PROBLEM SET 9, 18.155
DUE 9 DECEMBER, 2016

The last problem set! Laplacian on the sphere. This may be a bit challenging as it stands; hints will appear a bit later this week.

(1) Show that for each fixed $s \in \mathbb{C}$, any continuous function on the sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$ has a unique extension to a function on $\mathbb{R}^n \setminus \{0\}$ which is continuous and homogeneous of degree s.

Hint: The homogeneous extension of $u \in \mathcal{C}^0(\mathbb{S}^{n-1})$ is $\tilde{u}(x) = |x|^s u(\frac{x}{|x|})$.

(2) Prove that every continuous function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree s is the restriction to $\mathbb{R}^n \setminus \{0\}$ of a tempered distribution on $\mathbb{R}^n$.

Hint: Certainly the homogeneous function $f = |x|^s u(x/|x|)$ is a distribution in $\mathbb{R}^n \setminus \{0\}$ and if $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n \setminus \{0\})$ we can introduct polar coordinates to evaluate the pairing as an integral

$$f(\phi) = \int_0^\infty \int_{\mathbb{S}^{n-1}} r^s u(\omega)\phi(r\omega) r^{n-1} d\omega dr$$

$$= \int_0^\infty \int_{\mathbb{S}^{n-1}} r_+^{s+n-1} \Phi(r) dr,$$

$$\Phi(r) = \int_{\mathbb{S}^{n-1}} u(\omega)\phi(r\omega) r^{n-1} d\omega.$$

Now, notice that even for $\phi \in \mathcal{S}(\mathbb{R}^n)$ the function Φ is smooth on $[0, \infty)_r$ and Schwartz at infinity (even if u was a distribution on the sphere). So it suffices to show that for any s there is always a distribution on $\mathbb{R}$ with support in $R \geq 0$ which is equal to r^z in $r > 0$. We showed this earlier in the semester – it is r_+^z unless $z = s + n - 1$ is a negative integer in which case one needs to remove the pole in z, but such a distribution still exists.

(3) Show that

$$\mathcal{C}^\infty(\mathbb{S}^{n-1}) = \mathcal{C}^\infty(\mathbb{R}^n)\big|_{\mathbb{S}^{n-1}} \tag{1}$$

is a $\mathcal{C}^\infty$ structure on $\mathbb{S}^{n-1}$ making it a $\mathcal{C}^\infty$ manifold.

Hint: If you think of point sat $(0, 0, \ldots, 0, 1)$ on the sphere then the functions $x_1, ; x_{n-1} \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ define a coordinate patch as in the definition in $x_n > 0$. You can cover the sphere by 2^n such patches; this shows the first part of the definition. To get the 'maximality' part you just need to know that there is a partition of unity subordinate to this open cover and you can get this from a partition of unity on $\mathbb{R}^n$ subordinate to the corresponding conic open cover.

Show that for each $s \in \mathbb{C}$ $v \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ has a unique extension to an element of $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ which is homogeneous of degree s written below $v(\omega)|x|^s$.

(4) Check that the Euclidean Laplacian $\Delta = \sum_{j=1}^{n} -\partial_j^2$ can be written

$$\Delta = -\partial_r^2 - (n-1)\frac{1}{r}\partial_r + \frac{1}{r^2}\Delta_\omega,\ \Delta_\omega : \mathcal{C}^\infty(\mathbb{S}^{n-1}) \longrightarrow \mathcal{C}^\infty(\mathbb{S}^{n-1}) \tag{2}$$

in $|x| > 0$ if we identify $\mathbb{R}^n \setminus \{0\} = (0,\infty)_r \times \mathbb{S}^{n-1}_\omega$ where $r = |x|$ and $x = r\omega$.

Hint: You do *not* need to work out what Δ_ω looks like in local coordinates, just show that it is well-defined by this identity, i.e. that there is such a (differential) operator.

(5) Suppose that $\Delta_\omega v = \lambda v$, $\lambda \in \mathbb{C}$ and $v \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ show that $v|x|^s$ is a harmonic, i.e. in the null space of the Euclidean Laplacian, on $\mathbb{R}^n \setminus \{0\}$.

Hint: Just apply Δ to a homogeneous function of degree s which is an eigenfunction of Δ_ω on the sphere and see the condition that it be harmonic – it is a quadratic equation in s and λ.

(6) Show that the only such eigenfunctions of Δ_ω are the restrictions to $\mathbb{S}^{n-1}$ of a polynomial.

Hint: The extension result above shows that Δv, where $v = |x|^s u(x)$ is from the preceding question, is harmonic outside the origin. The tempered extension therefore has $\Delta\tilde{v}$ equal to a sum of derivatives of δ at the origin. You can argue that these have to vanish and so $\tilde{v}$ has to be a polynomial and this gives an integrality condition on λ.