CHAPTER 15

Fibrations and families

Lecture 15: 8 November, 2005

The usual geometric setting for the families version of the index theorem of Atiyah and Singer is in terms of operators on the fibres of a fibration. Thus, rather than simply consider a parameterized family $A_b \in \Psi^m(Z; E, F)$ depending on $b \in B$ we shall allow the family to be 'twisted' by diffeomorphisms of Z depending on B.

L15.1. Fibrations. Such twisting is to be interpreted in terms of a fibration of compact manifolds

(L15.1)
$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B. \end{tikzcd}$$

A fibration of compact manifolds is just a submersion. For simplicity of notation we will assume that the 'base' B is connected. Then a smooth map

(L15.2)
$$\phi : M \longrightarrow B$$

is a *submersion* if

(L15.3)
$$\phi_* : T_m M \longrightarrow T_{\phi(m)} B$$

is surjective for each $m \in M$.

THEOREM 9. *If $\phi : M \longrightarrow B$ is a smooth submersion between compact manifolds with B connected then*

1. *For each $b \in B$, $\phi^{-1}(b) = Z_b \subset M$ is an embedded compact submanifold diffeomorphic to a fixed manifold Z.*
2. *Each $b \in B$ has an open neighbourhood $b \in U_b \subset B$ such that there exists a diffeomorphism f_b giving a commutative diagramme*

(L15.4)
$$\begin{tikzcd} \phi^{-1}(U_b) \arrow[rr, "f_b"] \arrow[dr, "\phi"'] & & Z \times U_b \arrow[dl, "\pi_L"] \\ & U_b. & \end{tikzcd}$$

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(3) *For each intersectiong pair of such open sets it follows that there is a commutative diagramme*

(L15.5)
$$\begin{tikzcd} Z \times (U_b \cap U_{b'}) \arrow[dr, "\pi_L"'] & \phi^{-1}(U_b \cap U_{b'}) \arrow[l, "f_{b'}"'] \arrow[r, "f_b"] \arrow[d, "\phi"] & Z \times (U_b \cap U_{b'}) \arrow[dl, "\pi_L"] \\ & U_b \cap U_{b'} & \end{tikzcd}$$

which shows that $f_{b'b} = f_{b'} f_b^{-1} \in \mathcal{C}^\infty(U_b \cap U_{b'}; \mathrm{Difm}(Z))$ *is a smooth map into the diffeomorphisms of* Z *and also that the cocycle condition*

(L15.6)
$$f_{b''b'} f_{b'b} = f_{b''b} \text{ holds on } U_{b''} \cap U_{b'} \cap U_b.$$

Proof. (Brief) The implicit function theorem shows that $Z_b = \phi^{-1}(b)$ is an embedded compact submanifold of M. Indeed, if t_i are local coordinates near b on B then the $\phi^*(t_i)$ are defining functions for Z_b in M. One can choose commuting vector fields in $\phi^{-1}(U_b)$ for a sufficiently small neighbourhood U_b of b, T_i on $\phi^{-1}(U_b)$ such that $\phi_*(T_i) = \partial_{t_i}$ and then by integration along the T_i one can define f_b with $Z = Z_b$. Having done this on an open covering of B it follows that all the Z_b are diffeomorphic, so Z_b can be replaced by a fixed Z in (L15.4). This proves (1) and (2) and (3) follow directly from (2). □

One can recover the fibration, thought of here as a fibre bundle with fibre Z and structure group $\mathrm{Difm}(Z)$ (the diffeomorphism group of Z), from (2) and (3). If the maps $f_{b'b}$ can be chosen, for some covering of B, to lie in a subgroup $G \subset \mathrm{Difm}(Z)$ of the diffeomorphism group, then the structure group is 'reduced to G.'

Fibrations have various functoriality properties. The most important for us is that we may restrict to a submanifold of the base or more generally we may 'pull-back' a fibration.

Proposition 31. *If* $F : \tilde{B} \longrightarrow B$ *is any smooth map, with* $\tilde{B}$ *compact, and* $\phi : M \longrightarrow B$ *is a fibration, then*

(L15.7)
$$\tilde{M} = \{(m, \tilde{b}) \in M \times \tilde{B}; \phi(m) = F(\tilde{b})\} \text{ is an embedded submanifold of } M \times B \text{ and}$$
$$\tilde{\phi} : \tilde{M} \longrightarrow \tilde{B},\ \tilde{\phi}(m, \tilde{b}) = \tilde{b} \text{ is a fibration.}$$

Equally important is that the composite of two fibrations is a fibration.

Proposition 32. *If* $\phi' : M' \longrightarrow M$ *is a fibration with typical fibre* Z' *and* $\phi : M \longrightarrow B$ *is a fibration with typical fibre* Z *then* $\phi\phi' : M' \longrightarrow B$ *is a fibration with typical fibre* $Z \times Z'$.

It is also easy to see that the direct product of two fibrations, $\phi_i : M_i \longrightarrow B_i$, $i = 1, 2$ is a fibration

(L15.8)
$$\phi_1 \times \phi_2 : M_1 \times M_2 \longrightarrow B_1 \times B_2.$$

In the proof of the Atiyah-Singer index theorem discussed below, a given fibration is trivialized by embedding, so it is important to see that this is always possible.

Exercise 19. Given a fibration (of compact manifolds always) $\phi : M \longrightarrow B$ show that there is an embedding $e : M \longrightarrow \mathbb{S}^N \times B$ with range in $(\mathbb{S}^N \setminus \{p\}) \times B$

for fixed point $p \in \mathbb{S}^N$ such that

$$
\begin{tikzcd}
M \arrow[rr, "e"] \arrow[dr, "\phi"'] & & \mathbb{S}^N \times B \arrow[dl, "\pi_L"] \\
& B &
\end{tikzcd}
\tag{L15.9}
$$

commutes.

Hint: This is not hard, just use Whitney's theorem to embed M in a big sphere, staying away from one point, and then define e as the product of that embedding and ϕ. Check that this is an embedding and that (L15.9) holds.

L15.2. Pseudodifferential operators on the fibres. Next we turn to the definition of pseudodifferential operators 'on the fibres' of a fibration. We could proceed locally from (L15.4) and (L15.5), using the definition of pseudodifferential operators on Z depending on a parameter and then the invariance under diffeomorphisms to piece these together between patches. However we are in a position to proceed more directly than this.

The standard notation for pseudodifferential operators on the fibres of a fibration $\phi : M \longrightarrow B$ is $\Psi^m(M/B; E, F)$, where E and F are bundles over M. Note that the fibration ϕ does not appear explicitly in the notation, which is designed (I suppose) to suggest the the operators are on 'M/B' which does not mean anything but could only be interpreted as the fibre.

Given a fibration $\phi : M \longrightarrow B$ we first define the fibre-product of this fibration with itself, $M^2_\phi \longrightarrow B$. Namely, M^2_ϕ is the restriction of $M \times M$, as a fibration over $B \times B$, to the diagonal $B \equiv \mathrm{Diag} \subset B \times B$. The total space in then

$$
M^2_\phi = M \times_\phi M = \{(m, m') \in M \times M; \phi(m) = \phi(m')\}. \tag{L15.10}
$$

Thus the fibres of $M^2_\phi \longrightarrow B$, (where I use the same letter for the new fibration) are modelled on $Z \times Z$. Clearly the diagonal in M^2 is contained in M^2_ϕ where we may think of it as the 'fibre diagonal' Diag_ϕ so we have the embedded submanifold

$$
M \equiv \mathrm{Diag}_\phi \hookrightarrow M^2_\phi. \tag{L15.11}
$$

Definition 6. The space of pseudodifferential operators on the fibres of a fibration $\phi : M \longrightarrow B$ is identified as

$$
\Psi^m(M/B; E, F) = I^{m'}(M^2_\phi, \mathrm{Diag}_\phi; \mathrm{Hom}(E, F) \otimes \Omega_R), \ m' = m - \frac{1}{4} \dim B \tag{L15.12}
$$

for any complex vector bundles E and F over M; here Ω_R is the bundle of fibrewise densities on the right, discussed more below.

Note that if $M = Z \times B$ is 'trivial' and E, F are the lifts of bundles over Z then

$$
I^{m'}(M^2_\phi, \mathrm{Diag}_\phi; \mathrm{Hom}(E, F) \otimes \Omega_R) = \mathcal{C}^\infty(B; \Psi^m(Z; E, F)).
$$

Thus, locally over $U \subset B$ over which the fibration is trivial and so small that E and F are the pull-backs of their restrictions E_b and F_b to Z_b, $\Psi^m(M/B; E, F)$ reduces to $\mathcal{C}^\infty(U; \Psi^m(Z; E_b, F_b))$. From this, and the definition, we can deduce all the basic properties.

In general, elements of $\Psi^m(M/B; E, F)$ cannot be elements of $\Psi^m(M; E, F)$ since the latter have kernels singular only on the diagonal of M whereas the kernels of the fibrewise operators are supported on M^2_ϕ. In fact the only elements in both are fibrewise differential operators (assuming the dimension of the base is positive).

(1) (Action) The elements of $\Psi^m(M/B;E,F)$ are continuous linear operators

$$\Psi^m(M/B;E,F) \ni A : \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;F). \tag{L15.13}$$

Indeed, locally over any small open set, $U \subset B$, $u \in \mathcal{C}^\infty(M,E)$ becomes $u_b \in \mathcal{C}^\infty(U \times Z; E_b)$ and A maps this to $\mathcal{C}^\infty(U \times Z; F_b)$. Alternatively we can go back to the proof in the case of a single manifold and use the push-forward theorem.

These operators clearly act on the fibres. That is, if $u \in \mathcal{C}^\infty(M;E)$ and $u\big|_{Z_b} = 0$ then $Au\big|_{Z_b} = 0$. Hence A_b is well-defined by

$$A_b v = (Au)\big|_{Z_b},\ u \in \mathcal{C}^\infty(M;E) \text{ s.t. } u\big|_{Z_b} = v \text{ and } A_b \in \Psi^m(Z_b;E_b,F_b). \tag{L15.14}$$

(2) (Smoothing operators) The smoothing families are

$$\Psi^{-\infty}(M/B;E,F) = \mathcal{C}^\infty(M^2_\phi; \operatorname{Hom}(E,F) \otimes \Omega_R). \tag{L15.15}$$

(3) (Symbol map) For each point $b \in B$ the symbol of A_b, where $A \in \Psi^m(M/B;E,F)$, is an element of $\mathcal{C}^\infty(S^*Z_b; \hom(E_b,F_b) \otimes N_m)$ and in terms of a local trivialization of the fibration and bundles (i.e. local reduction to a product) depends smoothly on b. Let $T(M/B)$ be the subbundle of TM consisting of the vectors tangent to the fibre at each point. Thus, restricted to Z_b, $T(M/B)$ reduces to TZ_b. Let $T^*(M/B)$ be the dual bundle and $S^*(M/B)$ be the corresponding sphere bundle. Then, from the local properties, the symbol map becomes

$$\sigma_m : \Psi^m(M/B;E,F) \longrightarrow \mathcal{C}^\infty(S^*(M/B); \hom(E,F) \otimes N_m) \tag{L15.16}$$

where as usual, N_m is the bundle of functions homogeneous of degree m on $T^*(M/B)$ (as a bundle over $S^*(M/B)$).

(4) (Symbol sequence) The symbol map leads immediately to the short exact sequence

$$\Psi^{m-1}(M/B;E,F) \longrightarrow \Psi^m(M/B;E,F) \xrightarrow{\sigma_m} \mathcal{C}^\infty(S^*(M/B); \hom(E,F) \otimes N_m). \tag{L15.17}$$

(5) (Composition) Of course one of the most important properties of pseudodifferential operators is that they compose to give such operators. Agian it follows directly from the local picture, or using the same proofs as there but in the more global setting, that

$$\begin{gathered} A \in \Psi^m(M/B;F,G),\ B \in \Psi^{m'}(M/B;E,F) \Longrightarrow AB \in \Psi^{m+m'}(M/B;E,G) \text{ and} \\ \sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B). \end{gathered} \tag{L15.18}$$

(6) (Ellipticity) $A \in \Psi^m(M/B;E,F)$ is said to be elliptic (as a family) if each A_b is elliptic, which is the same as saying that the symbol has an inverse

$$(\sigma_m(A))^{-1} \in \mathcal{C}^\infty(S^*(M/B); \hom(F,E) \otimes N_m) \tag{L15.19}$$

Then, as in the case of a single operator, the ellipticity of A is equivalent to the existence of a two-sided parametrix $Q \in \Psi^{-m}(M/B;F,E)$ such that

$$QA - \operatorname{Id}_F \in \Psi^{-1}(M/B;F),\ AQ - \operatorname{Id}_E \in \Psi^{-1}(M/B;E). \tag{L15.20}$$

(7) (Asymptotic completeness) Using just the corresponding fact for conormal distributions we know that given a sequence $A_j \in \Psi^{m-j}(M/B;E,F)$ for $j \in \mathbb{N}_0$,

(L15.21)
$$\exists\ A \in \Psi^m(M/B;E,F) \text{ s.t. } A \sim \sum_j A_j \Longleftrightarrow A - \sum_{j=0}^N A_j \in \Psi^m(M/B;E,F)$$

and A is unique modulo $\Psi^{-\infty}(M/B;E,F)$.

Using these properties we can improve the parametrix for an elliptic operator from (L15.20). Namely let Q_0 be that operator, so

$$Q_0 A = \operatorname{Id} - R,\ R \in \Psi^{-1}(M/B;E). \tag{L15.22}$$

Then the formal Neumann series for $(\operatorname{Id} - R)^{-1}$, $\sum_j R^j$ is asymptotically summable, as is the product on the left with Q_0. Thus we can find

$$Q \sim \sum_j R^j Q_0 \in \Psi^{-m}(M/B;F,E) \Longrightarrow QA - \operatorname{Id}_E \in \Psi^{-\infty}(M/B;E). \tag{L15.23}$$

Similarly a right parametrix modulo smoothing operators can be constructed and shown to be equal to Q modulo smoothing, so Q also satisfies

$$AQ - \operatorname{Id}_F \in \Psi^{-\infty}(M/B;F).$$

L15.3. The analytic index. Now, we can proceed very much as in the case of the Toeplitz operators to discuss the families index theorem. Of course the geometry of the fibration M will cause complications. In fact we need another basic fact to proceed (the way I want, there are other approaches), namely a replacement for the projections on the first k terms in the Fourier series on the circle.

PROPOSITION 33. *For any fibration of compact manifolds, $\phi : M \longrightarrow B$ there is a sequence of projections $\pi_N \in \Psi^{-\infty}(M/B;E)$, for any vector bundle E, satisfying*

$$\operatorname{rank}(\pi_N) \le N,\ A\pi_N \longrightarrow A \text{ for any } A \in \Psi^{-\infty}(M/B;E,F) \tag{L15.24}$$

in terms of the usual topology on $\mathcal{C}^\infty(M^2_\phi; \operatorname{Hom}(E,F) \otimes \Omega_R)$.

Note that I am not assuming here that the projections are increasing, so it may be that $\pi_N \pi_{N+1} \neq \pi_N$ (and this product may not even be a projection).

PROOF. Missing – I do not yet have a reasonably elementary proof of this. There is one using Kuiper's theorem which I will resort to if necessary but I am still hoping to find something a bit better than that! It is pretty easy to do this in case of a product $Z \times B$ but the twisting of the bundle causes some trouble. $\square$

Assuming the existence of such a family of projections we can proceed as in the Toeplitz case to construct the analytic index. Thus, given an elliptic family $A \in \Psi^m(M/B;E,F)$ choose a parametrix $Q \in \Psi^{-m}(M/B;F,E)$ as above, i.e. satisfying (L15.23), so $QA = \operatorname{Id} - R$, with $R \in \Psi^{-\infty}(M/B;E)$. Then $R\pi_N \longrightarrow R$ for a family of projections as in Proposition 33 and hence for N sufficiently large, $(\operatorname{Id} - R(\operatorname{Id} - \pi_N))^{-1}$ exists and is of the form $\operatorname{Id} - S$ with $S \in \Psi^{-\infty}(M/B;E)$. Thus, for N sufficiently large,

$$(\operatorname{Id} - S)QA(\operatorname{Id} - \pi_N) = \operatorname{Id} - \Pi_N \tag{L15.25}$$

from which it follows that

$$\text{null}(A(\text{Id} - \pi_N)) = \text{Ran}(\pi_N) \text{ is a bundle over } B. \tag{L15.26}$$

It follows from this (just work locally as usual) that

$$\text{null}((\text{Id} - \pi_N)^* A^*) \text{ is a bundle over } B \tag{L15.27}$$

again of finite rank – for any choice of inner products and smooth densities used to define the adjoints. This latter bundle is a complement to the range of $A(\text{Id} - \pi_N)$ as a subbundle of $\mathcal{C}^\infty(M/B; E)$ thought of as a bundle over B.

PROPOSITION 34. *For any elliptic family in $\Psi^m(M/B; E, F)$ the symbol determines an element of $K^*_c(T^*(M/B))$ and the regularized null bundles in* (L15.26) *and* (L15.27) *determine an element of $K^0(B)$ and this correspondence projects to a well-defined map*

$$\text{ind}_\text{a} : K^0_c(T^*(M/B)) \longrightarrow K^0(B). \tag{L15.28}$$

PROOF. We need to show independence of the choice of π_N, independence of the choice of A, given the symbol, homotopy invariance under deformation of the symbol (which amounts to homotopy invariance for A) constancy of the index class under stablization and under composition of the symbol with bundle isomorphisms; of course we also need to check that every compactly supported K-class on $T^*(M/B)$ arises from a symbol. All of this is pretty straightforward and pretty much as in the Toeplitz case. □

Next time I will introduct the algebra of product-type pseudodifferential operators on a fibration which I will use to identify this analytic index map with the topological index map defined by embedding of the fibration. This is the index theorem of Atiyah and Singer.

15+. Addenda to Lecture 15

15+.1. Some more details.

15+.2. The analytic index map (L15.28).