CHAPTER 21

Families Atiyah-Singer index theorem

Lecture 21: 1 December, 2005

L21.1. Relative Chern character. For the Atiyah-Singer formula, we wish to associate with the symbol $[\sigma(A)] \in K_{\mathrm{c}}^0(T^*(M/B))$ of a family of elliptic operators a cohomology class $\mathrm{Ch}(\sigma(A)) \in H_{\mathrm{c}}^{\mathrm{evn}}(T^*(M/B))$. This enters crucially into the formula for the Chern character of the index bundle,

$$\mathrm{Ch}(\mathrm{ind}(A)) = \int \mathrm{Td} \cdot \mathrm{Ch}(\sigma(A)) \in H^{\mathrm{evn}}(B) \tag{L21.1}$$

where the integral is the pushforward operation for the overall fibration $T^*(M/B) \longrightarrow M \longrightarrow B$.

I will define this relative Chern character in the context of the interior of a compact manifold with boundary; the model case being $\overline{T^*(M/B)} \supset T^*(M/B)$. From a topological point of view there is not difficulty in defining this relative Chern character quite generally. This if the Chern character is defined for a general class of compact topological spaces then for non-compact spaces U with 1-point compactification ${}^1\overline{U}$ in this class one can (and indeed this is the standard way to do it) define the K-theory of U in terms of the K-theory of ${}^1\overline{U}$

$$K(U) = \mathrm{null}\left(K({}^1\overline{U}) \longrightarrow K(\{\mathrm{pt}\})\right) \tag{L21.2}$$

where the map is restriction to the point at infinity. Then if one has a topological Chern character the Chern character on $K(U)$ is defined as the composite.

However, I want a smooth version of this with explicit forms, since later I need to generalize the set up substantially. For the interior of a compact manifold with boundary, the definition (L21.2) reduces to the one I have been using. Namely elements of $K_{\mathrm{c}}(\mathrm{int}(X))$ are equivalence classes of pairs of bundles $[(E_+, E_-, a)]$ with a bundle isomorphism between then outside a compact set, i.e. in a neighbourhood of the boundary. In fact in this case we are free to assume that the bundles are smooth up to the boundary and a is just an identification of them over the boundary. For the moment however I will assume that a is defined near the boundary. The equivalence relation imposed identifies triples related by bundle isomorphisms and homotopies as previously discussed. So, we will associated a deRham form on $\mathrm{int}(X)$ with such a triple (and choice of connections) which vanishes near the boundary, and so defines a relative cohomology class, and show that this gives a map

$$\mathrm{Ch} : K_{\mathrm{c}}(\mathrm{int}(X)) \longrightarrow H_{\mathrm{c}}^{\mathrm{evn}}(\mathrm{int}(X)) = H^{\mathrm{evn}}(X, \partial X) \tag{L21.3}$$

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where the compact supported cohomology of the interior may be identified with the cohomology of X relative to its boundary.

Let $\nabla_\pm$ be connections on $E_\pm$. We can use the isomorphism a to relate a connection on E_+ to that on E_-. Thus, if $\rho \in \mathcal{C}^\infty_c(\mathrm{int}(X))$ is such that $1-\rho \in \mathcal{C}^\infty(X)$ has support in the neighbourhood of the boundary over which a is defined (and is an isomorphism) then

$$\nabla = \rho\nabla_+ + (1-\rho)a^{-1}\nabla_- a \tag{L21.4}$$

is a connection on E_+. The Chern form we consider is

$$\lambda = \mathrm{Tr}\left(\exp(\frac{i}{2\pi}\omega)\right) - \mathrm{Tr}\left(\exp(\frac{i}{2\pi}\omega_-)\right),\ \omega = \nabla^2,\ \omega_- = (\nabla_-)^2. \tag{L21.5}$$

That this is closed follows immediately from the discussion of last lecture. In this case $\lambda = 0$ as a form near the boundary and its class in $H^{\mathrm{evn}}(X, \partial X)$ is independent of choices. In fact I want to get a reasonably explicit formula for a representative of the class of λ which does not have the cut-off function in it.

First we need to compute the curvature of ∇. First recall that the connections $\nabla_\pm$ on $E_\pm$ determine a natural connection on $\hom(E_+, E_-)$ as a bundle over X. Namely, if a is such a homorphism then

$$(\nabla a)u = \nabla_-(au) - a(\nabla_+ u)\ \forall\ u \in \mathcal{C}^\infty(X; E_+) \tag{L21.6}$$

defines the connection which perhaps should be denoted ∇_{-+} since has nothing much to do with the ρ-dependent connection in (L21.4). In fact, we can express that connection in terms of it since

$$\nabla = \nabla_+ + (1-\rho)a^{-1}\nabla a \text{ on } \mathcal{C}^\infty(X; E_+). \tag{L21.7}$$

Thus the curvature of ∇, which is what appears in (L21.5) is

$$\begin{aligned} &\omega u = \nabla^2 u = (\omega_+ + (1-\rho)a^{-1}\nabla a)^2 u \\ &= \omega_+ + \nabla_+((1-\rho)a^{-1}(\nabla a)u) + (1-\rho)a^{-1}(\nabla a)\nabla_+ u + (1-\rho)^2(a^{-1}\nabla a)^2 u \Longrightarrow \\ &\quad \omega = -d\rho a^{-1}(\nabla a) + (1-\rho)\left(a^{-1}\omega_- a\right) + \rho\omega_+ - \rho(1-\rho)a^{-1}(\nabla a)a^{-1}(\nabla a). \end{aligned} \tag{L21.8}$$

Here I have used the identities

$$(\nabla^2 a)u = \omega_+ a - a\omega_- \text{ and } \nabla a^{-1} = -a^{-1}(\nabla a)a^{-1} \tag{L21.9}$$

which follow from the definitions.

Consider the form

$$\mathrm{Tr}\exp(\frac{i}{2\pi}\omega) = \sum_k \frac{i^k}{(2\pi)^k}\mathrm{Tr}(w^k). \tag{L21.10}$$

To remove ρ we will let it approach the characteristic function of the manifold. Choose a boundary defining function $x \in \mathcal{C}^\infty(X)$, $\partial X = \{x = 0\}$, $x \geq 0$, $dx \neq 0$ on ∂X and for $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(t) = 0$ in $t < \frac{1}{2}$, $\chi(t) = 1$ in $t \geq 1$, set

$$\rho = \chi(x/\epsilon). \tag{L21.11}$$

For $\epsilon > 0$ small enough ρ satisfies the conditions required above. The curvature form in (L21.8) can be written as the sum

$$\omega = \alpha + \frac{\chi'(x)}{\epsilon}dx\beta \tag{L21.12}$$

where α and β are localy integrable uniformly as $\epsilon \downarrow 0$. Inserting this into (L21.10) gives a similar decomposition

$$\operatorname{Tr}\exp(\frac{i}{2\pi}\omega) = A + \frac{\chi'(x/\epsilon)}{\epsilon} dx \wedge B \tag{L21.13}$$

where A and B have uniformly locally integrable coefficients.

LEMMA 34. *As $\epsilon \downarrow 0$ the form* (L21.13) *converges as a (supported) distibutional form on X to*

$$\operatorname{Tr}\exp(\frac{i}{2\pi}\omega_+) - \delta(x)dx \wedge \frac{i}{2\pi}\iota^*_{\partial X} \tag{L21.14}$$
$$\operatorname{Tr}\left(a^{-1}(\nabla a)\int_0^1 \exp\left(\frac{i}{2\pi}(s\left(a^{-1}\omega_- a\right) + (1-s)\omega_+ - s(1-s)a^{-1}(\nabla a)a^{-1}(\nabla a)\right)\right) ds.$$

PROOF. From (L21.8) the first term in (L21.13) converges in the sense of supported distributions to the first term in (L21.14) – that is after integrating agains a smooth section (up to the boundary) of the dual bundle tensored with the density bundle. Thus, it is only necessary to prove the convergence of the second term to the second term in (L21.14).

Expanding out the second term, using the trace identity to bring each possible factor $d\rho$ to the front, shows that B in (L21.13) is

$$B = \operatorname{Tr}\left(\sum_k \frac{i^k}{(2\pi)^k(k-1)!}((1-\rho)\left(a^{-1}\omega_- a\right) + \rho\omega_+ - \rho(1-\rho)a^{-1}(\nabla a)a^{-1}(\nabla a))^{k-1}\right)$$

The coefficient of B tends to $\delta(x)dx$, supported on the boundary and apart from the explicit dependence on ρ the form is uniformly smooth up to the boundary. Replacing the smooth coefficients in B by their restrictions to the boundary leaves an error of the form $x/\epsilon\chi'(x/\epsilon)dxB'$, with B' smooth, and this vanishes, as a distribution, in the limit as $\epsilon \to 0$. Thus we may assume that B coefficients in B are replaced by their restrictions to the boundary, extended to be independent of x in a product decomposition near the boundary. As a result the distribution limit is the same as the integral against a smooth x-independent factor. The x integral becomes

$$\frac{i}{2\pi}\operatorname{Tr}(a^{-1}(\nabla a)\int_0^{2\epsilon} \exp\left(\frac{i}{2\pi}(1-\rho)\left(a^{-1}\omega_- a\right) + \rho\omega_+ - \rho(1-\rho)a^{-1}(\nabla a)a^{-1}(\nabla a)\right) \chi'(\frac{x}{\epsilon})\frac{dx}{\epsilon} \tag{L21.15}$$

which reduces to (L21.14) after the change of variable $s = \chi(x/\epsilon)$. □

This gives the form on the relative Chern character (due, I believe, to Fedosov), as a distribution deRham class

$$\lambda = \operatorname{Tr}\exp(\frac{i}{2\pi}\omega_+) - \operatorname{Tr}\exp(\frac{i}{2\pi}\omega_-) - \delta(x)dx\wedge \tag{L21.16}$$
$$\frac{i}{2\pi}\iota^*_{\partial X}\operatorname{Tr}(a^{-1}(\nabla a)\int_0^1 \exp\left(\frac{i}{2\pi}(s\left(a^{-1}\omega_- a\right) + (1-s)\omega_+ - s(1-s)a^{-1}(\nabla a)a^{-1}(\nabla a))\right) ds.$$

Note that this sort of 'conormal representation' gives a cohomology class with an explicit transgression. That is, a distribution form

$$\alpha + \delta(x) dx \wedge \beta, \tag{L21.17}$$

where α and β are smooth forms, respectively up to and on the boundary, is closed *as a supported differential form* (dual to smooth sections) if and only if α is closed (and so defines an absolute cohomology class on X) and also

$$\iota^*_{\partial X}\alpha = d\beta \text{ on } \partial X. \tag{L21.18}$$

Note that this formula can be compared to the formula for last time for the Chern character in the Toeplitz case.

L21.2. Bott element.

21+. Addenda to Lecture 21