CHAPTER 10

Chern forms and the Fredholm determinant

Lecture 10: 20 October, 2005

I showed in the lecture before last that the topological group $G^{-\infty} = G^{-\infty}(Y;E)$ for any compact manifold of positive dimension, Y, and and bundle E, is an open subset of the (infinite dimensional) vector space $\Psi^{-\infty}(Y;E)$. I also, by fiat, declared it to be a classifying space for odd K-theory. This would not be sensible except of course that it is such a classifying space. If you consult a standard book on topology you will see that my claim amounts to the assertion

$$\pi_k(G^{-\infty}) = \begin{cases} 0 & k \text{ even} \\ \mathbb{Z} & k \text{ odd.} \end{cases} \tag{L10.1}$$

This result, which I will prove later, justifies my declaring that for any smooth compact manifold

$$K^{-1}(X) = [X, G^{-\infty}] \tag{L10.2}$$

is the abelain group of (smooth) homotopy classes of (smooth) maps.

Back to the statement that $G^{-\infty}$ is open in $\Psi^{-\infty}$, where I drop the qualifying space Y and bundle E since they are irrelevant. This means that I can happily treat $G^{-\infty}$ as a manifold. In fact the tangent space to $G^{-\infty}$ at a point $A = \operatorname{Id} + a$ (I will try to stick to this notation of A as the whole operator and a as the smoothing part) defined as usual as the equivalence classes of smooth curves $\operatorname{Id} + a_t$, $a_0 = a$, under tangency, is just $\Psi^{-\infty}$,

$$dA : T_A G^{-\infty} \ni [\operatorname{Id} + a_t] \longmapsto \frac{da_t}{dt}\Big|_{t=0} \in \Psi^{-\infty}. \tag{L10.3}$$

The notation 'dA' really comes from Lie group theory. In fact we may think of this map as defined on the whole of the tangent bundle to $G^{-\infty}$ and hence also

$$A^{-1}dA : TG^{-\infty} \longrightarrow \Psi^{-\infty}. \tag{L10.4}$$

This is the universal left-invariant 1-form on $G^{-\infty}$. Under left multiplication by $B \in G^{-\infty}$

$$L_B : G^{-\infty} \ni A \longmapsto BA \in G^{-\infty},\ L_B^*(A^{-1}dA) = A^{-1}B^{-1}BdA = A^{-1}dA. \tag{L10.5}$$

From this form we may construct the (Unnormalized) odd Chern forms

$$u_{2k-1} = \operatorname{Tr}\left((A^{-1}dA)^{2k-1}\right),\ k = 1, 2, \dots. \tag{L10.6}$$

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Here the product is interpreted as the product in $\Psi^{-\infty}$ followed by antisymmetrization, i.e. the wedge product. Written out more formally this is

(L10.7)
$$u_{2k-1}(b_1,\dots,b_{2k-1})=\sum_{\sigma\in\Sigma_{2k-1}}\operatorname{sgn}(\sigma)\operatorname{Tr}\left(A^{-1}b_{\sigma(1)}A^{-1}b_{\sigma(2)}\cdots A^{-1}b_{\sigma(2k-1)}\right)$$

where the sum is over the permutation group on $\{1,\dots,2k-1\}$. The smoothness of composition, inversion and the trace shows this to be a smooth form on $G^{-\infty}$. Of course this can also be done with an even number of factors but then the trace idenity shows that

(L10.8)
$$\operatorname{Tr}\left((A^{-1}dA)^{2k}\right)=\operatorname{Tr}\left((A^{-1}dA)\wedge(A^{-1}dA)^{2k-1})\right)=-\operatorname{Tr}\left((A^{-1}dA)^{2k-1})\wedge(A^{-1}dA)\right)=0$$

since an odd number of transpositions occur.

The forms u_{2k-1} are left invariant, from the left invariance of $A^{-1}dA$ but also right-invariant, since under $R_B : G^{-\infty}\ni A\longmapsto AB\in G^{-\infty}$, $R_B^*(A^{-1}dA)=B^{-1}(A^{-1}dA)B$.

Now the standard formula $dA^{-1}=-A^{-1}(dA)A^{-1}$ is justified here as usual by differentiation the equality of smooth functions $A^{-1}A=\operatorname{Id}$. Rewriting the definition

(L10.9)
$$u_{2k-1}=\operatorname{Tr}\left((A^{-1}dA)\wedge(A^{-1}dAA^{-1}dA)^{k-1}\right)=(-1)^{k-1}\operatorname{Tr}\left((A^{-1}dA)\wedge(dA^{-1}\wedge dA)^{k-1}\right).$$

Thus,

(L10.10)
$$du_{2k-1}=(-1)^{k-1}\operatorname{Tr}\left((dA^{-1}\wedge dA)\wedge(dA^{-1}\wedge dA)^{k-1}\right)=-\operatorname{Tr}\left((A^{-1}dA)2k\right)=0$$

and it follows that these forms are closed.

By definition in (L10.2), and odd K-class on a compact manifold X is represented by a smooth map $f : X\longrightarrow G^{-\infty}$. We may use f to pull back the forms u_{2k-1} to smooth forms on X. Since $df^*u_{2k-1}=f^*(du_{2k-1})$ these forms are necessarily closed.

PROPOSITION 22. *The deRham model of cohomology leads, for each $k\in\mathbb{N}$, to a well-defined and additive map*

$$U_{2k-1}: K^{-1}(X)\longrightarrow H^{2k-1}(X;\mathbb{C}). \tag{L10.11}$$

PROOF. The deRham cohomology class of the closed from f^*u_{2k-1} is constant under homotopy from $f : X\longrightarrow G^{-\infty}$ to $f : X\longrightarrow G^{-\infty}$. Indeed, such an homotopy is a smooth map $F:[0,1]\times X\longrightarrow G^{-\infty}$ with $F(0,\cdot)=f$ and $F(1,\cdot)=f'$. If $f_t=F(t,\cdot)$ then $dF^*u_{2k-1}=$ becomes the condition

$$\frac{df_t^*u_{2k-1}}{dt}=d_Xv_t\Longrightarrow f_1^*u_{2k-1}-f_0^*u_{2k-1}=dv,\ v=\int_0^1 v_t dt. \tag{L10.12}$$

Thus the map (L10.11) is well-defined. Its additivity follows from the discussion last time which shows that two maps $f_i : X\longrightarrow G^{-\infty}$, $i=1,2$ may be deformed homotopically to be each finite rank perturbations of the identity and in commuting $N\times N$ blocks in $G^{-\infty}$. For such maps the product $(f_1f_2)^*u_{2k-1}=f_1^*u_{2k-1}+f_2^*u_{2k-1}$ showing the additivity. □

Taking the correct constants in a formal sum

$$\sum_k c_k U_{2k-1} : K^{-1}(X) \longrightarrow H^{\mathrm{odd}}(X;\mathbb{C}) \tag{L10.13}$$

will give the 'odd Chern character' discussed later. Its range then spans $H^{\mathrm{odd}}(X;\mathbb{C})$ and its null space is the finite subgroup of torsion elements of $K^{-1}(X)$, those elements satisfying $p[f] = 0$ (represented by the constant maps) for some integer p depending on f.

Even Chern forms can be defined in the same way as forms on the group $G^{-\infty}_{(1)}$. Let me use the version of this group defined last time, were we consider (for some underlying manifold Y and bundle E) the space of smooth Schwartz maps

$$\text{(L10.14)}\quad G^{-\infty}_{(1)}(Y,E) =$$
$$\{a \in \mathcal{S}(\mathbb{R}_t; \Psi^{-\infty}(Y;E)) = \mathcal{S}(\mathbb{R}\times Y^2; \mathrm{Hom}(E)\otimes\pi_R^*\Omega_Y); \mathrm{Id} + a_t \in G^{-\infty}(Y;E)\ \forall\, t \in \mathbb{R}\}.$$

Then again $G^{-\infty}_{(1)}$ is an open subspace of $\mathcal{S}(\mathbb{R}\times Y^2; \mathrm{Hom}(E)\otimes\pi_R^*\Omega_Y)$ and we set

$$u_{2k} = \int_{\mathbb{R}} \mathrm{Tr}\left((A^{-1}dA)^{2k}(A^{-1}\frac{dA}{dt}\right) dt. \tag{L10.15}$$

Since we may regard $G^{-\infty}_{(1)}$ as a subset of $\mathcal{C}^\infty(\mathbb{R}; G^{-\infty})$ this may also be considered as the integral over $\mathbb{R}$ of the pullback of u_{2k+1}. In any case this is again a closed form, this can also be seen directly, and for the same reasons as in the odd case defines an additive map

$$K^{-2}(X) \longrightarrow H^{2k}(X;\mathbb{C}) \text{ for each } k \in \mathbb{N}_0. \tag{L10.16}$$

An appropriate combination of these forms gives the Chern character (now the 'usual' Chern character) which has image spanning over $\mathbb{C}$.

The simplest, and most fundamental, cases of these forms are the first odd Chern form

$$u_1 = \mathrm{Tr}(A^{-1}dA) \text{ on } G^{-\infty} \tag{L10.17}$$

and its integral in the even case

$$u_0 = \int_{\mathbb{R}} \mathrm{Tr}(A^{-1}\frac{dA}{dt})dt,\ A \in G^{-\infty}_{(1)}. \tag{L10.18}$$

Proposition 23. *The form $u_1/2\pi i$ is integral, i.e. for any smooth map $\gamma : \mathbb{S} \longrightarrow G^{-\infty}$,*

$$\int_\gamma u_1 \in 2\pi i\mathbb{Z}. \tag{L10.19}$$

Proof. We may prove this by finite rank approximation. Since the integral is a cohomological pairing, we know it is homotopy invariant. Thus it suffices to replace γ by an approximating loop which is a uniformly finite rank perturbation of the identity. Thus we can assume that $\gamma : \mathbb{S} \longrightarrow \mathrm{GL}(N,\mathbb{C})$ for some embedding of $\mathrm{GL}(N,\mathbb{C})$ in $G^{-\infty}$. Since the trace restricts in any such embedding we are reduced to the matrix case. Then (L10.19) follows from the standard formula for matrices that

$$d\log\det(A) = \mathrm{Tr}(A^{-1}dA) \tag{L10.20}$$

with the integer in (L10.19) being the variation of the argument of the determinant along the curve. □

Conversely we may use (L10.19) to conclude the the definition of the determinant on $G^{-\infty}$ which I proposed earlier,

$$\det(A) = \exp\left(\int_0^1 \operatorname{Tr}(A_t^{-1}\frac{dA_t}{dt}dt\right), \tag{L10.21}$$

where $t \to A_t$ is a curve in $G^{-\infty}$ from $A_0 = \operatorname{Id}$ to $A_1 = A$, does indeed lead to a well-defined function

$$\det : G^{-\infty} \longrightarrow \mathbb{C}. \tag{L10.22}$$

Indeed, such a curve exists, by the connectedness of $G^{-\infty}$ and two such curves differ by a closed curve (admittedly only piecewise smooth but that is not a serious issue).

Furthermore it follows directly from the definition that det is multiplicative. Namely for AB we may use the product A_tB_t of the curves connection the factors to the identity. Then

$$\begin{aligned}(A_tB+t)^{-1}d(A_tB_t) = B_t^{-1}dB_t + B_t^{-1}(A_t^{-1}dA_t)B_t \Longrightarrow \\ \operatorname{Tr}(A_tB+t)^{-1}d(A_tB_t) = \operatorname{Tr}(B_t^{-1}dB_t) + \operatorname{Tr}(A_t^{-1}dA_t)\end{aligned} \tag{L10.23}$$

from which it follows that $\det(AB) = \det(A)\det(B)$ as in the finite dimensional case. Of course this also follows by approximation, given the continuity of det which follows from the same formula.

In fact the Fredholm determinant in (L10.22) extends to a smooth map

$$\Psi^{-\infty}(Y;E) \ni A \longmapsto \det(\operatorname{Id}+A)\mathbb{C} \tag{L10.24}$$

which is non-vanishing precisely on $G^{-\infty}$.

++++ Add definition near zeros (this is a good exercise!)

Of course it follows from Propositon 23 that

$$\frac{u_0}{2\pi i} : G^{-\infty}_{(1)} \longrightarrow \mathbb{Z}. \tag{L10.25}$$

We shall see below that this can be interpreted as the simplest case of the index formula and that this map faithfully labels the components of $G^{-\infty}_{(1)}$.

Next I turn to the Toeplitz algebra. This algebra is the basic object which leads to a short exact sequence of groups

$$G^{-\infty} \longrightarrow G^0 \longrightarrow G^{-\infty}_{(1),-}[[\rho]] \sim G^{-\infty}_{(1),0}. \tag{L10.26}$$

Here I will not explain the whole notation for the moment, but the normal subgroup on the left is one of our 'smoothing groups', the central group is supposed to be contractible and the group on the right is homotopic to the identity component (this is the extra 0 subscript, meaning the index is zero in (L10.25)) of the loop group $G^{-\infty}_{(1)}$.

Now, this sequence is supposed to come, after some work, from the short exact sequence arises from the symbol of a pseudodifferential operator

$$\Psi^{-1}(X;\mathbb{C}^N) \longrightarrow \Psi^0(Z;\mathbb{C}^N) \longrightarrow \mathcal{C}^\infty(S^*Z;M(N,\mathbb{C})). \tag{L10.27}$$

For the moment I will ignore the difference between Ψ^{-1} and $\Psi^{-\infty}$, when taken into account this will lead to the 'formal power series' parameter ρ on the right in (L10.26) – there are other more serious problems to be dealt with! To get from

(L10.27) to (L10.26) we first want to consider the set of elliptic and invertible elements of $\Psi^0(Z;\mathbb{C}^N)$. If we consider the normal subgroup of invertible perturbations of the identity we arrive at

$$G^{-1}(Z;\mathbb{C}^N) \longrightarrow G^0(Z;\mathbb{C}^N) \longrightarrow \mathcal{C}^\infty(S^*Z;\mathrm{GL}(N,\mathbb{C})). \tag{L10.28}$$

Here
(L10.29)
$$G^{-1}(Z;\mathbb{C}^N) = \{\mathrm{Id}+A; A\in\Psi^{-1}(Z;\mathbb{C}^N),\ (\mathrm{Id}+A)^{-1} = \mathrm{Id}+B,\ B\in\Psi^{-1}(Z;\mathbb{C}^N)\},$$
$$G^0(Z;\mathbb{C}^N) = \{A; A\in\Psi^0(Z;\mathbb{C}^N) \text{ elliptic and } A^{-1}\in\Psi^0(Z;\mathbb{C}^N)\}$$

where we will finally replace the former by $G^{-\infty}$.

Now, in general the second map in (L10.28) is not surjective, since that would mean that every elliptic element can be perturbed to be invertible and we known that this means precisely that the index vanishes. Thus the index is the (only) obstruction to the exactness of (L10.28). Of course we want to discuss this in treating the index formula but for the moment I am after something else.

Namely, I would like to choose Z so that the central group in (L10.28) is contractible and the image group is essentially a $G^{-\infty}_{(1)}$. To arrange the latter we need to do two things. First we need to choose the manifold Z so that

$$S^*Z = \mathbb{S} \tag{L10.30}$$

and then to 'stabilize' things so that $\mathbb{C}^n$ is replaced by an infinite dimensional space in such a way that $\mathrm{GL}(N,\mathbb{C})$ becomes one of our $G^{-\infty}$ groups. This second step may seem the most daunting but it is not and I will discuss how to do this next time. So, let us think about how to arrange (L10.30). Of course the small problem here is that this is impossible, there is no such manifold. Indeed, it would have to be 1-dimensional and compact, hence just the circle if we demand it to be connected. However

$$S^*\mathbb{S} = \mathbb{S}\sqcup\mathbb{S} \tag{L10.31}$$

is the disjoint union of two copies of the circle.

There are two ways to overcome this problem (well I know a third which you can find in [**4**] if you look hard enough). Stated vaguely these are

(A) Replace the circle by the line $\mathbb{R}$ so that '$S^*\mathbb{R}$' is interpreted as the boundary of the radial compactification of $T^*\mathbb{R} = \mathbb{R}^2$ as a vector space (not a vector bundle over $\mathbb{R}$). In this sense we would arrive at (L10.30). I was going to do this in these lectures, and I may still do so. It requires going back to the beginning of the lectures and discussing a variant of the conormal distributions for subspaces of a vector space. This leads to the 'isotropic calculus' on $\mathbb{R}$ (or in fact on $\mathbb{R}^n$) which can be used to construct the sequence I am after.

(B) Kill off half of (L10.31) and work on the remaing half. This is what I will do, namely discuss the Toeplitz algebra and its variants. I find this approach less geometrically transparent but it has plenty of history behind it.

For the circle we can decomposte smooth functions as a direct sum

$$\mathcal{C}^\infty(\mathbb{S}) = \mathcal{C}^\infty_-(\mathbb{S}) + \mathcal{C}^\infty_+(\mathbb{S}) \tag{L10.32}$$

where these are limited by the Fourier coefficients

$$a \in \mathcal{C}^\infty_+(\mathbb{S}) \Longleftrightarrow a = \sum_{k\ge 0} a_k e^{ik\theta},\ \sum_{k\ge 0} |a_k| k^j < \infty\ \forall\ j. \tag{L10.33}$$

The Szegő projection is the linear map which excises the negative Fourier modes

$$S : \mathcal{C}^\infty(\mathbb{S}) \longrightarrow \mathcal{C}^\infty_+(\mathbb{S}),\ S(a) = \sum_{k\ge 0} a_k e^{ik\theta} \text{ if } a = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta}. \tag{L10.34}$$

Clearly this is a projection, $S^2 = S$ with null space $\mathcal{C}^\infty_-(\mathbb{S})$ and range $\mathcal{C}^\infty_+(\mathbb{S})$.

Note that one can always recover a compact manifold, Z, from $\mathcal{C}^\infty(Z)$ with its multiplicative structure. Namely the points of Z can be identified with the valuations on the ring, the linear maps $p : \mathcal{C}^\infty(Z) \longrightarrow \mathbb{C}$ such that $p(fg) = p(f)p(g)$. The space $\mathcal{C}^\infty_+(\mathbb{S})$ is a ring, as follows easily from the definition, but it is not the space of smooth functions on a manifold since the set of valuations actually recovers $\mathbb{S}$. Still, the idea is that we can think of this 'Hardy space' $\mathcal{C}^\infty_+(\mathbb{S})$ as the space of functions on 'half of $\mathbb{S}$.' Note that the Fourier parameter k is closely related to the dual variable on the fibres of the cotangent space $T^*\mathbb{S} = \mathbb{S}\times\mathbb{R}$ which indicates that S restricts to the 'positive half of the cotangent bundle.' More concretely

Lemma 21. *The Szegő projector* $S \in \Psi^0(\mathbb{S})$.

Consider the Toeplitz algebra

$$\mathcal{T} = \{A \in \Psi^0(\mathbb{S}); A = SAS\}. \tag{L10.35}$$

It is indeed a subalgebra of the algebra of pseudodifferential operators since

(L10.36)

$$A_1,\ A_2 \in \mathcal{T} \Longrightarrow S(A_1A_2) = S(SA_1S)(SA_2S)S = (SA_1S)(SA_2S) = A_1A_2.$$

To arrive at the algebra I will proceed in three steps.

(1) We need to replace $\Psi^0(\mathbb{S})$ by the corresponding algebra of operators 'valued in the smoothing operators' on some manifold Y. This can be identified with $\mathcal{C}^\infty(Y^2;\Psi^0(\mathbb{S}))$.
(2) The symbol space of this algebra consists of smooth functions on $S^*\mathbb{S} = \mathbb{S}\sqcup\mathbb{S}$ with values in $\mathcal{C}^\infty(Y^2)$. We will consider the subalgebra of functions which have (full) symbols vanishing to infinite order at one point $p \in \mathbb{S}_+$.
(3) We then consider the corresponding Toeplitz algebra SAS with A of this form and define G^0 to be the group of operators of the form $\mathrm{Id} + SAS$ which are elliptic on $\mathbb{S}_+$ and invertible.
(4) This group G^0 is actually contractible.

10+. Addenda to Lecture 10