PROBLEM SET 7, 18.155
DUE NOVEMBER 4, 2016

This week I ask you to think about *unbounded self-adjoint operators* (I mentioned these earlier). By definition this means a linear map

$$A : D(A) \longrightarrow H$$

where $D(A) \subset H$ is a dense linear subspace of a (separable) Hilbert space H and in addition we assume/require symmetry:

$$(Au, v) = (u, Av) \ \forall \ u, v \in D(A)$$

and an adjoint condition:

$$\text{if } v \in H \text{ and } D \ni u \longmapsto (Au, v) \text{ extends to be continuous on } H \text{ then } v \in D(A).$$

Q1 Show that if $P(\xi)$ is a real and elliptic polynomial of degree m in n variables and $V \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is real-valued then

$$P(D) + V(x) : H^m(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

is an unbounded self-adjoint operator (with $H^m(\mathbb{R}^n)$ as domain).

Hint: Use the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$ to get the symmetry condition. If $v \in L^2(\mathbb{R}^n)$ satisfies the adjoint condition then $(P(D) + V)v \in L^2(\mathbb{R}^n)$ and hence $v \in H^m(\mathbb{R}^n)$ by elliptic regularity.

Q2 If A is unbounded self-adjoint show that
(a) The graph of A is closed in $H \times H$.
(b) The spectrum, defined as

$$\operatorname{Spec}(A) = \{z \in \mathbb{C}; A + z \operatorname{Id} : D(A) \longrightarrow H \text{ is not a bijection}\}$$

is a closed subset of $\mathbb{R}$.
(c) $R(z) = (A + z \operatorname{Id})^{-1} \in \mathcal{B}(H)$ for $z \in \mathbb{C} \setminus \operatorname{Spec}(A)$ is holomorphic in z.

Hint:
(a) If $(v_n, A(v_n))$ in the graph and $(v_n, Av_n) \to (v, f)$ in $H \times H$ and $u \in D(A)$ then by the by the symmetry condition

$$(Au, v) = \lim_n (Au, v_n) = \lim_n (u, Av_n) = (u, f) \tag{1}$$

extends by continuity to $u \in H$ so $v \in D(A)$ and $Av = f$ (by symmetry) showing that the graph is closed.

(b) By symmetry if $u \in D(A)$ then (Au, u) is real. Certainly $A + z : D(A) \longrightarrow H$ and $\operatorname{Im}((A+z)u, u) = \operatorname{Im} z \|u\|^2$ so $A + z$ is injective. If $u_n \in D(A)$ and $f_n = Au_n \to f$ in H then $|\operatorname{Im} z| \|u_n\|^2 = \operatorname{Im}(f_n, u_n) \le \|f_n\| \|u_n\|$. Applying the same inequality to $u_n - u_m$ (and assuming z is not real) it follows that u_n is Cauchy and hence converges and so $(A + z)$ is a bijection from $D(A)$ to H. Thus the spectrum is contained in $\mathbb{R}$. Whenever $A + z$ is a bijection it has an 'alegbraic' inverse, $R(z)$, the graph of which is the 'reverse' of the graph of $A+z$ so as a map $H \longrightarrow H$ $R(z)$ is bounded since its graph is closed. If $z \notin \operatorname{Spec}(A)$ then $R(z)(A+z+t) = \operatorname{Id} + tR(z)$ and similarly for the opposite composition so for $|t|$ small $A+z+t$ is also a bijection. Thus the sectrum is closed and real.

(c) The Neumann series for $\operatorname{Id} + tR(z)$ converges for small t and shows that $R(z)$ is holomorphic on the open set where it is defined.

Q3 If A is unbounded self-adjoint let $R(i) = UB$ be the polar decomposition of $R(i)$.

(a) Show that $D(A) = \operatorname{Ran}(B)$.

(b) If E is the spectral subspace for B corresponding to the interval $(-\infty, \frac{1}{2}]$ then

$$\begin{gathered} D(A) = E^{\perp} + \operatorname{Ran}(B\big|_E), \\ A : E^{\perp} \longrightarrow E^{\perp} \text{ is bounded and self-adjoint} \\ A : \operatorname{Ran}(B\big|_E) \longrightarrow E \text{ is a bijection and} \\ A^{-1} : E \longrightarrow E \text{ is bounded and self-adjoint.} \end{gathered}$$

Hint:

(a) Since $R(i)$ has range $D(A)$ it follows from the proof of the polar decomposition that U is unitary – it is a bijection on H. Taking the adjoint $R(-i) = BU^*$ shows that the range of B is equal to the range of $R(-i)$ is $D(A)$. Now $R(i)$ commutes with A and hence with $A-i$ and so with $R(-i) = R(i)^*$ (so $R(i)$ is normal) and hence $R(i)$ commutes with $B^2 = R(-i)R(i)$ and so with B. [Maybe there is an easier way to see this!]

(b) Any bounded operator that commutes with B maps E to itself and $E^\perp$ to itself (from the definition of the E as a spectral subspace for B). It follows that $R(i)$ does this and is invertible on $E^\perp$ with inverse $A+i$; it follows that $E^\perp \subset D(A)$ and that $A : E^\perp \longrightarrow E^\perp$ is bounded and self-adjoint.

(c) I leave the last part to you!

Q4 Show how to define $f(A) \in \mathcal{B}(H)$ if A is unbounded self-adjoint and $f \in \mathcal{C}_0(\mathbb{R};\mathbb{R})$ is real valued and vanishes at infinity to give a continuous linear map

$\mathcal{C}_0(\mathbb{R};\mathbb{R}) \ni f \longrightarrow f(A) \in \mathcal{B}(H)$ s.t.

$$f(A)g(A) = (fg)(A),\ B = (A^2+1)^{-\frac{1}{2}}$$

where B is given in (Q3).

Hint: Define it separate on E and $E \perp$.

Q5 For the special case of (Q1), $\Delta + V$, $P(\xi) = |\xi|^2$, show that

$$\operatorname{Spec}(\Delta + V) \subset [M, \infty),\ M = \inf_{x \in \mathbb{R}^n} V(x).$$

Hint: Just use the fact that $V - M \geq 0$ to show that $A = \Delta - M + V$ is unbounded self-adjoint with domain $H^2(\mathbb{R}^n)$ and $(Au, u) \geq 0$ so A has no spectrum in $(-\infty, 0)$.

Q6-Opt Continue (Q5) to show that the part of the spectrum below 0 (if any) consists of a finite number of eigenvalues of finite multiplicity.

Q7-Opt Under the same conditions as (Q5) show that $\operatorname{Spec}(\Delta + V) \supset [0, \infty)$.