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Abstract. (1) The spaces of conormal functions $\Psi^m_{c,\lambda}(\mathbb{R}^{k+n}, \{z = 0\})$.
(2) Properties
(3) The global space $\Psi^m(M, S; V)$ for a closed embedded submanifold $S$ and (complex) vector bundle $V$.
(4) Elliptic Dirac operators $\partial : C^\infty(M; W) \to C^\infty(M; W)$ are Fredholm.

1. Hadamard regularization

I did not go through this in class.

Although I think ‘regularization’ of homogeneous functions to distributions by analytic continuation is the clearest approach let me present another, arguably more elementary approach.

Namely, we can regularize homogenous smooth functions $F_s(z) = a(\|z\|^s)$ on $\mathbb{R}^n \setminus \{0\}$ by considering the behaviour of the integral

$$ \int_{|z| > \epsilon} a(\hat{z})|z|^s \phi(z) dx, \quad \phi \in S(\mathbb{R}^n) $$

as $\epsilon \downarrow 0$. Writing this in polar coordinates we can carry out the integral over the sphere and examine

$$ \phi_a(|z|) = \int_{S^{n-1}} a(\|z\|) \phi(\|z\|) d\omega. $$

Now, as a function of $r$, $\phi(r\omega)$ is smooth down to $r = 0$ where it has Taylor series given in terms of the Taylor series of $\phi$:

$$ \sum_{k=0}^{\infty} \left( \sum_{\alpha = k} \phi_{\alpha} \omega^\alpha \right) r^k, \quad \phi(z) \simeq \sum_{\alpha} \phi_{\alpha} z^\alpha. $$

Thus, $\phi_a(r)$ is smooth down to $r = 0$ where it has Taylor series

$$ \phi_a(r) \simeq \sum_k c_{k,a} r^k, \quad \sum_{\alpha = k} \phi_{\alpha} \int_{S^{n-1}} a(\omega) \omega^\alpha d\omega. $$

The integral (1) can be written

$$ \int_1^\infty \phi_a(r) r^{n-1+s} dr = \int_1^\infty \phi_a(r) r^{n-1+s} dr + \int_1^1 \left( \phi_a(r) - \sum_{k=0}^{N} c_{k,a} r^k \right) r^{n-1+s} dr $$

$$ + \sum_{k=1}^{N} c_{k,a} \frac{1}{n+k-s} - \sum_{k=1}^{N} c_{k,a} \frac{\epsilon^{n+k-s}}{n+k-s} $$

Here the first integral on the right is convergent as is the second provided $N$ is large enough so

$$
(6) \quad s \notin -n - N_0 \implies
\tilde{F}_s(\phi) = \lim_{\epsilon \downarrow 0} \left( \int_{\epsilon}^{\infty} \phi_a(r)r^{n-1+s}dr + \sum_{k=1}^{N} c_{k,a} \frac{\epsilon^{n+k+s}}{n+k+s} \right)
= \int_{1}^{\infty} \phi_a(r)r^{n-1+s}dr + \int_{0}^{1} \left( \phi_a(r) - \sum_{k=0}^{N} c_{k,a} r^k \right) r^{n-1+s}dr + \sum_{k=1}^{N} \frac{c_{k,a}}{n+k+s},
N > -n - 1 - \Re s
$$

is well-defined independent of $N$ (because increasing $N$ beyond its minimal value changes the last sum in (5) by terms that vanish as $\epsilon \downarrow 0$).

**Lemma 1.** For $F_s(z) = a(\omega)|z|^s$ homogeneous of degree $s \notin -n - N_0$ on $\mathbb{R}^n \setminus \{0\}$, (6) defines a distribution $\tilde{F}_s \in S'(\mathbb{R}^n)$ which is homogeneous of degree $s$ in the sense that it satisfies Euler's condition

$$
(7) \quad (z \cdot \partial_z - s) \tilde{F}_s = 0.
$$

Conversely any distribution solution of this equation, for $s \notin -n - N_0$, arises this way.

**Proof.** The formal adjoint of $z\partial_z - s$ is $-z\partial_z - s - n$ so we wish to compute

$$
(8) \quad \tilde{F}_s((-z\partial_z - n - s)\phi), \ \phi \in S(\mathbb{R}^n).
$$

Replacing $\phi \in S(\mathbb{R}^n)$ by $z\partial_z \phi$ replaces $\phi_a(r)$ by $r d\phi_a(r)/dr$ and then integration by parts gives

$$
\int_{\epsilon}^{\infty} r \frac{d\phi_a(r)}{dr} r^{n-1+s}dr = -(n+s) \int_{\epsilon}^{\infty} \phi_a(r)r^{n-1+s}dr - \phi_a(\epsilon)\epsilon^{n+s}.
$$

From (4) the Taylor series of $z\partial_z \phi \simeq \sum_{\alpha} \alpha \phi_a z^\alpha$ so the constants for $z\partial_z \phi$ are

$$
(9) \quad c_{k,a}(z\partial_z \phi) = k c_{k,a}(\phi).
$$

Combining these formulæ

$$
(10) \quad \tilde{F}_s((-z\partial_z - s - n)\phi) = \lim_{\epsilon \downarrow 0} \left( \phi_a(\epsilon)\epsilon^{n+s} - \sum_{k=1}^{N} k c_{k,a} \frac{\epsilon^{n+k-s}}{n+k+s} - (n+s) \sum_{k=1}^{N} c_{k,a} \frac{\epsilon^{n+k-s}}{n+k+s} \right)
= \lim_{\epsilon \downarrow 0} \left( \phi_a(\epsilon) - \sum_{k=1}^{N} c_{k,a} \epsilon^k \right) \epsilon^{n-s} = 0.
$$
When $s = −n − p$ for $p ∈ N_0$ one of the terms has exponent $−1$ so the formula (5) is replaced by
\begin{equation}
\int_0^∞ \phi_a(r)r^{−1−p}dr = \int_1^∞ \phi_a(r)r^{−1−p}dr + \int_0^1 \left(\phi_a(r) − \sum_{k=0}^N c_{k,a}k^k\right)r^{−1−p}dr
\end{equation}
\begin{equation}
+ \sum_{k=1,|p|}^N c_{k,a} \frac{1}{k−p} − \sum_{k=1,|p|}^N c_{k,a} \frac{e^{n+k−s}}{n+k−s} − c_{p,a} \log \epsilon.
\end{equation}
The regularization works just as well by defining
\begin{equation}
\tilde{F}_{−n−p}(\phi) = \lim_{\epsilon \downarrow 0} \left(\int_0^∞ \phi_a(r)r^{n−1+s}dr + \sum_{k=1,|p|}^N c_{k,a} \frac{e^{n+k+s}}{n+k+s} + c_{p,a} \log \epsilon\right) \Rightarrow \tilde{F}_{−n−p} ∈ S(\mathbb{R}^n).
\end{equation}
However the distribution $\tilde{F}_{−n−p}$ satisfies
\begin{equation}
(z \cdot \partial_z + n_p)\tilde{F}_{−n−p} = \sum_{|\alpha|=p} d_\alpha D^\alpha_z \delta,
\end{equation}
where the coefficients $d_\alpha$ are linearly equivalent to the coefficients of the logarithmic term in (12)
\begin{equation}
\int_{S_{n−1}} a(\omega) \omega^\alpha d\omega, \ |\alpha| = p.
\end{equation}
Thus we get a homogeneous distribution if and only if these constants vanish.

**Proposition 1.** The space $M_s(\mathbb{R}^n)$ of homogeneous distributions on $\mathbb{R}^n$ of degree $s$, i.e. solutions of
\begin{equation}
(z\partial_z − s)u = 0
\end{equation}
is isomorphic to $C^∞(\mathbb{S}^{n−1})$ or each $s$; for $s \notin −n − N_0$ it is spanned by the homogeneous extensions of the $a(z)|z|^−s$ discussed above and for $s = −n − p, p ∈ N_0$, by the homogeneous extensions of the $a(z)|z|^{−n−p}$ for which (14) holds together with the derivatives $D^\alpha_z \delta$ for $|\alpha| = p$.

**Exercise 1.** Show that dropping the assumption of smoothness outside the origin doesn’t change things significantly -- the arguments above go through with $a ∈ C^∞(\mathbb{S}^{n−1})$, the distributions on the sphere.

Homogeneity shows that
\begin{equation}
D^\alpha_z : M_s(\mathbb{R}^n) → M_{s−|\alpha|}(\mathbb{R}^n),
\end{equation}
\begin{equation}
z^\beta : M_s(\mathbb{R}^n) → M_{s+|\beta|}(\mathbb{R}^n).
\end{equation}
This can be seen from the commutation results
\begin{equation}
D_j ∘ z\partial_z = (z\partial_z + 1) ∘ D_j, \ z_k ∘ z\partial_z = (z\partial_z − 1) ∘ z_k.
\end{equation}
Homogeneous functions are very global but since we have required that the elements of $M_s$ be smooth outside the origin, we can cut them off by taking $\chi ∈ C^∞_{\mathbb{S}^{n−1}}(\mathbb{R}^n), \ \chi = 1$ near 0 and then see that
\begin{equation}
(z \cdot \partial_z − s)u′ ∈ C^∞_{\mathbb{S}^{n−1}}(\mathbb{R}^n), \ u′ = \chi u, \ u ∈ M_s(\mathbb{R}^n).
\end{equation}
It is actually the solutions of (18) that we are really interested in.
Definition 1. Let \( Q'_s(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n) \) consist of the compactly supported distributions which are smooth outside the origin and satisfy (18).

Proposition 2. The quotient \( Q_s = Q'_s/(Q'_s \cap \mathcal{C}_c^\infty(\mathbb{R}^n)) \) is canonically isomorphic to \( M_s \) unless \( s = k \in \mathbb{N}_0 \) in which case it is spanned by the equivalent classes of the elements of \( M_s \)

\[
(19) \quad a(\omega)|z|^k, \quad \int_{\mathbb{S}^{n-1}} a(\omega)\omega^\alpha d\omega = 0, \quad \forall |\alpha| = k
\]

together with the locally integrable functions \( p(z) \log |z| \) where \( p(z) \) is a homogeneous polynomial of degree \( k \).

Proof. If \( g(z) \) is a homogeneous polynomial of degree \( k \) then

\[
(20) \quad (z\partial_z - s)p(z) = (k - s)p(z).
\]

So if \( s \notin \mathbb{N}_0 \) and \( f \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) with Taylor series \( f \simeq \sum z^\alpha \) at 0 using Borel’s Lemma we can find \( w \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) with Taylor series \( w \simeq \sum f(\alpha - s)z^\alpha \) and then

\[
(21) \quad (z\partial_z - s)w = f + g, \quad g \in \mathcal{C}_c^\infty(\mathbb{R}^n), \quad \partial^\alpha_z g(0) = 0 \quad \forall \alpha.
\]

Then \( g(r, \hat{z}) \in \mathcal{C}_c^\infty([0, \infty) \times \mathbb{S}^{n-1}) \) vanishes with all its derivatives at \( r = 0 \) and hence

\[
(22) \quad v(r, \hat{z}) = r^{-s} \int_0^r g(t, \hat{z})t^s dt \text{ satisfies } (r\partial_r - s)v(r, \hat{z}) = f(r, \hat{z})
\]

and \( v \) also is smooth and vanishes with all its derivatives at the origin. Thus in fact \( v(r, \hat{z}) = w'(r\hat{z}) \) with \( w' \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) (with Taylor series trivial at 0) and

\[
(23) \quad (z\partial_z - s)(w - w') = f
\]

has a smooth solution. Thus if \( u' \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) is smooth outside the origin and, as in (18), satisfies \( (z\partial_z - s)u' = f \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) then there exists \( u'' \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) such that

\[
(24) \quad (z\partial_z - s)u = 0, \quad u = u' - u''.
\]

The function \( u'' \) is uniquely determined by \( f \) since there are no smooth functions homogeneous of degree \( s \) and hence this constructs a map

\[
(25) \quad Q_s \ni u' \longmapsto u \in M_s.
\]

The null space consists of the smooth elements of \( Q'_s \) and the map is surjective since if \( u \in M_s \) and \( u' = \chi u \) with \( \chi \) a cut-off as above, then this construction reproduces \( u \). This proves the result for \( s \notin \mathbb{N}_0 \).

In case \( s = p \in \mathbb{N}_0 \) and \( f \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) the same argument applies to construct \( w \in \mathcal{C}_c^\infty(\mathbb{R}^n) \) such that

\[
(26) \quad (z\partial_z - p)w = f - f_p(z)
\]

where \( f_p(z) \) is the homogeneous part of degree \( p \) in the Taylor series of \( f \). Directly

\[
(27) \quad (z\partial_z - p)f_p(z) \log |z| = f_p(z)
\]

Thus \( u'' = u' - w - f_p(z) \log |z| \) is smooth away from the origin and satisfies

\[
(28) \quad (z\partial_z - p)u'' = 0.
\]

Thus \( u'' \) is a homogeneous function smooth outside the origin and this constructs a map \( Q'_p 
\rightarrow M_s + P_p(z) \log |z| \) with null space the smooth elements of \( Q'_p \). \( \square \)
The commutation identities (17) show that

\[ D_j : Q'_s \rightarrow Q'_{s-1}, \quad z_j : Q'_s \rightarrow Q'_{s+1} \]

and since they act on the smooth subspaces they act on the quotient spaces of microfunctions

\[ D_j : Q_s \rightarrow Q_{s-1}, \quad z_j : Q_s \rightarrow Q_{s+1}, \quad \forall s \in \mathbb{C}. \]

2. Classical conormal functions

Earlier I defined the spaces \( \Psi^m(\mathbb{R}^n, \{0\}) \) of classical conormal functions at a point, specifically the origin, in \( \mathbb{R}^n \); I have decided to change the notation to \( J^m(\mathbb{R}^n, \{0\}) \). These are Fréchet spaces so we can define smooth maps into them. Alternatively we can add parameters to the discussion above and consider distributions of compact support on \( \mathbb{R}_+^n \times \mathbb{R}_o^n \) which are smooth in \( z \neq 0 \) (in all variables) and satisfy

\[ (z\partial_z - s)u(y, z) = f(y, z) \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^n). \]

The arguments go through unchanged with parameters so \( C_c^\infty(\mathbb{R}^k; Q'_s(\mathbb{R}^n)) \) and its quotient are well-defined. Note that

\[ C_c^\infty(\mathbb{R}^k; Q'_s(\mathbb{R}^n)) \subset C(\mathbb{R}^k \times \mathbb{R}^n) \text{ if } \Re s > 0. \]

Combined with (29) we conclude that

\[ C_c^\infty(\mathbb{R}^k; Q'_s(\mathbb{R}^n)) \subset C^M(\mathbb{R}^k \times \mathbb{R}^n) \text{ if } \Re s > M. \]

Now we proceed to define classical conormal functions, now for the submanifold \( z = 0 \) as asymptotic sums, with respect to regularity, of the \( Q'_s \):

4. \( J^m(\mathbb{R}^{k+n}, \{z = 0\}) = \{u \in C_{c, \infty}^{-\infty}(\mathbb{R}^{k+n}); u|_{z \neq 0} \text{ is smooth, } \exists u_k \in C^\infty(\mathbb{R}^k; Q'_{s, m-n+k}(\mathbb{R}^n)), \quad k \in \mathbb{N}_0, \}

\[ u - \sum_{k=0}^{N} u_k \in C^M, \quad M \rightarrow \infty \text{ with } N \}

We can then see the dependence of \( M \) on \( N \) – namely to get an error with \( M \) derivatives we need \( N > M + \Re m + n \). We are most interested in the case that \( m \in \mathbb{Z} \).

We can deduce properties of these new spaces from those of the \( Q'_s \).

Proposition 3. For any \( m \in \mathbb{C} \)

\[ D_{yi} : J^m(\mathbb{R}^{k+n}, \{z = 0\}) \rightarrow J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \]

\[ D_{zj} : J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \rightarrow J^{m+1}_c(\mathbb{R}^{k+n}, \{z = 0\}) \]

\[ x_j : J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \rightarrow J^{m-1}_c(\mathbb{R}^{k+n}, \{z = 0\}) \]

\[ C^\infty(\mathbb{R}^{k+n}) \times J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \rightarrow J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \]

and if \( h : \Omega \rightarrow \Omega' \) is a diffeomorphism between open subsets of \( \mathbb{R}^{k+n} \) such that \( h(\{z = 0\} \cap \Omega) \subset \{z = 0\} \cap \Omega' \) then

\[ h^* : \{u \in J^m_c(\mathbb{R}^{k+n}, \{z = 0\}); \text{supp}(u) \subset \Omega' \} \rightarrow J^m_c(\mathbb{R}^{k+n}, \{z = 0\}). \]
Proof. The first result in (5) is a consequence of the fact that the $y$ coordinates are parameters in the definition of the $C_c^\infty(\mathbb{R}^n_0; Q_s^* (\mathbb{R}^n))$. Similarly the second and third mapping properties follow from (30) and the action on the remainder.

To show that $J^m(\mathbb{R}^{k+n}, \{ z = 0 \})$ is a module under the multiplicative action of $C^\infty(\mathbb{R}^{k+n})$ note that, again since the $y$ variables are parameters, this is certainly the case for $C^\infty(\mathbb{R}^n_0)$. For the general case, expand in Taylor series around $z = 0$:

$$b(y, z) = \sum_{|\alpha|<N} b_\alpha(y) z^\alpha + \sum_{|\alpha|=N} b'_\alpha(y, z) z^\alpha.$$  

Applying the first and third results shows that the terms in the finite sum map $J^m$ into $J^{m-|\alpha|}$ and the same is true for the $z^\alpha$ in the remainder term in (7). For $|\alpha|$ large enough $J^{m-|\alpha|} \subset C^M(\mathbb{R}^{k+n})$ which gives the defining condition for $bu \in J^m_c(\mathbb{R}^{k+n}, \{ z = 0 \})$.

Diffeomorphism invariance is a little more involved. From the preceding results we can decompose any $u \in J^m_c(\mathbb{R}^{k+n}, \{ z = 0 \})$ using a partition of unity and so it suffices to suppose that the support is near some point. We start by decomposing the diffeomorphism $h$. Since $h$ preserves $z = 0$ we can define a new diffeomorphism (near any point)

$$h_0(y, z) = (h(y, 0), z).$$

This acts only on the parameters so $h_0^*$ preserves the $J^m$. Thus we can replace $h$ by $h_0^{-1}h$ so now

$$h(y, z) = (y_s + \sum_j z_j a_{sj}(y, z), \sum_l b_{sl}(y, z) z_l).$$

The Jacobian at $(y, 0)$ is of the block form

$$\begin{pmatrix} \text{Id}_y & \ast \\ 0 & b_{*\ast}(y, 0) \end{pmatrix}$$

so $b$ is invertible and the linearization

$$h_1(y, z) = (y, \sum_j b_{sj}(y, 0) z_j)$$

is (locally) a diffeomorphism. The fact that $h_1^* : J^m \rightarrow J^m$ follows from the fact that the $y$s are parameters and linear transformations in $z$ act on the $Q_s^*$ – indeed the Euler vector field $z \cdot \partial_z$ is invariant under such (parameterized) linear transformations. Thus again we can replace $h$ by $h_1^{-1}h_0^{-1}h$ and so assume it has the form

$$h(y, z) = (y_s + \sum_j b_{sj} z_j, z_s + \sum_{jk} c_{sjk} z_j z_k).$$

The Jacobian is now the block upper triangular as in (10) with $b$ also the identity.

To proceed further we use and idea of M"oser. The diffeomorphism $h$ in (12) is so close to the identity that (locally near $z = 0$ which is all that matters) we can connect it to the identity by a path of diffeomorphisms

$$h_s(y, z) = (y_s + s \sum_j b_{sj} z_j, z_s + s \sum_{jk} c_{sjk} z_j z_k), \ s \in [0, 1].$$

Thus $h_0 = \text{Id}, h_1 = h$. 

Suppose that \( u : [0, 1] \to J^m_c(\mathbb{R}^{k+n}, \{ z = 0 \}) \) is a smooth path – really we can just add \( s \) as another parameter. Then consider the path, at this stage in the space of distributions
\[
h^*_s u_s.
\]
This is smooth in \( s \) and the chain rule gives a formula for the derivative
\[
\frac{d}{ds} h^*_s u_s = h^*_s (\frac{du_s}{ds} + V_s u_s).
\]
Here \( V_s \) is a smooth vector field which ‘generates the path of diffeomorphisms’. Since \( h_s \) is of the special form (13) we see that
\[
V_s = \sum_{i,j} e_{ij}(y, z) z_j \partial_{y_i} + \sum_{jkl} e'_{jkl}(y, z) z_j z_k \partial_{z_l}
\]
with smooth coefficients. Exploiting the earlier results we see that
\[
V_s : J^m_c([0, 1] \times \mathbb{R}^{k+n}, \{ z = 0 \}) \to J^{m-1}_c([0, 1] \times \mathbb{R}^{k+n}, \{ z = 0 \})
\]
‘lowers the order’ by (at least) one.

Now, the basic idea is to solve the equation
\[
\frac{du_s}{ds} + V_s u_s = 0.
\]
We don’t quite need to do this but instead let’s show that we can choose \( u_s \) so that, for any preassigned \( M \),
\[
\frac{du_s}{ds} + V_s u_s \in C^M, \ u_1 = u.
\]
Since the \( C^M \) spaces are preserved by diffeomorphism it then follows by integration that
\[
h_1^* u_1 - h_0^* u_0 = h^* u - u_0 \in C^M.
\]
This is just another way of saying that \( h^* u \in J^m_c(\mathbb{R}^{k+n}, \{ z = 0 \}) \).

So we are reduced to solving (18) for every large \( M \). We can do this by iteration. Make a first attempt by taking a constant curve
\[
u^0_s = u, \ s \in [0, 1] \implies \frac{du^0_s}{ds} + V_s u^0_s = v^1_s \in J^{m-1}_c([0, 1] \times \mathbb{R}^{k+n}, \{ z = 0 \})
\]
using (17). Now for the second try we simply integrate
\[
u^1_s = \int_s^1 v^1_s \in J^{m-1}_c(\mathbb{R}^{k+n}, \{ z = 0 \})
\]
which will abbreviate by saying it has order \( m - 1 \). Then
\[
u^2_s = 0, \ \frac{du^1_s}{ds} + V_s u^1_s = -v^1_s + v^2_s, \ v^2_s \in J^{m-2}_c([0, 1] \times \mathbb{R}^{k+n}, \{ z = 0 \})
\]
where the first term comes from the derivative and the second from the vector field. Clearly we can proceed iteratively and then sum the result to see that after sufficiently many steps
\[
u_s = \sum_l u^l_s \text{ satisfies (18)}.
\]
This we have proved the diffeomorphism invariance.
\[\square\]

We also want to know about integration.
Lemma 2. Integration in the last $n' < n = n'' + n'$ of the $z$ variables gives a continuous map

$$
\int dz_{n''+1} \cdots dz_n : J^m_c (\mathbb{R}^{k+n}, \{z = 0\}) \rightarrow J^m_c (\mathbb{R}^{k+n''}, \{z'' = 0\}).
$$

Existence is rather important too!

Proposition 4. If $u_j \in C^\infty_c (\mathbb{R}^j; Q'_{-m-n+j} (\mathbb{R}^n, \{0\})$ there exists $u \in J^m_c (\mathbb{R}^{k+n}, \{z = 0\})$ such that for given $M$,

$$
u - \sum_{j < N} u_j \in C^M (\mathbb{R}^{k+n})$$

for large enough $N$.

References

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