d+\mathbb{N}\} \longrightarrow \mathbb{C},\ d = \dim Z. \tag{L22.14}$$

Proof. From the discussion above, if we take the full symbol of $E(s)$ localized near the diagonal

$$\rho^{-s}a(s) = \mathcal{F}(G^*\left((1-\chi)E(s)\right) \tag{L22.15}$$

then $a(s) \in \mathcal{C}^\infty(\overline{T^*Z}; \hom(E))$ is holomorphic in $\Omega$ and

$$\operatorname{Tr}(E(s)) = \int_{T^*Z} \rho^{-s}\operatorname{tr}_E(a(s))\Omega^{2d} \text{ in } \Omega'. \tag{L22.16}$$

So we need only show that this integral extends meromorphically to $\Omega$ with the stated poles, since the uniqueness follows from the uniqueness of holomorphic extensions.

The integral (L22.16) can be decomposed using a partition of unity on $Z$ and the invariance of the trace under conjutation means that we may replace $E$ by a trivial bundle. Since the symbol $a(s)$ is itself holomorphic, the integral over any fixed compact region is holomorphic. Thus we may take $\rho = r = 1/R$ the inverse of a polar coordinate in $T^*Z \equiv U \times \mathbb{R}^d$ locally and reduce $\operatorname{Tr}(E(s))$ to a finite sum of integrals of the form

$$T_j(s) = \int_{S^*Z}\int_0^1 r^{-s}a(s,r,z,\omega)r^{-d-1}drdzd\omega. \tag{L22.17}$$

Here $a$ is a smooth function of all variales, down to $r = 0$ and holomorphic in $s$ and the singular factor comes for the usual formula for Lebesgue measure in polar coordinates, $R^{d-1}dR = -r^{-d-1}dr$. Here the local cutoff makes $a$ compactly supported in $z$ so the $z$ and $\omega \in \mathbb{S}^{d-1}$ integrals may be carried out, leaving the single integral

$$T_j(s) = \int_0^1 r^{-s}a'(s,r)r^{-d-1}drdzd\omega. \tag{L22.18}$$

The integral converges uniformly for $\operatorname{Re}(s) < -d$, which is the initial domain of it existence (inside $\Omega$). If $a' = r^k a''(s,r)$ where $a''(s,r)$ is also smooth and holomorphic in $s$ then the integral (L22.18) converges uniformly for $\operatorname{Re} s < -d+k$. Thus, if we replace $a'$ by its Taylor series at $r = 0$ to high order

$$a'(s,r) = \sum_{j=0}^{k-1} a'(s)_j r^j + r^k a''(s,r) \tag{L22.19}$$

we get just such a remainder term, so

$$T_j(s) - T_j'(s) = \sum_{j=0}^{k-1} a'(s)_j \int_0^1 r^{-s+j-d-1}dr = \sum_{j=0}^{k-1}\frac{a'(s)_j}{-s+j-d} \tag{L22.20}$$

with $T_j'(s)$ holomorphic in $\Omega \cap \{\operatorname{Re} s < -d+k\}$. This proves the stated meromorphy and shows that the extension only has simple poles and only at the points $s = -d+j$, $j \in \mathbb{N}_0$. □

**L22.4. Residue trace.** If we take an element $A \in \Psi^m(Z;E)$ for some $m \in \mathbb{Z}$ and a holomorphic family $E(s) \in \Psi^s(Z;E)$ satisfying (L22.11) then $A(s) = AE(s) \in \Psi^{s+m}(Z;E)$ and $\operatorname{Tr}(AE(s))$ can only have poles at the points $-d+m+\mathbb{N}_0$. Since $A(0) = A$ the pole at $s = 0$ is of particular interest. Wodzick observed that the residue is actually well-defined.

Proposition 48. *For any holomorphic family $A(s) \in \Psi^{m+s}(Z;E)$ the residue of the holomorphic extension of the trace from* $\operatorname{Re} s < -d$,

$$\operatorname{Tr}_{\mathrm{R}}(A(0)) = \lim_{s\to 0} s \operatorname{Tr}(A(s)) \tag{L22.21}$$

*is independent of the choice of $A(s)$, with $A(0) = A$, and so defines a continuous functional*

$$\operatorname{Tr}_{\mathrm{R}} : \Psi^m(Z;E) \longrightarrow \mathbb{C},\ m \in \mathbb{Z}, \tag{L22.22}$$

*which vanishes identically if $m < -d$ and satisfies*

$$[A,B] = 0\ \forall\ A \in Psi^m(Z;E),\ B \in \Psi^{m'}(Z;E),\ m,\ m' \in \mathbb{Z}. \tag{L22.23}$$

The functional (L22.21) is called the residue trace.

Proof. By Seeley's computation above, the residue in (L22.21) certainly exists.

To see that it does not depend on the holomorphic family of order $s$ chosen so that $A(0) = A$, suppose that $A'(s)$ is another such family. Thus $B(s) = A'(s) - A(s)$ is a holomorphic family of order $s$ such that $B(0) = 0$. Consider what this means. For the part away from the diagonal, the kernel as a family of smoothing operators must vanish at $s = 0$. By Taylors formula the kernel
$chiB(s) = sB'(s)$ where $B'(s)$ is also holomorphic. For the part near the diagonal, passing to the symbol $\rho^{-s}b(s)$ with $b$ holomorphic, it follows that $b(0) = 0$ and hence, from the same reasoning, that $b(s) = sb'(s)$. So in fact $B(s) = sB'(s)$ where $B'(s)$ is again a holomorphic family of order $s$. Now applying Seeley's computation again,

$$\operatorname{Tr}(B(s)) = s \operatorname{Tr}(B'(s)) \text{ is regular at } s = 0 \tag{L22.24}$$

since $\operatorname{Tr}(B'(s))$ can have at most a simple pole at the origin. Thus $\operatorname{Tr}_{\mathrm{R}}(A(0))$ defined by (L22.21) is indeed independent of the holomorphic family (of order $s$) used to define it.

In particular we may choose or basically family $E(s)$ satisfying (L22.11) and then

$$\operatorname{Tr}_{\mathrm{R}}(A) = \lim_{s\to 0} s \operatorname{Tr}(AE(s))\ \forall\ A \in \Psi^m(Z;E),\ m \in \mathbb{Z}. \tag{L22.25}$$

For a commutator,

$$\begin{aligned}\operatorname{Tr}_{\mathrm{R}}([A,B]) &= \lim_{s\to 0} s \operatorname{Tr}([A,B]E(s)) = \lim_{s\to 0} \operatorname{Tr}(ABE(s) - BAE(s)) \\ &= \lim_{s\to 0} s \operatorname{Tr}(A[B,E(s)]) - \lim_{s\to 0} s \operatorname{Tr}(B[A,E(s)]) = 0.\end{aligned} \tag{L22.26}$$

Here, $A[B,E(s)]$ and $B[A,E(s)]$ are both holomorphic families of order $s$ which vanish at $s = 0$ (since $E(0) = \mathrm{Id}$) so the residues must vanish. □

The discussion of Seeley's theorme above allows us to derive a formula for the residue trace. Namely, there can be no singularity in $\operatorname{Tr}(A(s))$ arising from the smoothing terms. I leave it as an exercise (probably disussed more in the addenda) to show that

$$\operatorname{Tr}_{\mathrm{R}}(A) = \int_{S^*M} \operatorname{tr}_E(a_{-d}) \tag{L22.27}$$

where $a_{-d}$ is the term or degree $-d$ in the expansion of the symbol, made into a density by multiplying by the term of homogeneity $d$ in the corresponding expansion of $\omega^d$. Note that this is true for the full symbol computed with respect to *any* normal fibration. Not that $a_{-d}$ is well-defined, but the integral of its bundle trace is.

**L22.5. Regularized trace.** As well as the residue trace we are interested in the regularization of the trace functional itself. Having chosen a holomorphic family $E(s)$ we set

$$\overline{\operatorname{Tr}}_E(A) = \lim_{s \to 0} \left( \operatorname{Tr}(AE(s) - \frac{1}{s} \operatorname{Tr}_{\mathrm{R}}(A) \right) \tag{L22.28}$$

where the limit exists exactly because we have removed the singular term. As the notation indicates this functional *does* depend on the family $E(s)$ chosen to define it; further more it is *not* a trace. Rather it is precisely the trace-defect which we want to compute.

As shown above, if $E_i(s) \in \Psi^s(Z;E)$, $i = 1, 2$, are two holomorphic families satisfying (L22.11) then

$$E_1(s) = E_2(s) = sB(s),\ B(s) \in \Psi^s(Z;E) \text{ holomorphic.} \tag{L22.29}$$

Thus, we can set $D(E_1, E_2) = B(0) \in \Psi^0(Z;E)$. Then
(L22.30)
$$\operatorname{Tr}(AE_2(s)) = \operatorname{Tr}(AE_1(s)) + s \operatorname{Tr}(AB(s)) \Longrightarrow \overline{\operatorname{Tr}}_{E_1}(A) = \overline{\operatorname{Tr}}_{E_2}(A) + \operatorname{Tr}_{\mathrm{R}}(AD(E_1, E_2)$$

since $AB(s)$ is a holomorphic family with value $AD(E_1, E_2)$ at $s = 0$. This shows (see the addenda):

Lemma 37. *The regularized traces, defined by* (L22.28) *on* $\Psi^{\mathbb{Z}}(Z, E)$*, by holomorphic families satisfying* (L22.11)*, form an affine space modelled on* $\Psi^0(Z, E)/\Psi^{-\infty}(Z;E)$.

**L22.6. Trace-defect formula.** There is another important operation [1] which arises from the properties of the holomorphic family satisfying (L22.11). Namely, as we have already remarked, $[A, E(s)] = sB(s)$ is a holomorphic family vanishing at the origin. Thus

$$D_E : \Psi^{\mathbb{Z}}(Z;E) \ni A \longmapsto [A, E(s)]/s\big|_{s=0} \in \Psi^{\mathbb{Z}}(Z;E) \tag{L22.31}$$

is a well-defined linear map.

Proposition 49. *The map* (L22.31) *is an exterior derivation mapping* $\Psi^m(Z;E)$ *to* $\Psi^{m-1}(Z;E)$ *for any* $m \in \mathbb{Z}$ *(actually for any* $m \in \mathbb{C}$*) which is well-defined up to interior derivations,*

$$D_{E_1}A = D_{E_2}A + [D(E_1, D_2), A] \tag{L22.32}$$

*and which is in fact the unique continuous exterior derivation (up to constant multiplies and addition of interior derivations).*

Proof. That $D_E$ is a derivation follow immediately from the identity

$$[AB, E(s)] = A[B, E(s)] + [A, E(s)]B. \tag{L22.33}$$

The difference formula (L22.32) follows from the definition of $D(E_1, E_2)$.

That $D_E$ is not itself an interior derivtion follows easily from the fact that The uniqueness is not so simple, maybe it will be/is discussed in the addenda. □

[1] which I did not quite emphasize enough during the lecture

Formally, $D_E A$ is the commutator $[\log Q, A]$ for some positive operator $Q$ of order 1 in the algebra. There is no such element, in the pseudodifferential algebra as it is defined above, so this is an exterior derivation (this is only supposed to be a plausibility argument). In fact it is easy enough to construct an operator which does represent the derivation as a commutator, it is just not in the algebra but rather is in an extension of the algebra.

One relationship that is easy to see is that the residue trace vanishes on the range of $D_E$ – of course it vanishes on the range of interior derivations by (L22.26)

$$\mathrm{Tr}_{\mathrm{R}}(D_E A) = 0 \ \forall \ A \in \Psi^{\mathbb{Z}}(Z; E). \tag{L22.34}$$

Indeed this just follows from the definition of $D_E$ in (L22.31) since

$$\mathrm{Tr}_{\mathrm{R}}(D_E A) = \lim_{s \to 0} s(\frac{[A, E(s)]}{s}) = 0. \tag{L22.35}$$

More importantly for computations in the sequel

Lemma 38. *For all* $A,\ B \in \Psi^{\mathbb{Z}}(Z; E)$,

$$\overline{\mathrm{Tr}}_E([A, B]) = \mathrm{Tr}_{\mathrm{R}}(BD_E A). \tag{L22.36}$$

Proof. By definition the regularized trace is the value at $s = 0$ of

$$\overline{\mathrm{Tr}}_E([A, B]) = \lim_{s \to 0} \mathrm{Tr}([A, B]E(s)) = \lim_{s \to 0} s \,\mathrm{Tr}(B\frac{[E(s), A]}{s}) = \mathrm{Tr}_{\mathrm{R}}(BD_E A), \tag{L22.37}$$

where there is no pole at the origin, since $\mathrm{Tr}_{\mathrm{R}}([A, B]) = 0$ and the identity

$$\mathrm{Tr}(ABE(s)) = \mathrm{Tr}(BE(s)A)$$

holds because it holds in the trace class region. □

**L22.7. The circle.** For pseudodifferential operators on the circle it is easy to make some of these operations explicit (this can in fact be done in general, although it is not necessarily enlightening). First we can take as our holomorphic family

$$E(s)e^{ik\theta} = (k^2 + 1)^{s/2} e^{ik\theta} \in \Psi^s(\mathbb{S}). \tag{L22.38}$$

That this can be checked following the arguments for the Szegő projector. Then the exterior derivation is seen to satisfy

$$\sigma_{m-1}(D_E A) = \pm r \partial_\theta \sigma_m(A),\ \forall \ A \in \Psi^m(\mathbb{S}) \tag{L22.39}$$

where the sign refers to the component of the cosphere bundle $S^*\mathbb{S} = \mathbb{S}_+ \sqcup \mathbb{S}_-$. The residue trace we already know to be

$$\mathrm{Tr}_{\mathrm{R}}(A) = \int_{\mathbb{S}_+} \sigma_{-1}(A) d\theta - \int_{\mathbb{S}_-} \sigma_{-1}(A) d\theta \ \forall \ A \in \Psi^{-1}(\mathbb{S}). \tag{L22.40}$$

**L22.8. Toeplitz $\eta$ forms.** Recall that on the Toeplitz smoothing group, stabilized by the smoothing operators on some other compact manifold,

$$G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(Z; E)) = \{a \in \Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(Z; E)); (\mathrm{Id} + a)^{-1} = \mathrm{Id} + b,\ b \in \Psi^{-\infty}_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(Z; E))\} \tag{L22.41}$$

is a classifying group for odd K-theory (in this approach by definition) and that the forms

$$\mathrm{Ch}_{\mathrm{odd}}(a) = \sum_{k=0}^{\infty} c_k \beta_{2k+1},\ \beta_{2k+1}(a) = \mathrm{Tr}\left(((\mathrm{Id} + a)^{-1} da)^{2k+1}\right). \tag{L22.42}$$

Consider the inclusion of $G^{-\infty}$ as a normal subgroup of the group of (stabilized) invertible Toeplitz operators of order 0 :

(L22.43)

$$G^0_{\mathcal{T}}(\mathbb{S};\Psi^{-\infty}(Z;E)) = \left\{A \in \Psi^0_{\mathcal{T}}(\mathbb{S};\Psi^{-\infty}(Z;E)); (\mathrm{Id}+A)^{-1} = \mathrm{Id}+B,\ B \in \Psi^0_{\mathcal{T}}(\mathbb{S};\Psi^{-\infty}(Z;E))\right\}.$$

The subgroup satifying the normalization condition $\sigma_0(A)(1) = 0$ is contractible, but for the moment we will ignore this.

Using the regularized trace introduced above, we can extend the forms in (L22.42) from the normal subgroup to the whole group. So we set

$$\eta = \sum_{k=0}^{\infty} c_k\eta_{2k+1},\ \eta_{2k+1}(A) = \overline{\mathrm{Tr}}((\mathrm{Id}+A)^{-1}dA)^{2k+1}) \tag{L22.44}$$

where we drop the suffix indicating the regularizing family, since we will just use (L22.38) for definiteness sake.

Proposition 50. *The forms in* (L22.44) *are well-defined on* $G^0_{\mathcal{T}}(\mathbb{S})$, *restrict to* $G^{-\infty}_{\mathcal{T}}(\mathbb{S})$ *to the forms in* (L22.42) *and are such that*

$$d\eta_{2k+1}(a) = \sigma_0^*(\beta_{2k} + d\gamma_{2k+1}),\ \textit{where} \tag{L22.45}$$
$$\beta_{2k}(b) = -\frac{1}{2}\int_{\mathbb{S}} \mathrm{Tr}\left((b^{-1}db)^{2k}b^{-1}\frac{\partial}{\partial\theta}b\right),$$
$$\gamma_{2k+1}(b) = \frac{1}{2}\int_{\mathbb{S}} \mathrm{Tr}\left((b^{-1}db)^{2k+1}b^{-1}\frac{\partial}{\partial\theta}b\right)$$

*are defined on the loop group*

$$\{b \in \mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Z;E); (\mathrm{Id}+b)^{-1} = \mathrm{Id}+b',\ b' \in \mathcal{C}^\infty(\mathbb{S};\Psi^{-\infty}(Z;E)\}. \tag{L22.46}$$

Proof. The functional $\overline{\mathrm{Tr}}$ is linear and continuous, so we can commute $d$ through it to compute $d\eta_{2k+1}$. The argument in (L22.44) can be written

$$((\mathrm{Id}+A)^{-1}dA)^{2k+1} = (-1)^k A^{-1}dA(d(A^{-1})dA)^k \tag{L22.47}$$

where we use the identity $dA^{-1} = -A^{-1}(dA)A^{-1}$. Thus, only the first factor in (L22.47) is not exact, so

$$d\eta_{2k+1}(A) = -\overline{\mathrm{Tr}}((A^{-1}dA)^{2k+2}). \tag{L22.48}$$

The argument can now be written as a 'supercommutator' – really it is a commutator when we take the antisymmetry of the exterior product into account. Namely

$$d\eta_{2k+1}(A) = -\frac{1}{2}\overline{\mathrm{Tr}}\left([A^{-1}dA,(A^{-1}dA)^{2k+1}]\right). \tag{L22.49}$$

Then, using a 'super' version of the trace defect formula we conclude that

$$d\eta_{2k+1}(A) = \frac{1}{2}\mathrm{Tr_R}\left((A^{-1}dA)^{2k+1}D_E(A^{-1}dA)\right). \tag{L22.50}$$

All the products $A^{-1}dA$ are of order zero and $D_E$ lower the order by one, so we know from (L22.40) that the residue trace here is just the integral of the principal symbol,

$$d\eta_{2k+1}(A) = \frac{1}{4\pi}\int_{\mathbb{S}}\sigma_{-1}\left((A^{-1}dA)^{2k+1}D_E(A^{-1}dA)\right)d\theta. \tag{L22.51}$$

Now, $D_E$ expands to

$$D_E(A^{-1}dA) = -A^{-1}(D_EA)A^{-1}dA + A^{-1}dD_EA. \tag{L22.52}$$

Now, using (L22.39) to evaluate the leading term in $D_E$, we arrive at (L22.45). Indeed, in the first term arising from inserting (L22.52) into (L22.51), the last factor $A^{-1}dA$ can be commuted to the front, giving $\beta_{2k}$. Similarly, in the term arising from the second part of (L22.52) the factor premultiplying $dD_E A$,

$$(A^{-1}dA)^{2k+1}A^{-1} = (-1)^{k+1}(dA^{-1}\cdot dA)^k dA^{-1}, \tag{L22.53}$$

is exact, so this reduces to $d\gamma_{2k+1}$ with $\gamma_{2k+1}$ as in (L22.45). $\square$

The pointed loop group, which is denote above

(L22.54)

$$G^{-\infty}_{(1)} = \{b \in \mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}(Z;E); b(1) = \mathrm{Id},\ (\mathrm{Id} + b)^{-1} = \mathrm{Id} + b',\ b' \in \mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}(Z;E)\}$$

is a normal subgroup of the full loop group in (L22.46) and it in turn has the normal subgroup of index 0 loops

$$G^{-\infty}_{(1),0}(Z;E) = \{b \in G^{-\infty}_{(1),0}(Z;E); \frac{1}{2\pi}\int_{\mathbb{S}} \mathrm{Tr}((\mathrm{Id} + b)^{-1}\frac{\partial}{\partial\theta}b)d\theta = 0\} \tag{L22.55}$$

which is the leading part of our classifying sequence. What has been shown above can be pictured like this

(L22.56)

$$\begin{tikzcd}
\mathrm{Ch}_{\mathrm{odd}} \arrow[d, dash, dotted] & \eta_{\mathrm{odd}} \arrow[l, "\big|_{G^{-\infty}}"'] \arrow[r, "d"] \arrow[d, dash, dotted] & \mathrm{Ch}_{\mathrm{evn}} + d\Gamma \arrow[dd, dash, dotted] \\
G^{-\infty}_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(Z;E)) \arrow[r] & G^0_{\mathcal{T}}(\mathbb{S}; \Psi^{-\infty}(\mathbb{S}; \Psi^{-\infty}(Z;E)) \arrow[d] & \\
& G^{-\infty}_{(1),0}(Z;E) + \rho\mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}(Z;E))[[\rho]] \arrow[r] & G^{-\infty}_{(1),0}(Z;E).
\end{tikzcd}$$

The fact that $d\eta_{2k+1}$ descends to the quotient group and represents there the (even) Chern character is what is fundamental. The fact that this differential actually lifts to the leading part of the quotient, and does not depend on the lower order symbols at all, is (a higher order extension of) the 'miracle of the loop group' of Pressley and Segal [**5**]. This latter behaviour does not carry over to higher dimensions or the 'geometric case'.

## 22+. Addenda to Lecture 22

**22+.1. Proof of Lemma 37.** We have already seen in (L22.30) that the difference between two regularized traces if given by $\mathrm{Tr}_{\mathrm{R}}(AD)$ where $D \in \Psi^0(Z;E)$ is the difference element discussed above. Since the residue trace vanishes on smoothing operators, this certainly vanishes if $D \in \Psi^{-\infty}(Z;E)$ and so only depends on the 'full symbol' element $D \in \Psi^0(Z;E)/\Psi^{-\infty}(Z;E)$. Every element $D$ can appear as $D(E_1, E_2)$ since if $E_1(s)$ is a given holomorphic family satisfying (L22.11) then $E_2(s) = E_1(s) + sDE_1(s)$ is another family of this type with $D(E_1, E_2) = D$. Finally the image of $D$ in $\Psi^0(Z;E)/\Psi^{-\infty}(Z;E)$ can be recoverd from the difference of the functionals, namely

(22+.57)

$$\Psi^{\mathbb{Z}}(Z;E) \ni A \longrightarrow \mathrm{Tr}_{\mathrm{R}}(AD) \text{ determines } [D] \in \Psi^0(Z;E)/\Psi^{-\infty}(Z;E) \text{ uniquely.}$$

To see this (and we need to use all the integral orders, or at least arbitrarily large ones) start with $A$ of order $-d$. Then the term in the symbol of $AD$ of order $-d$ is just the product of the principal symbols, so in this case

$$\mathrm{Tr}_{\mathrm{R}}(AD) = \int_{S^*M} \mathrm{tr}_E(\sigma_{-d}(A)\sigma_0(D)). \tag{22+.58}$$

If we think of $\sigma_0(D)$ as being a distribution on $S^*M$ this determines it, since $\mathrm{tr}(ab)$ is non-degenerate as a bilinear form on the fibre of $\hom(E)$ at each point. Thus $\sigma_0(D)$ can be recovered from the difference functional. This determines $D$ modulo $\Psi^{-1}(Z;E)$ so can subtract from $\mathrm{Tr}_{\mathrm{R}}(AD)$ the functional $\mathrm{Tr}_{\mathrm{R}}(A\tilde{D})$ where $\tilde{D} \in \Psi^0(Z;E)$ is some operator with the same principal symbol. Thus we can suppose that $D \in \Psi^{-1}(Z;E)$, or proceeding inductively that $D \in \Psi^{-k}(Z;E)$ and then repeat the argument, now with $A \in \Psi^{-d+k}(Z;E)$ so that the residue trace still comes out in terms of the principal symbol of $AD$. Thus, $D$ is indeed determined modulo $\Psi^{-\infty}(Z;E)$.