Abstract. I had originally intended to postpone the discussion of Lie algebroids but it now seems to me that it would be better to discuss the abstract notion now.

So a Lie algebroid over a manifold $M$ is the following data

(LA1) A (real) vector bundle $E \longrightarrow M$ over $M$
(LA2) A smooth bundle map $a : E \longrightarrow TM$ (called the anchor)
(LA3) A Lie algebra structure on $\mathcal{C}^\infty(M;E)$, the space of sections of $E$, so an antisymmetric map (bilinear over constants)

\[ [\cdot, \cdot] : \mathcal{C}^\infty(M;E) \times \mathcal{C}^\infty(M;E) \longrightarrow \mathcal{C}^\infty(M;E) \]

satisfying the Jacobi identity

\[ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \forall \quad X, Y, Z \in \mathcal{C}^\infty(M;E). \]

(LA4) The compatibility condition that

\[ a([X, fY]) = Xf \cdot a(Y) + fa([X, Y]) \quad \forall \quad X, Y \in \mathcal{C}^\infty(M;E), f \in \mathcal{C}^\infty(M). \]

Note that the smoothness in (LA2) means that $a \in \mathcal{C}^\infty(M;\text{Hom}(E,TM))$ so if $X \in \mathcal{C}^\infty(M;E)$ then $a(X) \in \mathcal{C}^\infty(M;TM)$ is a smooth vector field. It follows from (3) (taking $f = 1$) that $a$ is a Lie algebra map

\[ a([X, Y]) = [a(X), a(Y)]. \]

Thus the image $a(\mathcal{C}^\infty(M;E))$ is a Lie subalgebra of $\mathcal{C}^\infty(M;TM)$.

Here are some examples, including the ones of immediate interest.

- First a trivial example. If $\mathfrak{g}$ is a finite-dimensional (real) Lie algebra then it defines a Lie algebroid over a point.
- The obvious example of $E = TM$ with $a$ the identity maps
- Combining these two consider $E = TM \oplus \mathfrak{g}$ for a Lie algebra $\mathfrak{g}$. The Lie algebra structure on sections $X = V + v$, $v \in \mathcal{C}^\infty(M;\mathfrak{g})$ is

\[ [V + v, W + w] = [V, W]_TM + Vw - Wv + [v, w]_\mathfrak{g}. \]

I will talk about a special case of this below.

- Now, an important example that I have already been talking about.
  If $\phi : M \longrightarrow Y$ is a fibre bundle – for instance a submersion between compact manifolds – then we can take

\[ E_m = \text{Nul}(\phi_* : T_m M \longrightarrow T_{\phi(m)} Y) \implies E \subset TM \]

is the subbundle of vector fields tangent to the fibres of $\phi$. Thus the Lie algebra structure on

\[ \mathcal{C}^\infty(M;E) = \{ V \in \mathcal{C}^\infty(M;TM); V\phi^* f = 0 \quad \forall \quad f \in \mathcal{C}^\infty(Y) \} \]
is the restriction of the commutator for vector fields to the subspace — clearly $[X,Y]φ^*f = X(Yφ^*f) - Y(Xφ^*f) = 0$ is $X, Y ∈ C^∞(M;E)$. The anchor map is the inclusion map.

• A more substantial example, that is very closely related to the Dirac operator on (the radial compactification of) Euclidean space that we talked about earlier and that I still need to come back to.

Let $M$ be a (usually compact but it doesn’t matter here) manifold with boundary. Then we look at the space

$$V_b(M) = \{V ∈ C^∞(M;TM); V \text{ is tangent to the boundary}\}.$$  

(8)

This is somewhat similar to the preceding example in that there is always a ‘boundary defining function’ on a manifold with boundary — $x ≥ 0, ∂M = \{x = 0\}$ and $dx ≠ 0$ at $∂M$. Near a boundary point such a function can be extended to a local coordinate system

$$x, y_1, \ldots, y_{n-1}, \ n = \dim M$$

(9)

where the $y_i$’s are tangential coordinates — they induce coordinate on the boundary near the point. Then the defining condition in (8) can be written

$$Vx|_{x=0} = 0 \text{ or } Vx ∈ xc^∞(M).$$

(10)

Either way it is clear that $V_b(M)$ is a Lie subalgebra of $C^∞(M;TM)$. We need to find $E!$ Over the interior $E = TM$. Near a boundary point we can use coordinates of the form (9) with the boundary locally defined by $x = 0$. Then any smooth vector field on $M$ is locally of the form

$$V = a(x,y)∂_x + \sum_{i=1}^{n-1} b_i(x,y)∂_{y_i}.$$  

(11)

Since $∂_{y_i}x = 0$ and $∂_x x = 1$ this must satisfy

$$a(0,y) = 0 \implies a(x,y) = xα(x,y), \ α \text{ smooth}$$

(12)

if $V ∈ V_b(M)$. Conversely, if (11) holds near every boundary point then $V ∈ V_b(M)$. Thus $V ∈ V_b(M)$ if near every boundary point in coordinates (9) it is of the form

$$V = α(x,y)(x∂_x) + \sum_{i} b_i(x,y)∂_{y_i}.$$  

(13)

This shows how to define $E = bTM$. In coordinates a basis is $x∂_x, ∂_{y_i}$.

Exercise 1. Check what happens to this basis under change of coordinates between two systems of the form (9). The answer of course is that it defines a smooth bundle map between the local coordinate bases.

This is an important example (the ‘$b$’ stands for boundary). The Lie algebra $V_b(M) = C^∞(M; bTM)$ really does replace $C^∞(M;TM)$ when $∂M ≠ ∅$ in the sense that it is the Lie algebra of the diffeomorphism group.

• The example of immediate interest is somewhat between these two. Namely for a fibration $φ : X → Y$ between (let’s say compact but it is not crucial at this stage) manifolds we consider the ‘adiabatic algebroid’ on

$$M = X × [0,1).$$

(14)
So, the space of sections of the ‘putative’ bundle \( E \to M \) satisfies two conditions corresponding to the two fibrations \( \pi : M \to [0,1) \), and \( \phi : X \to Y \), fixing \( \phi : M \to Y \) where I don’t change the notation.

\[
\mathcal{V}_{\text{ad}}(M) = \{ V \in \mathcal{C}^\infty(M; TM); V\epsilon = 0 \text{ and } V(\phi^*f) \in \epsilon \mathcal{C}^\infty(M) \}.
\]

So, in \( \epsilon > 0 \) the second condition is void and the first means that \( V \) must be of the form

\[
V = \sum_k e_k(m,\epsilon)\partial_{m_k}
\]

just an \( \epsilon \) dependent vector field on \( M \). This corresponds to fibre vector fields for the fibration \( \pi \). Now, near any point we can introduce coordinates in \( X \), and hence \( M \), corresponding to the fibration \( \phi \) so

\[
V = \sum_{i=1}^k a_i(z,y,\epsilon)\partial_z + \sum_{j=1}^m b_j(z,y,\epsilon)\partial_y, \ m = \dim Y, \ k = \dim Z.
\]

Since \( \phi^*f \) is just a function of \( Y \) and \( \partial_z y_j = 0 \) the second condition in (15) just becomes

\[
b_j(z,y,0) = 0 \iff b_j(z,y,\epsilon) = \epsilon \beta_j(z,y,\epsilon)
\]

for smooth functions \( \beta_j \). Just as before we see that \( \mathcal{V}_{\text{ad}}(M) \) has basis over \( \mathcal{C}^\infty(M) \) namely

\[
E_m = \text{sp}\{\partial_z, \epsilon \partial_y\}.
\]

The Lie algebra structure is again ‘inherited’ from \( \mathcal{C}^\infty(M; TM) \).

That is enough for the moment but there are many more.

I did not do this in lecture but it is quite a good exercise in recalling the construction of the deRham complex. First note what happens for the differential of functions on the manifold \( M \). Let \( E^* \) be the dual bundle of \( E \), so also a smooth vector bundle over \( M \). Locally if \( e_i \) is a local basis for \( E \) over an open set of \( M \) then there is a canonically defined dual basis \( e_i^* \) of \( E^* \). The consider the sum

\[
du = \sum_i a(e_i)u \cdot e_i^*; \ u \in \mathcal{C}^\infty(M).
\]

I will not change the notation for \( d \), although I should call it maybe \( E^d \). It is a well-defined linear differential operator

\[
d : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M; E^*).
\]

I will write out the formulæ for the specific cases of interest.

The important part of the deRham complex is the extension to higher forms, and this really reduces to 1-forms. So we need to define \( d\alpha \) where \( \alpha \in \mathcal{C}^\infty(M; E^*) \) as an antisymmetric section \( d\alpha \in \mathcal{C}^\infty(M; E^* \wedge E^*) \). The key here is the standard formula for differential of a 1-form on a manifold:-

\[
d\alpha(V,W) = V(\alpha(W)) - W(\alpha(V)) - \alpha([V,W]), \ V, W \in \mathcal{C}^\infty(M; TM).
\]

Dropping to a coordinate basis you can check that this is correct – in particular it is linear over mutlication \( V \mapsto fV, f \in \mathcal{C}^\infty(M) \). So, we just extend (22) using the
Lie algebra structure on sections of $E$ and define

\begin{equation}
E^d \alpha(V, W) = E^d(\alpha(W))(V) - E^d(\alpha(V))(W) - \alpha([V,W]) \in C^\infty(M; E^2),
\end{equation}

$V, W \in C^\infty(M; E), \alpha \in C^\infty(M; E^*)$.

Here the $E^\Lambda^k$ are the totally antisymmetric parts of the $k$-fold tensor products of the $E^*$. Then

\begin{equation}
C^\infty(M) \xrightarrow{E^d} C^\infty(M; E^*) \xrightarrow{E^d} C^\infty(M; E^\Lambda) \xrightarrow{E^d} \ldots \xrightarrow{E^d} C^\infty(M; E^\Lambda^N) \xrightarrow{E^d} 0, \quad N = \text{rank } E
\end{equation}

is a complex.

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