Let me start with an example of a Lie algebra of smooth vector fields which is geometrically relevant but is not a Lie algebroid. Suppose $M$ is a manifold and $S$ is a closed embedded submanifold. For simplicity I will assume that $S$ is compact but that doesn’t really matter. So, near each point of $\bar{m} \in S$ there are local coordinates on $M$, $t_1, \ldots, t_k, s_1, \ldots, s_{n-k}$, in a coordinate neighbourhood $U \ni \bar{m}$ such that

$$S \cap U = \{ t_1 = \cdots = t_k = 0 \}.$$  

So $k$ is the codimension of $S$ and the $s_i$ give local coordinates on $U \cap S$.

Consider

$$W_S = \{ W \in C^\infty(M; TM); W \text{ is tangent to } S \}.$$  

This just means that in each coordinate system such that (1) holds $W s_i$ also vanishes at $S$. So consider the ideal

$$\mathcal{I}_S = \{ u \in C^\infty(M; TM); u \big|_S \in 0 \}.$$  

Then by definition

$$\text{for } V \in C^\infty(M; TM), \ W \in W_S \iff W \mathcal{I}_S \subset \mathcal{I}_S.$$  

If $k > 1$, so $S$ is not a hypersurface, the ‘issue’ is that $\mathcal{I}_S$ is not a principal ideal and consequently $W$ is not a Lie algebroid. This means that there is no vector bundle over $M$ of which $W_S$ it is all sections. Indeed in the local coordinates above

$$W \in W \iff W = \sum_{j=1}^{n-k} a_j(s, t) \partial_{s_j} + \sum_{i=1}^k b_i(s, t) t^i \partial_{t_i}$$  

where the coefficients are arbitrary smooth functions. The problem is that they are not linearly independent. Indeed if $c$ is any smooth function then adding $t_k c$ to $a_{kl}$ and subtracting $t_i c$ from $a_{kl}$ leaves $W$ unchanged.

However, we can resolve the Lie algebra to a Lie algebroid.

**Proposition 1.** If $S \subset M$ is a closed embedded submanifold of a manifold then there is a manifold with boundary $\tilde{M}$ and a smooth map (‘blow-down’) $\beta : \tilde{M} \longrightarrow M$ which restricts to a diffeomorphism of $\tilde{M} \setminus \partial \tilde{M}$ to $M \setminus S$ and is such that

$$\beta^* \mathcal{I}_S \text{ spans } \mathcal{I}_0(\tilde{M}) \text{ over } C^\infty(\tilde{M}).$$  

Furthermore, $\tilde{M}$ is essentially unique in the sense that given another such manifold and map $\beta' : \tilde{M}' \longrightarrow M$ the map $\beta^{-1} \beta' : \tilde{M}' \setminus \partial \tilde{M}' \longrightarrow \tilde{M} \setminus \partial \tilde{M}$ extends to a diffeomorphism $\tilde{M}' \longrightarrow \tilde{M}$. A smooth vector field on $M$ is $\beta$-related to a smooth...
vector field on $\tilde{M}$ if and only if it is tangent to $S$, the ‘lift’ is then unique and $W_S$ lifts to span $V_0(\tilde{M})$ over $C^\infty(\tilde{M})$.

I will denote the blown-up manifold by $[M; S]$. We can say more about it. First, we can identify the boundary explicitly. The ideal $I_s$ defines a subbundle of the cotangent bundle restricted to $S$,

$$\{df; f \in I_s\}|_S = N^*S \subset T^*_S M.$$  

This is the conormal bundle to $S$. In local coordinates as above this is the span of the $dt_i$ at each point of $s$. The annihilator is the tangent bundle of $S$

$$T_S M \supset TS = \{v \in T_S M; v \cdot df = 0, \ df \in N^*S\}.$$  

The normal bundle to $S$, $NS$ can be identified either as the dual bundle of $N^*S$ or, in view of (8), as the quotient

$$NS = T_S M / TS.$$  

As a vector bundle we can identify its ‘radial’ sphere bundle as the quotient by the $S^+$-action

$$SNS = (T\ S \setminus 0) / R^+$$  

so the space of half-lines through the origin in the fibres. Then

The boundary of $[M; S]$ is naturally identified with $SNS$, the spherical normal bundle to $S$.