CHAPTER 17

Product-type conormal distributions

Lecture 17: 15 November, 2005

The aim today is to complete the definition of the spaces of product-type conormal distributions, $I^{m,m'}(X,Y,Z;E)$ where $Z\subset Y\subset X$ are embedded compact submanifolds (of positive codimension!) As for the case where Z is absent, we shall find symbol maps which capture the leading singularity. Here there are two symbol maps, one corresponding to singularities on Z associated, perhaps unfortunately, with the first order m and the second one associated with the second order m' and the singularity on Y (which does include Z but the singularities captured by this second symbol are only those 'conormal to Y).'

The idea, as for $I^m(X,Y;E)$, is to reduce to the case of the normal bundle to Y. Here, however it is useful to discuss first the normal bundle to Z in X and how it is related to Y.

The most significant difference between the old spaces $I^m(X,Y;E)$ and the new $I^{m,m'}(X,Y,Z;E)$ is that the symbol map for Y itself takes values in conormal distributions. Recall that the old symbol map was

$$\sigma_m : I^{m'}(X,Y;E) \longrightarrow \mathcal{C}^\infty(SN^*Y;E_Y\otimes N_{-m'}) \tag{L17.1}$$

where SN^*Y is the sphere bundle (thought of as the compactifying surface at infinity for $\overline{N^*Y}$) of the conormal bundle to Y in X. In the present case this has a submanifold corresponding to Z, namely

$$SN^*_ZY=\bigcup_{z\in Z}SN^*_zY\subset SN^*Z \tag{L17.2}$$

just the union of the fibres over Z, i.e. the restriction of the bundle to Z; as indicated in (L17.2) this is a subbundle of the conormal bundle to Z itself. Then our modified 'Y-symbol' is to be part of a short exact sequence

(L17.3)
$$I^{m,m'-1}(X,Y,Z;E)\hookrightarrow I^{m,m'}(X,Y,Z;E)\xrightarrow{\sigma_Y} I^m(SN^*Y,SN^*_ZY;E_Y\otimes N_{-m'}),$$
$$\sigma_Y=\sigma_{Y,m,m'}.$$

There should be a picture here.

The other symbol, the Z-symbol, is more like the previous one

$$\sigma_m : I^m(X,Z;E)\longrightarrow \mathcal{C}^\infty(SN^*Z;E_Z\otimes N_{-m}). \tag{L17.4}$$

The extra singularity still shows up in the replacement for this map, in that SN^*Z is to be replaced by the part of the boundary of the relative compactification ${}^{N^*_ZY}\overline{N^*Z}$

0.7E; Revised: 29-11-2006; Run: November 29, 2006 which corresponds to it under the 'blow-down map'

$$\beta : {}^{N^*_Z Y}\overline{N^*Z} \longrightarrow \overline{N^*Z} \tag{L17.5}$$

discussed last time. I denoted the 'lift' or 'proper transform' of the boundary, SN^*Z, of the radial compactification under β as $[SN^*Z, SN^*_Z Y]$. Note that this is *not* the preimage under β. Rather it is the closure of the preimage of $SN^*Z \setminus SN^*_Z Y$. The notation $[X, Y]$ makes sense for any embedded submanifold of any manifold, but I am using it here without full explanantion – I will add something to the addenda about this. So the modified form of the symbol map for Z becomes the short exact sequence

$$\begin{aligned} I^{m-1,m'}(X,Y,Z;E) &\hookrightarrow I^{m,m'}(X,Y,Z;E) \\ &\xrightarrow{\sigma} \mathcal{C}^\infty([SN^*Z, SN^*_Z Y]; E_Z \otimes N_{-m,-m'}),\ \sigma = \sigma_{Y,m,m'}. \end{aligned} \tag{L17.6}$$

So, in this new setting, the Z-symbol is a smooth function (or section of a trivial line bundle) over a compact manifold with boundary.

The total symbol is the combination of these two. Even though each of these symbols is surjective there is a compatibility condition between them. Namely the symbol for Z in (L17.6) restricts to the boundary of the blown-up manifold to define the 'corner symbol'

$$\sigma_{Y,m,m'}\big|_{S^*N(SN^*_Z Y)},\ \partial[SN^*Z, SN^*_Z Y] \equiv S^*N(SN^*_Z Y). \tag{L17.7}$$

Here $S^*N(SN^*_Z Y)$ is the sphere bundle of the conormal bundle to $SN^*_Z Y$ as a submanifold of SN^*Y. On the other hand, this is exactly where the symbol of an element of the image of $\sigma_{Y,m,m'}$ lives. The compatibility condition between the two symbols is then precisely

$$\gamma_{m,m'}(u) = \sigma_{Y,m,m'}(u)\big|_{S^*N(SN^*_Z Y)} = \sigma_m(\sigma_{Y,m,m'}(u)). \tag{L17.8}$$

That is, together these two maps give one joint symbol map giving a short exact sequence

$$\begin{gathered} I^{m-1,m'-1}(X,Y,Z;E) \hookrightarrow I^{m,m'}(X,Y,Z;E) \xrightarrow{\sigma_{m,m'}} J^{m,m'}(Y,Z;E) \\ J^{m,m'}(Y,Z;E) = \big\{(a,v); a \in \mathcal{C}^\infty([SN^*Z, SN^*_Z Y]; E_Z \otimes N_{-m,-m'}), \\ v \in I^m(SN^*Y, SN^*_Z Y; E_Y \otimes N_{-m'}) \text{ s.t. } a\big|_{S^*N(SN^*_Z Y)} = \sigma_m(v)\big\}. \end{gathered} \tag{L17.9}$$

This does capture the 'full leading singularity' because

$$\bigcap_k I^{m-k,m'-k}(X,Y,Z;E) = \mathcal{C}^\infty(X;E) \tag{L17.10}$$

so in an iterative argument one would expect to finish up with smooth errors if all went well.

Note that we can also think in terms of the corner symbol in (L17.8) as being another symbol map. It corresponds to the short exact sequence

$$\begin{aligned} I^{m,m'-1}(X,Y,Z;E) + I^{m-1,m'}(X,Y,Z;E) &\hookrightarrow I^{m,m'}(X,Y,Z;E) \\ &\xrightarrow{\gamma_{m,m'}} \mathcal{C}^\infty(SN^*(SN^*_Z Y); E_Z \otimes N_{-m,-m'}). \end{aligned} \tag{L17.11}$$

So, it remains to define the spaces $I^{m,m'}(X,Y,Z;E)$ and prove all these things. For the most part this goes through following the earlier model for conormal distributions with respect to a single submanifold. I will therefore concentrate on the new twists which arise and relegate many of the proofs to the addenda.

We start with the model case where Z is a point and Y is a linear subspace of Euclidean space, $\{0\} \subset \mathbb{R}^k \subset \mathbb{R}^n$. Now, last time I recalled the definition of the relative compactification ${}^V\overline{W}$ where $V \subset W$ is a linear subspace of a vector space. We want to consider it here for the *dual* spaces. Thus $V = (\mathbb{R}^k)^\circ \subset \mathbb{R}^n = W$ is the inclusion $\mathbb{R}^{n-k} \subset \mathbb{R}^n$. Recall that there is a smooth map

$$({\rm L}17.12)\qquad {}^V\overline{W} \longrightarrow \overline{W}$$

so $\mathcal{C}^\infty({}^V\overline{W})$ is 'bigger than' $\mathcal{C}^\infty(\overline{W})$ in the sense that the latter is naturally included in the former. With these identifications we defined

$$({\rm L}17.13)\qquad I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\}) = \mathcal{F}^{-1}\left(\rho_W^{-M}\rho_V^{-M'}\mathcal{C}^\infty({}^V\overline{W})\right),\ M=m,\ M'=m'.$$

Here the defining function for the boundary of $\overline{W}$ pulls back under the map (L17.12) to $\rho_V\rho_W$ where both are elements of $\mathcal{C}^\infty({}^V\overline{W})$, the one, ρ_W, defining the 'main face' at infinity (the one whose image is the whole of the boundary of $\overline{W}$) and the other defining the 'product-type' face which corresponds to V, hence the notation ρ_V.

For simiplicity of notation, set $m = m' = 0$ so that the powers in (L17.13) are removed. In this case the two symbols of $u \in I^{0,0}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\})$ are just the restrictions of $\mathcal{F}(u)$ to the two faces $\rho_W = 0$ and $\rho_V = 0$. Referring back to the defining map for ${}^V\overline{W}$ in (L16.11) to see what these two boundary faces are with the compactification given by the closure of the image in (L16.12), corresponding to a choice of splitting $W = V \times U$. Here

$$({\rm L}17.14)\qquad \rho_W = t,\ \rho_V = s.$$

From (L16.12) we see that there is an identification

$$({\rm L}17.15)\qquad {}^V\overline{W} \supset \{s=0\} \longrightarrow \overline{U}\times SV,\ SV = \partial\overline{V}.$$

At least in coordinates, the 'Y-symbol' can therefore first be identified with an (arbitrary) element of $\mathcal{C}^\infty(\overline{U}\times SV)$. If we now take the inverse Fourier transform on U we will get an element of $I^m(U',\{0\})$ (ignoring as usual niceties about the shifts in the order of conormal distributions). Since V is the dual of $\mathbb{R}^k$, we may identify the dual, U' of U with V and hence identify $SV \times U'$ with $SN^*Y = \mathbb{R}^k \times \mathbb{S}^{n-k-1}$, the sphere bundle of the conormal bundle of $Y = \mathbb{R}^k$ in $\mathbb{R}^n$. With this identification, which we have to check behaves properly under linear transformations,

$$\begin{aligned}({\rm L}17.16)\quad \sigma_Y : I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\}) \ni u \longmapsto a = \mathcal{F}(u)\big|_{s=0} \longrightarrow \\ \mathcal{F}_u^{-1}(a) \in \mathcal{C}^\infty(\mathbb{S}^{n-k-1}; I^{m'}_{\mathcal{S}}(\mathbb{R}^k,\{0\}) = I^{m'}_{\mathcal{S}}(\mathbb{R}^k\times\mathbb{S}^{n-k-1},\{0\}\times\mathbb{S}^{n-k-1})\end{aligned}$$

is exactly what we have anticipated for the 'Y-symbol'. It is a conormal distribution on the spherical conormal bundle to Y with respect to the submanifold given by the fibre over $Z = \{0\}$.

We have already shown that any linear transformation of a real vector space W which preserves a subspace $V \subset W$ lifts to a diffeomorphism of $\overline{[V]}W$. If $L \in \mathrm{GL}(n,\mathbb{R})$ preserves the subspace $\mathbb{R}^k = \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ then for any $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$({\rm L}17.17)\qquad \mathcal{F}(L^*u) = |\det(L)|^{-1}(L^t)^{-1}\mathcal{F}(u).$$

Since the transpose preserves the annihlator $(\mathbb{R}^k)^\circ \subset \mathbb{R}^n$, we see directly from the definition that
(L17.18)
$$L^* : I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \longrightarrow I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \text{ if } L(\mathbb{R}^k) = \mathbb{R}^k,\ L \in \mathrm{GL}(n, \mathbb{R}).$$

Thus we do have linear invariance in the sense that under a general linear transformation

$$\text{(L17.19)} \qquad L^* : I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \longrightarrow I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n, L^{-1}\mathbb{R}^k, \{0\}),\ L \in \mathrm{GL}(n, \mathbb{R}).$$

The other symbol map is surjective, essentially by definition, onto the space $\mathcal{C}^\infty([SW, SV])$ where this manifold with boundary, which looks like $SU \times \overline{V}$, is identified (by definition) with $\{t = 0\}$ in ${}^V\overline{W}$. Then the properties of the symbol maps corresponding to (L17.3) and (L17.6) are the exactness of
(L17.20)
$$I^{0,-1}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \hookrightarrow I^{0,0}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \xrightarrow{\sigma_Y} I^{m'}_{\mathcal{S}}(\mathbb{R}^k \times \mathbb{S}^{n-k-1}, \{0\} \times \mathbb{S}^{n-k-1})$$

and

$$\text{(L17.21)} \qquad I^{-1,0}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \hookrightarrow I^{0,0}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \xrightarrow{\sigma_Z} \mathcal{C}^\infty([SW, SV]).$$

Namely, in each case, a function in $\mathcal{C}^\infty({}^V\overline{W})$ is in $\rho_Y \mathcal{C}^\infty({}^V\overline{W})$ or $\rho_Z \mathcal{C}^\infty({}^V\overline{W})$ if and only if it vanish on $s = \rho_Y = 0$ or $t = \rho_V = 0$ respectively, and this means exactly that the restriction of the function to that boundary face vanishes.

The joint symbol $\gamma(u)$ is obtained simply by combining of these two symbols. The only compatibility condition between them is that the have the same restriction to the corner of ${}^V\overline{W}$, which is $SU \times SV$ and which defines the 'corner symbol' $\gamma(u) \in \mathcal{C}^\infty(SU \times SV)$. The vanishing of the joint symbol means that $\mathcal{F}(u)$ can be written as stb with b smooth giving the analogue of (L17.9) in this vector space setting:-

$$\text{(L17.22)} \quad \begin{aligned} I^{m-1,m'-1}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \hookrightarrow I^{m,m'}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) &\xrightarrow{\sigma_{0,0}} J^{0,0}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k) \\ J^{0,0}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k) = \big\{(a, v); a \in \mathcal{C}^\infty&([SW, SV]), \\ v \in I^m(\mathbb{R}^k \times \mathbb{S}^{n-k-1}, \{0\} \times \mathbb{S}^{n-k-1}) &\text{ s.t. } a\big|_{SU \times SV} = \sigma(v)\big\}. \end{aligned}$$

The vanishing of the corner symbol γ implies that $\mathcal{F}(u) \in \mathcal{C}^\infty({}^V\overline{W})$ can be written as the sum of a smooth function vanishing at $t = 0$ and one vanishing at $s = 0$ (check this yourself!) giving the analogue of (L17.11)
(L17.23)
$$I^{0,-1}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) + I^{-1,0}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \hookrightarrow I^{0,0}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) \xrightarrow{\gamma_{0,0}} \mathcal{C}^\infty(SU \times SV).$$

Note that by iterative use of the 'Z-symbol' one would expect errors in

$$\text{(L17.24)} \qquad \bigcap_k I^{m-k,m'}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k, \{0\}) = I^{m'}_{\mathcal{S}}(\mathbb{R}^n, \mathbb{R}^k) = \mathcal{S}(\mathbb{R}^k; I^{m'}_{\mathcal{S}}(\mathbb{R}^{n-k}, \{0\}).$$

To see this equality note first that the middle space needs some comment even as regards its definition – simply because I did not define a 'tempered' conormal space in the general case of a subspace of a vector space (because I did not need it). However, the last equality serves as a reasonable definition of the middle space. If we take variables in $\mathbb{R}^n = \mathbb{R}^k_y \times \mathbb{R}^{n-k}_z$ then taking the Fourier transform to realize

the conormal distributions at the origin in $\mathbb{R}^{n-k}$ as symbols we get
(L17.25)
$$\mathcal{F}_{z\to\zeta}\mathcal{S}(\mathbb{R}^k_y; I^{m'}_{\mathcal{S}}(\mathbb{R}^{n-k}_z,\{0\})) = \mathcal{S}(\mathbb{R}^k_y;\rho^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^{n-k}_\zeta})) = \rho^\infty_y\rho_\zeta^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^k_y}\times\overline{\mathbb{R}^{n-k}_\zeta}).$$

Now, taking the Fourier transform in y gives again Schwartz functions in the dual variable η. Thus the right two spaces in (L17.24) can be identified under Fourier transform in all variables with

$$\rho^\infty_\eta\rho_\zeta^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^k_\eta}\times\overline{\mathbb{R}^{n-k}_\zeta}). \tag{L17.26}$$

Thus, it remains to see that this is the same as the space on the left in (L17.24). In fact, by definition, Fourier transform gives a symbol in

$$\rho_V^{-m'}\rho_W^\infty\mathcal{C}^\infty({}^V\overline{W}) \tag{L17.27}$$

so it remains to see that these two spaces are the same (as spaces of functions on $\mathbb{R}^n_{\eta,\zeta}$). This is Lemma 26 from last time (unproven then, with proof in the addenda).

As an intermediate case (which I did not have time to include in the lecture) suppose that $N \longrightarrow Z$ is a vector bundle over a compact manifold Z. I am thinking here of the normal bundle to Z as a submanifold of some compact manifold X. Then we replace Y by its linearization in N, so suppose that $M\subset N$ is a subbundle over Z. To fit a little with the earlier notation, let W be the dual bundle of N and V the annihilator of M in W. Then, for any bundle E over Z we wish to define, and explore the properties of, $I^{m,m'}_{\mathcal{S}}(N,M,Z;E)$. This is rather easy (which is why I skipped it), since we may always take local trivializations of the bundle N in which it becomes $\mathbb{R}^n\times O$ over an open subset $O\subset Z$ with the identification such that $M=\mathbb{R}^k\times O$. If we assume that E is trivial over O as well, then we are reduced to smooth functions on O with values in (the direct sum of rank E copies of) $I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\})$. From the linear invariance of $I^{m,m'}_{\mathcal{S}}(\mathbb{R}^n,\mathbb{R}^k,\{0\})$ the result space is independent of choice of trivialization and patches to give $I^{m,m'}_{\mathcal{S}}(N,M,Z;E)$.

We can alternatively proceed more directly, by taking the fibre Fourier transform on N and defining more directly
(L17.28)
$$I^{m,m'}_{\mathcal{S}}(N,M,Z;E) = \mathcal{F}^{-1}_{\text{fib}}\left(\rho_V^{-m'}\rho_W^{-m}\mathcal{C}^\infty({}^V\overline{W};E\otimes\Omega_{\text{fib}})\right),\ W=N^*,\ V=M^\circ.$$

Here we just observe that the fibre-by-fibre relative compactification gives a well-defined compact manifold with corners, fibred over Z so this makes good sense. Clearly this gives the same space as before. The second definition has the advantage that the symbol maps discussed above carry over directly and we get the analogous short exact sequences, except that everything is now fibred over Z.

Having briefly discussed the case of a bundle over Z we now consider the case of a bundle N over Y, a compact manifold with a given submanifold $Z\subset Y$. Of course, N will be the normal bundle for Y in a manifold X. Now the set up is completely global in Y and we will define the space by reduction to the previous case.

Lemma 28. *For a real vector bundle N over Y any submanifold $Z\subset Y$, identifed as a subset of the zero section of N, has a normal fibration F in N in which the image of the zero section of N lies in a subbundle $M_Y\subset NZ$.*

Given such a normal fibration, we can set, for any vector bundle over Y,

$$I_{\mathcal{S}}^{m,m'}(N,Y,Z;E) = I_{\mathcal{S}}^{m'}(N,Y;E) + F^*\{u \in I_{\mathcal{S}}^{m,m'}(NZ,M_Y,Z;E_F); \operatorname{supp}(u) \subset D(F) \subset NZ\}. \tag{L17.29}$$

Here $D(F)$ is the open neighbourhood of Z in NZ which is the image of F and E_F is a bundle over Z with an identification to E over $D'(F)$, the domain of F. Of course to use this as a definition we need to check that the right side is independent of the normal fibration. This follows the usual pattern and will be included in the addenda (when I get around to it).

Naturally we also wish to show that the symbol maps extend to these spaces and have the properties which will lead to those displayed above. In fact we are now in the general case, except for more coordinate invariance. That is, we need to show that we can set

$$I^{m,m'}(X,Y,Z;E) = \mathcal{C}^\infty(X;E) + G^*\{u \in I_{\mathcal{S}}^{m,m'}(NY,Y,Z;E_Y); \operatorname{supp}(u) \subset D(F) \subset NY\} \tag{L17.30}$$

where G is a normal fibration of Y in X with the usual identifications of bundles. Then the properties will reduce to the case in (L17.29). The main issues are to show that the symbol maps are well-defined, that they are surjective and that they have the null spaces as required to give the short exact sequences. For the symbol associated to Z this is rather clear. We already know it is unaffected by what happens away from Z so, apart from coordinate invariance, it drops back to the case (L17.28) of a bundle over Z where we already understand it.

So, it is more productive to talk about the Y symbol. This is global so needs to be discussed carefully. To see that it is well-defined we can proceed to make the decomposition in (L17.29) a little more definitive. Thus, we can choose a function $\psi \in \mathcal{C}_c^\infty(N)$ which is equal to one in a neighbourhood of Z and which has support in the domain of the normal fibration of Z in N. Then we the decomposition $u = \psi u + (1-\psi)u$ gives and element in $I^{m'}(N,Y;E)$ supported away from Z and an element in $v \in I_{\mathcal{S}}^{m,m'}(NZ,M_Y,Z;E_F)$ with compact support such that $\psi u = F^* v$. We may then define the symbol as the sum

$$\sigma_Y(u) = \sigma((1-\psi)u) + (F_*)^*\sigma_Y(v). \tag{L17.31}$$

These may both be directly interpreted as elements of the expected space

$$I^m(SN^*Y, SN_Z^*Y; E_Y \otimes N_{-m'}). \tag{L17.32}$$

Indeed the first term in (L17.31) is a smooth section of this bundle supported away from Z and the second is in this space from the discussion above. To prove that the result is well-defined we only need check that change of ψ does not affect the result. This just means showing that if u is supported away from Z but in the domain of the normal fibration then the two symbols are the same. This however follows from the definitions, which are the same away from Z.

This argument also shows surjectivity of σ_Y. Namely the second term in (L17.31) is of the form $\psi(F_*)^*\sigma_Y(v')$ and hence every conormal distribution arises this way. Conversely, if $\sigma_Y(u) = 0$ then u is certianly of order $m' - 1$ away from Z. Hence subtracting a term in $I^{m'-1}(X,Y;E)$ with support away from Z replaces u by a distribution supported in the domain of the normal fibration. Since its symbol can be computed from the bundle model and it vanishes by hypothesis, it must lie in $I^{m,m'-1}(X,Y,Z;E)$ and this proves the exactness of (L17.3).

The other claims may be established in much the same way.

The two inclusion which can be seen directly from the definition of product-type symbols are

$$\begin{gathered} I^m(X,Z;E) \subset I^{m,m}(X,Y,Z;E) \text{ for any } Y \supset Z, \\ I^{m'}(Y;E) \subset I^{-\infty,m'} \text{ for any } Z \subset Y. \end{gathered} \tag{L17.33}$$

There is a third inclusion which I will use below. Namely (I will put something about this back in the addenda) for any submanifold $Y \subset X$ of any manifold (compact for simplicity of notation) there is an inclusion

$$\mathcal{C}^{-\infty}(Y;E_Y \otimes \Omega(N^*Y)) \ni v \longrightarrow v \otimes \delta_Y \in \mathcal{C}^{-\infty}(X;E) \tag{L17.34}$$

corresponding to 'tensoring with a delta function in the normal direction' and the density factor is there because the delta 'function' wants to be a density in the normal direction (and Ω stands for the absolute value of the maximal exterior power of the dual of any real bundle). In local coordinates in which Y is given by $z_1 = \cdots = z_{n-k} = 0$, the map (L17.34) is just

$$v(y) \longmapsto v(y)\delta(z_1)\ldots\delta(z_{n-k}). \tag{L17.35}$$

I leave you to check that it is independent of coordinates.

Then there is an inclusion

$$I^{m'}(Y,Z;E_Y) \longrightarrow I^{0,m'}(X,Y,Z;E) \tag{L17.36}$$

which extends to the more obvious inclusion

$$\mathcal{C}^{\infty}(Y;E_Y \otimes \Omega(N^*Y)) \longrightarrow I^0(X,Y;E) \tag{L17.37}$$

and in which the '0' arises as the order of the delta function as a conormal distribution.

17+. Addenda to Lecture 17

17+.1. Linear invariance.