1. Euclidean Dirac operators

I want to connect with what you did in 18.155 – did everyone here take 155 – by talking about constant coefficient Dirac operators. Why are we interested in these? First, they are a ‘model’ for the variable coefficient case and secondly they are a model for non-compact manifolds. Let’s think about the second of these first although perhaps the first is more fundamental!

What is a constant-coefficient Dirac operator? It is a constant coefficient differential operator acting, on some $\mathbb{R}^n$, on vector-valued functions, so taking values in some $\mathbb{C}^N$, of first order and (for the moment at least) having no lower order terms:

\[(1) \quad \bar{\partial} = \sum_{j=1}^{n} \Gamma_j D_j.\]

Here $\Gamma_j$ are constant $N \times N$ matrices and the $D_j$ are the basic differential operators with and $i$:

\[(2) \quad D_j = \frac{1}{i} \frac{\partial}{\partial x_j}.\]

The $i$ is inserted (by some people, including me ..) so that the Fourier transform does not have an $i$:

\[(3) \quad \hat{D_j} u(\xi) = \xi_j \hat{u}\]

if for instance $u$ is a Schwartz function.

For a Dirac operator we further require that the square be scalar

\[(4) \quad \bar{\partial}^2 = \sum_{j,k=1}^{n} \Gamma_j \Gamma_k D_j D_k = 1d Q(D).\]

Here $Q$ must be a homogeneous polynomial of degree 2. There are two very standard special cases (and there are others), where $Q$ is positive-definite and where $Q \equiv 0$. Both are of interest but let’s concentrate on the ‘elliptic’ case where $Q$ is positive definite. We can always make a linear change of coordinates so that

\[(5) \quad Q = |\xi|^2.\]

Then $Q(D)$ is the standard Euclidean Laplacian.

Now, as you already know the condition (4) written in terms of the matrices as

\[(6) \quad \Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2 \delta_{ij} \]
is related to the Clifford algebra on $\mathbb{R}^n$. I will remind you of this in due course. For the moment it means that we can understand the Dirac operator in terms of the Laplacian.

Now, one attitude to $\partial$ is to think of it as a constant coefficient operator, which of course it is, and elliptic. What does this mean. Well it acts on the standard Sobolev spaces

$$\partial : H^{s+1}(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \forall k \in \mathbb{R}.$$  

As such it is a ‘closed’ operator. Maybe we should care about that, but not for the moment. The bad feature of (7) is that this operator/map is never Fredholm.

Remember that Fredholm is the infinite dimensional analog of invertible for a matrix. It means that at least you know the solvability properties modulo finite dimensional issues. So, why is (7) not Fredholm. Basically because the range is not closed. The easiest way to see this is to find related spaces on which it is Fredholm and then compare.

The ‘secret’ here is that the behaviour of $\partial$ and also $\Delta$ of course, is dominated by the homogeneity rather than the constancy of the coefficients. The latter lets us come to grips with the operator but it is the former that ‘makes things work’. For homogeneity we avoid the origin – since the problem is translation-invariant this just means that we stay away from a point. As you will see what it really means is that we think about what is happening ‘at infinity’.

If $u(tx) = t^\sigma u(x)$ is a smooth vector-valued function in $x \neq 0$ which is homogeneous of degree $r$ then

$$\partial u(t^\sigma x) = t^{\sigma-1} \partial u(x)$$

is homogeneous of degree $r - 1$. That is the problem with (7), our operator is ‘small at infinity’ in the sense that it makes things smaller there. Typical elements of the standard Sobolev spaces all have the same behaviour at infinity – they are roughly $L^2$.

A homogenous function of degree $\sigma$ is just the product of a homogeneous function of degree 0 (vector-valued) with $|x|^\sigma$. The natural thing to do then is to introduced polar coordinates in $\mathbb{R}^n$

$$x = r \omega, \ r = |x|, \ \omega = \frac{x}{|x|} \in S^{n-1}. $$

When we do this our Dirac operator becomes

$$\partial u(x) = M(\omega)(D_r + \frac{i}{r} \partial')$$

where $\partial'$ is a (Dirac, which is why I have written it in this weird way) operator on the sphere. Namely

$$\partial' v(\omega) = r M(\omega) \partial v(\omega)|_{r=1}$$

where on the left $v : S^{n-1} \to \mathbb{C}^N$ and on the right, $v$ is extended to be homogeneous of degree 0 – so of course the $r$ out the front does nothing (and $M^2 = \text{Id}$).

The form of $\partial$ in (10) suggests that we capture the fact that it is small at infinity by writing it (away from the origin) as

$$\partial = r^{-1}M(\omega)(rD_r + i\partial').$$

Taking this seriously we can define modified Sobolev spaces on $\mathbb{R}^n$ which take into account this structure. First we do something that is only book-keeping.
Namely we think of the Euclidean measure as weighted relative to an asymptotically homogeneous measure

\begin{equation}
|dx| = r^n \frac{dr}{r} d\omega, \quad \nu_b = \langle x \rangle^{-n} |dx| \simeq \frac{dr}{r} d\omega, \quad \langle x \rangle = (1 + |x|^2)^{1/2}.
\end{equation}

If we take $L^2$ functions with respect to this new measure and denote the space

\begin{equation}
L^2_b(\mathbb{R}^n) = \{ u \in L^2_{\text{loc}}(\mathbb{R}^n); \int |u|^2 \langle x \rangle^{-n} dx < \infty \}
\end{equation}

then we can write

\begin{equation}
L^2(\mathbb{R}^n) = \langle x \rangle^{-n/2} L^2_b(\mathbb{R}^n).
\end{equation}

So the ordinary $L^2$ space is quite a lot smaller.

Now, for the Sobolev part. If we look at the operators $rD_r$ and $D_\omega$ we can see that they are ‘the same size as’ the homogenous differential operators $x_j D_k$. We define for any integer $k \geq 0$,

\begin{equation}
H^k_b(\mathbb{R}^n) = \{ u \in L^2_b(\mathbb{R}^n); x^\alpha D^\beta u \in L^2_b(\mathbb{R}^n), \ |\alpha| = |\beta| \leq k \}.
\end{equation}

This is actually a cumbersome way if writing something pretty simple. The point is that we can now state a result which was already in the lectures last semester:

**Theorem 1.** An elliptic constant coefficient Dirac operator defines a Fredholm operator

\begin{equation}
\mathfrak{D} : \langle x \rangle^{-\sigma} H^k_b(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \langle x \rangle^{-\sigma-1} H^{k-1}_b(\mathbb{R}^n; \mathbb{C}^N) \forall k, \forall \sigma \notin \mathbb{N}_0 \cup (\mathbb{N}_0 + n - 1) \ (n \geq 2).
\end{equation}

The notation here is that

\begin{equation}
u_b : \langle x \rangle^{-\sigma} H^k_b(\mathbb{R}^n; \mathbb{C}^N) \iff \langle x \rangle^{\sigma} u \in H^k_b(\mathbb{R}^n; \mathbb{C}^N).
\end{equation}

The converse is also true in the sense that the operator is not Fredholm at these integer points.

How to understand, and of course also prove, this theorem. The idea here is that ‘elliptic regularity’ carries over to the derivatives $rD_r$ and $D_\omega$ so that we recover 1 derivative but lose a weight in solving the equation. If you think about it you will see what this cannot be captured in terms of the standard Sobolev spaces in (7).