CHAPTER 2

Conormal distributions at the origin

Lecture 2: 13 September, 2005

L2.1. Classical symbols. As indicated above, I will define the conormal distributions at the origin of a vector space, starting with $\mathbb{R}^n$, as the inverse Fourier transform of the spaces $\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$. Here $\rho \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$ is any boundary defining function. For a compact manifold with boundary X, a boundary defining function $\rho \in \mathcal{C}^\infty(X)$ is any function such that $\rho \geq 0$ everywhere, $\partial X = \{\rho = 0\}$ and $d\rho \neq 0$ on ∂X. Such a function always exists and any two are related by

$$\rho' = a\rho,\ 0 < a \in \mathcal{C}^\infty(X). \tag{L2.1}$$

For the radial compactification we know we can take as boundary defining function

$$Z_0 = \rho = \frac{1}{(1+|\xi|^2)^{\frac{1}{2}}} \tag{L2.2}$$

for any metric.

Then the space $\rho^{-m}\mathcal{C}^\infty(\overline{W})$ for any real vector space W is defined by

$$u \in \rho^{-m}\mathcal{C}^\infty(\overline{W}) \Longleftrightarrow \rho^m u \in \mathcal{C}^\infty(\overline{W}). \tag{L2.3}$$

Traditionally this is called the space of 'classical symbols or order m on W' and denoted $S^m_{\mathrm{cl}}(W)$. I will not use this notation (at least not much) because it is redundant and also there is some confusion in the literature between closely related, but different, spaces.

Now, for a little exercise in abstract nonsense, note that $\rho^{-m}\mathcal{C}^\infty(\overline{W})$ is the space of all global sections of a trivial line bundle, which we denote N_{-m}, so

$$\rho^{-m}\mathcal{C}^\infty(\overline{W}) = \mathcal{C}^\infty(\overline{W}; N_{-m}). \tag{L2.4}$$

Indeed, this is a direct consequence of the relation (L2.1) between any two defining functions. Thus, ρ^{-m} is a global section of this bundle for any boundary defining function ρ. If you want to be pedantic, the fibre at any point $q \in \overline{W}$ (including of course boundary points) may be defined to be

$$(N_{-m})_p = \rho^{-m}\mathcal{C}^\infty(\overline{W})/\mathcal{I}_p \cdot \rho^{-m}\mathcal{C}^\infty(\overline{W}) \tag{L2.5}$$

where $\mathcal{I}_p \subset \mathcal{C}^\infty(\overline{W})$ is the space of smooth functions which vanish at p. It is handy to have the notation (L2.4), and it is little more than notation, since it lets us push the 'weight' function ρ^{-m} into a bundle and 'hide' it.

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There are some rather obvious properties of these symbol spaces. Namely they multiply

$$\mathcal{C}^\infty(\overline{W};N_{-m})\cdot\mathcal{C}^\infty(\overline{W};N_{-m'})=\mathcal{C}^\infty(\overline{W};N_{-m-m'}),\ \forall\ m,m'\in\mathbb{R}. \tag{L2.6}$$

In particular they are all $\mathcal{C}^\infty(\overline{W})$-modules, corresponding to the case $m=0$ when N_{-m} is canonically trivial.

We also know that the action of $\mathrm{GL}(W)$ on W, and of W acting by translations, extends smoothly to $\overline{W}$ and necessarily maps the boundary onto itself. It follows that these actions extend to $\mathcal{C}^\infty(\overline{W})$, so G^* acts on $\mathcal{C}^\infty(\overline{W};N_{-m})$ for any m.

The name 'symbols' is related to the 'symbol estimates' that these functions satisfy. In the case of $\mathbb{R}^n$ we know that $(1+|\xi|^2)^{-\frac12}$ is a boundary defining function on $\overline{\mathbb{R}^n}$. Thus if $a\in\mathcal{C}^\infty(\overline{\mathbb{R}^n};N_{-m})$ then $\rho^m a$ is a bounded function and this reduces to

$$|a(\xi)|\le C(1+|\xi|^2)^{\frac{m}{2}}\Longleftrightarrow|a(\xi)|\le C'(1+|\xi|)^m\ \forall\ \xi\in\mathbb{R}^n. \tag{L2.7}$$

The second, simpler looking, form follows from the fact that $(1+|\xi|^2)^{\frac12}$ and $1+|\xi|$ are of the 'same size', meaning each is bounded above and below by some positive multiple of the other. The disadvantage of $1+|\xi|$ is that it is singular at the origin, but it is easier to write. Anyway we also know that $\xi^\alpha\partial_\xi^\beta a\in\rho^{-m+|\alpha|-|\beta|}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ and hence

$$|\xi^\alpha\partial_\xi^\beta a|\le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|+|\alpha|}\text{ if }a\in\mathcal{C}^\infty(\overline{\mathbb{R}^n};N_{-m}). \tag{L2.8}$$

This is an explicit form of the statement that differentiation by ξ lowers the order by 1 and multiplication by a polynomial raises the order by the order of the polynomial, i.e.

$$\begin{aligned}\partial_{\xi_i}:\mathcal{C}^\infty(\overline{\mathbb{R}^n};N_{-m})&\longrightarrow\mathcal{C}^\infty(\overline{\mathbb{R}^n};N_{-m+1})\\ \xi_i\times:\mathcal{C}^\infty(\overline{\mathbb{R}^n};N_{-m})&\longrightarrow\mathcal{C}^\infty(\overline{\mathbb{R}^n};N_{-m-1}).\end{aligned} \tag{L2.9}$$

The symbol estimates (L2.8), even if valid for all α and β, do not imply that $a\in\mathcal{C}^\infty(\overline{\mathbb{R}^n};N_{-m})$. Some discussion of the extent to which they are weaker and why they usually appear centrally in a treatment of microlocal analysis can be found in the addenda below. The present treatment avoids the use of these larger spaces of symbols 'with bounds', although they still have their place.

One thing that does follow easily from (L2.8) is that symbols of arbitrarily low order are Schwartz functions

Lemma 4. *On any real vector space and for any $m\in\mathbb{R}$,*

$$\bigcap_{N\in\mathbb{N}}\rho^{-m+N}\mathcal{C}^\infty(\overline{W})=\mathcal{S}(W). \tag{L2.10}$$

Proof. Since we can always replce m in (L2.8) by $m-N$ it follows that if a is in the intersection in (L2.10) then

$$\sup_\xi|\xi^\alpha\partial_\xi^\beta a|<\infty\ \forall\ \alpha,\beta\Longrightarrow a\in\mathcal{S}(W). \tag{L2.11}$$

The converse statement also follows, namely it suffices to show that $\mathcal{S}(W)\subset\mathcal{C}^\infty(\overline{W})$. □

Returning to the general properties of the classical symbol spaces, there is a short exact sequence which will turn out to be of fundamental importance later. Namely, for any m

$$0 \longrightarrow \mathcal{C}^\infty(\overline{W}; N_{-m+1}) \longrightarrow \mathcal{C}^\infty(\overline{W}; N_{-m}) \longrightarrow \mathcal{C}^\infty(\mathbb{S}W; N_{-m}) \longrightarrow 0. \tag{L2.12}$$

In future I will often leave out the zeros at the ends of such short exact sequences. The claim of exactness is just that the second map is injective, the third is surjective and the range of the second is exactly the null space of the third. If we use ρ^{-m} to trivialize the bundle N_{-m} then this just reduces to the short exactness[1] of

$$\rho\mathcal{C}^\infty(\overline{W}) \longrightarrow \mathcal{C}^\infty(\overline{W}) \longrightarrow \mathcal{C}^\infty(\mathbb{S}W). \tag{L2.13}$$

This in turn means the restriction map to the bounding sphere is surjective and that a smooth function is of the form ρf for another smooth f if and only if it vanishes at the boundary; this is a form of Taylor's theorem.

There is an equally important but more complicated version of this called 'asymptotic completeness' of the spaces or 'asymototic summability' of series of symbols.

Proposition 5. *[Asymptotic Completeness] If $a_k \in \mathcal{C}^\infty(\overline{W}; N_{-m+k})$, is* any *sequence then there exists an element $a \in \mathcal{C}^\infty(\overline{W}; N_{-m})$ such that*

$$a - \sum_{k=0}^{N} a_k \in \mathcal{C}^\infty(\overline{W}; N_{-m+N+1}) \ \forall\ N \in \mathbb{N}. \tag{L2.14}$$

Proof. We can multiply everything by ρ^m to reduce to the case $m = 0$. Then it is a form of Borel's Lemma. Namely it follows from the fact that for any (compact) manifold with boundary X and any sequence $b_k \in \rho^k\mathcal{C}^\infty(X)$, $k \in \mathbb{N}_0$, there exists an element $b \in \mathcal{C}^\infty(X)$ such that

$$b - \sum_{k=0}^{N} b_k \in \rho^{N+1}\mathcal{C}^\infty(X) \ \forall\ N \in \mathbb{N}. \tag{L2.15}$$

This in turn can be reduced to the corresponding local statement for a hypersurface $z_1 = 0$ in $\mathbb{R}^n$ and then to the 1-dimensional case, with smoothness in parameters – this is the setting for the original Lemma of É. Borel. Namely that the sequence of derivatives of a smooth function at a fixed point is unconstrained, i.e. if c_k is any sequence of complex numbers then there exists a smooth function $u \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\frac{d^k u}{dx^k}(0) = c_k \ \forall\ k. \tag{L2.16}$$

Let me at least remind you of how this is proved – an extension of this argument leads to (L2.15). Namely one forces the Taylor series to converge, of course without constraints on the c_k's in (L2.16) it will not converge of its own volition! So, choose a cut-off function $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ which is 1 in $|x| < \frac{1}{2}$ and vanishes in $|x| > 1$. Then consider the series of smooth functions

$$b(x) = \sum_k \frac{c_k x^k}{k!}\chi(\frac{x}{\epsilon_k}), \tag{L2.17}$$

[1] Meaning supply your own zeroes at the ends.

where $\epsilon_k > 0$ is a sequence which tends to 0. It follows that the series is finite in any region $x \geq x_0 > 0$ so converges to a smooth function in $x > 0$. In fact it is easy to make it converge uniformily in $x \geq 0$. Indeed, the size of the kth term is

$$|c_k|\epsilon_k^k/k! \tag{L2.18}$$

since the cutoff vanishes when $x > \epsilon_k$. Now the ϵ_k just need to be chosen to vanish rapidly enough and the series will converge uniformly and absolutely. A similar choice allows the series for the derivatives of any order to be made to converge and then a diagonalization arument gives convergence in $\mathcal{C}^\infty(\mathbb{R})$. It follows that the sum satisfies (L2.16). □

The relationship (L2.14) is usually written

$$a \sim \sum_{k=0}^{\infty} a_k. \tag{L2.19}$$

Notice that $a \in \mathcal{C}^\infty(\overline{W}; N_{-m})$ is not uniquely determined by this condition. Any other element a' satisfying (L2.14) in place of a is such that $a'-a \in \mathcal{C}^\infty(\overline{W}; N_{-m+N})$ for all N which is to say $a' - a \in \mathcal{S}(W)$, so the 'asymptotic sum' is determined up to a rapidly decreasing 'error.'

L2.2. Classical conormal distibutions. Now, we are finally in a position to define the 'classical conormal distributions' on $\mathbb{R}^n$ with respect to the origin

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) = \mathcal{F}^{-1}\left(\rho^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^n})\right),\ m = m' + \frac{n}{4}. \tag{L2.20}$$

As promised these are just the inverse Fourier transforms of our symbol spaces. Notice however that I have shifted the 'order' on the left by a constant that depends on the dimension only. This 'normalization' is for reasons related to the 'principle of stationary phase' that will not show up for quite a long time, but leaving it out will cause more confusion than putting it in.

The simplest nontrivial example of a conormal distribution with respect to the origin of $\mathbb{R}^n$ is the Dirac delta 'function', the inverse Fourier transform of the constant function 1. According to (L2.20) it has 'order $n/4$' (however this is just a choice of normalization and doesn't correspond to a meaningful regularity statement)

$$\delta_0 \in I^{\frac{n}{4}}_{\mathcal{S}}(\mathbb{R}^n, \{0\}). \tag{L2.21}$$

However this is almost enough to allow one to remember the normalization (which I have a hard time doing)!

So, what are the basic properties. Certainly (L2.20) defines a space of tempered distributions

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \subset \mathcal{S}'(\mathbb{R}^n). \tag{L2.22}$$

Since $\rho\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \subset \mathcal{C}^\infty(\overline{\mathbb{R}^n})$ it follows that

$$I^{m-k}_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \subset I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \text{ if } k \in \mathbb{N}. \tag{L2.23}$$

Now the estimate (L2.7) shows that for the Fourier transform

$$\text{if } u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \text{ then } a = \mathcal{F}(u) \in \rho^{-m+\frac{n}{4}}\mathcal{C}^\infty(\overline{\mathbb{R}^n}),$$
$$\text{so } -m + \frac{n}{4} > n \Longrightarrow a \in L^1(\mathbb{R}^n)$$

and then the inverse Fourier transform

$$u(z) = (2\pi)^{-1} \int_{\mathbb{R}^n} a(\zeta) d\zeta, \ |u(z)| \le (2\pi)^{-n} \int_{\mathbb{R}^n} |a(\zeta)| d\zeta$$

is bounded. It is in fact also continuous (by the continuity-in-the-mean of L^1 functions) and vanishes at infinity so

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \subset \mathcal{C}^0_0(\mathbb{R}^n) \text{ if } m < -\frac{3n}{4}, \tag{L2.24}$$

the space of continuous functions which vanish at infinity.

Since the right hand side in (L2.20) is a space of smooth functions with some growth it is reasonable to expect the elements of the space on the left to be smooth with some localized singularities. That is indeed the case and we will show that

$$I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\})\big|_{\mathbb{R}^n \setminus \{0\}} \subset \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}), \tag{L2.25}$$

so the only singularities in an element of $I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\})$ are at the origin, i.e. $\operatorname{sing\,supp}(u) \subset \{0\}$. We will actually prove something even stronger.

Lemma 5. *If $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ is equal to 1 in a neighbourhood of the origin then*

$$u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \Longrightarrow (1-\chi)u \in \mathcal{S}(\mathbb{R}^n). \tag{L2.26}$$

Proof. The Fourier transform has the property that

$$\mathcal{F}(z^\alpha D^\beta_z u) = (-D_\zeta)^\alpha (\zeta^\beta \mathcal{F}(u)) \tag{L2.27}$$

where $D_{z_k} = \frac{1}{i}\partial_{z_k}$ takes care of the factors of i. Recalling (L2.9) for the symbol spaces (and of course (L2.20)) we see that

$$z^\alpha D^\beta_z : I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \longrightarrow I^{m+|\beta|-|\alpha|}_{\mathcal{S}}(\mathbb{R}^n, \{0\}), \tag{L2.28}$$

just the opposite of the symbols spaces, so that differentiation raises the order but multiplication by a monomial lowers the order by the degree. Combining this with (L2.24) we conclude that

$$u \in I^m_{\mathcal{S}}(\mathbb{R}^n, \{0\}) \Longrightarrow z^\alpha D^\beta_z u \in \mathcal{C}^0_0(\mathbb{R}^n) \text{ if } |\alpha| > m + |\beta| + \frac{3n}{4}. \tag{L2.29}$$

So, adding a large number of terms we see that

$$|z|^{2N} u \in \mathcal{C}^p_0(\mathbb{R}^n) \text{ is bounded with its first } p \text{ derivatives if } 2N > m + p + \frac{3n}{4}. \tag{L2.30}$$

Now, multiplying by the cutoff $(1-\chi)$ the same is true of $(1-\chi)u$, However, $|x|^{2N}$ then does not vanish on the support, so we conclude that

$$|D^\beta_z ((1-\chi)u)| \le C_{N,p}(1+|z|)^{-2N}, \ 2N > m + p + \frac{3n}{4}, \ |\beta| \le p. \tag{L2.31}$$

Since m is fixed, we can simply take N very large and hence conclude that $(1-\chi)u \in \mathcal{S}(\mathbb{R}^n)$ which was the claim. □

This is the reason for the suffix $\mathcal{S}$ in the definition (L2.20); these distributions are rapidly decaying at infinity with all derivatives, it is just that they may be singular in a very specific way at the origin.

Next time I will talk more about invariance, showing that for a vector space there is an invariant version of the Fourier transform giving an isomorphism

$$\mathcal{F} : \mathcal{S}(W) \longrightarrow \mathcal{S}(W'; \Omega W') \tag{L2.32}$$

onto the Schwartz space of densities. In any case it is pretty clear that $I^m(\mathbb{R}^n, \{0\})$ is invariant under the action of $\mathrm{GL}(n,\mathbb{R})$ since

$$\mathcal{F}(G^*u) = ((G^{-1})^t)^*\mathcal{F}u \cdot |\det G|^{-1},\ G \in \mathrm{GL}(n,\mathbb{R}). \tag{L2.33}$$

We will eventually need more invariance than this, namely that the nature of the singularity at the origin is the same in any coordinates based at the origin.

2+. Addenda to Lecture 2

2+.1. Borel's lemma. Let me go a little further with the proof of Borel's lemma. As noted above, the series (L2.17) converges uniformly, with all derivatives, on compact subsets of in $|x| > 0$ if we simply require $\epsilon_k \to 0$. The estimates (L2.18) can be extended to the derivatives. Namely for any $j \geq k$ (only to avoid complications with indices)

$$D_x^j\left(\frac{c_k x^k}{k!}\chi(\frac{x}{\epsilon_k})\right) = \sum_{p=0}^{j}\binom{j}{p}\frac{c_k x^{k-j+p}}{(k-j+p)!}\epsilon_k^{-p}\chi^{(p)}(\frac{x}{\epsilon_k}) \Longrightarrow \tag{2+.34}$$
$$|D_x^j\left(\frac{c_k x^k}{k!}\chi(\frac{x}{\epsilon_k})\right)| \leq C_{k,j}\epsilon_k^{k-j}$$

where $C_{k,j}$ is a constant that does not depend on ϵ_k. It follows that if we choose

$$\epsilon_k < 2^{-k}/(1 + C_{k,j})\ \forall\ k > j,\ \forall\ j \tag{2+.35}$$

then the series of jth derivatives converges absolutely and uniformly for all x. The important point here is that making (2+.35) hold for all j represents only a finite number of conditions on each ϵ_k, namely there are conditions only for $0 \leq j \leq k$. Thus choosing each ϵ_k to be small enough the series (L2.17) converges uniformly, will all its derivatives. The sum is therefore a smooth function and it satisfies (L2.16).

A similar argument applies in more variables. If $u_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is a sequence with each element supported in a fixed compact set K then choosing $\epsilon_k > 0$ small enough ensures that

$$u(x,y) = \sum_k \frac{u_k(y)x^k}{k!}\chi(\frac{x}{\epsilon_k}) \in \mathcal{C}_c^\infty(\mathbb{R}^n) \tag{2+.36}$$

converges absolutely and uniformly with all its partial derivatives and satisfies

$$\partial_x^k u(0,y) = u_k(y)\ \forall\ k. \tag{2+.37}$$

Indeed, we simply have to arrange that all the differentiated series, with both x and y derivatives, converge absolutely and uniformly. The x derivatives behave exactly as before and the y derivatives fall on the u_k only. Thus we can arrange that the series for $\partial_x^k\partial_y^\alpha u$ converges by choosing

$$\epsilon_k < \epsilon_{k,j,\alpha}\ \forall\ k > j + |\alpha|. \tag{2+.38}$$

Here $\epsilon_{k,j,\alpha}$ is the same constant as in (2+.35) except that the $|c_k|$'s leading to the bound are replaced by the supremums of the $\partial_y^\alpha u_k$. Again the important point is that the convergence of each of the series is determined by what happens from some (any) finite point onwards. Thus we only need impose the bound on ϵ_k for $k > j + |\alpha|$ as in (2+.38). So again this is only a finite number of conditions on each ϵ_k but implies the uniform convergence of the series for all partial derivatives, so (2+.37) follows.

The general case now follows by use of a partition of unity to reduce the problem to a finite number applications of the construction above on $\mathbb{R}^n$.

2+.2. Symbols with bounds. As remarked above, the 'symbol estimates' (L2.8) do not imply that $a \in \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$. To understand a little better what they do mean, first observe that the case $m = 0$ is fundamental since

$$a \text{ satifies (L2.8)} \iff (1+|\xi|^2)^{-m/2}a \text{ satifies (L2.8) with } m = 0. \tag{2+.39}$$

In fact the estimates with $\alpha \neq 0$ in (L2.8) are redundant, since they follow from those with $\alpha = 0$. It is also possible to reorganize these estimates as follows.

Exercise 11. Show (probably using induction) that the estimates (L2.8) for $m = 0$ are equivalent to the statements

$$\left(\prod_{j=1}^{N} V_{k_j l_j}\right) a \in L^\infty(\mathbb{R}^n),\ V_{kl} = \xi_k \partial_{\xi_l} \tag{2+.40}$$

for all N and all integer sequences k_j, l_j (including implicitly the case of no factors at all).

The operators V_{kl} are the linear vector fields on $\mathbb{R}^n$ and we know from § L1.2 that these lift to $\overline{\mathbb{R}^n}$ to span, near infinity, all vector fields tangent to the boundary.

Definition 2. On any compact manifold with boundary X let $\mathcal{V}_\mathrm{b}(X)$ denote the Lie algebra of all those smooth vector fields on X which are tangent to the boundary and define

$$\mathcal{A}(X) = \{a \in \mathcal{C}^\infty(\operatorname{int} X) \cap L^\infty(X); V_1 \cdots V_N a \in L^\infty(X),\ \forall\ V_i \in \mathcal{V}_\mathrm{b}(X),\ \forall\ N\}\,. \tag{2+.41}$$

Using the discussion of compactification last time, try your hand at a proof of

Proposition 6. *The symbol estimates* (L2.8) *are equivalent to requiring* $a \in \rho^{-m}\mathcal{A}(\overline{\mathbb{R}^n})$.

2+.3. Density and approximation. It is quite usual to replace the classical spaces by the larger spaces (with weaker topology) introduced above

$$\rho^{-m}\mathcal{C}^\infty(\overline{W}) \subset \rho^{-m}\mathcal{A}(\overline{W}). \tag{2+.42}$$

One reason for this is that it allows density arguments to be used.

Lemma 6. *For any* $a \in \rho^{-m}\mathcal{C}^\infty(\overline{W})$ *there exists a sequence* $a_k \in \mathcal{S}(W)$ *such that*

$$\begin{gathered} a_k \text{ is bounded in } \rho^{-m}\mathcal{A}(\overline{W}) \text{ and} \\ a_k \longrightarrow a \text{ in the topology of } \rho^{-m'}\mathcal{A}(\overline{W})\ \forall\ m' > m. \end{gathered} \tag{2+.43}$$

Proof. In fact we can take the sequence to be in $\mathcal{C}^\infty_c(W) \subset \mathcal{S}(W)$. Namely, if $\rho \in \mathcal{C}^\infty(\overline{W})$ is a defining function for 'infinity' and $\phi \in \mathcal{C}^\infty_c(0,\infty)$ has $\rho(x) = 1$ in $x > 1$ then

$$a_k = \phi(k\rho)a \in \mathcal{C}^\infty_c(W) \tag{2+.44}$$

has the desired properties. Indeed the result is equivalent to the special case $m = 0$ applied to $\rho^m a$. Thus we may assume that $a \in \mathcal{C}^\infty(\overline{W})$ in which case it follows directly from the definition, (2+.44), that a_k is bounded in $L^\infty(W)$ and that for any $\epsilon > 0$, $\rho^\epsilon a_k \longrightarrow \rho^\epsilon a$ in $L^\infty(\overline{W})$. These are the first estimates corresponding to (2+.43), which is the same statement after applying any number of smooth vector fields V_i tangent to the boundary of $\overline{W}$. Thus it is enough to check that for such vector fields and any $\epsilon > 0$,

(2+.45)
$$V_1 \dots V_N a_k \text{ is bounded in } L^\infty(W) \text{ and } \rho^\epsilon V_1 \dots V_N a_k \longrightarrow \rho^\epsilon a \text{ in } L^\infty(W).$$

This in turn follows by observing the boundedness of all the terms arising from differentiating the cut-off $\phi(k\rho)$ and the fact that they are supported arbitrarily close to the boundary (so when multiplied by ρ^ϵ each of them tends to zero). □

Note that you cannot do much better than this, namely $\mathcal{S}(W)$ is certainly not dense in $\rho^{-m}\mathcal{C}^\infty(\overline{W})$ in our 'classical symbol topology' (just the topology of $\mathcal{C}^\infty(\overline{W})$ on $\rho^m a$) – in fact it is a closed subspace!

2+.4. Asymptotic summation. If one wishes to use these larger symbol spaces, $\rho^{-m}\mathcal{A}(\overline{W})$ (which by the way would normally be denoted $S^m_{1,0}(W)$, with the $1,0$ suffix being a special case of a more general ρ, δ notation) then one needs to check various properties of it. Essentially by definition $\xi^\alpha \partial^\beta_\xi$ maps $\rho^{-m}\mathcal{A}(\overline{\mathbb{R}^n})$ into $\rho^{-(m+|\alpha|-|\beta|)}$ with $|\alpha| = \alpha_1 + \dots + \alpha_n$. Slightly more serious is the analogue of Borel's lemma, which is

Proposition 7. *[Asymptotic summability]. If $a_j \in \rho^{-m_j}\mathcal{A}(\overline{W})$ is a sequence with $m_j \to -\infty$ then there exists $a \in \rho^{-M}\mathcal{A}(\overline{W})$ where $M = \max_j m_j$ such that*

$$a - \sum_{j \le N} \in \rho^{-M(N)}\mathcal{A}(\overline{W}),\ M(N) = \max_{j > N} m_j,\ \forall\ N. \tag{2+.46}$$

Sketch only. The same method as for Borel's lemma, based on (2+.36), works. □

2+.5. Homogeneity and conormality. It is natural to ask exactly what these conormal distributions, both 'classical' and corresponding to symbols with bounds, are like. In the classical case it is possible to see quite explicitly the local behaviour of the singularity at the origin.

Lemma 7. *If $a \in \rho^{-m}(\overline{\mathbb{R}^n})$ with $m \notin \mathbb{Z}$ then there exists a sequence of functions $u_k \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $k \in \mathbb{N}_0$, such that the inverse Fourier transform*

(2+.47)
$$u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}a(\xi)d\xi \in \mathcal{S}'(\mathbb{R}^n) \text{ satisfies}$$
$$u - \sum_{0 \le k \le N} |x|^{-m-n+k}u_k(\frac{x}{|x|}) = U_N\big|_{x\ne 0},\ U_N \in \mathcal{C}^{N-n-[m]}(\mathbb{R}^n),\ N > n + [m]$$

where $[m]$ is the integral part of m.

In fact the coefficients u_k in the expansion (2+.47) are completely determined by a (remember that m is not an integer here) and conversely they completely determine the singularity of a in the sense that two classical symbols a and a' giving the same expansions differ by an element of $\mathcal{S}(\mathbb{R}^n)$. There is in fact no mystery about the u_k, they can be computed by formally substituting the Taylor series expansion of a at infinity, so

$$(2+.48)\quad a - \sum_{0\le k<N} |\xi|^{m-k} a_k(\frac{\xi}{|\xi|})| \le C|\xi|^{m-N} \text{ in } |\xi|>1 \Longrightarrow$$
$$u_k(\mu) = (2\pi)^{-n}\gamma_{m-k}\int_{\mathbb{S}^{n-1}} e^{i\mu\cdot\omega} a_k(\omega)d\omega,\ \mu\in\mathbb{S}^{n-1}$$

for certain constants γ_{m-k} which I leave you to evaluate.

In the case of integral m the result is almost the same, but a little more complicated. The expansion of a, in (2+.48) is always the same. However the expansion of u depends a little on how big the integer m is. If $m\le -n$, so $m=-n-p$ for some non-negative integer p, then we need to replace (2+.47) by

$$(2+.49)\quad u - p_N(x)\log|x| - \sum_{0\le k\le N} |x|^{-m-n+k} v_{-m-n+k}(\frac{x}{|x|}) = U_N\big|_{x\neq 0},$$
$$U_N \in \mathcal{C}^{N-n-[m]}(\mathbb{R}^n),\ N>n+[m]$$

where p_N is a fixed formal power series starting with terms of homogeneity at least $-m-n$ in x truncated at level N,

$$(2+.50)\qquad p_N(x) = \sum_{-m-n\le|\alpha|<N} p_\alpha x^\alpha$$

where the p_α are constants independent of N, and the u_k are smooth functions on the sphere which satisfy the constraints

$$(2+.51)\qquad \int_{\mathbb{S}^{n-1}} v_{-q}(\omega)\omega^\alpha d\omega = 0,\ |\alpha|\le q.$$

All such functions occur in, and are determined by, these expansions and again the singularity of u is determined by then expansion. The normalization (2+.51) means that there are no polynomials in the expansion in (2+.49), which is naturel since these do not correspond to singularities at the origin for u. The corresponding singular terms occur with the logarithmic coefficient.

When $-n<m<0$ the expansion is the same, except there are additional terms of homogeneity between 0 and $-m-n$ which are subject to no constraints. When m is a non-negative integer the are terms which do not appear in the expansion (which is in $x\neq 0$ where u is smooth) but correspond to the delta functions at the origin. Thus, the expansion of a has a unique polynomial part with inverse Fourier transform a sum of derivatives of the delta function. So one can consider $a\in\rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ without polynomial part. Then there is an expansion just as in (2+.49) except that the terms now of non-negative integral homogeneity must satisfy the same integral constraints as in (2+.51).

One way to make the relationship between homogeneity and conormality explicit is to check

Lemma 8. *Any distribution on $\mathbb{R}^n$ which is smooth outside the origin and 'homogeneous modulo $\mathcal{C}^\infty$' of some degree, i.e.*

$$(2+.52)\qquad u(tx) = t^h u(x) + F(t,x),\ t>0, x\in\mathbb{R}^n \text{ with } F\in\mathcal{C}^\infty((0,\infty)\times\mathbb{R}^n)$$

is equal to a (classical) conormal distribution in a neighbourhood of 0. *Conversely, finite sums $\psi_k u_k + \psi$ with the u_k of this form and 'homogeneous' of degree $m-k$*

with the ψ_k, $\psi \in \mathcal{S}(\mathbb{R}^n)$ *are dense in the space of classical conormal distributions of order* $-m + \frac{3n}{4}$.

2+.6. Blow up of the origin. The operation of 'blowing up a submanifold' is in many senses dual to the process of compactification discussed last time. For one thing it is related to maps *into* the space in question, rather than maps from the space into a compactification. Thus for a vector space W, the space '$[W, \{0\}]$', which is 'W blown up at the origin' is associated to a map, namely polar coordinates

$$\beta : [0, \infty) \times \mathbb{S}^{n-1} \ni (r, \omega) \longmapsto r\omega \in \mathbb{R}^n. \tag{2+.53}$$

Here we can think of the sphere as the usual 'Euclidean sphere of radius 1'

$$\mathbb{S}^{n-1} = \{z \in \mathbb{R}^n; |z| = 1\}. \tag{2+.54}$$

At any point of $\mathbb{S}^{n-1}$ there are 'projective coordinates'. Namely at each point there can be at most one component z_j with $z_j^2 = 1$ and any $n - 1$ components, not including one with $z_j^2 = 1$, give local coordinates. This is just the implicit function theorem since

$$\sum_j z_j dz_j = 0 \tag{2+.55}$$

is the only constraint on the differentials, so any $n - 1$ of them are independent unless they include a dz_j with $z_i = 0$ for all $i \neq j$ (which means $z_j^2 = 1$ and $dz_j = 0$ on $\mathbb{S}^{n-1}$).

Thus the smoothness of (2+.53) follows from the smoothness of the components, as functions on $\mathbb{S}^{n-1}$. It is surjective, since $0 \in \mathbb{R}^n$ is the image of $\{0\} \times \mathbb{S}^{n-1}$ and any other point $0 \neq z \in \mathbb{R}^n$ is the image of $(|z|, z/|z|)$. In fact this shows that β is a diffeomorphism of $(0, \infty) \times \mathbb{S}^{n-1}$ onto $\mathbb{R}^n \setminus \{0\}$, with the inverse being $r = |z|$, $\omega = z/|z|$.

Now, the standard action of the orthogonal group on the sphere, which is induced from the action on $\mathbb{R}^n$, commutes with β

$$\beta(r, O\omega) = O\beta(r, \omega) \ \forall \ r \in [0, \infty), \ \omega \in \mathbb{S}^{n-1}.$$

Just as for the map defining radial compactification, it is important to know that a general element of $\mathrm{GL}(n, \mathbb{R})$ lifts under β.

Lemma 9. *There is a smooth action of* $\mathrm{GL}(n, \mathbb{R})$ *on* $[0, \infty) \times \mathbb{S}^{n-1}$ *which is intertwined with the standard action on* $\mathbb{R}^n$ *by* β :

$$\begin{tikzcd} {[0,\infty) \times \mathbb{S}^{n-1}} \arrow[r, "\beta"] \arrow[d, "\tilde{A}"'] & \mathbb{R}^n \arrow[d, "A"] \\ {[0,\infty) \times \mathbb{S}^{n-1}} \arrow[r, "\beta"'] & \mathbb{R}^n. \end{tikzcd} \tag{2+.56}$$

Proof. See the discussion in the case of radial compactification. The Lie algebra of $\mathrm{GL}(N, \mathbb{R})$ consists of the linear vector fields $z_i\partial_j$. Each of these is homogeneous of degree 0 under the homotheity $z \longmapsto sz$, $s \in (0, \infty)$. Since β is a diffeomorphism, there is a unique smooth vector field V_{ij} on $(0, \infty) \times \mathbb{S}^{n-1}$ such that $\beta_*(V_{ij}) = z_i\partial_j$ at each point. Thus $V_{ij} = a(r, \omega)\partial_r + V'_{ij}(r)$ where $V'_{ij}(r)$ is a smooth vector field on the sphere, depending smoothly on $r \in (0, r)$. By the homogeneity $a(r, \omega) = ra(1, \omega)$ and V'_{ij} is independent of r. Thus

$$V_{ij} = a(\omega)r\partial_r + V'_{ij}(\omega) \tag{2+.57}$$

extends to be smooth down to $r = 0$ (and tangent to $r = 0$). As in the case of compactification this shows that any element $A \in \mathrm{GL}(n, \mathbb{R})$ lifts to a smooth diffeomorphism $\tilde{A}$ of $[0, \infty) \times \mathbb{S}^{n-1}$ and in fact gives a smooth action

$$\mathrm{GL}(n, \mathbb{R}) \times [0, \infty) \times \mathbb{S}^{n-1} \longrightarrow [0, \infty) \times \mathbb{S}^{n-1}. \tag{2+.58}$$

□

Continuing to follow the discussion of the radial compactification, this shows that we may define $[W, \{0\}]$ as a manifold associated to the principal $\mathrm{GL}(n, \mathbb{R})$-space, $P(W)$, of bases of W. Thus $\mathrm{GL}(n, \mathbb{R})$ acts on $P(W)$ by replacing a basis by the corresponding linear combination of its elements and $\mathrm{GL}(W)$ acts on it by acting on the elements of the basis. From this abstract point of view we may set

$$[W, \{0\}] = \big(P(W) \times [0, \infty) \times \mathbb{S}^{n-1}\big) \,/\, \mathrm{GL}(n, \mathbb{R}). \tag{2+.59}$$

Exercise 12. Show that $[W, \{0\}]$ is a manifold with boundary, diffeomorphic to $[0, \infty) \times \mathbb{S}^{n-1}$, that $\mathrm{GL}(W)$ acts smoothly on it and that there is a smooth map (the blow-down map)

$$\beta : [W, \{0\}] \longrightarrow W \tag{2+.60}$$

which intertwines the actions, maps the boundary to $\{0\}$ and is a diffeomorphism of the interior to $W \setminus \{0\}$.

As with conormal distributions, I will show later how to extend this notion to blowing up an embedded submanifold of a given manifold, by passing through the special case of blowing up the zero section of a vector bundle. It is also convenient to have a concrete realization of the blown-up space.

Exercise 13. Define the sphere of W to be

$$\mathbb{S}W = (W \setminus \{0\})/\mathbb{R}^+ \tag{2+.61}$$

where $\mathbb{R}^+$ acts by multiplication. Show that $\mathbb{S}W$ is a smooth compact manifold diffeomorphic to $\mathbb{S}^{n-1}$, $n = \dim W$, that there is is a natural diffeomorphism $\mathbb{S}W \longrightarrow \partial[W, \{0\}]$ and a unique $\mathcal{C}^\infty$ structure on the disjoint union so that

$$[W, \{0\}] = (W \setminus \{0\}) \sqcup \mathbb{S}W. \tag{2+.62}$$