PROBLEM SET 6, 18.155
DUE 28 OCTOBER, 2016

Now that I have written it out, this problem set seems pretty easy! Maybe you should try Q6 or Q7 as well. You might also enjoy reading about 'Eilenberg's Swindle', it is pretty uncommon to see this terminology for a theorem!

A bounded operator, $P \in \mathcal{B}(H)$, on a (separable, infinite dimensional) Hilbert space is semi-Fredholm if it has closed range and one (at least) of $\mathrm{Nul}(P)$ and $\mathrm{Nul}(P^*)$ is finite dimensional.

Q1 If $R \subset H$ is a closed subspace with a finite dimensional orthocomplement, and $R \subset R_1 \subset H$ is another subspace, show that R_1 is closed and has finite dimensional complement.

Q2 Suppose P is semi-Fredholm and has range with finite dimensional complement, show that

(a) There exists a right inverse, an operator $Q \in \mathcal{B}(H)$ such that

$$PQ = \mathrm{Id} - \Pi_{\mathrm{Nul}(P^*)},\ \Pi_{\mathrm{Nul}(P^*)} \text{ a projection of finite rank.}$$

(b) Conversely, show that if $P \in \mathcal{B}(H)$ and there exist $Q' \in \mathcal{B}(H)$ which is a right parametrix modulo compact operators in the sense that

$$PQ' = \mathrm{Id} - K,\ K \in \mathcal{K}(H), \tag{1}$$

then P is semi-Fredholm with range of finite codimension.

Q3 Using such a parameterix (or otherwise ...) show that the set of semi-Fredholm operators is open.

Hint: For the ones with closed range of finite codimension, expand the composite $(P + A)Q$ and observe that $(\mathrm{Id} + AQ)$ is invertible if $\|A\|$ is small enough; apply the inverse on the right (to the identity you found) and deduce the result from the previous question. For the others, think adjoints.

Q4 Show that P is Fredholm iff P and P^* are operators as in Q2 and conclude from Q3 that the set of Fredholm operators is open.

Q5 Show that if $A \in \mathcal{B}(H)$ is self-adjoint, Fredholm but not invertible, then its index vanishes and 0 is an isolated point in its spectrum.

Q6-optional Show that $\mathrm{GL}(H)$ is a (path) connected metric space (it is actually contractible.)

Hint: Use the polar decomposition to reduce the question to $\mathrm{U}(H)$, the group of unitary operators. To connect a unitary operator to the identity through unitary (or just invertible) operators use the spectral theorem to write the self-adjoint part as the difference of two non-negative self-adjoint operators which commute with each other and with the antisymmetric part of the unitary operator. Use these to 'pry it apart'.

Q7-optional Show that an operator is Hilbert-Schmidt (resp of trace class) iff the eigenvaluses (repeated with multiplicty) of its self-adjoint and anti-self-adjoint parts each from a sequence in l^2 (resp l^1); define general Schatten ideals by replacing this by l^p and show that these are Banach spaces with respect to an appropriately chosen norm.