18.155 Pset 7 Partial Solutions

Ethan Y. Jaffe

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3. (a) I'll sketch two methods of doing this.

   i. We first show that U is unitary. Indeed, U is an isometrey from $H = \overline{\operatorname{im} B} \to \overline{\operatorname{im} R(i)} = H$.[1] Thus
   So, $D(A) = \operatorname{im} R(-i) = \operatorname{im} R(i)^* = \operatorname{im} BU^* = \operatorname{im} B$.

   ii. That $D(A) \subseteq \operatorname{im} B$ is clear:

   $$R(-i) = R(i)^* = BU^*,$$

   and so $D(A) = \operatorname{im} R(-i) \subseteq \operatorname{im} B$. For the reverse inculusion, observe that for $u \in D(A), v \in H$,

   $$\begin{aligned}\langle Au, Bv\rangle &= \langle (A-i)u, Bv\rangle + i\langle u, Bv\rangle \\ &= \langle (A-i)u, B^*v\rangle + i\langle u, Bv\rangle \\ &= \langle (A-i)u, (U^*R(i))^*v\rangle + i\langle u, Bv\rangle \\ &= \langle (A-i)u, R(i)^*U^*v\rangle + i\langle u, Bv\rangle \\ &= \langle u, U^*v\rangle + i\langle u, Bv\rangle,\end{aligned}$$

   and so the functional $u \mapsto \langle Au, Bv\rangle$ has a bounded extension to H, and so $Bv \in D(A)$.[2]

   (b) We will start off by proving a few facts about spectral subspaces. These are all easy to prove if one has a Borel functional calculus coming from a projection-valued measure, as briefly mentioned in the L14 notes. However we will argue in a more direct way from the definition of $E = E_{1/2}$ given therein.

   Lemma 3.1. *Suppose L is a bounded operator commuting with B. Then $L : E \to E$ and $E^\perp \to E^\perp$. In particular, this is true of $L = B$.*

[1] This argument has many forms. The present one, which I think is the shortest, is due to Donghao Wang.
[2] The argument written like this was taken from Julien Clancy.

Proof. To show that $L : E \to E$,we need to show that if $\chi \in C_c^0((1/2,\infty))$, and $\langle u, \chi(B)v\rangle = 0$ for all v, then $\langle Lu, \chi(B)v\rangle = 0$, too. Since L commutes with $\chi(B)$ taking adjoints shows that this last expression is $\langle u, \chi(B)Lv\rangle = 0$.

We will give two arguments to show that $L : E^\perp \to E^\perp$. One simple, and another one which will also be used below.

i. Since B is self-adjoint[3], L^* commutes with B, too. If $u \in E^\perp$, then for all v,

$$\langle Lu, \chi(B)v\rangle = \langle u, \chi(B)L^*v\rangle = 0,$$

so $L : E^\perp \to E^\perp$, too.

ii. It is a general fact that

$$\left(\bigoplus_j C_j\right)^\perp = \bigcap_j C_j^\perp,$$

and taking complements on both sides,

$$\overline{\bigoplus_j C_j} = \left(\bigcap_j C_j^\perp\right)^\perp.$$

In particular this works if $C_j = \mathrm{im}(\chi(B))^\perp$. So suppose $u \in E^\perp$. This then happens if and only if

$$u = \lim_n \sum_{j_n} \chi_{j_n}(B)v_{j_n},$$

where $\chi_{j_n} \in C_c^0(1/2,\infty)$. Since L commutes with $\chi_{j_n}(B)$, it follows that $Lu \in E^\perp$. So $L : E^\perp \to E^\perp$.

□

Lemma 3.2. *B is invertible from $E^\perp \to E^\perp$.*

Proof. Set $\varphi \in C_0^0(1/3,\infty)$ to be any function satisfying $\varphi(t) = 1/t$ for $t \geq 1/2$. Then

$$\chi(B)\varphi(B)B = B\varphi(B)\chi(B) = \chi(B)$$

(since $t\varphi(t)\chi(t) = \chi(t)$), and $\varphi(B)$ is bounded. If $u \in E^\perp$, then by the above,

$$u = \lim_n \sum_{j_n} \chi_{j_n}(B)v_{j_n} = B\varphi(B)\left(\lim_n \sum_{j_n} \chi_{j_n}(B)v_{j_n}\right) = B\varphi(B)(u).$$

Similarly, $\varphi(B)Bu = u$. Since $\varphi(B)$ commutes with B, it preserves $E^\perp$, and so $\varphi(B)|_{E^\perp} = B|_{E^\perp}^{-1}$. □

[3] The argument written like this was taken from Jesse Freeman.

Lemma 3.3. *Suppose $B-\lambda$ is invertible. Then if $\chi \in C_c^0(\mathbf{R})$ is supported in a sufficiently small neighbourhood of λ, $\chi(B)=0$.*

Proof. For simplicity, we assume $\lambda = 0$. Set $\varphi(t) = \chi(1/t)$. Then $\varphi \in C_b^0(\mathbf{R})$ (after setting $\varphi(0) = 0$). We claim that $\chi(B) = \varphi(B^{-1})$. It clearly holds for $\chi = 1$. We show it for $\chi(t) = (t-z)^{-1}$ for $z \in \mathbf{C} \setminus \mathbf{R}$. This will be sufficient since by the Stone-Weierstrass theorem for functions vanishing at infinity, such functions are dense in $C_0^0(\mathbf{R})$. Indeed, taken $\chi_n \to \chi$ uniformly. Then $\varphi_n \to \varphi$ uniformly, too. Then $\chi_n(B) \to \chi(B)$ and $\varphi_n(B) \to \varphi(B)$ since the assignment is continuous, and so $\chi(B) = \varphi(B)$.
If $\chi(t) = (t-z)^{-1}$, the $\varphi(B^{-1}) = B^{-1}(1-zB^{-1})^{-1}$. This is a well-defined operator since z is not realy and B^{-1} is self-adjoint, so $1/z$ is not in its spectrum. It is each to check that $B-z$ is an inverse to this. Since we already know $\chi(B-z) = (B-z)^{-1}$, we deduce that $\chi(B) = \varphi(B^{-1})$.
Now, if χ is supported close to 0, φ is supported close to infinity. By definition of the functional calculus, as soon as φ is not supported in $[-\|B^{-1}\|, \|B^{-1}\|]$, $\varphi(B^{-1})$ vanishes, and thus so does $\chi(B)$.[4] □

Lemma 3.4. $\|B|_E\| \leq 1/2$.

Proof. Let $\varphi \in C_c^0(\mathbf{R})$ be identically 1 on $[-\varepsilon, 1/2]$ and supported in $[a,b] \supseteq [0,1/2]$. Set $\psi(t) = t\varphi(t)$. By the functional calculus, $\|\psi(B)\| \leq \max(|a|,|b|)$. We will show that in fact $\psi(B) = B|_E$. Approximating $1_{[0,1/2]}$ with $\varphi \in C_c^0(\mathbf{R})$ then proves the lemma.
If $v \in E$, then for $\chi \in C_c^0([1/2,\infty))$, $\chi(B)v \in E$ since $\chi(B)$ commutes with B. But

$$\langle \chi(B)v, \chi(B)v \rangle = 0$$

since $\chi(B)v \in E$. So $v = (1-\chi)(B)v$. Let $\kappa \in C^0(\mathbf{R})$ be any function which is 0 for $t \leq a$, and 1 for $t \geq 0$. We want to show that $\kappa(B)v = v$. Indeed, B is positive so $B - t$ is invertible for all $a < 0$. Using a partition of unity and the previous lemma, we check that $(1-\kappa)(B)v = 0$, which is sufficient.
So, $v = (\kappa(1-\chi))(B)v$ if $v \in E$. We may choose κ, χ so that $\kappa(1-\chi)$ is 1 on $[a,b]$, and so $(\kappa(t)(1-\chi(t)))\psi(t) = t(\kappa(t)(1-\chi(t)))$. It follows immediately that

$$Bv = B(\kappa(1-\chi))(B)v = \psi(B)v.$$

□

We now return to proving the four required facts.
First, we need to show that $\operatorname{im} B = E^\perp + \operatorname{im}(B|_E)$. It is clear that $\operatorname{im} B = \operatorname{im}(B|_{E^\perp}) + \operatorname{im}(B|_E)$. By definition, $B : E^\perp \to E^\perp$. Now B is invertible on $E^\perp$, so $\operatorname{im}(B|_{E^\perp}) = E^\perp$.

[4] It would be interesting to give an elementary proof which just used the elementary properties of the functional calculus (i.e. without using the Borel functional calculus) rather than its construction.

For the rest, we begin by showing that various operators commute with each other. It is clear that, at least on $D(A)$, $A-i$ and $R(i)$ commutes, and thus so do A and $R(i)$, and thus so do $R(-i)$ and $R(i)$. Since $R(-i)=R(i)^*$, we conclude that $R(i)$ and $R(i)^ast$ commute, at least on $D(A)$. Since both are bounded, they commute on H. In particular, $R(i)$ commutes with $R(i)^*R(i)=B^2$, and thus $R(i)$ commutes with $\sqrt{B^2}=B$, too. The same reasoning show that A commutes with B, at least on $D(A)$.

In particular, $R(i)$ preserves E, and $E^\perp$. Thus $R(i):E^\perp\to E^\perp$ is bounded. We next show that $A-i:E^\perp\to E^\perp$. The domain makes sense by part (a). We now show that the codomain is correct. If $u\in E^\perp$, write $(A-i)u=v+w$, where $v\in E^\perp$ and $w\in E$. Applying $R(i)$ to both sides and using that $R(i)$ maps E to E and $E^\perp$ to $E^\perp$ means that $u-R(i)v=R(i)w$, where the left-hand side is in $E^\perp$, and the right-hand side is in E. Thus $R(i)w=0$, and $w=0$, i.e. $(A-i)u=v\in E^\perp$.

Thus $A-i=R(i)^{-1}$ as a map $E^\perp\to E^\perp$ is bounded by the open mapping theorem, and as a consequence so is A. That A is self-adjoint follows by symmetry on $D(A)$ and that A is bounded.

Next we show that $A:\mathrm{im}(B|_E)\to E$. Suppose $u\in E$, and $v\in E^\perp$. Then

$$\langle ABu,v\rangle=\langle Bu,Av\rangle$$

by symmetry (since $E^\perp$, $\mathrm{im}(B|_E)\subseteq D(A)$). But the first argument is in E and the second in $E^\perp$, and so $ABu\in E^{\perp\perp}=E$. We just need to show that it is bijective. First, notice that the graph $\Gamma(A|_{\mathrm{im}(B|_E)})$ is closed, being $\Gamma(A)\cap\{E\times H\}$.

We give two arguments to show that $A:\mathrm{im}(B|_E)\to E$ is a bijection, with bounded inverse, which both ultimately depend on the same observation. We observe for future reference that $\mathrm{im}(B|_E)$ is dense in E. This is because if it were, then $D(A)$ would also not be dense. Also, $D(A)\cap E=\mathrm{im}(B|_E)$. This is because $D(A)=\mathrm{im}(B|_E)+E^\perp$ is an orthogonal decomposition.

i. First, notice that the graph $\Gamma(A|_{\mathrm{im}(B|_E)})$ is closed, being $\Gamma(A)\cap\{E\times H\}$. Next, observe that, at least on $D(A)$,

$$(AB)^2=A^2B^2=1-B^2.$$

Indeed, this follows since

$$(A^2+1)B^2=(A+i)(A-i)BB=1.$$

Both $1-B^2$ and $(AB)^2$ make sense as maps $E\to E$. We thus wish to prove that they are the same as maps $E\to E$.

$1-B^2$ is clearly bounded as a map on E. Since A has closed graph on $\mathrm{im}(B|_E)$ and B is bounded, $AB:E\to E$ has closed graph, and is thus bounded. So $(AB)^2$ extends to a bounded operator, too. Since $\mathrm{im}(B|_E)$ is dense, we therefore have that $(AB)^2 = 1 - B^2$ as maps $E \to E$, and not just on $D(A)$. But $\|B\| \leq 1/2$ on E by the lemma. So $1 - B^2 = (AB)^2$ is invertible $E \to E$, and so is AB. This means that $A : \mathrm{im}(B|_E) \to E$ is bijective.

Its inverse A^{-1} maps $E \to E$. Its graph is closed, being just a homeomorphism $((x, y) \mapsto (y, x))$ applied to the graph of $A|_{\mathrm{im}(B|_E)}$, which we know to be closed. Thus A^{-1} is bounded by the closed graph theorem.

ii. Observe[5]

$$A = (A + i)(1 - iR(i)),$$

at least as maps $D(A) \to H$. Like in the first proof, $\|R(i)\| \leq \|B\| < 1$ on E. So $(1 - iR(i))$ is invertible from $E \to E$. Moreover, $(1 - iR(i)) : \mathrm{im}(B|_E) \to \mathrm{im}(B|_E)$ since it commutes with B.

A simple formal computation shows that $(1 - iR(i))^{-1}R(-i)A$ is equal to the identity map, and the formal computation is valid when the domain is $D(A) \cap E = \mathrm{im}(B|_E)$. So

$$(1 - iR(i))^{-1}R(-i)A = 1_{\mathrm{im}(B|_E)},$$

and A is injective.

Likewise a formal computation shows that $A(1-iR(i))^{-1}R(-i)$ is equal to the identity map. The formal computation is valid on the preimage of $\mathrm{im}(B|_E)$ under $(1-iR(i))^{-1}R(-i)$. Since $(1-iR(i))^{-1}$ is a bijection mapping $\mathrm{im}(B|_E)$ to itself and $R(-i)$ is a bijection $E \to E \cap D(A) = \mathrm{im}(B|_E)$, the preimage is just all of E. So

$$A(1 - iR(i))^{-1}R(-i) = 1_E.$$

Thus A is surjective, with bounded inverse $(1 - iR(i))^{-1}R(-i) : E \to E$ (one can also show directly self-adjointness, but we will not do that here).

Since A is symmetric on $\mathrm{im}(B|_E)$, to show that A^{-1} is self-adjoint, we need only use that $\mathrm{im}(B|_E)$ is dense in E

4. We give two arguments for this.

   i. Define $f(A) : E^\perp \to E^\perp$ using the usual functional calculus. Set $g(t) = f(1/t)$. Then g is bounded and continuous (really extends to be bounded and continuous). Set $f(A) : E \to E$ by $f(A) = g(A^{-1})$, where the latter is defined using the functional calculus. $f(A)$ is a bounded operator because $g(A^{-1})$ is. Observe that this defines $f(A) : H = E + E^\perp \to H$, which may seem odd since we may have $D(A) \subsetneq H$, but this is in fact what happens so long as f is bounded. That the assignment is an algebra homomorphism and is continuous follows from that for the ordinary calculus, check on $E, E^\perp$ separately.

[5] The idea of this argument is due to Donghao Wang, Tim Large and Sarah Tammen. A small modification was needed to provided to have a complete proof.

Lastly, we need to show that $B = (A^2+1)^{-1/2} = (x^2+1)^{-1/2}(A)$. Since square roots are unique, $B^2 = R(i)R(-i)$, by the homomorphism properties of the functional calculus it suffices to check that $R(\mp i) = (x \pm i)^{-1}(A)$. On $E^\perp$ this is clear since we in fact have a polynomial functional calculus which is consistent with the bounded functional calculus (in the sense that $(fp)(A) = f(A)p(A)$ for a bounded f and polynomial p).

On E, we use our definition:

$$(x \pm i)^{-1}(A) = \left(\frac{x}{1 \pm ix}\right)(A^{-1}) = \frac{A^{-1}}{1 \pm iA^{-1}},$$

which makes sense as an operator on E (notice that $(1+iA^{-1})^{-1}$ is well-defined since A is self-adjoint and i is not in its spectrum). Notice that the second equality follows since we are using the functional calculus on A^{-1}, which we already know to behave how we need it to. That this is actually $R(\mp i)$ is easily checked by multiplying on the left and right by $(A \pm i) : \mathrm{im}(B|_E) \to E$ (and using arguments similar to problem 3 to justify that we expect $R(\mp i) : E \to \mathrm{im}(B|_E)$).

Notice that we used a complex-valued functional calculus here. Of course this follows from the real-valued calculus, or alternatively the proof of the real calculus also applies to the complex-valued functional calculus if one uses an extension of Weierstrass approximation.

ii. We use that unitary operators also have a functional calculus.[6] The proof of this is analogous to the one that self-adjoint operators have a functional calculus, except one uses the unit circle $S^1 \subseteq \mathbf{C}$ instead of an interval $[-\|A\|, \|A\|]$ in the proof.

Observe that $(A-i)R(-i)$ is a bijection $H \to H$. Also, it is an isometry. Indeed for $u \in H$, using that $A+i$ and $A-i$ commute with each other and are each others adjoints,

$$\langle (A-i)R(-i)u, (A-i)R(-i)u\rangle = \langle (A+i)R(-i)u, (A+i)R(-i)u\rangle = \langle u, u\rangle.$$

Thus $(A-i)R(-i)$ is an isometry.

Let $\varphi(x) = \frac{x-i}{x+i}$. One checks that $\varphi : \mathbf{R} \to S^1$ is continuous, with inverse $\psi : S^1 \setminus \{1\} \to \mathbf{R}$ given by $\psi(y) = i\frac{y+1}{1-y}$. If $f \in C_c^0(\mathbf{R})$, set $\Phi(f) \in C^0(S^1)$ by $\Phi(f)(y) = f(\psi(y))$, and $\Phi(f)(1) = 0$. We define

$$f(A) := \Phi(f)((A-i)R(-i)).$$

It is then clear from the functional calculus from unitary operators that the assignment is a continuous homomorphism.

Like above we check that $R(\mp i) = (x \pm i)^{-1}(A)$. Using our definition,

$$(x-i)^{-1}(A) = \frac{1-(A-i)R(-i)}{2i(A-i)R(-i)}.$$

[6]This argument is due mostly to Kaavya Valiveti.

Observe that

$$(A-i)R(-i) = (A+i)R(-i) - 2iR(-i),$$

so the above is

$$((A-i)R(-i))^{-1}R(-i) = R(i).$$

Similarly,

$$(x+i)^{-1}(A) = \frac{1-(A-i)R(-i)}{2i} = R(-i).$$