NOTES FOR 18.156 SPRING 2020

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Abstract.

1. Euclidean Dirac operators

Rough notes for Lecture 1, 18.156 Spring 2020

I want to connect with what you did in 18.155 – did everyone here take 155 – by talking about constant coefficient Dirac operators. Why are we interested in these? First, they are a 'model' for the variable coefficient case and secondly they are a model for non-compact manifolds. Let's think about the second of these first although perhaps the first is more fundamental!

What is a constant-coefficient Dirac operator? It is a constant coefficient differential operator acting, on some $\mathbb{R}^n$, on vector-valued functions, so taking values in some $\mathbb{C}^N$, of first order and (for the moment at least) having no lower order terms:

3.2.2020.1
$$\eth = \sum_{j=1}^{n} \Gamma_j D_j. \tag{1}$$

Here Γ_j are constant $N \times N$ matrices and the D_j are the basic differential operators with and i :

3.2.2020.2
$$D_j = \frac{1}{i}\partial_{x_j}. \tag{2}$$

The i is inserted (by some people, including me ..) so that the Fourier transform does not have an i :

3.2.2020.5
$$\widehat{D_j u}(\xi) = \xi_j \hat{u} \tag{3}$$

if for instance u is a Schwartz function.

For a Dirac operator we further require that the square be scalar

3.2.2020.3
$$\eth^2 = \sum_{j,k=1}^{n} \Gamma_j \Gamma_k D_j D_k = \operatorname{Id} Q(D). \tag{4}$$

Here Q must be a homogeneous polynomial of degree 2. There are two very standard special cases (and there are others), where Q is positive-definite and where $Q \equiv 0$. Both are of interest but lets concentrate on the 'elliptic' case where Q is positive definite. We can always make a linear change of coordinates so that

3.2.2020.4
$$Q = |\xi|^2. \tag{5}$$

Then $Q(D)$ is the standard Euclidean Laplacian.

Now, as you already know the condition (4) [3.2.2020.3] written in terms of the matrices as

3.2.2020.6
$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij} \tag{6}$$

is related to the Clifford algebra on $\mathbb{R}^n$. I will remind you of this in due course. For the moment it means that we can understand the Dirac operator in terms of the Laplacian.

Now, one attitidue to $\eth$ is to think of it as a constant coefficent operator, which of course it is, and elliptic. What does this mean. Well it acts on the standard Sobolev spaces

3.2.2020.7
$$\eth : H^{s+1}(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n)\ \forall\ k \in \mathbb{R}. \tag{7}$$

As such it is a 'closed' operator. Maybe we should care about that, but not for the moment. The bad feature of (7) is that this operator/map is *never* Fredholm.

Rememeber that Fredholm is the infinite dimensional analog of invertible for a matrix. It means that at least you know the solvability properties modulo finite dimensional issues. So, why is (7) *not* Fredholm. Basically because the range is not closed. The easiest way to see this is to find related spaces on which it *is* Fredholm and then compare.

The 'secret' here is that the behaviour of $\eth$ and also Δ of course, is dominated by the homogeneity rather than the constancy of the coefficients. The latter lets us come to grips with the operator but it is the former that 'makes things work'. For homogeneity we avoid the origin – since the problem is translation-invariant this just means that we stay away from a point. As you will see what it really means is that we think about what is happening 'at infinity'.

If $u(tx) = t^\sigma u(x)$ is a smooth vector-valued function in $x \neq 0$ which is homogeneous of degree r then

3.2.2020.8
$$\eth u(t^\sigma x) = t^{\sigma-1}\eth u(x) \tag{8}$$

is homogeneous of degree $r-1$. That is the problem with (7), our operator is 'small at infinity' in the sense that it makes things smaller there. Typical elements of the standard Sobolev spaces all have the same behaviour at infinity – they are roughly L^2.

A homogenous function of degree σ is just the product of a homogneneous function of degree 0 (vector-valued) with $|x|^\sigma$. The natural thing to do then is to introduced polar coordinates in $\mathbb{R}^n$

3.2.2020.9
$$x = r\omega,\ r = |x|,\ \omega = \frac{x}{|x|} \in \mathbb{S}^{n-1}. \tag{9}$$

When we do this our Dirac operator becomes

3.2.2020.10
$$\eth u(x) = M(\omega)(D_r + \frac{i}{r}\eth') \tag{10}$$

where $\eth'$ is a (Dirac, which is why I have written it in this weird way) operator on the sphere. Namely

3.2.2020.11
$$\eth' v(\omega) = rM(\omega)\eth v(\omega)\big|_{r=1} \tag{11}$$

where on the left $v : \mathbb{S}^{n-1} \longrightarrow \mathbb{C}^N$ and on the right, v is extended to be homogeneous of degree 0 – so of course the r out the front does nothing (and $M^2 = \mathrm{Id}$).

The form of $\eth$ in (10) suggests that we capture the fact that it is small at infinity by writing it (away from the origin) as

3.2.2020.12
$$\eth = r^{-1}M(\omega)\left(rD_r + i\eth'\right). \tag{12}$$

Taking this seriously we can define modified Sobolev spaces on $\mathbb{R}^n$ which take into account this structure. First we do something that is only book-keeping.

Namely we think of the Euclidean measure as weighted relative to an asymptotically homogeneous measure

3.2.2020.13 (13)
$$|dx| = r^n \frac{dr}{r} d\omega,\ \nu_b = \langle x \rangle^{-n} |dx| \simeq \frac{dr}{r} d\omega,\ \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

If we take L^2 functions with respect to this new measure and denote the space

3.2.2020.14 (14)
$$L^2_{\mathrm{b}}(\mathbb{R}^n) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n); \int |u|^2 \langle x \rangle^{-n} dx < \infty\}$$

then we can write

3.2.2020.15 (15)
$$L^2(\mathbb{R}^n) = \langle x \rangle^{-n/2} L^2_{\mathrm{b}}(\mathbb{R}^n).$$

So the ordinary L^2 space is quite a lot smaller.

Now, for the Sobole part. If we look at the operators rD_r and D_ω we can see that they are 'the same size as' the homogenous differential operators $x_j D_k$. We define for any integer $k \ge 0$,

3.2.2020.16 (16)
$$H^k_{\mathrm{b}}(\mathbb{R}^n) = \{u \in L^2_{\mathrm{b}}(\mathbb{R}^n); x^\alpha D^\beta u \in L^2_{\mathrm{b}}(\mathbb{R}^n),\ |\alpha| = |\beta| \le k\}.$$

This is actually a cumbersome way if writing something pretty simple. The point is that we can now state a result which was already in the lectures last semester:

3.2.2020.18 **Theorem 1.** *An elliptic constant coefficient Dirac operator defines a Fredholm operator*

3.2.2020.19 (17)
$$\eth : \langle x \rangle^{-\sigma} H^k_{\mathrm{b}}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \langle x \rangle^{-\sigma-1} H^{k-1}_{\mathrm{b}}(\mathbb{R}^n; \mathbb{C}^N)\ \forall\, k,\ \forall\, \sigma \notin -\mathbb{N}_0 \cup (\mathbb{N}_0 + n - 1)\ (n \ge 2).$$

The notation here is that

3.2.2020.20 (18)
$$u \in \langle x \rangle^{-\sigma} H^k_{\mathrm{b}}(\mathbb{R}^n; \mathbb{C}^N) \Longleftrightarrow \langle x \rangle^{\sigma} u \in H^k_{\mathrm{b}}(\mathbb{R}^n; \mathbb{C}^N).$$

The converse is also true in the sense that the operator is not Fredholm at these integer points.

How to understand, and of course also prove, this theorem. The idea here is that 'elliptic regularity' carries over to the derivatives rD_r and D_ω so that we recover 1 derivative but lose a weight in solviing the equaition. If you think about it you will see what this cannot be captured in terms of the standard Sobolev spaces in (7). 3.2.2020.7

2. Rough notes for Lecture 2
18.156 Spring 2020

2.1. **Results.**

L2.1 **Proposition 1.** *An elliptic Dirac operator with constant coeffficients*

L2.2 (1)
$$\eth = \sum_{j=1}^n \Gamma_j D_j,\ \Gamma_j \Gamma_k + \Gamma_k \Gamma_j = \delta_{jk}$$

has a unique homogenous fundamental solution

L2.3 (2)
$$E_\eth = c_n \sum_{j=1}^n \Gamma_j \frac{z_j}{|z|^n}.$$

The space of smooth functions on $\mathbb{R}^n \setminus \{0\}$ which are homogeneous of degree $s \in \mathbb{C}$ is isomorphich to $\mathcal{C}^\infty(\mathbb{S}^{n-1})$

L2.4
$$f(z) = |z|^s g(\frac{z}{|z|}),\ |z|^s = \exp(s \log |z|) \tag{3}$$

for the standard branch of the logarithm. Thes functions satisfy Euler's identity

L2.5
$$(z\partial_z - s)f = 0 \text{ on } \mathbb{R}^n \setminus \{0\}. \tag{4}$$

Use (4) as the definition of homogeneity for distributions but now over the whole of $\mathbb{R}^n$.

L2.6 **Proposition 2.** *The space Q_s of homogeneous distributions of degree s on $\mathbb{R}^n$ which are smooth outside the origin restricts to $\mathbb{R}^n \setminus \{0\}$ isomorphically to (3) for $s \notin \{-n, -n-1, \dots\}$ and for these exceptional values there is a short exact sequence*

L2.7
$$\{D_z^\alpha \delta; |\alpha| = k\} \longrightarrow Q_{-n-k} \longrightarrow \\ \{g \in \mathcal{C}^\infty(\mathbb{S}^{n-1}; \int g(\omega)\omega^\alpha d\omega = 0 \textit{ if } |\alpha| = k\},\ k \in \mathbb{N}_0. \tag{5}$$

For good or evil, this exceptional set of negative integal powers occurs frequently.

L2.8 **Proposition 3.** *The Fourier transform defines an isomorphism for each $s \in \mathbb{C}$*

L2.9
$$\mathcal{F} : Q_s \longrightarrow Q_{-s-n}. \tag{6}$$

The space of 'microfunctions' on an open set $U \subset \mathbb{R}^n$ is the quotient

L2.10
$$\mathcal{M}(U) = \mathcal{C}^{-\infty}(U)/\mathcal{C}^\infty(U) \tag{7}$$

of the distributions modulo the smooth functions. We do not really use this space, since it does not have a good topology for instance, but it is useful to think about its elements as being 'naked singularities'.

The only homogeoeous smooth functions on $\mathbb{R}^n \setminus \{0\}$ with smooth extensions across 0 are the polynomials

L2.11
$$P_k \subset Q_k. \tag{8}$$

Observe that these play a role in (5) as the Fourier transforms of the delta distributions in Q_{-k-n}. Taking the Fourier transform then gives a somewhat different version of (5). We will discuss these things a bit later.

3. Discussion

To see Propositin 2 we start with the 1-dimensional case. In fact we consider only distributions supported in $[0, \infty)$. For $\operatorname{Re} s > -1$ the function

L2.12
$$x_+^s = \begin{cases} = \exp(s \log x) & x > 0 \\ 0 & x < 0 \end{cases} \tag{1}$$

is locally integrable across the origin since

L2.13
$$|x_+^s| = |x|^{\operatorname{Re} s} \text{ in } x > 0. \tag{2}$$

Thus it defines a tempered distribution

L2.14
$$\langle x_+^s, \phi \rangle = \int_0^\infty x^s \phi(x) dx,\ \phi \in \mathcal{S}(\mathbb{R}). \tag{3}$$

If we take $\operatorname{Re} S > 0$ then x_+^s is continuous across the origin and for $\operatorname{Re} s > 1$ it is once continuously differentiable on the whole line. Moreover it satisfies

L2.15 (4)
$$\frac{d}{dx}x_+^s = sx_+^{s-1},\ \operatorname{Re} s > 1.$$

This is therefore true distributionally as well so after pairing

L2.16 (5)
$$\langle x_+^s, \phi\rangle = -\frac{1}{s+1}\langle x_+^{s+1}, \frac{d\phi}{dx}\rangle\ \forall\ \text{Res} > 1,\ \phi \in \mathcal{S}(\mathbb{R})$$

where the sign comes from integration by parts.

Now, keeping ϕ fixed we can regard the paring in (L2.14 3) as a function of s. As such it is holomorphic in $\operatorname{Re} s > -1$.

4. Notes for Lecture 3
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Abstract. Despite saying I would not do this yet, I will go through a direct proof that the fundamental solution of an elliptic Dirac operator with constant coefficients defines a convolution operator which give a 2-sided inverse showing that

BASE.1 (1)
$$\eth : \langle z\rangle^{-\sigma}H_{\mathrm{b}}^{\infty}(\overline{\mathbb{R}^n}) \longrightarrow \langle z\rangle^{-\sigma-1}H_{\mathrm{b}}^{\infty}(\overline{\mathbb{R}^n}),\ 0 < \sigma < n-1.$$

Depending how much time I have left, I will then start to talk about parametrices for elliptic Dirac operators with variable coefficients, how we can construct them and how this shows that these operators are Fredholm.

So today's theorems:
Schur's Lemma, see below.

L3.2a **Theorem 2.** *If $\mathcal{E}$ and $\mathcal{F}$ are Fréchet spaces and $A : \mathcal{E} \longrightarrow CF$ is a continouous and surjective linear map then A is open: $A(O) \subset \mathcal{F}$ is open for each $O \subset \mathcal{E}$ open.*

L3.3a **Theorem 3.** *The fundamental solution of an elliptic, constant coefficient, Dirac operator defines a continuous linear map*

L3.4 (2)
$$E_{\eth}* : \langle x\rangle^{-\sigma-1}H_{\mathrm{b}}^{\infty}(\mathbb{R}^n) \longrightarrow \langle x\rangle^{-\sigma}H_{\mathrm{b}}^{\infty}(\mathbb{R}^n),\ 0 < \sigma < n-1.$$

L3.5 **Corollary 1.** *An elliptic Dirac operator with constant coefficients defines an isomorphsim inverse to (L3.4 2).*

L3.7 **Theorem 4.** *For any integer k let P_k be the space of polynomials homogeneous of degree k (so trivial for $k < 0$) and set*

L3.6 (3)
$$M_k = \{u \in \mathcal{S}'(\mathbb{R}^n); (z \cdot \partial_z - k)u \in P_k\},\ U_k = M_k/P_k$$

Then $\eth : U_k \longrightarrow U_{k-1}$ is an isomorphism for each $k > -n+1$.

L3.8 **Theorem 5** (Borel again)**.** *Let $u_n \in M_{k+n}$ be any sequence of 'homogeneous singularities' for $k \in \mathbb{Z}$ then there exists $u \in \mathcal{S}'(\mathbb{R}^n)$ of compact support such that*

L3.9 (4)
$$u = \sum_{n<N} u_n \in \mathcal{C}^{N-|k|-1}(\mathbb{R}^n)\ \forall\ N > |k|$$

and u is determined modulo $\mathcal{C}_c^{\infty}(\mathbb{R}^n)$.

L3.10 **Theorem 6.** *Every generalized Dirac operator with smooth variable coefficients in an open set $U \subset \mathbb{R}^n$*

L3.11
$$P(z, D_z) = \sum_{j=1}^{n} \Gamma_j(z) D_j + M(z),\ \Gamma_j(z)\Gamma_k(z) + \Gamma_k\Gamma_j = g_{jk}(z)\,\mathrm{Id},\ g_* >> 0 \tag{5}$$

has a parameterix

L3.12
$$\begin{gathered} E(z,y) \in L^1_{\mathrm{loc}}(U \times U; M(N)),\ U \in \mathcal{C}^\infty(U \times U \setminus \mathrm{Diag}; M(N)), \\ P(z, D_z)E(z,y) = \delta(z-y) + F(z,y),\ F \in \mathcal{C}^\infty(U \times U; M(N)). \end{gathered} \tag{6}$$

5. Invertibility of Euclidean $\eth$

Let me prove a weak version of part of the isomorphism/Fredholm theorem for Euclidean Dirac operator. Recall that, last time, I wrote out an explicit fundamental solution:

10.2.2020.1
$$\eth = \sum_{j=1}^{n} \Gamma_j D_j \Longrightarrow E_\eth = c_n \sum_{j=1}^{n} \Gamma_j \frac{x_j}{|x|^n} \text{ satisfies } \eth E_\eth = \delta_0. \tag{1}$$

What I want to show pretty directly is that

10.2.2020.2
$$\eth : \langle x\rangle^{-\sigma} H^\infty_{\mathrm{b}}(\mathbb{R}^n) \longrightarrow \langle x\rangle^{-\sigma-1} H^\infty_{\mathrm{b}}(\mathbb{R}^n) \text{ is an isomorphism if } 0 < \sigma < n-1. \tag{2}$$

It may be slightly counter-intuitive that it is easier to prove this version than the isomorphism-between-Sobolev spaces version. You judge!

The first step is to show the boundedness of

10.2.2020.3
$$E_\eth * : \langle x\rangle^{-\sigma-1} L^2_{\mathrm{b}}(\mathbb{R}^n) \longrightarrow \langle x\rangle^{-\sigma} L^2_{\mathrm{b}}(\mathbb{R}^n),\ 0 < \sigma < n-1. \tag{3}$$

For this we use Schur's Lemma, also called 'Schur's Criterion' to distinguish it from the lemma about irreducible representations.

10.2.2020.4 **Lemma 1** (Schur's Criterion)**.** *If (M, ν) is a measure space and $k : M \times M \longrightarrow \mathbb{C}$ is a measurable function such that*

10.2.2020.5
$$\sup_{x\in M} \int_M |k(x,y)|\nu(y),\ \sup_{y \in M} \int_M |k(x,y)|\nu(x) < \infty \tag{4}$$

then the integral operator

10.2.2020.6
$$K : f \longmapsto \int_M k(x,y) f(y) \nu(y) \tag{5}$$

is bounded on $L^2(M)$.

It is a useful criterion because it does not require (i.e. does not imply) that K is compact.

Proof. Take an element, $u \in L^2(M)$, in the dense subspace of simple functions, finite sums of complex multiples of charactetistic functions of sets of finite measure. The integrals below then exist and applying Cauchy-Schwarz

L3.2
$$\begin{aligned} |\int K(z,y)v(y)\nu(y)|^2 &\le \int |K(z,y)|\nu(y) \int |K(z,y)||v(y)|^2\nu(y) \\ &\le (\sup_z \int |K(z,y)|\nu(y)) \int |K(z,y)||v(y)|^2\nu(y). \end{aligned} \tag{6}$$

Now integrating in z, using Fubini's theorem and applying the other estimate gives boundedness with

L3.3
$$\|K\|^2 \le \sup_z \int |K(z,y)|\nu(y) \sup_y \int |K(z,y)|\nu(z) \tag{7}$$

□

Recall that to understand the spaces $H^k_{\mathrm{b}}(\mathbb{R}^n)$ we can think of $\mathbb{R}^n$ as compactified to a ball – by introducing 'inverted polar coordinates' near infinity to make it look like $[0,\epsilon)_x \times \mathbb{S}^{n-1}_\omega$, $x = 1/|x|$, $\omega = x/|x|$. The convolution kernel

10.2.2020.7
$$k(y,z) = \frac{y_i - z_i}{|y-z|^n} \tag{8}$$

is measurable – being continuous away from $y = z$.

We can divide K into four piece corresponding to $|z| < R$, $|y| < R$ and the opposite. The finite part is clear enough since

10.2.2020.8
$$|k(y,z)| \le C|y-z|^{-n-1} \tag{9}$$

is integrable in $|y| < R$ with the result bounded in $|z| < R+1$ and conversely. So this leaves three pieces, two of which are the same by symmetry.

When one or both variables is near infinity we need to remember the weight. In $|y|$, $|z| > 1$ The boundedness in (3) localized to $|y| < R$ and $|z| > 2R+1$ for some $R > 0$ is equivalent to the boundedness of the operator with kernel

10.2.2020.9
$$\chi_{|y|<R}k(y,z)|z|^{-\sigma-1} \text{ on } L^2_{\mathrm{b}}(\mathbb{R}^n). \tag{10}$$

Since we can assume that $|z|$. Bounding the kernel by

10.2.2020.10
$$\chi_{|y|<R}|k(y,z)| \le C|z|^{-n-\sigma}\chi_{|z|>2R+1} \tag{11}$$

shows that Schur's criterion holds precisely for $\sigma < n-1$ where the end point corresponds to integrability near infinity (with respect to $drd\omega/r$).

So it remains to consider the region where both variables are near infinity. The difficulty here is that there is 'underdermined behaviour'. If y, $z \to \infty$ in *different* direction then the norm of the difference $|y-z|$ is bigger than a small multiple of either, but near the diagonal it can be much larger.

One way to resolve this, which I advocate, is by 'blowing up the corner'. If we simply bound the kernel by $C|z-y|^{-n+1}$ we see that near infinity in both variables it is bounded by

10.2.2020.11
$$|\frac{\omega_1}{x_1} - \frac{\omega_2}{x_2}|^{-n+1} \tag{12}$$

with $\omega_i \in \mathbb{S}^{n-1}$ and x_1, $x_2 \ge 0$.

Blowing up the corner means introducing polar coordinates in x_1, x_2 but since they are both non-negative this is equivalent to introducting projective coordinates

10.2.2020.12
$$t = x_1 + x_2 \ge 0,\ s = \frac{x_1 - x_2}{x_1 + x_2} \in [-1,1],\ x_1 = \frac{1}{2}t(1+s),\ x_2 = \frac{1}{2}t(1-s). \tag{13}$$

Then the function in (12) becomes

10.2.2020.13
$$2^{-n+1}t^{-n+1}(1+s)^{-n+1}(1-s)^{-n+1}|(1-s)\omega_1 - (1+s)\omega_2|^{-n+1}. \tag{14}$$

The last part corresponds to the norm of a finite vector which can be written

10.2.2020.14
$$(\omega_1 - \omega_2) - s(\omega_1 + \omega_2) \tag{15}$$

where these two terms are orthogonal. So this can only vanish if both terms vanish. The first of course only vanishes where $\omega_1 = \omega_2$ but then $\omega_1 + \omega_2 \neq 0$ so we actually see that

10.2.2020.15
$$|(1-s)\omega_1 - (1+s)\omega_2| > c(|\omega_1 - \omega_2|^2 + |s|^2)^{\frac{1}{2}}, \ c > 0 \tag{16}$$

is essentially a distance.

Geometrically we have the following picture (add picture!)

How does this help? Well, one thing it suggests is that we can usefully cut the original kernel up into two pieces, $k_1 + k_2$ with one part, k_1, supported near $\omega_1 = \omega_2$, $s = 0$ which is the diagonal and the other vanishing near there. If we do this then this second part, k_2, will satisfy bounds from (14) where the last part is bounded, i.e.

10.2.2020.16
$$|k_2(y,z)| \le C|y|^{-n+1}|z|^{-n+1}. \tag{17}$$

The same argument as before gives us boundedness as in (3).

So, what about the part k_2 supported near the diagonal? On the support, $|s| \le \frac{1}{2}$ for instance, so only the first and last factors in (16) arise. In this region, near infinity, x_1 and x_2 are roughly the same size, and the same as t. We can write the kernel in terms of polar coordinates as

10.2.2020.17
$$K_2 f(x_1, \omega) = \int x_1^{-\sigma} \kappa(x_1, x_2, \frac{x_1 - x_2}{x_1 + x_2}, \omega_1, \omega_2) f(x_2, \omega_2) x_2^{\sigma+1} \frac{dx_2}{x_2} d\omega_2 \tag{18}$$

and one half of Schur's criterion becomes

10.2.2020.18
$$\sup_{x_1, \omega_1} \int |x_1^{-\sigma} x_2^{\sigma+1} \kappa| \frac{dx_2}{x_2} d\omega_2 < \infty. \tag{19}$$

We can change the variable of integration to s where $x_2 = x_1(1-s)/(1+s)$ so

10.2.2020.19
$$|\frac{dx_2}{x_2}| = |d\frac{1-s}{1+s}| \tag{20}$$

which is bounded by a constant multple of measure $|ds|$ on the support of the kernel which is in the region $|s| < \frac{1}{2}$. Ultimately then the boundedness on

6. Parametrix

Suppose now we have a variable coefficient first order elliptic differential operator $\eth$ acting on vector bundle V over a manifold M. What would we expect of a 'parametrix'. This is supposed to be an integral oerator which effecitvely inverts the differential operator. From what I said last time (but did not yet prove). What we look for is a kernel which is a section of the 'exterior' homomorphism bundle

10.2.2020.21
$$\mathrm{Hom}(V)_{x,y} = V_x \otimes V'_y = \mathrm{hom}(V_x, V_y) \text{ over } M \times M. \tag{1}$$

We look for a section

10.2.2020.20
$$K : \tag{2}$$

7. Notes for Lecture 4
18.156 Spring 2020

Abstract.

L4.1a **Proposition 4.** *The space of homogeneous microfunctions, of degree s, smooth outside the origin consists of the quotient by $\mathcal{C}_c^\infty(\mathbb{R}^n)$ of the space*

L4.2 (1) $K_c^s(\mathbb{R}^n) = \{u \in \mathcal{C}^{-\infty}(\mathbb{R}^n); u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}),\ (z \cdot \partial_z - s)u \in \mathcal{C}_c^\infty(\mathbb{R}^n)\}.$

From last time

L4.7 (2) $$M^s(\mathbb{R}^n) \equiv K_c^s(\mathbb{R}^n)/\mathcal{C}_c^\infty(\mathbb{R}^n).$$

L4.3 *Definition* 1. The space $\Psi_c^t(\mathbb{R}^n; \{0\})$ of classical conormal functions of order $t \in \mathbb{C}$ (note the weird normalization) consists of those elements $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ such that there is a sequence $v_k \in K_c^{-t-n+k}(\mathbb{R}^n)$ $k \in \mathbb{N}_0$ such that for each N

L4.4 (3) $$u - \sum_{k=0}^{M} v_k \in \mathcal{C}_c^N(\mathbb{R}^n)$$

for M large enough.

L4.5a **Lemma 2.** *For any sequence $v_k \in K_c^{-t-n+k}(\mathbb{R}^n)$ there exists $u \in \Psi_c^t(\mathbb{R}^n; \{0\})$ satisfying (3) and consequently*

L4.6 (4) $$\Psi_c^t(\mathbb{R}^n; \{0\})/\mathcal{C}_c^\infty(\mathbb{R}^n) \equiv \bigoplus_k M^{-s-n+k}(\mathbb{R}^n).$$

(1) Finish boundedness on weighted L^2 of $E_{\eth}*$.
(2) Isomorphism on homogeneous microfunctions.
(3) Classical conormal distributions at a point.
(4) Local parametrix construction.
(5) Global parametrices and Fredholm property.

8. Notes for Lecture 5, 18.156, 20 February, 2020

Abstract. (1) The spaces of conormal functions $\Psi_c^m(\mathbb{R}^{k+n}_{(y,z)}, \{z=0\})$.
(2) Properties
(3) The global space $\Psi^m(M, S; V)$ for a closed embedded submanifold S and (complex) vector bundle V.
(4) Elliptic Dirac operators $\eth : \mathcal{C}^\infty(M; W) \longrightarrow \mathcal{C}^\infty(M; W)$ are Fredholm.

9. Hadamard regularization

I did not go through this in class.

Although I think 'regularization' of homogeneous functions to distributions by analytic continuation is the clearest approach let me present another, arguably more elementary approach.

Namely, we can regularize homogenous smooth functions $F_s(z) = a(\omega)|z|^s$ on $\mathbb{R}^n \setminus \{0\}$ by considering the behaviour of the integral

21.2.2020.1 (1) $$\int_{|z|>\epsilon} a(\hat{z})|z|^s \phi(z) dx,\ \phi \in \mathcal{S}(\mathbb{R}^n)$$

as $\epsilon \downarrow 0$. Writing this in polar coordinates we can carry out the integral over the sphere and examine

21.2.2020.2 (2) $$\phi_a(|z|) = \int_{\mathbb{S}^{n-1}} a(\omega)\phi(|z|\omega) d\omega.$$

Now, as a function of r, $\phi(r\omega)$ is smooth down to $r=0$ where it has Taylor series given in terms of the Taylor series of ϕ :

21.2.2020.3
$$\sum_{k=0}^{\infty}\left(\sum_{\alpha=k}\phi_\alpha\omega^\alpha\right)r^k,\ \phi(z)\simeq\sum_{\alpha}\phi_\alpha z^\alpha. \tag{3}$$

Thus, $\phi_a(r)$ is smooth down to $r=0$ where it has Taylor series

21.2.2020.4
$$\phi_a(r)\simeq\sum_k c_{k,a}r^k,\ c_{k,a}=\sum_{\alpha=k}\phi_\alpha\int_{\mathbb{S}^{n-1}}a(\omega)\omega^\alpha d\omega. \tag{4}$$

The integral (21.2.2020.1 1) can be written

21.2.2020.5
$$\begin{aligned}\int_\epsilon^\infty\phi_a(r)r^{n-1+s}dr=\int_1^\infty\phi_a(r)r^{n-1+s}dr+\int_\epsilon^1\left(\phi_a(r)-\sum_{k=0}^N c_{k,a}r^k\right)r^{n-1+s}dr\\+\sum_{k=1}^N c_{k,a}\frac{1}{n+k-s}-\sum_{k=1}^N c_{k,a}\frac{\epsilon^{n+k-s}}{n+k-s}\end{aligned} \tag{5}$$

Here the first integral on the right is convergent as is the second provided N is large enough so

21.2.2020.6
$$\begin{gathered}s\notin-n-\mathbb{N}_0\Longrightarrow\\ \tilde F_s(\phi)=\lim_{\epsilon\downarrow0}\left(\int_\epsilon^\infty\phi_a(r)r^{n-1+s}dr+\sum_{k=1}^N c_{k,a}\frac{\epsilon^{n+k+s}}{n+k+s}\right)\\ =\int_1^\infty\phi_a(r)r^{n-1+s}dr+\int_0^1\left(\phi_a(r)-\sum_{k=0}^N c_{k,a}r^k\right)r^{n-1+s}dr+\sum_{k=1}^N\frac{c_{k,a}}{n+k+s},\\ N>-n-1-\operatorname{Re}s\end{gathered} \tag{6}$$

is well-defined independent of N (because increasing N beyond its minimal value changes the last sum in (21.2.2020.5 5) by terms that vanish as $\epsilon\downarrow0$).

21.2.2020.7 **Lemma 3.** *For $F_s(z)=a(\omega)|z|^s$ homogeneous of degree $s\notin-n-\mathbb{N}_0$ on $\mathbb{R}^n\setminus\{0\}$, (21.2.2020.6 6) defines a distribution $\tilde F_s\in\mathcal{S}'(\mathbb{R}^n)$ which is homogeneous of degree s in the sense that it satisfies Euler's condition*

21.2.2020.8
$$(z\cdot\partial_z-s)\tilde F_s=0. \tag{7}$$

Conversely any distribution solution of this equation, for $s\notin-n-\mathbb{N}_0$, arises this way.

Proof. The formal adjoint of $z\partial_z-s$ is $-z\partial_z-s-n$ so we wish to compute

21.2.2020.12
$$\tilde F_s((-z\partial_z-n-s)\phi),\ \phi\in\mathcal{S}(\mathbb{R}^n). \tag{8}$$

Replacing $\phi\in\mathcal{S}(\mathbb{R}^n)$ by $z\partial_z\phi$ replaces $\phi_a(r)$ by $rd\phi_a(r)/dr$ and then integration by parts gives

$$\int_\epsilon^\infty r\frac{d\phi_a(r)}{dr}r^{n-1+s}dr=-(n+s)\int_\epsilon^\infty\phi_a(r)r^{n-1+s}dr-\phi_a(\epsilon)\epsilon^{n+s}.$$

From (21.2.2020.4 4) the Taylor series of $z\partial_z\phi\simeq\sum_\alpha\alpha\phi_\alpha z^\alpha$ so the constants for $z\partial_z\phi$ are

21.2.2020.10
$$c_{k,a}(z\partial_z\phi)=kc_{k,a}(\phi). \tag{9}$$

Combining these formulæ

21.2.2020.11 (10)

$$\tilde{F}_s((-z\partial_z - s - n)\phi) = \lim_{\epsilon \downarrow 0}\left(\phi_a(\epsilon)\epsilon^{n+s} - \sum_{k=1}^{N} k c_{k,a}\frac{\epsilon^{n+k-s}}{n+k+s} - (n+s)\sum_{k=1}^{N} c_{k,a}\frac{\epsilon^{n+k-s}}{n+k+s}\right)$$
$$= \lim_{\epsilon \downarrow 0}\left(\phi_a(\epsilon) - \sum_{k=1}^{N} c_{k,a}\epsilon^k\right)\epsilon^{n-s} = 0.$$

□

When $s = -n - p$ for $p \in \mathbb{N}_0$ one of the terms has exponent -1 so the formula (5) is replaced by

21.2.2020.13 (11)

$$\int_\epsilon^\infty \phi_a(r) r^{-1-p} dr = \int_1^\infty \phi_a(r) r^{-1-p} dr + \int_\epsilon^1 \left(\phi_a(r) - \sum_{k=0}^{N} c_{k,a} r^k\right) r^{-1-p} dr$$
$$+ \sum_{k=1,\neq p}^{N} c_{k,a}\frac{1}{k-p} - \sum_{k=1,\neq p}^{N} c_{k,a}\frac{\epsilon^{n+k-s}}{n+k-s} - c_{p,a}\log\epsilon.$$

The regularization works just as well by defining

21.2.2020.14 (12)

$$\tilde{F}_{-n-p}(\phi) = \lim_{\epsilon \downarrow 0}\left(\int_\epsilon^\infty \phi_a(r) r^{n-1+s} dr + \sum_{k=1,\neq p}^{N} c_{k,a}\frac{\epsilon^{n+k+s}}{n+k+s} + c_{p,a}\log\epsilon\right) \Longrightarrow \tilde{F}_{-n-p} \in \mathcal{S}(\mathbb{R}^n).$$

However the distribution $\tilde{F}_{-n-p}$ satisfies

21.2.2020.15 (13)
$$(z \cdot \partial_z + n_p)\tilde{F}_{-n-p} = \sum_{|\alpha|=p} d_\alpha D_z^\alpha \delta$$

where the coefficients d_α are linearly equivalent to the coefficients of the logaritmic term in (12)

21.2.2020.16 (14)
$$\int_{\mathbb{S}^{n-1}} a(\omega)\omega^\alpha d\omega,\ |\alpha| = p.$$

Thus we get a homogeneous distribution if and only if these constants vanish.

21.2.2020.17 **Proposition 5.** *The space $M_s(\mathbb{R}^n)$ of homogeneous distributions on $\mathbb{R}^n$ of degree s, i.e. solutions of*

21.2.2020.18 (15)
$$(z\partial_z - s)u = 0$$

is isomorphic to $\mathcal{C}^\infty(\mathbb{S}^{n-1})$ or each s; for $s \notin -n - \mathbb{N}_0$ it is spanned by the homogeneous extensions of the $a(\hat{z})|z|^s$ discussed above and for $s = -n - p$, $p \in \mathbb{N}_0$, by the homogeneous extensions of the $a(\hat{z})|z|^{-n-p}$ for which (14) holds together with the derivatives $D_z^\alpha\delta$ for $|\alpha| = p$.

21.2.2020.19 *Exercise* 1. Show that dropping the assumption of smoothness outside the origin doesn't change things significantly – the arguments above go through with $a \in \mathcal{C}^{-\infty}(\mathbb{S}^{n-1}$, the distributions on the sphere.

Homogeneity shows that

21.2.2020.20 (16)
$$D_z^\alpha : M_s(\mathbb{R}^n) \longrightarrow M_{s-|\alpha|}(\mathbb{R}^n),$$
$$z^\beta : M_s(\mathbb{R}^n) \longrightarrow M_{s+|\beta|}(\mathbb{R}^n).$$

This can be seen from the commutation results

21.2.2020.21 (17)
$$D_j \circ z\partial_z = (z\partial_z + 1) \circ D_j,\ z_k \circ z\partial_z = (z\partial_z - 1) \circ z_k.$$

Homogeneous functions are very global but since we have required that the elements of M_s be smooth outside the origin, we can cut them off by taking $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$, $\chi = 1$ near 0 and then see that

21.2.2020.22 (18)
$$(z \cdot \partial_z - s)u' \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n),\ u' = \chi u,\ u \in M_s(\mathbb{R}^n).$$

It is actually the solutions of (18) that we are really interested in.

21.2.2020.23 *Definition* 2. Let $Q'_s(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$ consist of the compactly supported distributions which are smooth outside the origin and satisfy (18).

21.2.2020.24 **Proposition 6.** *The quotient* $Q_s = Q'_s/(Q'_s \cap \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n))$ *is canonically isomorphic to* M_s *unless* $s = k \in \mathbb{N}_0$ *in which case it is spanned by the equivalent classes of the elements of* M_s

21.2.2020.25 (19)
$$a(\omega)|z|^k,\ \int_{\mathbb{S}^{n-1}} a(\omega)\omega^\alpha d\omega = 0,\ \forall\ |\alpha| = k$$

together with the locally integrable functions $p(z)\log|z|$ *where* $p(z)$ *is a homogneous polynomial of degree* k.

Proof. If $q(z)$ is a homogeneous polynomial of degree k then

L4.5 (20)
$$(z\partial_z - s)p(z) = (k - s)p(z).$$

So if $s \notin \mathbb{N}_0$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ with Taylor series $f \simeq \sum_\alpha z^\alpha$ at 0 using Borel's Lemma we can find $w \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$ with Taylor series $w \simeq \sum_\alpha /(|\alpha_{-}s)z^\alpha$ and then

L4.6a (21)
$$(z\partial_z - s)w = f + g,\ g \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n),\ \partial_z^\alpha g(0) = 0\ \forall\ \alpha.$$

Then $g(r\hat{z}) \in \mathcal{C}_{\mathrm{c}}^\infty([0,\infty) \times \mathbb{S}^{n-1}$ vanishes with all its derivatives at $r = 0$ and hence

L4.7a (22)
$$v(r,\hat{z}) = r^{-s}\int_0^r g(t,\hat{z})t^s dt \text{ satisfies } (r\partial - s)v(r,\hat{z}) = f(r,\hat{z})$$

and v also is smooth and vanishes with all its derivatives at the origin. Thus in fact $v(r,\hat{z}) = w'(r\hat{z})$ with $w' \in \mathcal{C}^\infty(\mathbb{R}^n)$ (with Taylor series trivial at 0) and

L4.8 (23)
$$(z\partial_z - s)(w - w') = f$$

has a smooth solution. Thus if $u' \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$ is smooth outside the origin and, as in (18), satisfies $(z\partial_z - s)u' = f \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$ then there exists $w'' \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

L4.9a (24)
$$(z\partial_z - s)u = 0,\ u = u' - w''.$$

The function w'' is uniquely determined by f since there are no smooth functions homogeneous of degree s and hence this constructs a map

L4.10 (25)
$$Q_s \ni u' \longmapsto u \in M_s.$$

The null space consists of the smooth elements of Q'_s and the map is surjective since if $u \in M_s$ and $u' = \chi u$ with χ a cut-off as above, then this construction reproduces u. This proves the result for $s \notin \mathbb{N}_0$.

In case $s = p \in \mathbb{N}_0$ and $f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ the same argument applies to construct $w \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

L4.11
$$(z\partial_z - p)w = f - f_p(z) \tag{26}$$

where $f_p(z)$ is the homogeneous part of degree p in the Taylor series of f. Directly

L4.12a
$$(z\partial_z - p)f_p(z)\log|z| = f_p(z) \tag{27}$$

Thus $u'' = u' - w - f_p(z)\log|z|$ is smooth away from the origin and satisfies

L4.13a
$$(z\partial_z - p)u'' = 0. \tag{28}$$

Thus u'' is a homogeneous function smooth outside the origin and this constructs a map $Q'_p \longrightarrow M_s + P_p(z)\log|z|$ with null space the smooth elements of Q'_p. □

The commutation identities (21.2.2020.21 17) show that

L4.18
$$D_j : Q'_s \longrightarrow Q'_{s-1},\ z_j : Q'_s \longrightarrow Q'_{s+1} \tag{29}$$

and since they act on the smooth subspaces they act on the quotient spaces of microfunctions

L4.15
$$D_j : Q_s \longrightarrow Q_{s-1},\ z_j : Q_s \longrightarrow Q_{s+1},\ \forall\ s \in \mathbb{C}. \tag{30}$$

10. Classical conormal functions

Earlier I defined the spaces $\Psi^m(\mathbb{R}^n, \{0\})$ of classical conormal functions at a point, specifically the origin, in $\mathbb{R}^n$; I have decided to change the notation to $J^m(\mathbb{R}^n, \{0\})$. These are Fréchet spaces so we can define smooth maps into them. Alternatively we can add parameters to the discussion above and consider distributions of compact support on $\mathbb{R}^k_y \times \mathbb{R}^n_z$ which are smooth in $z \neq 0$ (in all variables) and satisfy

L4.16a
$$(z\partial_z - s)u(y,z) = f(y,z) \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^k \times \mathbb{R}^n). \tag{1}$$

The arguments go through unchanged with parameters so $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^k; Q'_s(\mathbb{R}^n))$ and its quotient are well-defined. Note that

L4.17a
$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^k; Q'_s(\mathbb{R}^n)) \subset \mathcal{C}(\mathbb{R}^k \times \mathbb{R}^n) \text{ if } \operatorname{Re} s > 0. \tag{2}$$

Combined with (L4.18 29) we conclude that

L5.19
$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^k; Q'_s(\mathbb{R}^n)) \subset \mathcal{C}^M(\mathbb{R}^k \times \mathbb{R}^n) \text{ if } \operatorname{Re} s > M. \tag{3}$$

Now we proceed to define classical conormal functions, now for the submanifold $z = 0$ as asymptotic sums, with respect to regularity, of the Q'_s :

L4.1
$$\begin{aligned} J^m_{\mathrm{c}}(\mathbb{R}^{k+n}_{y,z}, \{z=0\}) = \{&u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^{k+n});\\ &u\big|_{z\neq 0} \text{ is smooth, } \exists\ u_k \in \mathcal{C}^\infty(\mathbb{R}^k; Q'_{-m-n+k}(\mathbb{R}^n)),\ k \in \mathbb{N}_0,\\ &u - \sum_{k=0}^N u_k \in C^M,\ M \to \infty \text{ with } N\}. \end{aligned} \tag{4}$$

We can then see the dependence of M on N – namely to get an error with M derivatives we need $N > M + \operatorname{Re} m + n$. We are most interested in the case that $m \in \mathbb{Z}$.

We can deduce properties of these new spaces from those of the Q'_s.

L5.20 **Proposition 7.** *For any* $m \in \mathbb{C}$

L5.21 (5)
$$\begin{gathered} D_{y_l} : J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \longrightarrow J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \\ D_{z_j} : J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \longrightarrow J^{m+1}_c(\mathbb{R}^{k+n}, \{z = 0\}) \\ \times z_j : J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \longrightarrow J^{m-1}_c(\mathbb{R}^{k+n}, \{z = 0\}) \\ \mathcal{C}^\infty(\mathbb{R}^{k+n}) \times J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \longrightarrow J^m_c(\mathbb{R}^{k+n}, \{z = 0\}) \end{gathered}$$

and if $h : \Omega \longrightarrow \Omega'$ *is a diffeomorphism between open subsets of* $\mathbb{R}^{k+n}$ *such that* $h(\{z = 0\} \cap \Omega) \subset \{z = 0\} \cap \Omega'$ *then*

L5.22 (6)
$$h^* : \{u \in J^m_c(\mathbb{R}^{k+n}, \{z = 0\}); \operatorname{supp}(u) \subset \Omega'\} \longrightarrow J^m_c(\mathbb{R}^{k+n}, \{z = 0\}).$$

Proof. The first result in (5) is a consequence of the fact that the y coordinates are parameters in the definition of the $\mathcal{C}^\infty(\mathbb{R}^k_y; Q'_s(\mathbb{R}^n)$. Similarly the second and third mapping properties follow from (30) and the action on the remainder.

To show that $J^m_c(\mathbb{R}^{k+n}, \{z = 0\}))$ is a module under the multiplicative action of $\mathcal{C}^\infty(\mathbb{R}^{k+n})$ note that, again since the y variables are parameters, this is certainly the case for $\mathcal{C}^\infty(\mathbb{R}^k_y)$. For the general case, expand in Taylor series around $z = 0$:

L5.25 (7)
$$b(y, z) = \sum_{|\alpha|<N} b_\alpha(y) z^\alpha + \sum_{|\alpha|=N} b'_\alpha(y, z) z^\alpha.$$

Applying the first and third results shows that the terms in the finite sum map J^m into $J^{m-|\alpha|}$ and the same is true for the z^α in the remainder term in (7). For $|\alpha|$ large enough $J^{m-|\alpha|} \subset \mathcal{C}^M(\mathbb{R}^{k+n})$ which gives the defining condition for $bu \in J^m_c(\mathbb{R}^{k+n}, \{z = 0\})$.

Diffeomorphism invariance is a little more involved. From the preceding results we can decompose any $u \in J^m_c(\mathbb{R}^{k+n}, \{z = 0\})$ using a partition of unity and so it suffices to suppose that the support is near some point. We start by decomposing the diffeomorphism h. Since h preserves $z = 0$ we can define a new diffeomorphism (near any point)

L5.26 (8)
$$h_0(y, z) = (h(y, 0), z).$$

This acts only on the parameters so h^*_0 preserves the J^m. Thus we can replace h by $h_0^{-1} h$ so now

L5.27 (9)
$$h(y, z) = (y_* + \sum_j z_j a_{*j}(y, z), \sum_l b_{*l}(y, z) z_l).$$

The Jacobian at $(y, 0)$ is of the block form

L5.28 (10)
$$\begin{pmatrix} \mathrm{Id}_y & * \\ 0 & b_{**}(y, 0) \end{pmatrix}$$

so b is invertible and the linearization

L5.29 (11)
$$h_1(y, z) = (y, \sum_j b_{*j}(y, 0) z_j)$$

is (locally) a diffeomorphism. The fact that $h^*_1 : J^m \longrightarrow J^m$ follows from the fact that the ys are parameters and linear transformations in z act on the Q'_s – indeed the Euler vector field $z \cdot \partial_z$ is invariant under such (parameterized) linear transformations. Thus again we can replace h by $h_1^{-1}h_0^{-1}h$ and so assume it has the form

L5.30
$$h(y,z) = (y_* + \sum_j b_{*j}z_j, z_* + \sum_{jk} c_{*jk}z_jz_k). \tag{12}$$

The Jacobian is now the block upper triangular as in (10) with b also the identity.

To proceed further we use and idea of Möser. The diffeomorphism h in (12) is so close to the identity that (locally near $z = 0$ which is all that matters) we can connect it to the identity by a path of diffeomorphisms

L5.31
$$h_s(y,z) = (y_* + s\sum_j b_{*j}z_j, z_* + s\sum_{jk} c_{*jk}z_jz_k),\ s \in [0,1]. \tag{13}$$

Thus $h_0 = \mathrm{Id}$, $h_1 = h$.

Suppose that $u_s : [0,1]_s \longrightarrow J_{\mathrm{c}}^m(\mathbb{R}^{k+n}, \{z = 0\})$ is a smooth path – really we can just add s as another parameter. Then consider the path, at this stage in the space of distributions

L5.32
$$h_s^* u_s. \tag{14}$$

This is smooth in s and the chain rule gives a formula for the derivative

L5.33
$$\frac{d}{ds}h_s^*u_s = h_s^*(\frac{du_s}{ds} + V_su_s). \tag{15}$$

Here V_s is a smooth vector field which 'generates the path of diffeomorphisms'. Since h_s is of the special form (13) we see that

L5.34
$$V_s = \sum_{i,j} e_{ij}(y,z)z_j\partial_{y_i} + \sum_{jkl} e'_{jkl}(y,z)z_jz_k\partial_{z_l} \tag{16}$$

with smooth coefficients. Exploiting the earlier results we see that

L5.35
$$V_s : J_{\mathrm{c}}^m([0,1]\times\mathbb{R}^{k+n}, \{z = 0\}) \longrightarrow J_{\mathrm{c}}^{m-1}([0,1]\times\mathbb{R}^{k+n}, \{z = 0\}) \tag{17}$$

'lowers the order' by (at least) one.

Now, the basic idea is to solve the equation $\frac{du_s}{ds} + V_su_s = 0$. We don't quite need to do this but instead lets show that we can choose u_s so that, for any preassigned M,

L5.36
$$\frac{du_s}{ds} + V_su_s \in \mathcal{C}^M,\ u_1 = u. \tag{18}$$

Since the $\mathcal{C}^M$ spaces are preserved by diffeomorphism it then follows by integration that

L5.37
$$h_1^*u_1 - h_0^*u_0 = h^*u - u_0 \in \mathcal{C}^M. \tag{19}$$

This is just another way of saying that $h^*u \in J_{\mathrm{c}}^m(\mathbb{R}^{k+n}, \{z = 0\})$.

So we are reduced to solving (18) for every large M. We can do this by iteration. Make a first attempt by taking a constant curve

L5.38
$$u_s^0 = u,\ s \in [0,1] \Longrightarrow \frac{du_s^0}{ds} + V_su_s^0 = v_s^1 \in J_{\mathrm{c}}^{m-1}([0,1]\times\mathbb{R}^{k+n}, \{z = 0\}) \tag{20}$$

using (17). Now for the second try we simply integrate

L5.39
$$u_s^1 = \int_s^1 v_s^1 \in J_{\mathrm{c}}^{m-1}(\mathbb{R}^{k+n}, \{z = 0\}) \tag{21}$$

which will abbreviate by saying it has order $m-1$. Then

L5.40 (22)
$$u_1^1 = 0,\ \frac{du_s^1}{ds} + V_s u_s^1 = -v_s^1 + v_s^2,\ v_s^2 \in J_c^{m-2}([0,1]\times\mathbb{R}^{k+n}, \{z=0\})$$

where the first term comes from the derivative and the second from the vector field. Clearly we can proceed iteratively and then sum the result to see that after sufficiently many steps

L5.41 (23)
$$u_s = \sum_l u_s^l \text{ satisfies (18).}$$

This we have proved the diffeomorphism invariance. □

We also want to know about integration.

L5.23 **Lemma 4.** *Integration in the last $n' < n = n'' + n'$ of the z variables gives a continuous map*

L5.24 (24)
$$\int dz_{n''+1}\cdots dz_n : J_c^m(\mathbb{R}^{k+n}, \{z=0\}) \longrightarrow J_c^m(\mathbb{R}^{k+n''}, \{z''=0\}).$$

Existence is rather important too!

L5.42 **Proposition 8.** *If $u_j \in \mathcal{C}_c^\infty(\mathbb{R}^j; Q'_{-m-n+j}(\mathbb{R}^n, \{0\})$ there exists $u \in J_c^m(\mathbb{R}^{k+n}, \{z = 0\})$ such that for given M,*

L5.43 (25)
$$u - \sum_{j<N} u_j \in \mathcal{C}^M(\mathbb{R}^{k+n})$$

for large enough N.

11. Notes for Lecture 6
18.156 Spring 2020

Abstract. (1) Integration (invariantly push-forward) and restriction (so pull-back) of conormal functions.
(2) Symbol(s)
(3) Global parametrices and Fredholm property (finally).

Did I do this already?

22.2.2020.4 **Lemma 5.** *If $u \in Q'_s(\mathbb{R}^n, \{0\})$ and $v \in Q'_{s'}(\mathbb{R}^n, \{0\})$ then $u * v \in Q'_{s+s'+n}$.*

Note that this is consistent with the fact that $\delta_0 *$ is the identity operator.

Proof. Consider the relationship between the Euler operator $E = z\cdot\partial_z$ and convolution. Since

22.2.2020.1 (1)
$$(\partial_{z_j} + \partial_{y_j})u(z-y) = 0$$

we can write, for the convolution of two smooth functions of compact support

22.2.2020.2 (2)
$$\begin{aligned} E(u*v(z)) &= \int E(u(z-y))v(y)dy \\ &= \int \left((Eu)(z-y))u(y) + u(z-y)((E+n)v)(y)\right) dy. \end{aligned}$$

Extending by continuity the same formula holds for compactly supported distributions and hence

22.2.2020.3 (3)
$$(E-s)(u*v) = ((E-s_1)u)*v + u*((E-s_2+n)v),\ s_1+s_2 = s.$$

It follows that if $(E-s)u \in \mathcal{C}^\infty_{\mathrm{c}}$ and $(E-s')v \in \mathcal{C}^\infty_{\mathrm{c}}$ then $(E-s-s'-n)(u*v) \in \mathcal{C}^\infty_{\mathrm{c}}$. □

Another thing I meant to do earlier was to write down a differential characterization of $J^m(\mathbb{R}^n, \{0\})$.

L4.8a **Proposition 9.** *An element $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n))$ is in $J^m(\mathbb{R}^n,\{0\})$ if and only if for each $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that*

L4.9
$$\prod_{k=0}^{N}(z\cdot\partial_z + m + n - k)u \in \mathcal{C}^M(\mathbb{R}^n). \tag{4}$$

Proof. Show that $(z\partial_z - s)v = \mathcal{C}^M$ has a solution in $\mathcal{C}^M$ near 0 if $\operatorname{Re} s > M$. □

We can reduce the composition formula for pseudodifferential operators to convolution.

L4.10a **Proposition 10.** *For any m, $m' \in \mathbb{C}$,*

L4.11a
$$\Psi^m_c(\mathbb{R}^n) \circ \Psi^{m'}_c(\mathbb{R}^n) \subset \Psi^{m+m'}_c(\mathbb{R}^n). \tag{5}$$

Proof. By definition, an element of $\Psi^m_{\mathrm{c}}(\mathbb{R}^n)$ is an operator with Schwartz kernel $k \in J^m(\mathbb{R}^{2n}_{y,z};\{y=z\})$. The composition of operators – and we need to think below about why this is defined but let's just proceed somewhat formally – is defined by

L4.12
$$k(z,z') = \int k'(z,y)k''(y,z')dy \tag{6}$$

where the issue is the meaning of the product of distributions. Now, we can make a change of variables to bring out the various diagonals, where the singularities are located:

L4.13
$$z'-z = Z,\ y-z = Y \tag{7}$$

So introducting the variables z, Z and Y as coordinates on $\mathbb{R}^{3n}$ we see that

L4.16
$$k'(z,y) = k'(z,z+Y) = \tilde{k}'(z,Y),\ k''(y,z') = \tilde{k}''(y,z'-y) = \tilde{k}''(z+Y,Z-Y) \tag{8}$$

where $\tilde{k}'(z,Y) \in J^m(\mathbb{R}^{2n}_{z,Y},\{Y=0\})$ and $\tilde{k}''(z,W) \in J^{m'}(\mathbb{R}^{2n}_{z,W},\{W=0\})$.

The last expression in (8) deserves some comment. If we go from $\tilde{k}''(z,W)$ to $\tilde{k}''(z+Y,W)$ we are simply introducing an additional parameter so we get an element of $J^{m'}(\mathbb{R}^{3n}_{z,Y,W},\{W=0\})$.

Now we can expand this in Taylor series at $Y=0$ giving

L4.17
$$\tilde{k}''(z+Y,W) = \sum_{|\alpha|\le N} Y^\alpha e_\alpha(z,W) + \sum_{|\alpha|=N+1} Y^\alpha f_\alpha(z,Y,W) \tag{9}$$
$$e_\alpha \in J^{m'}(\mathbb{R}^{2n}_{z,W},\{W=0\}),\ f_\alpha \in J^{m'}(\mathbb{R}^{3n}_{z,Y,W},\{W=0\})$$

Thus the composition (6) is reduced to a sum

L4.18a
$$\sum_{|\alpha|\le N} Y^\alpha \tilde{k}'(z,Y)e_\alpha(z,Z-Y)dY + \sum_{|\alpha|=N+1} Y^\alpha \tilde{k}'(z,Y)f_\alpha(z,Y,Z-Y)dY \tag{10}$$

The first sum consists of parameterized convolutions of an element of $J^{m-|\alpha|}$ and $J^{m'}$ and in the second sum the first term is in $\mathcal{C}^M$ for N large enough. □

12. Notes for Lecture 7: Elliptic families 18.156 Spring 2020

Abstract. Since everyone indicated some familiarity with the Hodge theorem I will proceed a little more generally.
(1) Submersions and fibrations
(2) Fibre bundles, metrics and connections
(3) Families of pseudodifferential operators
(4) Fredholm theory for an elliptic familiy
(5) Hodge theorem and fibre harmonic forms

First let's make sure that we have the basic properties of pseudodifferential operators sorted out. If M is a compact manifold (it works fine for non-compact manifolds but with annoying details) and ν is a positive smooth density on M then

L4.17b (1)
$$A \in \Psi^m(M; V, W) \Longleftrightarrow k_A \in J^m(M^2, \mathrm{Diag}; W_x \otimes V'_y)$$
$$A : \mathcal{C}^\infty(M; V) \longrightarrow \mathcal{C}^\infty(M; W),\ (Av)(x) = \int k_A(x, y)v(y)\nu(y).$$

The mapping property follows by pairing which shows that

L4.18b (2)
$$k_A(x, y) \cdot v(y) \in J^m(M^2, \mathrm{Diag}; W_x) \Longrightarrow \int_M k_A(x, y)v(y)\nu(y) \in \mathcal{C}^\infty(W).$$

The important symbol sequence is exact:

L4.19 (3)
$$\Psi^{m-1}(M; V, W) \longrightarrow \Psi^m(M; V, W) \overset{\sigma}{\longrightarrow} R^m(T^*M \setminus \{0\}; \hom(V, W))$$

where R^m is the space of maps on $T^*M \setminus \{0\}$ which are homogeneous of degree m taking values in the vector bundle $\hom(V, W)$.

These spaces are multiplicative

L4.20 (4)
$$\Psi^m(M; V, W) \circ \Psi^{m'}(M; W, U) \subset \Psi^{m+m'}(M; V, U),\ \sigma(A \circ B) = \sigma(A) \circ \sigma(B).$$

I have not yet shown the boundedness properties

(1) Boundedness $\Psi^m(M; V, W) \in \mathcal{B}(L^2(V); L^2(W))$ if $\operatorname{Re} m \le 0$.
(2) Compactness $\Psi^m(M; V, W) \in \mathcal{K}(L^2(V); L^2(W))$ if $\operatorname{Re} m < 0$.
(3) Hilbert-Schmidt$\Psi^m(M; V, W) \in \mathrm{HS}(L^2(V); L^2(W))$ if $\operatorname{Re} m < -\frac{1}{2}\dim M$.
(4) Trace class $\Psi^m(M; V, W) \in \mathrm{TC}(L^2(V); L^2(W))$ if $\operatorname{Re} m < -\dim M$.

Let me go through the proof of boundedness since it illustrates how the symbol map can be moved judiciously. First we apply Schur's criterion to get the weaker statement that if $\operatorname{Re} m < 0$ then $B \in \Psi^m(M; V, W)$ is bounded. Then consider

L4.21 (5)
$$A^*A \in \Psi^0(M; V),\ \sigma(A^*A) = \sigma(A)^*\sigma(A)(x, \xi) \in \hom(V)$$

is homogeneous of degree 0. Then if $C > 0$ is large enough

L4.22 (6)
$$C\,\mathrm{Id} - \sigma(A)^*\sigma(A)(x, \xi) >> 0$$

as a family of matrices. It therefore has a smooth, homogeneous square root $s(x, \xi) \in \hom(V)$. By the surjectivity of the symbol sequence we can choose $S \in \Psi^0(M; V)$ with $\sigma(S) = s$. It follows that

L4.23 (7)
$$\sigma(C - A^*A - S^*S) = C\,\mathrm{Id} - \sigma(A)^*\sigma(A) - s^*s = 0 \Longrightarrow B = C - A^*A - S^*S \in \Psi^{-1}(M; V) \text{ is bounded on } L^2(V).$$

Now, rewriting and pairing with $\phi \in \mathcal{C}^\infty(M; V)$ gives

L4.24 (8)
$$\|A\phi\|_{L^2} = \langle A^*A\phi, \phi\rangle = C\|\phi\|^2_{L^2} - \|S\phi\|^2_{L^2} + \langle B\phi, \phi\rangle \le (C + C')\|\phi\|^2_{L^2}$$

giving the continuity on L^2.

13. Notes for Lecture on 10 March, 18.156 Spring 2020

Abstract. I had originally intended to postpone the discussion of *Lie algebroids* but it now seems to me that it would be better to discuss the abstract notion now.

A Lie algebroid over a manifold M is the following data

(LA1) A (real) vector bundle $E \longrightarrow M$ over M

(LA2) A smooth bundle map $a : E \longrightarrow TM$ (called the anchor)

(LA3) A Lie algebra structure on $\mathcal{C}^\infty(M; E)$, the space of sections of E, so an antisymmetric map (bilinear over constants)

11.3.2020.1 (1)
$$[\cdot,\cdot] : \mathcal{C}^\infty(M; E) \times \mathcal{C}^\infty(M; E) \longrightarrow \mathcal{C}^\infty(M; E)$$

satisfying the Jacobi identity

11.3.2020.2 (2)
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \ \forall \ X, Y, Z \in \mathcal{C}^\infty(M; E).$$

(LA4) The compatibility condition that

11.3.2020.3 (3)
$$a([X, fY]) = Xf \cdot a(Y) + fa([X, Y]) \ \forall \ X,\ Y \in \mathcal{C}^\infty(M; E),\ f \in \mathcal{C}^\infty(M).$$

Note that the smoothness in (LA2) means that $a \in \mathcal{C}^\infty(M; \mathrm{Hom}(E, TM))$ so if $X \in \mathcal{C}^\infty(M; E)$ then $a(X) \in \mathcal{C}^\infty(M; TX)$ is a smooth vector field. It follows from 11.3.2020.3 (3) (taking $f = 1$) that a is a Lie algebra map

11.3.2020.4 (4)
$$a([X, Y]) = [a(X), a(Y)].$$

Thus the image $a(\mathcal{C}^\infty(M; E))$ is a Lie subalgebra of $\mathcal{C}^\infty(M; TM)$.

Here are some examples, including the ones of immediate interest.

- First a trivial example. If $\mathfrak{g}$ is a finite-dimensional (real) Lie algebra then it defines a Lie algebroid over a point.
- The obvious example of $E = TM$ with a the identity maps
- Combining these two consider $E = TM \oplus \mathfrak{g}$ for a Lie algebra $\mathfrak{g}$. The Lie algebra structure on sections $X = V + v$, $v \in \mathcal{C}^\infty(M; \mathfrak{g})$ is

11.3.2020.5 (5)
$$[V + v, W + w] = [V, W]_{TM} + Vw - Wv + [v, w]_{\mathfrak{g}}.$$

I will talk about a special case of this below.
- Now, an important example that I have already been talking about.

If $\phi : M \longrightarrow Y$ is a fibre bundle – for instance a submersion between compact manifolds – then we can take

11.3.2020.6 (6)
$$E_m = \mathrm{Nul}(\phi_* : T_m M \longrightarrow T_{\phi(m)} Y) \Longrightarrow E \subset TM$$

is the subbundle of vector fields tangent to the fibres of ϕ. Thus the Lie algebra structure on

11.3.2020.7 (7)
$$\mathcal{C}^\infty(M; E) = \{V \in \mathcal{C}^\infty(M; TM); V\phi^* f = 0 \ \forall \ f \in \mathcal{C}^\infty(Y)\}$$

is the restriction of the commutator for vector fields to the subspace – clearly $[X, Y]\phi^* f = X(Y\phi^* f) - Y(X\phi^* f) = 0$ is $X,\ Y \in \mathcal{C}^\infty(M; E)$. The anchor map is the inclusion map.

- A more substantial example, that is very closely related to the Dirac operator on (the radial compactification of) Euclidean space that we talked about earlier and that I still need to come back to.

  Let M be a (usually compact but it doesn't matter here) manifold with boundary. Then we look at the space

  11.3.2020.8
  $$\mathcal{V}_{\mathrm{b}}(M) = \{V \in \mathcal{C}^\infty(M;TM); V \text{ is tangent to the boundary}\}. \tag{8}$$

  This is somewhat similar to the preceeding example in that there is always a 'boundary defining funtion' on a manifold with boundary – $x \in \mathcal{C}^\infty(M)$ such that $x \ge 0$, $\partial M = \{x = 0\}$ and $dx \ne 0$ at ∂M. Near a boundary point such a function can be extended to a local coordinate system

  11.3.2020.9
  $$x,\ y_1, \dots, y_{n-1},\ n = \dim M \tag{9}$$

  where the y_i's are tangential coordinates – they induce coordinate on the boundary near the point. Then the defining condition in (8) can be written

  11.3.2020.10
  $$Vx\big|_{x=0} = 0 \text{ or } Vx \in x\mathcal{C}^\infty(M). \tag{10}$$

  Either way it is clear that $\mathcal{V}_{\mathrm{b}}(M)$ is a Lie subalgebra of $\mathcal{C}^\infty(M;TM)$. We need to find E! Over the interior $E = TM$. Near a boundary point we can use coordinates of the form (9) with the boundary locally defined by $x = 0$. Then any smooth vector field on M is locally of the form

  11.3.2020.11
  $$V = a(x,y)\partial_x + \sum_{i=1}^{n-1} b_i(x,y)\partial_{y_i}. \tag{11}$$

  Since $\partial_{y_i} x = 0$ and $\partial_x x = 1$ this must satisfiy

  11.3.2020.12
  $$a(0,y) = 0 \Longrightarrow a(x,y) = x\alpha(x,y),\ \alpha \text{ smooth} \tag{12}$$

  if $V \in \mathcal{V}_{\mathrm{b}}(M)$. Conversely, if (11) holds near very boundary point then $V \in \mathcal{V}_{\mathrm{b}}(M)$. Thus $V \in \mathcal{V}_{\mathrm{b}}(M)$ if near every boundary point in coordinates (9) it is of the form

  11.3.2020.13
  $$V = \alpha(x,y)(x\partial_x) + \sum_i b_i(x,y)\partial_{y_j}. \tag{13}$$

  This shows how to define $E = {}^{\mathrm{b}}TM$. In coordinates a basis is $x\partial_x,\ \partial_{y_i}$.

  11.3.2020.14
  Exercise 2. Check what happens to this basis under change of coordinates between two sysmtems of the form (9). The answer of course is that it defines a smooth bundle map between the local coordinate bases.

  This is an important example (the 'b' stands for boundary). The Lie algebra $\mathcal{V}_{\mathrm{b}}(M) = \mathcal{C}^\infty(M; {}^{\mathrm{b}}TM)$ really does replace $\mathcal{C}^\infty(M;TM)$ when $\partial M \ne \emptyset$ in the sense that it is the Lie algebra of the diffeomorphism group.
- The example of immediate interest is somewhat between these two. Namely for a fibration $\phi : X \longrightarrow Y$ between (let's say compact but it is not crucial at this stage) manifolds we consider the 'adiabatic algebroid' on

  11.3.2020.15
  $$M = X \times [0,1)_\epsilon. \tag{14}$$

  So, the space of sections of the 'putative' bundle $E \longrightarrow M$ satisfies two conditions corresponding to the two fibrations $\pi : M \longrightarrow [0,1)_\epsilon$ and $\phi : X \longrightarrow Y$, fixing $\phi : M \longrightarrow Y$ where I don't change the notation.

  11.3.2020.16
  $$\mathcal{V}_{\mathrm{ad}}(M) = \{V \in \mathcal{C}^\infty(M;TM); V\epsilon = 0 \text{ and } V(\phi^* f) \in \epsilon\mathcal{C}^\infty(M)\}. \tag{15}$$

So, in $\epsilon > 0$ the second condition is void and the first means that V must be of the form

11.3.2020.17 (16)
$$V = \sum_k e_k(m,\epsilon)\partial_{m_k}$$

just an ϵ dependent vector field on M. This corresponds to fibre vector fields for the fibration π. Now, near any point we can introduce coordinates in X, and hence M, corresponding to the fibration ϕ so

11.3.2020.18 (17)
$$V = \sum_{i=1}^{k} a_i(z,y,\epsilon)\partial_{z_i} + \sum_{j=1}^{m} b_j(z,y,\epsilon)\partial_{y_j},\ m = \dim Y,\ k = \dim Z.$$

Since $\phi^* f$ is just a function of Y and $\partial_{z_i} y_j = 0$ the second condition in (15) 11.3.2020.16
just becomes

11.3.2020.19 (18)
$$b_j(z,y,0) = 0 \Longleftrightarrow b_j(z,y,\epsilon) = \epsilon\beta_j(z,y,\epsilon)$$

for smooth functions β_j. Just as before we see that $\mathcal{V}_{\mathrm{ad}}(M)$ has basis over $\mathcal{C}^\infty(M)$ namely

11.3.2020.20 (19)
$$E_m = \mathrm{sp}\{\partial_{z_i},\ \epsilon\partial_{y_j}\}.$$

The Lie algebra structure is again 'inherited' from $\mathcal{C}^\infty(M;TM)$.

That is enough for the moment but there are many more.

I did not do this in lecture but it is quite a good exercise in recalling the construction of the deRham complex. First note what happens for the differential of functions on the manifold M. Let E^* be the dual bundle of E, so also a smooth vector bundle over M. Locally if e_i is a local basis for E over an open set of M then there is a canonically defined dual basis e_i^* of E^*. The consider the sum

11.3.2020.21 (20)
$$du = \sum_i a(e_i)u \cdot e_i^*,\ u \in \mathcal{C}^\infty(M).$$

I will not change the notation for d, although I should call it maybe ${}^E d$. It is a well-defined linear differential operator

11.3.2020.22 (21)
$$d : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M;E^*).$$

I will write out the formulæ for the specific cases of interest.

The important part of the deRham complex is the extension to higher forms, and this really reduces to 1-forms. So we need to define $d\alpha$ where $\alpha \in \mathcal{C}^\infty(M;E^*)$ as an antisymmetric section $d\alpha \in \mathcal{C}^\infty(M;E^* \wedge E^*)$. The key here is the standard formula for differential of a 1-form on a manifold:-

11.3.2020.23 (22)
$$d\alpha(V,W) = V(\alpha(W)) - W(\alpha(V)) - \alpha([V,W]),\ V,\ W \in \mathcal{C}^\infty(M;TM).$$

Dropping to a coordinate basis you can check that this is correct – in particular it is linear over mutlication $V \longmapsto fV$, $f \in \mathcal{C}^\infty(M)$. So, we just extend (22) using the Lie algebra structure on sections of E and define

11.3.2020.24 (23)
$$\begin{aligned}{}^E d\alpha(V,W) = {}^E d(\alpha(W))(V) - {}^E d(\alpha(V))(W) - \alpha([V,W]) \in \mathcal{C}^\infty(M;{}^E\Lambda^2),\\ V,\ W \in \mathcal{C}^\infty(M;E),\ \alpha \in \mathcal{C}^\infty(M;E^*).\end{aligned}$$

Here the ${}^E\Lambda^k$ are the totally antisymmetric parts of the k-fold tensor products of the E^*. Then

11.3.2020.25 (24)

$$\mathcal{C}^\infty(M) \xrightarrow{{}^E d} \mathcal{C}^\infty(M;E^*) \xrightarrow{{}^E d} \mathcal{C}^\infty(M;{}^E\Lambda) \xrightarrow{{}^E d} \dots \xrightarrow{{}^E d} \mathcal{C}^\infty(M:{}^E\Lambda^N) \xrightarrow{{}^E d} 0,\ N = \operatorname{rank} E$$

is a complex.

14. Preliminary notes for Lecture 11, 12 March 18.156 Spring 2020

Abstract. Blow-up.

Let me start with an example of a Lie algebra of smooth vector fields which is geometrically relevant but is not a Lie algebroid. Suppose M is a manifold and S is a closed embedded submanifold. For simplicity I will assume that S is compact but that doesn't really matter. So, near each point of $\bar m \in S$ there are local coordinates on M, $t_1, \dots, t_k$, $s_1, \dots, s_{n-k}$, in a coordinate neighbourhood $U \ni \bar m$ such that

12.3.2020.1 (1)
$$S \cap U = \{t_1 = \dots = t_k = 0\}.$$

Here k is the codimension of S and the s_i give local coordinates on $U \cap S$.

Consider

12.3.2020.2 (2)
$$\mathcal{W}_S = \{W \in \mathcal{C}^\infty(M;TM); W \text{ is tangent to } S\}.$$

This just means that in each coordinate system such that (12.3.2020.1 1) holds Ws_i also vanishes at S. So consider the ideal

12.3.2020.3 (3)
$$\mathcal{I}_S = \{u \in \mathcal{C}^\infty(M;TM); u\big|_S = 0\}.$$

Then by definition

L11.2 (4)
$$\text{for } V \in \mathcal{C}^\infty(M;TM),\ W \in \mathcal{W}_S \Longleftrightarrow W\mathcal{I}_S \subset \mathcal{I}_S.$$

If $k > 1$, so S is not a hypersurface, the 'issue' is that $\mathcal{I}_S$ is *not* a principal ideal and consequently $\mathcal{W}$ is not a Lie algebroid. This means that there is no vector bundle over M of which $\mathcal{W}_S$ it is all sections. Indeed in the local coordinates above

12.3.2020.4 (5)
$$W \in \mathcal{W} \Longleftrightarrow W = \sum_{j=1}^{n-k} a_j(s,t)\partial_{s_j} + \sum_{i=1}^{k} b_{il}(s,t)t_i\partial_{t_l}$$

where the coefficients are arbitrary smooth functions. The problem is that they are not linearly independent. Indeed if c is any smooth function then adding t_kc to a_{il} and subtracting t_ic from a_{kl} leaves W unchanged.

However, we can *resolve* the Lie algebra to a Lie algebroid.

12.3.2020.5 **Proposition 11.** *If $S \subset M$ is a closed embedded submanifold of a manifold then there is a manifold with boundary $\tilde M$ and a smooth map ('blow-down') $\beta : \tilde M \longrightarrow M$ which restricts to a diffeomorphism of $\tilde M \setminus \partial \tilde M$ to $M \setminus S$ and is such that*

12.3.2020.6 (6)
$$\beta^*\mathcal{I}_S \text{ spans } \mathcal{I}_\partial(\tilde M) \text{ over } \mathcal{C}^\infty(\tilde M).$$

Furthermore, $\tilde M$ is essentially unique in the sense that given another such manifold and map $\beta' : \tilde M' \longrightarrow M$ the map $\beta^{-1}\beta' : \tilde M' \setminus \partial\tilde M' \longrightarrow \tilde M \setminus \partial \tilde M$ extends to a diffeomorphism $\tilde M' \longrightarrow \tilde M$. A smooth vector field on M is β-related to a smooth vector field on $\tilde M$ if and only if it is tangent to S, the 'lift' is then unique and $\mathcal{W}_S$ lifts to span $\mathcal{V}_{\mathrm{b}}(\tilde M)$ over $\mathcal{C}^\infty(\tilde M)$.

I will denote the blown-up manifold by $[M;S]$. We can say more about it. First, we can identify the boundary explicitly. The ideal $\mathcal{I}_S$ defines a subbundle of the cotangent bundle restricted to S,

L11.3
$$\{df; f\in\mathcal{I}_S\}\big|_S = N^*S\subset T^*_SM. \tag{7}$$

This is the conormal bundle to S. In local coordinates as above this is the span of the dt_i at each point of S. The annihilator is the tangent bundle of S

L11.4
$$T_SM\supset TS=\{v\in T_SM; v\cdot df=0,\ df\in N^*S\}. \tag{8}$$

The normal bundle to S, NS can be identified either as the dual bundle of N^*S or, in view of (8), as the quotient

L11.5
$$NS=T_SM/TS. \tag{9}$$

As a vector bundle we can identify its 'radial' sphere bundle as the quotient by the $\mathbb{S}^+$-action

L11.6
$$\mathbb{S}NS=(TS\setminus 0_S)/\mathbb{R}^+ \tag{10}$$

so the space of half-lines through the origin in the fibres. Then

L11.8
$$\partial[M;S]=\mathbb{S}NS. \tag{11}$$

From this point of view what we are claiming is that the disjoint union,

L.58
$$\mathbb{S}NS\sqcup(M\setminus S) \tag{12}$$

has a unique structure of a manifold with boundary such that the inclusion map of the interior extends to a smooth map β with the listed propeties. The topological structure is easy enough to see.

So, how to prove Proposition 11? It is I think important to understand that it is related to several other results which are of independent interest – meaning that I think they are important. These include the 'Collar Neighbourhood Theorem' and the 'Linearization of vector fields'. It is very closely related to the notion of polar coordinates too, as we shall see. Let me emphasize the relationship to linearization. You might like to look up the 'Sternberg Linearization Theorem'.

So, suppose $V\in\mathcal{C}^\infty(M;TM)$ is a smooth, real, vector field on a manifold M and that it vanishes at a closed embedded (let's say compact as usual) submanifold S. Now it follows from the fact that V vanishes at S that

L.58
$$V:\mathcal{C}^\infty(M)\longmapsto\mathcal{I}_S\subset\mathcal{C}^\infty(M) \tag{13}$$

in fact that is exactly what vanishing means. In particular

L.59
$$V:\mathcal{I}_S\longrightarrow\mathcal{I}_S,\ V:\mathcal{I}_S^2\longrightarrow\mathcal{I}_S^2. \tag{14}$$

If $f\in\mathcal{I}_S$ then $df\big|_S\in N^*S$ can only depend on df because of (23). Thus it defines a bundle map

L.60
$$L_V:N^*S\longleftrightarrow N^*S. \tag{15}$$

This is called the linearlization of V at S.

We can ask a question such as, if two vector fields have the same linearization, is there a diffeomorphism near S mapping one to the other? This is pretty clearly quite unreasonable but less so if the linearization is an isomorphism. We are interested in a particular case

L.61 *Definition* 3. A vector field on M is said to be *radial at S* (by me, other people say Euler-like for instance) is its linearization is the identity map.

In local coordinates t, s as in (1) the vector field V takes the form

L.62
$$V=\sum_{i,j=1}^{k}t_ia_{ij}(t,s)\partial_{t_j}+\sum_{i=1}^{k}\sum_{l=1}^{n-k}t_ib_{il}(t,s)\partial_{s_l} \tag{16}$$

since the coefficients must vanish at $t=0$. The linearization just picks out the $a_{ij}(0,s)$ so our assumption is that

L.63
$$V=\sum_i t_i\partial_{t_i}+V',\ V'=\sum_{i,j,q=1}^{k}t_it_qa_{ijq}(t,s)\partial_{t_j}+\sum_{i=1}^{k}\sum_{l=1}^{n-k}t_ib_{il}(t,s)\partial_{s_l}. \tag{17}$$

A local answer to the question above in this case is provided by

L.65 **Proposition 12.** *For a vector field (17) near $t=0$, $s=0$ on $\mathbb{R}^n$ there exist smmoth functions in an neighbourhood of* 0, $T_i=t_i+e_i$, $i=1,\dots,k$, $S_l=s_l+f_l$, $l=1,\dots,n-k$, $e_i\in\mathcal{I}_S^2$, $f_l\in\mathcal{I}_S$ *such that*

L.64
$$VT_i=T_i,\ VS_l=0. \tag{18}$$

We can introduce T_i, S_l as local coordinates and then $V=\sum\limits_i T_i$

pa_{T_i} so indeed in this case we can locally reduce such a vector field to its linear part.

Proof. The condition on V', the difference in (17) between V and its linearization, means that

L.66
$$V':\mathcal{I}_S^k\longrightarrow\mathcal{I}_S^{k+1}\ \forall\ k. \tag{19}$$

This suggests that we look for a solution of (18) in Taylor series in t. Thus, suppose we have found

L.67
$$T_i^{(p)}=t_i+\sum_{2\le|\alpha|\le p+1}t^\alpha a_\alpha(s),\ S_l^{(p)}=s_i+\sum_{1\le|\alpha|\le p}t^\beta b_\beta(s) \tag{20}$$
$$\text{s.t. } VT_i^{(p)}-T_i^{(p)}\in\mathcal{I}_S^{p+2},\ VS_l\in\mathcal{I}_S^{p+1}.$$

We can certainly do this for $p=0$ by starting with t_i and s_l. Now, proceed by induction. To get $T_i^{(p+1)}$ and $s_l^{(p+1)}$ from their predecessors we add a term

L.68
$$E_{p+1}=\sum_{|\alpha|=p+2}t^\alpha a_\alpha(s),\ F_{p+1}=\sum_{|\beta|=p+1}t^\beta b_\beta(s) \tag{21}$$

respectively. From (17) and (19) we see that

L.69
$$VE_{p+1}-(p+2)E_{p+1}\in\mathcal{I}_S^{p+3},\ VF_{p+1}-(p+1)F_{p+1}\in\mathcal{I}_S^{p+2} \tag{22}$$

where the error terms are what we want on (20).

□

So, how to prove Proposition 11? Let me try an approach I have not used before and see how it is received.

Consider the notion of a quadratic defining function for S. By this I mean a smooth function $\gamma\in\mathcal{C}^\infty(M)$, $\gamma\ge0$, such that $S=\{\gamma=0\}$ and near a point of S in coordinates in which (1) holds, $\gamma\ge\epsilon|t|^2$ for some $\epsilon>0$. Now, such a function exists satisfying the opposite estimate $\gamma\le|t|^2/\epsilon$ (for small enough ϵ) as well. Indeed, cover M by a partition of unity with elements with supports either disjoint ftom M or contained in a coordinate patch (1) and sum over the partition of unity with coefficient one for the former and a factor of the locally defined $|t|^2$ for the latter. It follows that any two such are bounded by positive multiples of each other.

Now choose one such function and consider the functions

L.59 (23) $$r = \gamma^{\frac{1}{2}},\ \mathfrak{f}r,\ f \in \mathcal{I}_S,\ \mathcal{C}^\infty(M).$$

The claim is that these are define functions on the set (13) and that the generate a $\mathcal{C}^\infty$ structure on it. All of them are smooth on $M \setminus \mathbb{S}NS$ so we only have to define them on the 'boundary' $\mathbb{S}NS$. We take r to vanish there and the last, the smooth functions on M we assign the value on S, pulled back to $\mathbb{S}NS$.

15. Hodge theorem

If M is a smooth manifold then the deRham operator acts on smooth forms over it, let me recall the setup even though I assume you are somewhat familiar with it.

The basic object associated to M is the space $\mathcal{C}^\infty(M)$ of smooth (say real-valued for the moment) functions on M. These are defined by reference to local coordinates.

It is usual to take the tangent bundle as the 'fundamental' vector bundle over M. If $m \in M$ then the tangent fibre at m consists of the derivations at m, the linear maps

L.1 (1) $$v : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}$$

such that

L.2 (2) $$v(fg) = f(m)v(g) + g(m)v(f).$$

This is a linear condition in v, so these form a vector space T_mM. Moreover, for the constant function 1, it follows that $v(1) = 2v(1)$ so in fact v annihilates all constant functions. Similarly, if f and g both vanish at m then $v(fg) = 0$. It follows that if $\chi \in \mathcal{C}^\infty(M)$ is equal to 1 near m then $v\chi = 0$ and $v(f) = v(\chi f)$ for all $f \in \mathcal{C}^\infty(M)$. By Taylor's theorem, applied in local coordinates z_j based at m any smooth function can be written in the form

L.3 (3) $$f = f(m) + \sum_j c_j z_j + \sum_{jk} a_{jk}(z) z_j z_k \text{ in } |z| < \epsilon,\ a_{jk} \text{ smooth},\ c_j = \frac{\partial f}{\partial z_j}(0).$$

Inserting a suitable cutoff

L.4 (4) $$v(f) = v(\chi f) = \sum_j c_j v(\chi z_j).$$

Thus T_mM has dimension $n = \dim M$ and is spanned by the derivations given by the derivatives with respect to local coordinates

L.5 (5) $$\partial_j(m) f = \frac{\partial f}{\partial z_j}(0).$$

Conversely we can consider the ideal $\mathcal{I}_m \subset \mathcal{C}^\infty(M)$ of functions vanishing at m and its 'square' $\mathcal{I}_m^2$ consisting of the finite sums of products of two smooth functions both vanishing at m. Again reference to local coordinates shows that

L.6 (6) $$T_m^*M = \mathcal{I}_m/\mathcal{I}_m^2 \text{ is spanned by the } z_j\chi.$$

Then from (5) these vector spaces are canonically dual to each other

L.7 (7) $$T_mM \times T_m^*M \ni (v, [f]) \longrightarrow v(f) \in \mathbb{R}.$$

There is a canonically defined surjective map

L.8
$$d : \mathcal{C}^\infty(M)) \ni f \longmapsto [f - f(m)] = df(m) \in T^*_m M. \tag{8}$$

Thus local coordinates z_j fix dual bases ∂_j of $T_m M$ and dz_j (denoting $d(\chi z_j)$ for a suitable cutoff). Translating these coordinates give a basis nearby and hence local trivializations of

L.9
$$TM = \bigcup_{m \in M} T_m M,\ T^*M = \bigcup_{m \in M} T^*M. \tag{9}$$

Change of coordinates induces dual linear changes of these bases in terms of the Jacobian matrix showing TM and T^*M to be smooth vector bundles over M.

One useful way to think about vector bundles such as TM is in terms of the associated principal bundle. For a fibre $T_m M$ consider all the linear isomorphisms F_m to $\mathbb{R}^n$. We can post-compose these with isomorphism of $\mathbb{R}^n$, which is to say elements of $\mathrm{GL}(n, \mathbb{R})$. This action is free and transitive on F_m. We can identify the elements of F_m with the bases of $T_m M$ which they define by pulling back the standard basis of $\mathbb{R}^n$. In terms of a local basis this all extends smoothly to a neighbourhood of m giving the frame bundle

L.10
$$F = \bigcup_m F_m \longrightarrow M \tag{10}$$

the structure of a fibre bundle. Thus the local smooth sections of F are smooth bases of TM and two such are related by a unique smooth map into $\mathrm{GL}(n, \mathbb{R})$. I will write such a principal bundle as

L.11
$$\begin{tikzcd} \mathrm{GL}(n, \mathbb{R}) \arrow[r, no head] & F \arrow[d] \\ & M. \end{tikzcd} \tag{11}$$

One property of F is that one can recover the tangent bundle from it, at least up to bundle isomorphism. Thus consider the standard representation of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ and take the quotient of $F \times \mathbb{R}^n$ by the action of $\mathrm{GL}(n, \mathbb{R})$, $(f, gv) \simeq (gf, v)$, $f \in F$, $v \in \mathbb{R}^n$, $g \in \mathrm{GL}(n, \mathbb{R})$. This recovers the tangent bundle

L.12
$$TM = F \times_{\mathrm{GL}(n, \mathbb{R})} \mathbb{R}^n. \tag{12}$$

If one takes the dual representation of $\mathrm{GL}(n, \mathbb{R})$ one gets the cotangent bundle.

One reason for going through this setup is that it makes the tensor bundles rather clear. Namely the tensor products of representations is a representation. The k-fold tensor products of the dual representation on $\mathbb{R}^n$ is generally not irreducible, and one of the invariant subspaces consists of the totall antisymmetric elements, the space of which may be written $\lambda^k \mathbb{R}^n \subset (\mathbb{R}^n)^{\otimes k}$. If carry out the construction of 'induced bundles' as in (L.12 12) this gives us the form bundles

L.13
$$\Lambda^k M \longrightarrow M,\ \Lambda^1 M = T^*M,\ \Lambda^k M = \{0\},\ k > n. \tag{13}$$

Thus, local coordinates z_j in an open set U induce the basis dz_j of $T^*_U M$ and these in turn define the local smooth bases

L.14
$$\begin{gathered} dz^\alpha = dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_k},\ \alpha_1 < \alpha_2 < \ldots \alpha_k, \\ dz^\alpha = \sum_{\sigma \in \Sigma_k} \mathrm{sgn}(\sigma) dz_{\alpha_{\sigma(1)}} \otimes \cdots \otimes dz_{\alpha\sigma(k)}. \end{gathered} \tag{14}$$

There is no unamimity on the normalization of the basis here but this convention means that pairing with a tangent vector is easy

L.15 (15)
$$dz^{\alpha}(\partial_{z_{\alpha_1}}, \cdots) = dz^{\alpha'},\ \alpha = (\alpha_1, \alpha').$$

The deRham differential in (8) extends to the deRham complex

L.16 (16)
$$\mathcal{C}^{\infty}(M) \xrightarrow{d} \mathcal{C}^{\infty}(M; T^*M) \xrightarrow{d} \mathcal{C}^{\infty}(M; \Lambda^2 M) \xrightarrow{d} \cdots$$
$$\cdots \xrightarrow{d} \mathcal{C}^{\infty}(M; \Lambda^{n-1} M) \xrightarrow{d} \mathcal{C}^{\infty}(M; \Lambda^n M) \xrightarrow{d} 0.$$

At the level of the basis (14) this is fixed by the action on coefficients

L.17 (17)
$$d(\sum_{\alpha} f_{\alpha} dz^{\alpha}) = \sum_{\alpha} \sum_{j} \frac{\partial f_{\alpha}}{\partial z_j} dz_j \wedge dz^{\alpha}$$

where

L.18 (18)
$$dz_j \wedge dz^{\alpha} = \begin{cases} 0 & \text{if } j \in \alpha \\ (-1)^k dz^{\alpha(j)} & \alpha'(j) = \alpha \cup \{j\},\ k = \max_l(\alpha_l < j). \end{cases}$$

That $d^2 = 0$ follows immediately (in local coordinates) from the symmetry of the second derivatives of a function. Moreover directly from (18)

L.19 (19)
$$d(f \wedge g) = df \wedge g + (-1)^k f \wedge dg$$

if f is a k form. Conversely, (19) forces (17). The fact that the wedge product, being given by antisymmetrization of the tensor product, is independent of coordinates and that d on functions is also independent of coordinates shows tht d itself is independent of coordinates and hence globally defined as in (16).

L.20 **Theorem 7** (deRham). *For a smooth manifold the deRham cohomology, the hypercohomology of (16)*

L.21 (20)
$$H^k_{\mathrm{dR}}(M) = \{u \in \mathcal{C}^{\infty}(M; \Lambda^k M); du = 0\}/d\mathcal{C}^{\infty}(M; \Lambda^{k-1} M)$$

is naturally isomorphic to your favourite cohomology theory (with real/complex coefficients) – for instance singular or Čech.

Now, our real interest here is in Hodge theory, for this we suppose that M is equipped with a Riemannian structure. Thus on each fibre of $T_m M$ we assume that a positive definite symmetric inner product, g_m, has been chosen. This constitues a section of the tensor product $T^*M \otimes T^*M$ – since the fibres form the bilinear maps from $T_m M$ to $\mathbb{R}$ – and we assume that this is smooth. In terms of local coordinates this means that

L.22 (21)
$$g_m = \sum_{i,j=1}^{n} g_{ij}(z) dz_i \otimes dz_j,\ g_{ij} = g_{ji} \text{ smooth.}$$

This means that the notion of an orthonormal basis of $T_m M$ makes sense at and – by Gram-Shmidt – such a basis can be chosen smoothly near a given point.

L.23 **Proposition 13.** *The choice of a Riemannian structure on M is equivalent to the choice of a reduction of the principal frame bundle (11) to a (smooth) principal bundle for the compact group* $\mathrm{O}(n) \subset \mathrm{GL}(n)$

L.24
$$\begin{tikzcd} \mathrm{O}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{O}} \arrow[d] \\ \mathrm{GL}(n) \arrow[r, dash] & F \arrow[d] \\ & M. \end{tikzcd} \tag{22}$$

I mention this abstract way of looking at things since the representations of $\mathrm{GL}(n)$ involved in the definition of T^*M and the tensor bundles all restrict to representations of $\mathrm{O}(n)$ preserving induced Euclidean metrics. This means that all these bundles, particularly the Λ^*M, carry metrics induced from the metric on TM. The same is true of the density bundle and so a Riemannian metric induces a positive-definite density on M which I will denote as

L.25
$$|dg| \in \mathcal{C}^\infty(M; \Omega M). \tag{23}$$

Note that we do not need M to be oriented for this to exist and be smooth since it is defined by

L.26
$$|dg|(v_1, \dots, v_n) = 1 \text{ if } v_1, \dots, v_n \in T_m M \text{ is an orthonormal basis} \tag{24}$$

independently of orientation.

The inner product on Λ^* and this density define a pre-Hilbert space structure on $\mathcal{C}^\infty(M; \Lambda^*)$ where the (real) inner product is

L.27
$$\langle\langle u, v \rangle\rangle = \int_M \langle u, v \rangle |dg|. \tag{25}$$

The deRham operator then has a formal adjoint with respect to this inner product

L.28
$$\langle\langle du, v \rangle\rangle = \langle\langle u, \delta v \rangle\rangle,\ \delta : \mathcal{C}^\infty(M; \Lambda^k) \longrightarrow \mathcal{C}^\infty(M; \Lambda^{k-1}) \ \forall\ k. \tag{26}$$

L.29 **Theorem 8** (Hodge Theorem)**.** *On any compact manifold, for any choice of Riemann metric, the null space of $d + \delta$ is graded and naturally isomorphic to the deRham cohomology*

L.30
$$\{u \in \mathcal{C}^\infty(M; \Lambda^*); (d + \delta)u = 0\} = \sum_k H^k_{\mathrm{Ho}}(M),\ H^k_{\mathrm{Ho}}(M) \simeq H^k_{dR}(M). \tag{27}$$

Proof. The firt step is to check the ellipticity of the Hodge operator $d + \delta$. Since d is a first order operator we can compute its symbol directly from (17):-

L.31
$$\sigma(d)(\xi) = i\xi \wedge \text{ at } \xi \in T^*_m M. \tag{28}$$

The adjoint of the symbol is the symbol of the adjoint so

L.32
$$\sigma(\delta) = -i\iota(\xi) \text{ at } \xi \in T^*_m M \tag{29}$$

where ι evaluates a k-form at the dual vector, $v_\xi \in T_m M$ with respect to the metric.

It follows that $d + \delta$ is at least a generalized Dirac operator – it is in fact a Dirac operator – since the square of its symbol is

L.33
$$\sigma(d + \delta)^2 = (i\xi \wedge -i\iota(\xi))^2 = \xi \wedge \iota(\xi) + \iota(\xi)\xi\wedge = |\xi|^2 \,\mathrm{Id}\,. \tag{30}$$

This is easily seen by by using an orthonormal basis in which $\xi = |\xi| e_i$.

From the ellipticity of $d+\delta$ it follows that

L.34 (31) $$\mathcal{C}^\infty(M;\Lambda^*) = \operatorname{Nul}(d+\delta) + (d+\delta)\mathcal{C}^\infty(M;\Lambda^*)$$

since $(d+\delta)^* = d+\delta$. Now in fact (31) can be rewritten as the Hodge decomposition for each k

L.35 (32) $$\begin{gathered}\mathcal{C}^\infty(M;\Lambda^k) = H^k_{\text{Ho}}(M) \oplus d\mathcal{C}^\infty(M;\Lambda^{k-1}) \oplus \delta\mathcal{C}^\infty(M;\Lambda^{k+1}),\\ H^k_{\text{Ho}}(M) = \{u \in \mathcal{C}^\infty(M;\Lambda^k); du = 0,\ \delta u = 0\}.\end{gathered}$$

To see this, note first that

L.36 (33) $$\operatorname{Nul}(d+\delta) = \{u \in \mathcal{C}^\infty(M;\Lambda^*); du = 0,\ \delta u = 0\} = \bigoplus_k H^k_{\text{Ho}}(M)$$

in terms of (27). Certainly the right side is contained in the central space and if $(d+\delta)u = 0$ then

L.37 (34) $$0 = \langle\langle (d+\delta)u, du\rangle\rangle = \|du\|^2 + \langle\langle u, d^2u\rangle\rangle \Longrightarrow du = 0.$$

Thus the null space in (31) decomposes into terms of each degree and it follows that a k-form can be written uniquely as

L.38 (35) $$\mathcal{C}^\infty(M;\Lambda^k) \ni u = h + dv + \delta v,\ h \in H^k_{\text{Ho}}(M)$$

for some smooth form v. This proves the existence and uniqueness of the Hodge decomposition (32), since dv snd δv are orthogonal.

The Hodge isomorphism follows directly. Indeed if $u \in \mathcal{C}^\infty(M;\Lambda^k)$ is closed then it is orthogonal to the range of δ so (35) defines a map

L.39 (36) $$\{u \in \mathcal{C}^\infty(M;\Lambda^k); du = 0\} \ni u \longmapsto h \in H^k_{\text{Ho}}(M)$$

which is surjective and has null space $d\mathcal{C}^\infty(M;\Lambda^{k-1})$ proving (27). □

16. Spectrum of the Laplacian

Probably nore detail would be useful!

In the discussion of the Hodge theorem above we used the exitence of a parameterix $E \in \Psi^{-1}(M;\Lambda^*)$ for $d+\delta$. We can extract much more information than this from it. For one thing

L.40 **Theorem 9.** *The Hodge operator $d+\delta$ and the Laplacian $\Delta = (d+\delta)^2$ on a compact Riemannian manifold have compact self-adjoint generalized inverses and consequently discrete spectrum of finite multiplicty. The non-zero eigenvalues of $d+\delta$ are the positive and negative square-roots of the non-zero (necessarily positive) eigenvalues of Δ.*

Proof. The Laplacian is a little easier to deal with, since it preserves the degree of forms, i.e. it is graded. If E is a parameterix for $d+\delta$ then E^2 is a parameterix for $\Delta = d\delta + \delta d$. Since both $d+\delta$ and Δ are 'formally' self-adjoint on $= \mathcal{C}^\infty(M;\Lambda^*)$ it follows that E can be replaced by $\frac{1}{2}(E+E^*)$ as a parametrix and now, since it is bounded, self-adjointness in the usuall sense follow by density of $\mathcal{C}^\infty(M;\Lambda^*)$ in $L^2(M;\Lambda^*)$.

For simplicity we can consider $\Delta + \operatorname{Id}$ and conclude that this is inverible on $\mathcal{C}^\infty(M;\Lambda^*)$ and has as inverse a compact, positive self-adjoint operator $F \in \Psi^{-2}(M;\Lambda^*)$

which is also graded with respect to form degree (by uniqueness). This F has discrete spectrum, of finite multiplicity with no null space. Conversely the spectrum of $\varDelta$ is discrete of the form

L.41 (1) $$0 = \mu_0 < \mu_1 < \mu_2 \dots \mu_k \to \infty$$

all corresponding to smooth eigenforms. It is more usual to repeat the eigenvalues according to multiplicity – we know that the multiplicity of 0 is the Betti number, the dimension of the cohomology.

For the Hodge operator □

A great deal is known about the spectrum of the Laplacian, but not as much as one might hope. For instance it is certainly not clear when a sequence of λ_j's is the spectrum of a Riemann manifold.

17. Semiclassical limit

One case of the adiabatic limit is where the fibration of the manifold is trivial, in the sense that it is the identity – so the fibres are actually points. This actually arises in quantum mechanics where the eigenvalues typically correspond to energy levels – for instance of the electrons in a molecule. A simple model (really on Euclidean space) is

L.42 (1) $$\epsilon^2 \varDelta + V$$

Let's consider this where $\varDelta = \delta d$ is the Laplacian on functions a compact manifold, M, and V is a real potential on M. Here the factor ϵ^2 is some form of Planck's constant and is, in practice, very small. Mathematically we can ask: What happens to the eigenvalues as $\epsilon \to 0$? This is often called a 'semiclassical limit' refering to the passage from a quantum to a classical (meaning in this case commutative, i.e. constant coefficient) system.

The addition of V as a scalar potential does not formlly do much to the spectrum – adding a constant shifts it up or down and a variable potential distorts it but leaves the essential nature, discrete of finite multiplicity and tending to infinity, unchanged. Since $\varDelta$ is formally positive, the smallest eigenvalue of the operator (1) satisfies

L.43 (2) $$\lambda_0 \ge \inf_M V.$$

So as $\epsilon \downarrow 0$ we might expect the eigefunction to concentrate near the set where V takes its minimum. This I will discuss in some detail below.

For the moment let's assume we are interested in what happens to the inverse to (1) away from the spectrum and specifically we will assume that

L.46 (3) $$\inf_M V > 0$$

and we try to understand the behaviour of $(\epsilon^2 \varDelta + V)^{-1}$ as $\epsilon \downarrow 0$. You might ask: In what sense does the inverse tend to V^{-1} as $\epsilon \downarrow 0$? Clearly this can not be a uniform limit since for $\epsilon > 0$ the inverse is a compact operator on L^2.

If we set $\epsilon = 0$ then the inverse, V^{-1}, is a local operator where as for $\epsilon > 0$ it is smooth away from the diagonal and conormal at it. It is therefore rather natural to expect different behaviour at and away from the diagonal and to see this we consider the semiclassical double space (on which we hope to make sense of the inverse operator)

L.47
$$M[2, \mathrm{scl}] = [M^2 \times [0,1)_\epsilon ; \mathrm{Diag} \times \{0\}] \tag{4}$$

in which the diagonal is blown up at $\epsilon = 0$.

The new boundary face introduced by the blow-up is a closed ball bundle as a bundle over the diagonal – which is diffeomorphic to M. If we introduce z_j as local coordinates in the left factor of M and the same coordinates as z'_j in the right factor then coordinates in the interior of this ball bundle are

L.48
$$Z_j = \frac{z_j - z'_j}{\epsilon},\ z'_j \tag{5}$$

where we could just as well use z_j in place of z'_j but it is more convenient to use the right-coordinates since the vector fields are acting mainly on the left. Near the boundary of the ball bundle, which is a sphere bundle, we can use

L.49
$$\hat{Z} = \frac{Z}{|Z|} = \frac{z - z'}{|z - z'|},\ |z - z'|,\ \eta = \epsilon/|z| \tag{6}$$

where the first are 'coordinates' on the sphere – we have to drop an appropriate element to get coordinates.

Now we can see what happens to the 'semiclassical vector fields' acting on the left factor of M. Namely in local coordinates they are the

L.50
$$\epsilon \partial_{z_j} = \partial_{Z_j}. \tag{7}$$

The lifted diagonal – the closure of the inverse image of the part of the diagonal outside $\epsilon - 0$ – is $Z = 0$ in the coordinates (5) so these vector fields are transversal to it, then span its normal bundle, which is really the notion of normal ellipticity.

Thus on each of the fibres of the front face of the blow up, our degenerated differential operator $\epsilon^2 \Delta + V$ is resolved to the non-degenerate constant coefficient operator

L.51
$$\Delta_{\mathrm{Eu}} + V(m) = \sum_j D_j^2 + V(m) \tag{8}$$

where m is the point (m, m) in the diagonal that we are looking at. So we are back to one of the simple cases we talked about earlier, a fibrewise family of elliptic operators. Since $v > 0$ we know what the fundamental solution will look like, namely

L.52 (9) $(\Delta_{\mathrm{Eu}} + 1)^{-1}$ is classical conormal at 0 and Schwartz near infinity.

With this in mind we can guess the form of, and then check, the behaviour of kernel of the inverse of $\Delta + V$ once we take into account the scaling of the Dirac distribution in terms of the coordinates (5) near the diagonal

L.53
$$\delta(z - z') = \epsilon^{-n} \delta(Z). \tag{10}$$

L.54 **Proposition 14.** *Given (3) the semiclassical opeator (1) has inverse with kernel of the form*

L.55
$$k \epsilon^{-n} |dg(z')| \tag{11}$$

where k is a locally integrable function on $M[2, \mathrm{scl}]$ which is an elliptic conormal distribution of order -2 at the (lifted) diagoanal, smooth up to all boundary faces and vanishes rapidly at the lift of the face $\epsilon = 0$.

Proof. The main step here is to understand the geometry of the space constructed in (4). The blow-down map is smooth as are the projections so there is a composite 'stretched projection'

L.70 (12) $$\tilde{p}i_R = \pi_R \circ \beta : M[2, \mathrm{scl}] \longrightarrow M \times [0,1)_\epsilon.$$

The important point to note is that this is a fibration near the lifted diagonal. Indeed, this is certainly the case away from $\epsilon = 0$. In terms of the local coordinates (5) pi_L is

$$(Z, z', \epsilon) \longmapsto (z', \epsilon)$$

which is a local fibration.

Combined with (7) it follows that we can solve away the singularity of (10) with a term $\epsilon^{-n}k_1(Z, z', \epsilon)$ which is conormal at $Z = 0$ and smooth in the parameteris z', ϵ. Here we may assume that k_1 is supported in $|Z| < \delta$ for any small positive δ. This means that we have solved the problem up to an error term

L.72 (13) $$(\Delta + V)E = \delta|dg'| + R'$$

where $R' = \epsilon^{-n}r(Z, z', \epsilon)|dg'|$ with r smooth on $M[2, \mathrm{scl}]$ and supported near the lifted diagonal.

To remove this term we argue in terms of Taylor series near the front fact. On each fibre we are reduced to the constant coefficient problem

$$(\Delta_{\mathrm{Eu}} + V(m))k_2(Z, m) = r(Z, m, 0) \in \mathcal{C}^\infty_{\mathrm{c}}(TM).$$

This indeed has a unique solution which lies in the Schartz space on the fibres of TM forming the front face. Said differently, it is a smooth function on the ball bundle which vanishes to infinite order at the boundary. So in fact we can make a first choice of $k_2 \in \mathcal{C}^\infty(M[2, \mathrm{scl}]$ which vanishes at the 'old' boundary and which removes the part of the error term replacing (13) by

$$(\Delta + V)(E + E') = \delta|dg'| + R_2', \;\; R_2' = \epsilon^{-n:}r_2'(Z, z', \epsilon)|dg'$$

where r_2' also vanishes to infinite order at the old boundary. Since ϵ commutes with the problem we can proceed inductively, solving the slight generalization of (17), where the error is now in $\mathcal{S}(TM)$, still with a Schwartz solution, and summing the resulting power series we get an improved parameterix with

L.75 (14) $$(\Delta + V)(E + E') = \delta|dg'| + R'', \;\; R'' = r''|dg'|, \;\; r'' \in \epsilon^\infty \mathcal{C}^\infty(M^2 \times [0,1)).$$

Here we use the fact that a smooth function on $M[2, \mathrm{scl}]$ which vanishes to infinite order at the two boundaries above $\epsilon = 0$ is actually smooth on $M^2 \times [0,1)$ and vanishes to infinite order at $\epsilon = 0$.

Now, we are actually finishes since such an error terms, $\mathrm{Id} + R''$ is invertible for small $\epsilon > 0$ with inverse of the same form – simply by summing the Nuemann series. □

So we see how the transition from an elliptic conormal function to a local operator at $\epsilon = 0$ takes place. To see how V^{-1} arises as the inverse at $\epsilon = 0$ we can check that

L.56 **Lemma 6.** *If $f \in \mathcal{C}^\infty(M \times [0,1)_\epsilon)$ then the unique solution of $(\Delta + V)u = f$ (subject to (3)) is smooth on $M \times [0,1)$ and*

L.57 (15) $$u\big|_{\epsilon=0} = V^{-1}f\big|_{\epsilon=0}.$$

18. Semiclassical ground state

Again we can add an appropriate constant and normalize things so that

L.44 (1) $$\inf V = 0.$$

To avoid subsequent complications we will assume that V has a non-degenerate minimum, for the moment just at one point $m_0 \in M$, and in local coordinates based at that point

L.45 (2) $$V(0) = 0,\ V \geq \delta |x|^2,\ \delta > 0.$$

19. Adiabatic Limit

First consider the direct sum of two vector spaces $V + W$, which is the product with component-wise linear operations. The tensor spaces of this sum decompose and the one we are particularly interested in is the exterior algebra for which there is natural isomoprhism

11.4.2020.1 (1) $$\Lambda^*(V + W) = \Lambda^* V \otimes \Lambda^* W.$$

This corresponds to the expansion in terms of a basis, v_i of V and w_j of W

11.4.2020.2 (2) $$\Lambda^*(V + W) \ni u = \sum_{\alpha,\beta} u_{\alpha,\beta}(v_1^{\alpha_1} \wedge v_k^{\alpha_k}) \otimes (w_1^{\beta_1} \wedge w_l^{\beta_l}).$$

Here the α_i and β_j are either 0 or 1, i.e. the sum is over subsets of the indices.

If we consider the product of two manifolds $X \times Y$ then the tangent space at each point is the direct sum $T_x X + t_y Y$ and the form bundles (with V, W the cotangent spaces) are then tensor products

11.4.2020.3 (3) $$\Lambda^*(X \times Y) = \Lambda^* X \otimes \Lambda^* Y.$$

If we think about forms then this decomposition give a global tensor product decomposition

11.4.2020.4 (4) $$\mathcal{C}^\infty(X \times Y) = \mathcal{C}^\infty(X \times Y; \Lambda^* X) \otimes_{\mathcal{C}^\infty(X \times Y)} \mathcal{C}^\infty(X \times Y; \Lambda^* Y).$$

Here the spaces of sections on the right are really the sections over $X \times Y$ of the pull-backs of the form bundles under the two projections to X and Y and the tensor product is over $\mathcal{C}^\infty(X \times Y)$ – meaning the coefficients only appear once since we need to impose the equivalence relation

11.4.2020.5 (5) $$fv \otimes w = v \otimes fw,\ f \in \mathcal{C}^\infty(X \times Y),\ v \in \mathcal{C}^\infty(X \times Y; \Lambda^* X),\ w \in \mathcal{C}^\infty(X \times Y; \Lambda^* Y).$$

In this sense d decomposes as the sum of d_X acting on the left factor and d_Y on the right, except that the second differential has to be 'moved across' the term v in 11.4.2020.5 (5). So if we let N be the *number operator* on $\Lambda^* X$, defined to evaluate to the degree of a form on a basis then

11.4.2020.6 (6) $$d = d_X + (-1)^N d_Y,\ d_X d_Y = d_Y d_X.$$

To avoid constantly writing this sign set $\tilde{d}_Y = (-1)^N d_Y$ so that

11.4.2020.7 (7) $$d = d_X + \tilde{d}_Y,\ d_x \tilde{d}_Y + \tilde{d}_Y d_X = 0.$$

Now, take a product metric $g = g_X + g_Y$ on $X \times Y$, just the sum of a metric on $T_x X$ and $T_y Y$ at each point and consider the Hodge theory. It follows directly that the Hodge operator can be written

11.4.2020.8 (8) $$d + \delta = d_X + \delta_X + \tilde{d}_Y + \tilde{\delta}_Y.$$

The Hodge decomposition

11.4.2020.11 (9)
$$\mathcal{C}^\infty(X\times Y;\Lambda^*)=H^*_{\mathrm{Ho}}(X\times Y)\oplus(d_X+\tilde{d}_Y)\mathcal{C}^\infty(X\times Y;\Lambda^*)\oplus(\delta_X+\tilde{\delta}_Y)\mathcal{C}^\infty(X\times Y;\Lambda^*)$$

in this case has a further refinement. The anticommutation relations $d_X\tilde{\delta}_Y+\tilde{\delta}_Y d_X=0$ etc., show that $d_X+\delta_X$ and $\tilde{d}_Y+\tilde{\delta}_Y$ anti-commute and so have orthogonal images on a given smooth form

11.4.2020.9 (10)
$$\langle\langle(d_X+\delta_X)u,(\tilde{d}_Y+\tilde{\delta}_Y)u\rangle\rangle=-\langle\langle(\tilde{d}_Y+\tilde{\delta}_Y)u,(d_X+\delta_X)u\rangle\rangle=0$$

since the inner product is symmetric on real forms. Thus

11.4.2020.10 (11)
$$(d+\delta)u=0\Longrightarrow(d_X+\delta_X)u=0=(\tilde{d}_Y+\tilde{\delta}_Y)u\Longrightarrow d_Xu,\delta_Xu,\tilde{d}_Yu,\tilde{\delta}_Yu=0.$$

An harmonic form is therefore harmonic in both senses and from this the decompositon of the Hodge cohomology follows:-

11.4.2020.12 (12)
$$H^*_{\mathrm{Ho}}(X\times Y)=H^*_{\mathrm{Ho}}(X)\otimes H^*_{Ho}(Y),\ g=g_X+g_Y.$$

If one takes a non-product metric then the isomorphism to the tensor product of the cohomologies does not follows so easily!

We want to extend this discussion of Hodge cohomology to the case of a fibration of manifolds $\phi:M\longrightarrow Y$. There is a notion of 'partial product metric' here, which usually called a 'Riemannian submersion' really refering to a metric g on M and a metric g_Y on Y. We know that the null space of ϕ_* is the space of fibre-tangent vectors, naturally the tangent space $T_mZ_{\phi(m)}$ to the fibre $Z_{\phi(m)}=\phi^{-1}(\phi(m))$ through each point. The metric g certainly restricts to a metric on each of these spaces, so defining a fibre metric. The choice of g also induces an orthogonal decomposition at each point $m\in M$,

11.4.2020.13 (13)
$$T_mM=T_mZ_{\phi(m)}\oplus(T_mZ_{\phi(m)})^\perp.$$

Then $\phi_*:(T_mZ_{\phi(m)})^\perp\longrightarrow T_{\phi(m)}Y$ is an isomorphism at each point. The condition that g correspond to a Riemannian submersion is the requirement that under this ismorphism

11.4.2020.14 (14)
$$g\big|(T_mZ_{\phi(m)})^\perp=g_Y.$$

This is really just the requirement that the metric at the image point induced by the restriction of g to orthcomplement of the fibre tangent should be independent of the point in the fibre from which it comes. Thus it determiens g_Y.

11.4.2020.15 *Exercise* 3. Amuse yourself by checking that Riemannian fibrations always exist for any given metric on the base and any smooth fibre metric. Just choose a smooth complementary 'horizontal' bundle $H\subset TM$ so that

11.4.2020.16 (15)
$$T_mM=T_mZ_{\phi(m)}\oplus H_m\ \forall\ m\in M.$$

Then define a metric on the fibres of H by pulling back a fixed metric g_Y by ϕ_* and then making (15) orthogonal determines a Riemannian submersion.

Even if one takes a metric of this type, the Hodge theory on M is not so easily related to the Hodge theory on Y and on the fibres. Instead, we look at an adiabatic limit (family of) metric(s)

11.4.2020.17 (16)
$$g_{\mathrm{ad}}=\epsilon^{-2}\phi^*g_Y+h.$$

Here h could be a metric on M but it only needs to be a symmetric 2-tensor on M which restricts to a be positive definite on the fibre tangent bundle and so induces a fibre metric. Although I will not bother with this too much we could also allow h to depend on ϵ as a parameter.

As already discussed, such an adiabatic metric is a metric on the adiabatic Lie algebroid but it is a special one and we want to look at its algebraic, and then its analytic properties. Recall that the adiabatic vector fields are smooth vector fields on the 'single space'

11.4.2020.18 (17)
$$\begin{gathered} M[1] = M \times [0,1)_\epsilon, \\ \mathcal{V}_{\mathrm{ad}}(M) = \{V \in \mathcal{C}^\infty(M[1]; TM[1]); V\epsilon \equiv 0,\ V \text{ is tangent to the fibres of } \phi \text{ at } \epsilon = 0\}. \end{gathered}$$

I think of this as a 'structure on M' although the vector fields are really on $M[1]$. In view of its obvious significance we set

11.4.2020.19
$$M_0 = \partial M[1] = \{\epsilon = 0\} \tag{18}$$

which is just a copy of M, the most relevant one in fact.

We already know that there are local coordinates near each point $m \in M$, y and z where the y are pulled back coordinates from Y under ϕ and so locally ϕ is the map $\phi(y, z) = y$. In terms of such coordinates we also know that the adiabatic vector fields have a local basis,

11.4.2020.20
$$\epsilon \partial_{y_j},\ \partial_{z_l}. \tag{19}$$

This is what makes $\mathcal{V}_{\mathrm{ad}}(M)$ into a Lie algebroid (on $M[1]$)) in that there is no relation between the (11.4.2020.20 19) which therefore span a vector bundle ${}^{\mathrm{ad}}T$ over $M[1]$ of which

11.4.2020.21
$$\mathcal{V}_{\mathrm{ad}}(M) = \mathcal{C}^\infty(M[1]; {}^{\mathrm{ad}}T) \tag{20}$$

is the space of all smooth sections. Of course $\mathcal{V}_{\mathrm{ad}}(M)$ is also a Lie algebra. Since the elements of $\mathcal{V}_{\mathrm{ad}}(M)$ are smooth vector fields they define sections of the ordinary tangent bundle of $M[1]$ – in fact they define sections of the fibre tangent bundle for the fibration $M[1] \longrightarrow [0,1)$ but I don't want to complicate the notation by including this. Anyway we get a natural smooth bundle map

11.4.2020.22
$${}^{\mathrm{ad}}T \longrightarrow TM[1] \tag{21}$$

which is injective with image of codimension one for $\epsilon > 0$ but the rank drops to $\dim Z$ at $\epsilon = 0$. In fact we see that the null space of this map is a smooth bundle over M_0 of rank $\dim Y$.

On the othe hand we also have the map ϕ_* extended to $M[1]$. Form $TM[1]$ this is always surjective of course. But composed with the map (11.4.2020.22 21) it is surjective for $\epsilon > 0$ but its rank drops to zero at $\epsilon = 0$. In fact we can see from the basis (11.4.2020.20 19) that every semiclassical vector field on Y, so locally in the span of $\epsilon\partial_{y_j}$. What we conclude is that the projection map in $\epsilon > 0$ extends to a smooth surjective bundle map

11.4.2020.23
$${}^{\mathrm{ad}}TM[1] \longrightarrow {}^{\mathrm{sl}}TY[1] \tag{22}$$

where the semiclassical bundle over $Y[1] = Y \times [0, 1)$ is the adiabatic bundle in the case that the fibration of Y is the trivial, identity, fibre bundle (so has fibres which are points).

This in turn shows that over the boundary the adiabatic bundle decomposes naturally

11.4.2020.24
$${}^{\mathrm{ad}}TM_0 = {}^{\phi}TM_0 \oplus {}^{\mathrm{sl}}TY. \tag{23}$$

This raises the question: Precisely what is a linear differential operator (with smooth coeffficents) acting between sections of two vector bundles (over the same manifold M)? So, it is a linear map

12.4.2020.1
$$P : \mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^\infty(M;W) \tag{24}$$

which we could assume to be continuous, but this follows in any case from the two condition we require:-

12.4.2020.2 (25)
$$P \text{ is local } u \in \mathcal{C}^\infty(M;V),\ u = 0 \text{ over } \Omega \subset M \text{ open } \Longrightarrow Pu = 0 \text{ over } \Omega$$
$$P(fu) = fPu + L(df)u,\ \forall\ u \in \mathcal{C}^\infty(M;V),\ u \in \mathcal{C}^\infty(M) \text{ with } L \in \mathcal{C}^\infty(M; \hom(T^*M \otimes V; W)).$$

If one introduces local coordinates z_j then over the coordinate patch L reduces to $n = \dim M$ bundle maps $L_j \in \hom(V, W)$

12.4.2020.3
$$L = \sum_{j=1}^n dz_j \otimes L_j. \tag{26}$$

From this we can make a local differential operator

12.4.2020.4
$$\tilde{P} = \sum_{j=1}^n L_i \partial_{z_j}. \tag{27}$$

This satisfies a local version of the second condition in (12.4.2020.2 25)

$$\tilde{P}fu = f\tilde{P}u + \sum_j L_j \partial_j f u = f\tilde{P}u + L(df)u.$$

So the difference $\tilde{M} = P - \tilde{P}$ commutes with multiplication by $f \in \mathcal{C}^\infty_{\mathrm{c}}(U)$, supported in the coordinate patch. Thus $\tilde{M}$ is actually a bundle map from V to W, at least over the coordinate patch U (since the value of $\tilde{M}u$ at a point m can only depend on $u(m)$. Thus in fact $P = \tilde{P} + \tilde{M}$ is locally a differential operator in the obvious coordinate sense.

Thus (12.4.2020.2 25) does imply the local coordinate form

12.4.2020.6
$$P = \sum_{j=1}^n L_i \partial_{z_j} + M. \tag{28}$$

The locality actually follows from the second condition if it is global as stated.

If we take a Lie algebroid, such as the adiabatic or semiclassical ones we are currently considering with associate tangent bundle $\tau \longrightarrow M$, then we can define a differential operator in this sense to be 'generated by the Lie algebroid' if (12.4.2020.2 25) holds and we require that

12.4.2020.7
$$L \in \mathcal{C}^\infty(M; \hom(\tau^* \otimes V; W)). \tag{29}$$

20. For 28 April: Model problems

I will start today by reviewing our understanding of 'model problems', for Dirac operators and in particular the Hodge operator, and then start to discuss how we can use them to construct parameterices for more complicated cases – so much of this material I have touched on before but I felt it would be wise to clarify it.

20.1. **A compact Riemann manifold.** Here I have discussed the Hodge operator $\eth = d + \delta$ acting on $\mathcal{C}^\infty(M; \Lambda^*)$ and shown it to be Fredholm with generalized inverse a compact self-adjoint operator, E_0, acting on $L^2(M; \Lambda^*)$ – which is a pseudodifferential operator of order -1. This $E_0 \in \Psi^{-1}(M; \Lambda^*)$ satisfies

L.76 (1)
$$(d + \delta)E_0 = \operatorname{Id} - \pi_{\mathrm{Ho}}$$

where π_{Ho} is the self-adjoint projection onto the null space, the finite-dimensional space of harmonic forms.

L.77 **Proposition 15.** *The Hodge operator on a compact manifold (without boundary) has resolvent $R(\zeta) = ($*
$eth - \zeta)^{-1}$ a meromorphic family with values in $\Psi^{-1}(M; \Lambda^*)$ and the with symbol indepenent of ζ; the poles of $R(\zeta)$ are of all simple, on the real axis, and of finite rank with the the non-zero poles in pairs $\pm\lambda_i$, the residue at $\zeta = \lambda_i$ is the finite rank self-adjoint projection on the eigenspace.

Proof. We can use the generalized inverse to construct the resolvent. Set $E = E_0 + \pi_{\mathrm{Ho}}$. It follows from (1) that

L.79 (2)
$$(\eth - \zeta)E = \operatorname{Id} - \pi_{\mathrm{Ho}} - \zeta E.$$

The modified E is injective and it follows that for $\operatorname{Im}\zeta \neq 0$ the operator in the right in (2) is invertible. Indeed, it is of the form $\operatorname{Id} + K$ for K compact and is injective – since

L.80 (3)
$$\operatorname{Im}\langle(\operatorname{Id} - \pi_{\mathrm{Ho}} - \zeta E)u, Eu\rangle = -\operatorname{Im}\|Eu\|^2.$$

Thus it is invertible and, from the propeties of compact operators, is a meromorphic function of ζ with finite rank poles and values in $\Psi^{-1}(M; \Lambda^*)$. Then similar statements are true of

L.82 (4)
$$R(\zeta) = E(\operatorname{Id} - \pi_{\mathrm{Ho}} - \zeta E)^{-1}.$$

The formal self-adjointness of $\eth$ on $\mathcal{C}^\infty(M; \Lambda^*)$ shows that

L.81 (5)
$$\operatorname{Im}\langle(\eth - \zeta)u, u\rangle = -\operatorname{Im}\|u\|^2_{L^2}$$

for $u \in \mathcal{C}^\infty(M; \Lambda^*)$ and hence

L.83 (6)
$$\|R(\zeta)\|_{L^2} \le |\operatorname{Im}\zeta|^{-1}.$$

Thus the poles can only be simple (in the sense that they have singular part $M(\zeta - \lambda_j)^{-1}$) with residues the projections onto the corresponding eigenforms – which form an orthonormal basis of $L^2(M; \Lambda^*)$ since they are also the eigenfunctions of the compact self-adjoint operator E_0. □

This is all true for an self-adjoint elliptic operator of positive order on a compact manifold. However in the case of the Hodge operator we can go further using the Hodge deocmposition. Namely an eigenform of $\eth$ must also be an eigenform for $\eth^2 = \Delta$ and this preserves form degree, so the decomposition by degree also gives eigenforms with the same eigenvalue. For non-zero eigenvalues these must be orthogonal to the harmonic forms and so decompose into exact and coexact parts. These subspaces are preserved by Δ so all there is a basis of eigenforms for Δ which are each closed or coclosed. It follows that each eigenform for $d + \delta$ decomposes into closed and coclosed terms in each degree, which are eigenforms for Δ. Consider a coclosed form v of degree k obtained in this manner, δv cannot vanish, since it is not harmonic, and must be an eigenform for Δ; similarly for a closed form. Thus there is an orthonormal basis of the eigenforms for $\eth_{\mathrm{Ho}}$ formed from two-dimensional subspaces of $\mathcal{C}^\infty(M;\Lambda^*)$ spanned by

L.84
$$u \in \mathcal{C}^\infty(M;\Lambda^k),\ v \in \mathcal{C}^\infty(M;\Lambda^{k+1}),\ du = \lambda_j v,\ \delta v = \lambda_j u \\ \Longrightarrow \eth(u+v) = \lambda_j(u+v),\ \eth(u-v) = \lambda_j(u-v). \tag{7}$$

Thus the non-zero eigenvalues come in pairs $\pm\lambda_j$.

20.2. **Fibre families.** A generalization that we have already considered is the family of Hodge operators on the fibres of a fibration, $\phi : M \longrightarrow Y$ (of compact manifolds without boundary) corresponding to a choice of a smooth family of fibre metrics.

The fibre tangent bundle ${}^{\phi}TM$ is a smooth bundle over M which restricts to each fibre $Z_y = \phi^{-1}(y)$, to be naturally isomorphic to TZ_y and a fibre family of metrics is just a smooth family of Euclidean metrics on the fibres of ${}^{\phi}TM$. Then for each $y \in Y$, the discussion above applies and the main thing to think about is smoothness in Y.

In fact we showed that the generalized inverse is smooth – this is special to a family like the Hodge operators for which the null space has constant dimension – could be constructed as a smooth family in the sense that the kernel is a classical conofmal distribution on the diagonal of the 'double space'

L.85
$$M[2,\phi] = M \times_\phi M \tag{8}$$

the fibre product. This is the natural bundle over Y with fibre over y just $Z_y \times Z_y$.

If we proceed to the construction of the resolvent as before

21. For 30 April, 2020: b-metrics

I started this last time but have not written up everything.

I have just four more lectures including today – when I will talk about the Hodge operator for a b-metric on a compact manifold with boundary. The question is what I should do in the remaining three lectures. What I had proposed was to try to cover what I have set up so far, which would mean after today

- Scattering metrics – Euclidean space
- Adiabatic limit
- Witten deformation

The alternative would be that I proceed to write this up anyway but that I talk a little about index theory in these three lectures, it is up to you.

Last time I started talking about exact b-metrics which exist on any compact manifold with boundary and take the form

30.4.2020.1
$$g = \frac{dx^2}{x^2} + h(y,dy) + xg' \text{ in } x < \epsilon. \tag{1}$$

This is in terms of some product decomposition near the boundary with x a boundary defining function and h a metric on the boundary extended to be constant along the normals in the product decomposition. We allow a perturbation right up to the boundary where I will require g' to be a smooth 2-cotensor in the ordinaty sense, so involving dx, dy_j.

In the model case, where $g' = 0$, I worked out the form of the Hodge operator in terms of the decomposition of forms

30.4.2020.2 (2)
$$ {}^{\mathrm{b}}\Lambda^* = \Lambda^* Y + \frac{dx}{x} \otimes \Lambda^* Y,\ Y = \partial M $$

as

30.4.2020.3 (3)
$$ \eth = \begin{pmatrix} \eth_Y & -x\partial_x \\ x\partial_x & -\eth_Y \end{pmatrix} $$

This corresponds to a suspended family, with one Euclidean variable, if we replace x by $\xi = \log x$.

Long ago (it seems) I introduced the natural Sobolev spaces with respect to the Lie algebroid $\mathcal{V}_{\mathrm{b}}(M)$ on a compact manifold with boundary, in this case,

30.4.2020.4 (4)
$$ H^k_{\mathrm{b}}(M; {}^{\mathrm{b}}\Lambda^*) = \{u \in L^2_{\mathrm{loc}}(M \backslash \partial M; \Lambda^*); \int |V_1 \dots V_j u|^2 \frac{dx}{x} dh < \infty;\ j \le k,\ V_i \in \mathcal{V}_{\mathrm{b}}(M)\}. $$

Here I really mean the action of differential operators made up out of b-vector fields of order up to k. We can do this for all real values of the order without too much trouble. We also need the weighted versions of these spaces

30.4.2020.6 (5)
$$ x^t H^k_{\mathrm{b}}(M; {}^{\mathrm{b}}\Lambda^*). $$

Now, essentially by definition

30.4.2020.5 (6)
$$ \eth : x^t H^{k+1}_{\mathrm{b}}(M; {}^{\mathrm{b}}\Lambda^*) \longrightarrow x^t H^k_{\mathrm{b}}(M; {}^{\mathrm{b}}\Lambda^*) $$

is bounded for any k and $t \in \mathbb{R}$ since conjugation by x^t preserves the b-differential operators.

30.4.2020.7 **Theorem 10.** *The Hodge operator for an exact b-metric* (1) *is Fredohom as an operator* (6) *for all k and for all $t \in \mathbb{R} \setminus \mathrm{Spec}(\eth_Y)$.*

Proof. This, I hope, follows a by-now-standard format. We construct a parameterix on the space $M[2, \mathrm{b}] = [M^2; (\partial M)^2]$ obtained from the ordinary product by blowing up the corner.

The Fredholm statement is equivalent to the claim that the conjugate operator $x^{-t}\eth x^t$ is Fredholm on the unweighted b-Sobolev spaces. If we ignore the non-product terms xg' in the metric the 'model' operator is conjugated to

30.4.2020.8 (7)
$$ \eth^t = \begin{pmatrix} \eth_Y & -x\partial_x - t \\ x\partial_x + t & -\eth_Y \end{pmatrix}. $$

This commutes with $\eth_Y$ so we can decompose it according to the spectral decomposition of $\eth_Y$ (usually called 'separation of variables') giving the ordinary differential operators

30.4.2020.9 (8)
$$ \begin{pmatrix} \lambda & -x\partial_x - t \\ x\partial_x + t & -\lambda \end{pmatrix}. $$

on two copies of the eigenspace with eigenvalue λ. Introducing $\xi = \log x$ and taking the Fourier transform reduces this to a matrix with determinant

30.4.2020.10 (9)
$$ \det \begin{pmatrix} \lambda & -i\eta - t \\ i\eta + t & -\lambda \end{pmatrix} = (i\eta + t)^2 - \lambda^2 = (i\eta + t + \lambda)(i\eta + t - \lambda) $$

which is non-vanishing for all real η, the dual variable to ξ, unless $t = \pm\lambda$. Since the spectrum of $\eth_Y$ is symmetric this just means $t \in \mathrm{Spec}(\eth_Y)$.

Under the assumptions that $t \notin \operatorname{Spec}(\partial_Y)$ it follows that the operator (30.4.2020.8 7) has a global fundamental solution on $Y \times (0, \infty)$. This is the inverse Fourier transform, with variable ξ replaced by $\log x$ of a locally integrable function on $Y^2 \times \mathbb{R}_\eta$ which is smooth away from the 'diagonal' and Schwartz near infinity. This is really the inverse Mellin transform

30.4.2020.11
$$E_0(y, t', S) = \frac{1}{2\pi} \int S^{i\eta} \begin{pmatrix} \eth_Y & -x\partial_x - t \\ x\partial_x + t & -\eth_Y \end{pmatrix}^{-1} d\eta. \tag{10}$$

Here I have written the variable as S because in the parametrix we replace x by $S = x/x'$. Again the expansion near $|\eta| \to \infty$ in the eigenbasis shows that the inverse there is an asymptotic sum of exponentially decreasing terms corresponding to the eigenvalues and so the inverse Fourier transform is exponentially decaying and the inverse Mellin transform, just thinking in terms of $x = \exp(\xi)$, is bounded near infinity by some fixed positive power

30.4.2020.12
$$|E_0(y, t', S)| \le C \min(S^\delta, S^{-\delta}) \text{ as } S \to 0 \text{ or } S \to \infty. \tag{11}$$

The same is true for all y, y' and $s\partial_S$ derivatives. The power here is determined by the distance of t from the spectrum of ∂_Y.

Now, we can use this fundamental solution to construct a parameterix for $\eth^t$. I omit the details at the moment but we use this fundamental solution at the leading term at the front face $M[2, \mathrm{b}]$. It is consistent with the (unique modulo smoothing) classical parameterix across the diagonal and vanishes at the old faces, corresponding to $S = 0$, ∞ in (30.4.2020.11 10). The remainder term for this parameterix vanishes to one order higher at the front face and has similar vanishing at the old faces and is otherwise smooth.

An application of Schur's criterion shows that this error term is compact – it is bounded and is the limit of the compact operators obtained by cutting off near the boundaries. Since we have a parameterix for the adjoint, and hence both a left and right parameterix, if follows that $\eth^t$ is Fredholm as claimed. □

Now of course the question is, what is the index? The first question we ask is about the dependence on t. This is variously called a 'jumps' formula or a 'wall-crossing formula' in various contexts.

30.4.2020.13 **Proposition 16.** *As a function on $\mathbb{R} \setminus \operatorname{Spec}(\eth_Y)$ the index is constant on intervals and decreases across each point of $\operatorname{Spec}(\eth_Y)$ by the dimension of the null space of*

30.4.2020.14
$$\begin{pmatrix} \eth_Y & -t \\ t & -\eth_Y \end{pmatrix} \text{ on } \mathcal{C}^\infty(Y; \Lambda^* \oplus \Lambda^*) \tag{12}$$

which is to say the dimension of the eigenspace with eigenvalue $t \neq 0$ and by $2 \dim H_{\mathrm{Ho}}(Y)$ for $t = 0$.

Proof. □

22. Index theory

In these last three lectures I will try to give an introduction to index theory and some outline of the methods that can, and I think should, be used even though in some cases the 'details'; have not been worked out – i.e. the methods are conjectural (although the results themselves are all well-established).

Some of the early and highly influential results include these two:

3.5.2020.1 **Theorem 11** (Toeplitz)**.** *For* $a \in \mathcal{C}(\mathbb{T}; \mathrm{GL}(n; \mathbb{C}))$

3.5.2020.2
$$\pi a \pi : H \longrightarrow H \text{ is Fredholm and } \operatorname{ind}(\pi a \pi) = -\operatorname{wn}\det(a) \tag{1}$$

where $\pi : L^2(\mathbb{T}; \mathbb{C}^N) \longrightarrow H$ *is the self-adjoint projection onto the Hardy space, obtained by dropping the negative Fourier coefficients.*

Proof. We can give a rather direct proof. □

3.5.2020.3 **Theorem 12** (Hirzebruch)**.** *The signature of a compact oriented* $4d$*-manifold, defined as the signature of the symmetric form*

3.5.2020.4
$$H^{2d}(M) \times H^{2d}(M) \ni (\alpha, \beta) \longmapsto \int_M \alpha \wedge \beta \tag{2}$$

is the Hirzebruch L-genus $\int_M L$ *where* L *is a universal form on* M *given in terms of the curvature of a Riemannian metric on* M*.*

More explicitly L is given as a polynomial in the Pontryagin classes of M – which are classes in degrees $4k$ on an oriented manifold associated to the tangent bundle, or more canonically to the principal $\mathrm{SO}(n)$-bundle of oriented orthonormal frames. It is associated with the formal power series

3.5.2020.21
$$Q(x) = \frac{x}{\tanh x} = 1 + O(x) \tag{3}$$

by the 'usual prescription'. Take the product of n, the dimension of the manifold, copies of Q in different variables and express this as a function of p_i, the basic symmetric polynomials, the term of a given homogeneity k gives the corresponding polynomail $L_k(p_*)$.

In the first case we have a 'geometric operator' and we find what is a topological formula for the index – namely $\det a$ is a map into $\mathbb{C}^*$ which is a $K(Z, 1)$, a classifying space for 1-dimensional cohomology. Said differently, corresponding the deRham class on $\mathbb{T}$ defined in case a is smooth by $(2\pi i)^{-1} \operatorname{tr}(a^{-1} da)$ integrates to the winding number. So the index is the integral over $\mathbb{T}$ of a cohomology class made from a – it can always be smoothed.

In the second case we do not have any Fredholm operator at all it would seem. However we know the connection between the Hodge operator $d + \delta$ and cohomology given by the Hodge theorem. I did not talk about Poincaré duality but now is as good a time as any. A Riemann metric defines a bundle isomorphism '$*$' on forms (so this is at the level of the cotangent space at each point as an oriented Eucludean space) by

3.5.2020.5
$$\begin{gathered} * : \mathcal{C}^\infty(M; \Lambda^k) \longrightarrow \mathcal{C}^\infty(M; \Lambda^{n-k}),\ n = \dim M, \\ \alpha \wedge *\beta = \langle \alpha, \beta \rangle dg \ \forall\ \alpha, \beta\ k\text{-forms} \end{gathered} \tag{4}$$

On an oriented orthonormal basis it is defined by $*e^I = (-1)^{\operatorname{sgn}(I,J)} e^J$ where $I \sqcup J = \{1, \dots, n\}$ and sgn is the signature of the rearrangement. It follows that $*^2 = (-1)^{k(n-k)}$ on k-forms.

Now the adjoint of d is given in terms of $*$ by

3.5.2020.6
$$\delta = (-1)^{n(k-1)+1} * d* \tag{5}$$

on k-forms as follows by using the integrated identity from (3.5.2020.5 4) twice:

3.5.2020.7
$$(du, v) = \int du \wedge *v = (-1)^k \int u \wedge d * v = (-1)^k (u, *^{-1} d * v) \tag{6}$$

where v is a k-form.

It follows that $*$ maps harmonic k-forms to harmonic $(n-k)$-forms and implements Poincaré duality on a compact oriented manifold:-

3.5.2020.8
$$*: H^k_{\mathrm{Ho}}(M) \longrightarrow H^{n-k}_{\mathrm{Ho}}(M). \tag{7}$$

Now, assuming I have the exponents right when $n = 4d$

3.5.2020.9
$$\tau = i^{k(k-1)+2d} * \text{ on } k-\text{forms} \tag{8}$$

is a self-adjoint involution and it anti-commutes with $\eth_{\mathrm{Ho}}$. It follows that with respect to the decomposition into ± 1-eigenspaces for τ

3.5.2020.10
$$\eth_{\mathrm{Ho}} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},\ \operatorname{sgn}(M) = \operatorname{ind}(D_+) = -\operatorname{ind}(D_-). \tag{9}$$

Thus Hirzebruch signature formula can be regarded as an index theorem

3.5.2020.11
$$\operatorname{sgn}(M) = \operatorname{ind}(D_+) = \int L(M). \tag{10}$$

These two theorems have fundamental, vast and overlapping generalizations.

Let me first discuss the generalization, given by Atiyah and Singer, to the signature theorem. Consider a vector bundle $V \longrightarrow M$ over a compact manifold M. Suppose that V is a smooth Clifford module, meaning that each fibre has a linear action by the Clifford algebra of the cotangent bundle. Recall that the latter is defined as the quotient of the full tensor algebra by the ideal generated by the relations

$$\alpha \otimes \beta + \beta \otimes \alpha = 2\langle \alpha, \beta \rangle \operatorname{Id},\ \alpha, \beta \in T^*_m M.$$

This bundle of algebras is locally trivialized by taking an orthonomral basis of T^*M and has the same dimension, 2^n as the exterior algebra (to which it is isomorphic as a vector bundle).

I will give some examples of such Clifford modules but we know one already, namely the (complex) exterior algebra to which the symbol of $d + \delta$ gives a Clifford module structure – any linear map from T^*M into the homomorphism bundle $\hom V$, which satisfies (3.5.2020.12 22) generates a Clifford action.

Now, we shall impose some conditions on the Clifford module V – which is the bundle with its Clifford action. First we want the action to be unitary for an Hermitian fibre metric, h, on V. Secondly we want the Clifford module to have a connection $\nabla : \mathcal{C}^\infty(M; V) \longrightarrow \mathcal{C}^\infty(T^*M \otimes V)$ which is Hermitian (also called unitary) and Clifford

3.5.2020.13
$$\begin{gathered} V(u,v)_h = (\nabla_V u, v) + (u, \nabla_V v) \\ \nabla_V(\operatorname{cl}(\alpha)u) = \operatorname{cl}(\nabla_V \alpha)u + \operatorname{cl}(\alpha)\nabla_V u, \\ \alpha \in \mathcal{C}^\infty(T^*M),\ u \in \mathcal{C}^\infty(M; V),\ V \in \mathcal{C}^\infty(MTM). \end{gathered} \tag{11}$$

There always is such a connection.

Given this data we can define a corresponding Dirac operator on sections of V :

3.5.2020.14
$$\eth : \mathcal{C}^\infty)M; V) \longrightarrow \mathcal{C}^\infty(M; V),\ \eth u = \operatorname{cl} \circ \nabla u. \tag{12}$$

In the definition ∇ maps u to a section of $T^*M \otimes V$ and then cl 'contracts' this to a section of V by the action of T^*M on V. Written out in terms of any basis of vector fields v_i and metric dual basis α_i of 1-forms this becomes locally

3.5.2020.15
$$\eth u = \sum_i \operatorname{cl}(\alpha_i)\nabla_{v_i} u. \tag{13}$$

In case M had dimension $2d$ the Clifford algebra on each fibre is graded in that the product

3.5.2020.16 (14) $$Z = i^d \operatorname{cl}(e_1)\operatorname{cl}(e_2)\cdots\operatorname{cl}(e_{2d}) \text{ satisifes } Z^2 = \operatorname{Id},\ \operatorname{cl}(e_i)Z + Z\operatorname{cl}(e_i) = 0.$$

Thus the Clifford action by any 1-form anticommutes with Z so maps the $+1$-eigenspace V^+ to the -1-eigenspace V^-. However the Clifford property of the connection means that it acts separately on sections of the subbundles $V^{\pm}$. Thus

3.5.2020.17 (15) $$\eth = \begin{pmatrix} 0 & \eth_- \\ \eth_+ & 0 \end{pmatrix},\ V = V^+ \oplus V^-.$$

Since while $\eth$ itself indices of $\eth_+$ and $\eth_-$ sum to zero.

3.5.2020.18 **Theorem 13** (Atiyah+Singer)**.** *The index of a Dirac operator in the sense discussed above is*

3.5.2020.19 (16) $$\operatorname{ind}(\eth_+) = \int_M \operatorname{Ch}'(V)\widehat{A}(M).$$

Here $\widehat{A}$ is another characteristic class given as a polynomial in the Pontryagin classes this time associated to the formal power series

3.5.2020.20 (17) $$\frac{x/2}{\cosh x/2}$$

Now, before talking about the other facto, $\operatorname{Ch}'(V)$, in the index formula (3.5.2020.19 16) let me talk about a special case. After all the question arises – how can be construct Clifford modules?

One way to do this is to look at spin structures. On an oriented Riemann manifold there is a natural principle bundle

3.5.2020.22 (18) $$\begin{tikzcd} \mathrm{SO}(n) \arrow[r, dash] & F_{\mathrm{SO}} \arrow[d] \\ & M. \end{tikzcd}$$

The fibre at a point $m \in M$ is the collection of oriented, orthonormal bases, frames, of the tangent bundle. Two such are related by a linear transformation in $\mathrm{SO}(n)$. In fact this action on the fibre is free and transitive, each fibre *is* $\mathrm{SO}(n)$ except there is no base point. You can recover the tangent bundle from the frame bundle by using the standard action of $\mathrm{SO}(n)$ to generate a vector bundle with fibre

3.5.2020.23 (19) $$T_m = (F_{\mathrm{SO}})_m \times_{\mathrm{SO}(n)} \mathbb{R}^n$$

where we take the quotient by the action of $\mathrm{SO}(n)$ on both factors. This is a general construction of a vector bundle from a principal G-bundle and a representation.

Now, consider $\mathrm{SO}(n)$ for $n \geq 3$ (the 2-d case is special). If you consider continuous paths in $\mathrm{SO}(n)$ beginning and ending at the identity then the fall into two classes, the ones which are contractible and the ones which are not. This corresponds to the fact that the homotopy group $\Pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$. This in turn means that $\mathrm{SO}(n)$ has a double cover, the spin group

3.5.2020.24 (20) $$\mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n).$$

Namely just define $\mathrm{Spin}(n)$ to be the set of continous paths $[0,1] \longrightarrow \mathrm{SO}(n)$ beginning at Id and identify two if they have the same endpoint (giving the map to

SO(n)) and the 'join' – out one and back along the other – is contractible. The group stucture comes from pointwise composition of paths.

You can see this rather explictly for $n = 3$. Then SO(3) is 3-d real projective space and Spin(3) $= \mathbb{S}^3$ is the natural dobule cover.

Now, the question of whether M, whichi is orinted and has a Riemannian structure, is *spin* is whether one can consistently double the fibres of F_{SO} in this way. Namely we want the bigger bundle to form a fibration $F_{\text{Spin}} \longrightarrow M$ which factors through a 2-1 projection onto F_{SO} and has a fibre wise (free and transitive) action of Spin(n) so that we get a commutative diagram

3.5.2020.25 (21)
$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, dash] \arrow[d] & F_{\mathrm{Spin}} \arrow[d] \\ \mathrm{SO}(n) \arrow[r, dash] & F_{\mathrm{SO}} \arrow[d] \\ & M \end{tikzcd}$$

covering (18). This is not always possible, just as there are non-orientable manifolds. It is a standard result that such a spin structure exists if and only if the second Stieffel-Whitney class $w_2 \in H^2(M; \mathbb{Z}_2)$ vanishes. Then there are as many different spin stuctures as there are elements of $H^1(M; \mathbb{Z}_2)$.

If we have a manifold with a spin structure, a double cover as in (21), then we can construct a Clifford module, the *spinor bundle* over M. This is an associated bundle to F_{Spin} in the sense discussed above in terms of the 'spin representation' of Spin(n). This can be constructed conveniently using the structure of the Clifford algebra. Let's restrict attention to even dimension $n = 2d$ for reasons alluded to above. Then a form of 'Bott periodicity' is that the complexified Clifford algebra on $\mathbb{R}^{2d}$ is isomorphic to a matrix algebra

3.5.2020.26 (22)
$$\mathbb{C}\mathrm{l}(2d) = \mathrm{Cl}(2d) \otimes \mathbb{C} \simeq \mathrm{GL}(2^d; \mathbb{C}).$$

This follows by induction over n from a direct proof for $d = 1$ which also shows that $\mathbb{C}\mathrm{l}(2d) \simeq \mathrm{GL}(2) \otimes \mathbb{C}\mathrm{l}(2(d-1))$.

The other ingredient is that we can find $\mathrm{Spin}(2d) \subset \mathbb{C}\mathrm{l}(2d)$ as a subgroup of the group of invertible elements. This means that (22) gives a representation, a group homomorphism of Spin($2d$) into GL(2^d). So, as noted above, this means we construct a bundle

3.5.2020.27 (23)
$$S = F_{\text{Spin}} \times_{\text{Spin}(2d)} \mathbb{C}^{2^d}$$

and the Clifford module structure comes along with the construction.

The representation of Spin($2d$) is unitary and there is a 'Levi-Civita' connection on S which is automatically a unitary Clifford connection. So this defines the spin Dirac operator

3.5.2020.28 (24)
$$\eth_{\text{Spin}} : \mathcal{C}^\infty(M; S) \longrightarrow \mathcal{C}^\infty(M; S)$$

as a self-adjoint, $\mathbb{Z}_2$-graded differntial operator. Thus Theorem 13 applies. In fact this in some sense a 'generating' example in that the formula (16) reduces in this case to

3.5.2020.29 (25)
$$\mathrm{ind}(\eth^+_{\text{Spin}}) = \int_M \widehat{A} = \widehat{A}(M).$$

At the time that Atiyah and Singer proved (25) it was know that the '$\widehat{A}$ genus' $\widehat{A}(M)$ was a rational number attached to each compact oriented manifold (it reverses sign under change of orientation). It was also known that in the case of a spin manifold it was an integer, as follows of course from (25). What this formula does is give a clear and compleling explanation (as well as a proof) of this.

Once we have one Clifford module we can construct others. In fact if V is a Clifford module and W is any smooth complex vector bundle then $V \otimes W$ is also a Clifford module, where the Clifford action just looks like the identity on W. Moreover, if W has an hermitian inner product and a unitary connection then, together with the data on V, these give a unitary Clifford connection on $V \otimes W$. This is one example of 'twisting' (although lots of things are described as 'twisted'). So if we start from the spin bundle S and twist by some bundle W we get a new Dirac opeator $\eth_{\mathrm{Spin},W}$ and then Atiyah and Singer proved the more general formula

3.5.2020.30
$$\operatorname{ind}(\eth^+_{\mathrm{Spin},W}) = \int_M \operatorname{Ch}(W)\widehat{A}. \tag{26}$$

Here $\operatorname{Ch}(W)$ is the Chern character of the bundle. Namely if the curvature of the connection on W defined by its action on any section w

3.5.2020.33
$$\omega(X,Y)w = \nabla_X\nabla_Y w - \nabla_Y\nabla_X w - \nabla_{[X,Y]}w \tag{27}$$

it follows that $\omega \in \mathcal{C}^\infty(M;\Lambda^2 \otimes \hom(W))$ since $\omega(X,Y)(fw) = f\omega(X,Y)fw$ for any $f \in \mathcal{C}^\infty(M)$ then

3.5.2020.31
$$\operatorname{Ch}(W) = \sum_j \frac{1}{j!(2\pi i)^j} \operatorname{tr}_W(\omega \wedge \omega \cdots \wedge \omega) \tag{28}$$

is a sum (formally infinite) of closed $2j$-forms.

Now, going back to the general formula (16) the form $\operatorname{Ch}'(V)$, the twisting Chern character, is defined by the same formula (28) except that ω is replaced by ω' defined by decomposing

3.5.2020.32
$$\hom(V) = \hom'(V) \otimes \mathbb{C}\mathrm{l}(M) \tag{29}$$

with $\hom'(V)$ having fibre the homomorphisms which commute with the Clifford action. This corresponds to the fact that locally any Clifford module decomposes as $V = W \otimes S$ for a local spinor bundle.

The generalization of the Toeplitz index theorem made by Atiyah and Singer is to elliptic pseudodifferential operators. We now consider two complex vector bundles, V and W over a compact manifold M and a pseudodifferential operator

5.5.2020.3
$$A : \mathcal{C}^\infty(M;V) \longrightarrow \mathcal{C}^\infty(M;W). \tag{30}$$

Recall that we defined the space of pseudodifferential operators $\Psi^m(M;V,W)$, of order m, in terms of their Schwartz kernels

5.5.2020.1
$$\Psi^m(M;V,W) = J^m(M^2, \mathrm{Diag}; \operatorname{Hom}(V,W) \otimes \Omega_R). \tag{31}$$

The 'right density' factor is needed to integrate and the boundle $\operatorname{Hom}(V,W)$ over M^2 has fibre $\hom(V_y, W_x) = W_x \otimes V'_y$ at (x,y). The action of A on $u \in \mathcal{C}^\infty(M;V)$ is given by pulling u back to a section of π_R^*V over M^2, multiplying the kernel by it 'on the right' which gives an element of $J^m(M^2, \mathrm{Diag}; \pi_L^*V \otimes \Omega_R)$ which can then be pushed forward to a section of $\mathcal{C}^\infty(M;W)$ corresponding to integration over the right factor of M.

The symbol of A was defined in terms of the Fourier transform of the classical conormal singularity at the diagonal and this determines the leading part and so gives a short exact sequence

5.5.2020.2
$$\Psi^{m-1}(M;V,W) \longrightarrow \Psi^m(M;V,W) \xrightarrow{\sigma} \mathcal{C}^\infty_m(T^*M\setminus 0_M;\pi^*\hom(V,W)) \tag{32}$$

where the image space consists of the smooth maps, homogeneous of degree m, from $T^*M\setminus 0_M$, the complement of the zero section of this vector bundle, into the pull-back of $\hom(V,W)$ from the base. Thus on the fibre at each $m\in M$ the symbol is a map from $T^*_mM\setminus 0\longrightarrow \hom(V_m,W_m)$ which is homogeneous of degree m.

Ellipticity of A is equivalent to everywhere invertibility of $\sigma(A)$. This implies in particular that V and W must have the same rank and is equivalent to the fact that π^*V and π^*W are isomorphic as bundles over $T^*M\setminus 0_M$ – with the isomorphism given by $\sigma(A)$. If they are isomorphic then there is an isomorphism of any given homogeneity (just take a metric and restrict a given isomorphism to the unit sphere bundle and then extend it to be homogeneous). We know that the ellipticity of A implies that it is Fredholm as an operator (5.5.2020.3 30). The converse is not true (look up hypoellipticity!) but ellipticity is equivalent to A being Fredholm as an operator on Sobolev spaces

5.5.2020.4
$$A: H^{s+\operatorname{Re} m}(M;V)\longrightarrow H^s(M;W) \tag{33}$$

for one s and hence all $s\in\mathbb{R}$.

So, the question is – what is the index of A? Since perturbations by lower order terms induce compact perturbations in (5.5.2020.4 33) we can see that the index must be determined by the 'geometric data' M, V, W and $\sigma(A)$. It is always possible to find an invertible psuedodifferential operator of any desired order with a scalar symbol on V or W. To do this for order $p>0$ just take $|\xi|^p\operatorname{Id}$ for a Riemann metric as symbol and add a big positive constant to $\frac12(B+B^*)$ where B has symol $|\xi|^p$ and check invertibility. The inverse works for $p<0$. This allows us to multiply A by an invertible operator such that the composite is of order 0 – so we may as well assume that $\sigma(A)$ is homogeneous of degree 0.

5.5.2020.5 **Theorem 14** (Atiyah+Singer)**.** *The index of an elliptic operator $A\in\Psi^0(M;V,W)$ is*

5.5.2020.6
$$\operatorname{ind}(A)=\int_{T^*M}\widetilde{\operatorname{Ch}}(\sigma(A))\operatorname{Td}(M). \tag{34}$$

Here $\operatorname{Td}(M)$ is another characteristic class on M given by the same prescription as above but now for the power series

5.5.2020.7
$$\frac{x}{1-e^{-x}} \tag{35}$$

and $\widetilde{\operatorname{Ch}}(\sigma(A)$ is a cohomology characters with compact support on T^*M obtained as the the difference of Chern classes of a bundle $E(V,W,\sigma)$ and π^*W.

Let me comment on the bundle E. This is a vector bundle over T^*X which is isomorphic to V near 0_M, the zero section, and isomorphic to W outside a compact set – so in $|\xi|>C$ for any Riemann metric and some $C>0$. In fact we can use σ, the symbol, to define a subbundle of $\pi^*V\oplus\pi^*W$ with fibre at each point (x,ξ) the image of the fibre $V_x \ni v$

5.5.2020.8 (36) $E(x,\xi)$ is the image of $V \ni v \longmapsto$

$$\begin{cases} v & |\xi| \le 1 \\ \cos(\pi(|\xi|-1)v + \sin(\pi(|\xi|-1))a(x,\xi)v & 1 \le |\xi| \le 2 \\ a(x,\xi)v & |\xi| \ge 2. \end{cases}$$

So the fibre is just V_x for $|x| \le 1$ and is W_x for $|x| \ge 2$ since $a(x,\xi)$ is surjective. I have only made it continuous but smoothing the rotation a little easily makes the subbundle smooth. Now, if we give V and W connections we can easily construct a connection on E which reduces to that of W in $|x| > 3$. Then the Chern characters, of E and W are equal as forms outside a compact set and

5.5.2020.9 (37) $$\widetilde{\mathrm{Ch}}(\sigma(A)) = \mathrm{Ch}(E) - \mathrm{Ch}(W) \in \mathcal{C}^\infty_{\mathrm{c}}(T^*M; \Lambda^{\mathrm{even}}).$$

The difference of Chern characters here represents a property of the Chern character, namely that

L.86 (38) $$\mathrm{Ch}(E_1 \oplus E_2) = \mathrm{Ch}(E_1) + \mathrm{Ch}(E_2)$$

How to think about these theorems. Note that the Dirac theorem is contained in the pseudodifferential one, except that the answer is different. Of course it only *looks* different. So one approach is just to prove the more general formula and then work out how it specializes. On the other hand there is a rather direct, essentially computational (as opposed to the earlier arguments which can be described as 'by exhaustion'), proof of the Dirac theorem and with some topological gymnastics this can be used to deduce the general case from the particular.

Still, it is important to understand what these theorems are 'really about'. This can be seen by looking at one of our generalizations of the pseudodifferential calculus above – the simplest one looking at families of operators with respect to a fibration $\phi : M \longrightarrow Y$ of compact manifolds. Then we know what a smooth family of operator $A \in \Psi^m_\phi(M; V, W)$ is, for bundles V and W over M. This just amounts to a pseudodifferential operator acting from sections of the restriction of V to sections of the restriction of W for each fibre Z_y and 'varying smoothly' with $y \in Y$. In particular the symbol of A is just the collection of symbols so there is no question about ellipticity, it just means the invertibility of the symbol $\sigma(A)$ as a homogeneous section of the homomorphism bundle over the complement of the origin in the cotangent bundle of each fibre.

Such an elliptic family has finite dimensional null space and a finite dimensional complement to the range over each fibre. In the case of family of Hodge operators corresponding to a family of fibre metrics we know, via the Hodge theorem, that the dimension of the null spaces is constant and hence they form a bundle over Y. Note that all *sections* of this bundle are elements of the null space of the family, so this null space is not by any means finite-dimensional in general. What happens in the general case of an elliptic family?

L.87 **Proposition 17.** *An ellitpic family of smoothing operators $A \in \Psi^m_\phi(M; V, W)$ can be perturbed by a smoothing family $B \in \Psi^{-\infty}_\phi(M; V, W)$ so that the fibre null spaces of $A + B$ have constant dimension and hence form a vector bundle over the base of the fibration, the same is then true for the adjoint $(A^* + B^*)$ with respect to any positive definite smooth inner products.*

Thus the range of the family $A+B$ in $\mathcal{C}^\infty(M;W)$ has a complement which is all the smooth sections of a bundle over Y.

Of course the question immediately arises as to how much the null bundles of $A+B$ and A^*+B^* depend on B. It is easy enough to see that small changes of B lead to null bundles which are isomorphic (as do changes of the inner products used to define the adjoints). Certainly this is not the case for big changes. The null bundle is a bundle over Y where the fibre is embedded smoothly in $\mathcal{C}^\infty(Z_y;V_y)$ for each $y\in Y$. Suppose we had another bundle $F\longrightarrow Y$ with such an embedding (in fact the embedding part is always possible). Suppose additionally that the range of the embedding of F does not meet the null space of $A+B$. Then consider the (family of) orthogonal projection(s) π_F onto F as a subbundle of $\mathcal{C}^\infty(M,V)$ for some choice of inner products. This is a finite rank smoothing operator so $(A+B)(\operatorname{Id}-\pi_F)=A+B-(A+B)\pi_F)$ is another smoothing perturbation of A and clearly its null space is $(A+B)\oplus F$ where E. The range has been reduced by $(A+B)\pi_F$ which is F embedded as a subbundle of $\mathcal{C}^\infty(M;W)$. This is really the general case.

L.88 **Proposition 18.** *If B_1 and B_2 are two smoothing perturbations of an elliptic family A with both perturbed families having fibre null spaces of constant dimension then there exists a bundle F over Y, and a bundle isomorphism over Y*

L.89
$$\operatorname{null}(A+B_1)+\operatorname{null}(A^*+B_2^*)+F\simeq\operatorname{null}(A+B_2)+\operatorname{null}(A^*+B_1^*)+F. \tag{39}$$

This suggests the construction, due to Grothendieck, of a linear space (actually a ring) from the vector bundles over a compact manifold – this can be generalized enormously.

L.90 *Definition* 4. The (complex topological) K-group $K^0(Y)$ of a compact manifold has elements the equivalence classes of pairs of (smooth) vector bundles (N^+,N^-) over Y under the equivalence relation $(N_1^+,N_1^-)\simeq(N_2^+,N_2^-)$ If there exist a bundle F and bundle isomorphism

L.91
$$N_1^++N_2^-+F\simeq N_2^++N_1^-+F. \tag{40}$$

This equivalence relation is called 'stable isomorphism'.

One thing to note or recall is that any vector bundle over a compact manifold can be complemented to a trivial bundle. The basic step here is to embed it as a subbundle of a trivial bundle, then take the orthocomplement with respect to an Hermitian structure.

L.92 **Lemma 7.** *For any complex vector bundle E over a compact manifold Y there is a smooth bundle injection*

L.93
$$E\longrightarrow Y\times\mathbb{C}^N \tag{41}$$

for a sufficiently large N.

Proof. This is elementary. By definition a vector bundle is locally trivial, so Y is covered by open sets over which E has a smooth basis. The compactness means that we can take a finite cover by open sets U_k over each of which there is a smooth basis $e_{i,k}$ where $1\le i\le r$ runs up to the rank of E. Now, a point in E over U_k is a finite linear combination of the $e_{i,k}$ and so defines the trivialization map

L.94
$$\mu_k:E\big|_{U_k}\longrightarrow U_k\times\mathbb{C}^r. \tag{42}$$

Choose a partition of unity ψ_k subordinate to the open cover and then defined

L.95 (43)
$$\mu : E \ni v \longmapsto \oplus_k \psi_k(\pi(v))v \in \bigoplus_k \mathbb{C}^r.$$

This is a smooth injective bundle map into the trivial bundle of rank rp where p is the number of elements of the cover. $\square$

Since we can add a complementary bundle for E to both sides of (40) this means the equivalance relation is the same if we require E to be trivial.

One of the objects that survives in $K^0(Y)$ is the rank of the bundle

L.96 (44)
$$\dim : K^0(Y) \longrightarrow \mathbb{Z},\ \dim(N^+, N^-) = \operatorname{rank}(N^+) - \operatorname{rank}(N^-).$$

For the case that Y is a point this is all that survives

L.97 (45)
$$\dim : K^0(\mathrm{pt}) \longrightarrow \mathbb{Z} \text{ is an isomorphism.}$$

As I said above, the tensor product of bundles also survives in K-theory to turn $K^0(Y)$ into a ring but we need to use the 'graded' tensor product

L.98 (46)
$$(N_1^+, N_1^-) \otimes (N_2^+, N_2^-) = (N_1^+ \otimes N_2^+ + N_1^- \otimes N_2^-, N_1^+ \otimes N_2^- + N_1^- \otimes N_2^+).$$

Now $K^0(Y)$ defines (the even part of) a generalized cohomology theory. What does this mean, well look it up. Anyway from the point of view of index theory it elucidates what is going on in the index theorem. This is made even more relevant by the

L.100 **Theorem 15** (Atiyah+Hirzebruch). *The Chern character extends to a multiplicative map*

L.101 (47)
$$\mathrm{Ch} : K^0(Y) \longrightarrow H^{even}_{\mathrm{dR}}(Y)$$

which is an isomorphism modulo torsion, i.e. passing to $K^0(Y) \otimes \mathbb{R} \longrightarrow H^{even}_{\mathrm{dR}}(Y)$ gives an isomorphism.

Now, so far I have only talked about K-theory for a compact manifold whereas in (37) we have two bundles over the non-compact manifold T^*M. The standard definition of the K-theory over a non-compact manifold is in terms of pairs of bundles which are isomorphic outside a compact set, with a given isomorphism. This is usually couched in terms of the null space of the map from the K-theory of the one-point compactification to the integers given by the dimension at the point at infinity. What requires a little work, but is not really hard, is that in our case of a bundle ${}^\phi T^*M$, the dual of ${}^\phi TM$, the triple (V, W, a) of a pair of bundles over the base, Y, and an isomorphism between them 'near infinity' on ${}^\phi TM$ determine the full K-theory in this sense. The families index we have discussed above, defined in terms of the regularization of the null bundle for an elliptic family defined by a, is independent of choices as an element of K-theory and defines a map

L.102 (48)
$$\mathrm{ind} : K^0({}^\phi T^*M) \longrightarrow K^0(Y).$$

The index theorem in this case states that this map is 'the natural one'

L.103 **Theorem 16** (Atiyah+Singer). *The families index map (48) is the natural push-forward, or Gysin, map in K-theory defined by the fibre-orientation of ${}^\phi T^*M$ given by the natural class of almost complex structures.*

How to come to grips with such a theorem? Clearly one needs to understand what thus Gysin, or wrong-way, map is. This is the standard proof, to follow the construction if the Gysin map in terms of quantization, which is the (highly ill-defined) process of turning a symbol into an operator. Rather than do that directly, I will proceed to manipulate the problem to make it easier to handle analytically.

The first step is to consider the *odd* version of K-theory. For regular cohomology, of instance deRham cohomology

L.104
$$H^*_{\mathrm{dR}}(Y \times \mathbb{T}) = H^*_{\mathrm{dR}}(Y) \otimes H^*(\mathbb{T}) \tag{49}$$

for the circle $\mathbb{T}$ and any manifold Y. Moreover we know that $H^*_{\mathrm{dR}}(\mathbb{T})$ is $\mathbb{R} \oplus \mathbb{R}$ with a generator in dimension 0 and one in dimension 1. So in particular

L.105
$$H^{\mathrm{even}}_{\mathrm{dR}}(Y \times \mathbb{T}) = H^{\mathrm{even}}_{\mathrm{dR}}(Y) \oplus H^{\mathrm{odd}}_{\mathrm{dR}}(Y) = H^*_{\mathrm{dR}}(Y) \tag{50}$$

and we use this as a guide by anticipating that

L.106
$$K^0(Y \times \mathbb{T}) = K^0(Y) + K^1(Y). \tag{51}$$

To actually define $K^1(Y)$ this way we need to check some things. First we have two maps $\pi : Y \times \mathbb{T} \longrightarrow Y$, just projection, and $i : Y \longrightarrow Y \times \mathbb{T}$ being the inclusion as $Y \times \{1\}$ for instance. Pull-back under these maps gives

L.107
$$K^0(Y) \xrightarrow{\pi^*} K^0(Y \times \mathbb{T}) \xrightarrow{i^*} K^0(Y) \tag{52}$$

and the composite is the identity, since $\pi \circ i = \mathrm{Id}$. Thus the second map is surjective and the first map must be injective. We define

L.108
$$K^1(Y) = (i^* : K^0(Y \times \mathbb{T}) \longrightarrow K^0(Y)). \tag{53}$$

To justify (L.106 51) we need to show that $K^1(Y)$ is a complement to $K^0(Y)$ in $K^0(Y \times \mathbb{T})$. This we can see directly and in the process more is revealed.

L.109 **Proposition 19.** *The odd K-theory, $K^1(Y)$, of a manifold is naturally isomorphic to the space of equivalence classes of smooth maps $Y \longrightarrow \mathrm{GL}(n, \mathbb{C})$, equal to the identity outside a compact set, under smooth homotopy and stabilization by the inclusion $\mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(n+N, \mathbb{C})$ by adding an identity block.*

Note that this is one justification for Atiyah's statement that K-theory is the topolgy of the unitary group (where by polar decomposition $\mathrm{GL}(n, \mathbb{C})$ is homotopy equivalent to $\mathrm{U}(n)$).

For a non-compact manifold all the homotopies etc should reduce to the identity outside a compact set.

Proof. □

In some sense the identification (L.108 53) is 'half-way' to Bott periodicity:

L.116 **Proposition 20** (Bott periodicity)**.** *For any manifold the stanard orientation of $\mathbb{R}^2$ induces a natural isomorphism*

L.117
$$K^0(Y \times \mathbb{R}^2) = K^0(Y). \tag{54}$$

It is actually important here that we include non-compact manifolds, both for applications and also for the proof because of Atiyah's 'rotation trick'.

Proof. There are three steps here, the first being Proposition 19 which gives an isomorphism

L.118 (55)
$$K^0(Y\times\mathbb{R}^2)\longmapsto K^1(Y\times\mathbb{R})$$

for any manifold Y. The secons step is to define a map

L.119 (56)
$$K^1(Y\times\mathbb{R})\longrightarrow K^0(Y).$$

For this we can use a families version of 'Toeplitz quantization' as in Proposition 11.

That is, we consider a smooth map $a : Y\times\mathbb{R}\longrightarrow \mathrm{GL}(N,\mathbb{C})$ for a manifold Y and some N, with $a=\mathrm{Id}$ on the complement of $K\times[-T,T]$ for some $K\Subset Y$ and $T\in\mathbb{R}$. We can reinterpret this as a smooth map $Y\times\mathbb{T}\longrightarrow \mathrm{GL}(N,\mathbb{C})$ which reduces to the identity on $Y\times N$ for a neighbourhood N of $1\in\mathbb{T}$ and outside $K\times\mathbb{T}$ for some compact subset of Y. Then passing to the Toeplitz family

L.120 (57)
$$\pi_H a\pi_H$$

□

In fact from an analytic point of view we can do even better than Proposition 19. For any complex vector bundle W over a compact manifold Z set

L.112 (58)
$$G^{-\infty}(Z;W)=\{\mathrm{Id}+A\in\mathcal{B}(L^2(Z;W)); A\in\Psi^{-\infty}(Z;W),\ \mathrm{Id}+A \text{ is injective}\}.$$

In fact injectivity here is equivalent to invertibility with inverse of the same form, $\mathrm{Id}+B$ with $B\in\Psi^{-\infty}(Z;W)$, so this is a group. In fact we can see that all the finite-dimensional general linear groups inject into it

L.113 (59)
$$\mathrm{GL}(N;\mathbb{C})\hookrightarrow G^{-\infty}(Z;W)\ \forall\ N$$

for instance by taking the matrices to act on an N-dimensional space of eigenfunctions of some elliptic differential or pseudodifferential operator.

L.114 **Proposition 21.** *The groups $G^{-\infty}(Z;W)$ are classifying spaces for odd K-theory: For any compact manifold with corners there is a natural isomorphism*

L.115 (60)
$$K^1(Y)=\mathcal{C}^\infty(Y;G^{-\infty}(Z;W))/\simeq$$

where the equivalence relation is smooth homotopy – two smooth maps are homotopic in this sense if they are the boundary maps for some smooth map $[0,1]\times Y\longrightarrow G^{-\infty}(Z;W)$.

Proof. Again this is relatively elementary. Using the convergence of the Fourier-Bessel series for the eigenbasis of an elliptic operator we can see that any element of $\mathcal{C}^\infty(Y;G^{-\infty}(Z;W))$ is homotopic to one into some $\mathrm{GL}(N;\mathbb{C})$ as in (59) and that this induces stable homotopy equivalence. □

Now finally, I can get to the point of this excursion. Namely

L.110 **Proposition 22.** *Semiclassical quantization on the fibres of a fibration of compact manifolds defines a map*

L.111 (61)
$$\mathrm{ind}_{\mathrm{sl}} : K^1({}^\phi T^*M)\longrightarrow K^1(Y).$$

Well, that is a little bald. Here what we are really saying is that

References

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